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ANTICIPATED BACKWARD STOCHASTIC VOLTERRA INTEGRAL EQUATIONS WITH JUMPS AND APPLICATIONS TO DYNAMIC RISK MEASURES*

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Abstract In this paper, we focus on anticipated backward stochastic Volterra integral equations (ABSVIEs) with jumps. We solve the problem of the well-posedness of so-called M-solutions to this class of equation, and analytically derive a comparison theorem for them and for the continuous equilibrium consumption process. These continuous equilibrium consumption processes can be described by the solutions to this class of ABSVIE with jumps. Motivated by this, a class of dynamic risk measures induced by ABSVIEs with jumps are discussed.

Key words anticipated backward stochastic Volterra integral equations; comparison theorems; dynamic risk measures

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1 Introduction

Since 1990, when general nonlinear backward differential stochastic equations (BSDEs) were first investigated in the seminal paper [17], BSDEs have attracted the attention of many researchers. For example, [3, 30] developed BSDEs in terms of Poisson jumps, while a class of anticipated BSDEs, where the anticipated generator includes not only the present values of solutions but also the future values of solutions, were studied in [19]. Moreover, these works proved a comparison theorem and a duality between these equations and the stochastic differential delay equations in [19]. For more on BSDEs, see [5, 6, 25, 29] and the references therein.

Backward stochastic Volterra integral equations (BSVIEs), as a natural extension of BSDEs, were studied in [40, 42], where the author was concerned with the following equations:

$$Y(t)=\xi(t)+\int_t^T f\left(t,s,Y(s),Z(t,s),Z(s,t)\right)\mathrm{d}s-\int_t^T Z(t,s)\mathrm{d}W(s),\quad t\in[0,T]. \tag{1.1}$$

The notion of so-called M-solutions to (1.1) was introduced in [40]. This work also established the well-posedness of (1.1) and a maximum principle for optimal controls of stochastic integral equations. The equation (1.1), independent of $Z(s,t)$, was first discussed in [15], and was further extended in terms of BSVIEs with jumps under non-Lipschitz coefficients in [24, 37], and for path-dependent BSVIEs with jumps by [16], where path-dependence means that the terminal condition and generator of BSVIE depend on a path of the càdlàg process. Furthermore, a comparison theorem and a class of dynamic risk measure induced by path-dependent BSVIEs with jumps was derived in [16]. Recently, [21] studied BSVIEs with jumps for some general filtration, which further extended the related theory, including regarding the existence, uniqueness and corresponding comparison result, for BSVIEs with jumps under some general filtration setting, while the existence and uniqueness of anticipated BSVIEs and some comparison theorems were established in [38]. For more related results on BSVIEs, see [11, 26, 28, 31, 35, 43].

In addition to the mathematical extension of BSVIEs, the optimal control theory for BSVIEs is an interesting field and has investigated things, such as stochastic optimal control for forward-backward stochastic Volterra integral equations, see [27, 28, 36]. Recently, under an infinite horizon setting, some stochastic control problems for BSVIEs were established in [12]; for time-inconsistent stochastic optimal control problems, see [33].

As a mathematical finance application for BSDEs, risk measures to quantify the riskiness associated with financial positions were introduced in an axiomatic way in the seminal papers [2, 7, 9]. These researches have been extensively pursued, from a static to a dynamic framework. Particularly in the BSDE setting, some continuous consumption processes/dynamic risk measures, such as some cash flow processes, can be described by the solutions to a kind of classical BSDE which first was established by [17]. Furthermore, some continuous consumption processes/dynamic risk measures, based on this kind of classical BSDEs by random variables ξ (called financial positions, such as some $\mathcal{F}_T$-measurable payoffs of certain European type contingent claims), indicate some time-consistency; see [4, 6, 22, 23]. Meanwhile, BSDEs, like the theory of conditional g-expectation introduced by [18], are a powerful tool for studying some continuous consumption processes/dynamic risk measures; see [10, 14, 39]. For measuring the riskiness of some consumption processes $\{X(t),t\in[0,T]\}$, the corresponding dynamic risk measures for processes, developed in [13, 20], can be described by the solutions to the following BSDEs:

$$Y(t)=\xi+\int_t^T g(s,Y(s)+X(s),Z(s))\mathrm{d}s-\int_t^T Z(s)\mathrm{d}B(s),\qquad t\in[0,T].$$

Compared with the classical BSDE case, BSVIEs (1.1) were used to define risk measures by stochastic processes $\{\xi(t), t\in[0,T]\}$ (called position processes, not $\mathbb{F}$-adapted, only $\mathcal{F}_T$-measurable) in [34, 41]. Furthermore, some general continuous-time equilibrium dynamic risk measures and equilibrium recursive utility processes were proposed in [32]. For the dynamic risk measures induced by BSVIEs with jumps, see [1, 16], and for more studies on risk measures, see [5, 8].

In this paper, we consider the equation

$$\begin{cases} Y(t)=\xi(t)+\int_t^T g\left(t,s,Y(s),Z(t,s),Z(s,t),K(t,s,\cdot),K(s,t,\cdot),Y(s+\delta(s)),Z(t,s+\gamma(s)),\right.\\ \qquad\qquad \left.Z(s+\gamma(s),t),K(t,s+\zeta(s),\cdot),K(s+\zeta(s),t,\cdot)\right)\mathrm{d}s-\int_t^T Z(t,s)\mathrm{d}B(s)\\ \qquad\qquad -\int_t^T\int_{\mathbb{R}^*}K(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e),\quad t\in[0,T];\\ Y(t)=\xi(t),\quad t\in[T,T+K];\\ Z(t,s)=\eta(t,s),\quad K(t,s,\cdot)=\theta(t,s,\cdot),\quad (t,s)\in[0,T+K]^2\setminus[0,T]^2, \end{cases}\tag{1.2}$$

where the terminal condition $\xi(\cdot)$ and the generator g satisfy some given conditions. The above equation is called an anticipated backward stochastic Volterra integral equation (ABSVIE) with jumps. To the best of our knowledge, no study concerning ABSVIEs with jumps has thus far been available. However, in [16, 21], equation (1.2), independently of anticipated terms, demonstrated special cases of BSVIEs. Compared with the BSVIEs in [28, 38]. The main difficulty encountered in this paper is dealing with the jump process and anticipated terms in order to solve (1.2). Thanks to the techniques of [19, 38], we overcome this difficulty.

BSVIEs are important from a pure mathematical point of view, as well as from an application point of view. In financial markets they can be applied to risk measures, capital allocation and so on. For example, in derivative securities, an investor may hold a combination of certain European type contingent claims (which are mature at time T, and thus are only $\mathcal{F}_T$-measurable), some bonds, stocks, and so on, and hence jump dynamics, which might be caused by policy interference, natural accidents, and so on, indeed exist. Meanwhile, for an investor, the wealth process $Y(\cdot)$ may be predicted on the basis of a small time in the future. That is why the generator g of (1.2) involves both present and future information. Compared with the BSVIEs cases in [1, 16, 34, 41], we consider not only the current riskiness of the wealth process but also the corresponding possible riskiness in the future. Thus we will make a more accurate assessment of the riskiness, and that may have broad applications in terms of financial markets. Motivated by this, we construct dynamic risk measures by means of ABSVIE (1.2).

In this paper, we study a class of ABSVIEs with jumps. First, we establish the well-posedness of ABSVIEs with jumps in the sense of an M-solution. Second, a comparison theorem for ABSVIEs with jumps is provided. Finally, we study the dynamic risk measures induced by

ABSVIEs with jumps. The coherent and convex dynamic risk measures induced by ABSVIEs with jumps are discussed. The results obtained extend the results of [1, 35, 38].

The rest of this paper is organiged as follows: In Section 2, we introduce some preliminary notations and some results on BSVIEs with jumps. In Section 3, we prove the existence and uniqueness of solutions for ABSVIEs with jumps. Section 4 contains a comparison theorem for ABSVIEs with jumps. Dynamic risk measures for ABSVIEs with jumps are provided in Section 5.

2 Preliminaries

In this section, we will introduce some preliminaries. Let $(\Omega,\mathcal{F},\mathrm{P})$ be a probability space, and let $T>0$ be a given terminal time. Let $\{B(t),t\in[0,\infty)\}$ be a d-dimensional Brownian motion on this space and let $N(\mathrm{d}t,\mathrm{d}e)$ be a Poisson random measure with the compensator $\nu(\mathrm{d}e)\mathrm{d}t$ such that ν is a σ-finite measure on $\mathbb{R}^*:=\mathbb{R}^l\setminus\{0\}$. Let $\widetilde{N}(\mathrm{d}t,\mathrm{d}e)=N(\mathrm{d}t,\mathrm{d}e)-\nu(\mathrm{d}e)\mathrm{d}t$ be the compensated Poisson random measure. Let $\mathbb{F}=\{\mathcal{F}_t,t\geq 0\}$ be the natural filtration generated by $\{B(t),t\in[0,\infty)\}$ and $\{N(\mathrm{d}t,\mathrm{d}e),t\in[0,\infty),e\in\mathbb{R}^*\}$. Let $\triangle$ and $\triangle^c$ be given by

$$\triangle:=\left\{(t,s)\in[0,T]^2\mid t\leq s\right\}\qquad\text{and}\qquad\triangle^c:=\left\{(t,s)\in[0,T]^2\mid t>s\right\}.$$

For any positive integer m and $z\in\mathbb{R}^m$, $|z|$ denotes its Euclidean norm, and we also define that $|A|=\sqrt{TrAA^*}$ for any $m\times d$ matrix A. Thus we will introduce some spaces for any $t\in[0,T+K]$ and a Euclidean space $\mathbb{H}$.

- $L^2_{\mathcal{F}_t}(\Omega;\mathbb{H}):=\Big\{\xi:\Omega\to\mathbb{H}\mid\xi\text{ is }\mathcal{F}_t\text{-measurable},E[|\xi|^2]<\infty\Big\}$.
- $L^2_{\mathcal{F}_T}([0,T+K];\mathbb{H}):=\Big\{\xi:\Omega\times[0,T+K]\to\mathbb{H}\mid\xi(t)\text{ is }\mathcal{F}_{T\vee t}\text{-measurable},$
  $E\Big[\int_0^{T+K}|\xi(t)|^2\mathrm{d}t\Big]<\infty\Big\}$.
- $L^2_{\mathbb{F}}([0,T];\mathbb{H}):=\Big\{h:\Omega\times[0,T]\to\mathbb{H}\mid h(\cdot)\text{ is }\mathbb{F}\text{-adapted},E\Big[\int_0^T|h(t)|^2\mathrm{d}t\Big]<\infty\Big\}$.
- $L^2_{\nu}(\mathbb{H}):=\Big\{K:\mathbb{R}^*\to\mathbb{H}\mid K(\cdot)\text{ is }\mathcal{B}(\mathbb{R}^*)\text{-measurable},\|K\|^2_\nu:=\int_{\mathbb{R}^*}|K(e)|^2\nu(\mathrm{d}e)<\infty\Big\}$.
- $L^2_N([0,T];\mathbb{H}):=\Big\{K:\Omega\times[0,T]\times\mathbb{R}^*\to\mathbb{H}\mid s\to K(s,\cdot)\text{ is }\mathbb{F}\text{-predictable on }[0,T],$
  $E\Big[\int_0^T\int_{\mathbb{R}^*}|K(s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\Big]<\infty\Big\}$.
- $L^2_{\mathbb{F}}(\triangle;\mathbb{H}):=\Big\{Z:\Omega\times\triangle\to\mathbb{H}\mid s\to Z(t,s)\text{ is }\mathbb{F}\text{-predictable on }[t,T],$
  $E\Big[\int_0^T\int_t^T|Z(t,s)|^2\mathrm{d}s\mathrm{d}t\Big]<\infty\Big\}$.
- $L^2_{\mathbb{F}}([0,T]^2;\mathbb{H}):=\Big\{Z:\Omega\times[0,T]^2\to\mathbb{H}\mid s\to Z(t,s)\text{ is }\mathbb{F}\text{-predictable on }[0,T],$
  $E\Big[\int_0^T\int_0^T|Z(t,s)|^2\mathrm{d}s\mathrm{d}t\Big]<\infty\Big\}$.
- $L^2_N(\triangle;\mathbb{H}):=\Big\{K:\Omega\times\triangle\times\mathbb{R}^*\to\mathbb{H}\mid s\to K(t,s,\cdot)\text{ is }\mathbb{F}\text{-predictable on }[t,T],$
  $E\Big[\int_0^T\int_t^T\int_{\mathbb{R}^*}|K(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\mathrm{d}t\Big]<\infty\Big\}$.
- $L^2_N([0,T]^2;\mathbb{H}):=\Big\{K:\Omega\times[0,T]^2\times\mathbb{R}^*\to\mathbb{H}\mid s\to K(t,s,\cdot)\text{ is }\mathbb{F}\text{-predictable on }[0,T],$
  $E\Big[\int_0^T\int_0^T\int_{\mathbb{R}^*}|K(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\mathrm{d}t\Big]<\infty\Big\}$.

$$\mathcal{H}^2_\triangle[0,T]:=L^2_{\mathbb{F}}([0,T];\mathbb{H})\times L^2_{\mathbb{F}}(\triangle;\mathbb{H})\times L^2_N(\triangle;\mathbb{H}),$$
$$\mathcal{H}^2[0,T]:=L^2_{\mathbb{F}}([0,T];\mathbb{H})\times L^2_{\mathbb{F}}([0,T]^2;\mathbb{H})\times L^2_N([0,T]^2;\mathbb{H}).$$

Similarly, we can define $L^2_{\mathcal{F}_t}(\Omega; L^2_\nu(\mathbb{H}))$, $L^2_{\mathcal{F}_T}([0,T];\mathbb{H})$, $L^2_{\mathbb{F}}([0,T+K];\mathbb{H})$, $L^2_{\mathbb{F}}([0,T+K]^2;\mathbb{H})$, $L^2_N([0,T+K];\mathbb{H})$, $L^2_N([0,T+K]^2;\mathbb{H})$, $\mathcal{H}^2_\triangle[0,T+K], \mathcal{H}^2[0,T+K]$, etc.

For convenience, we recall the known results for BSVIEs with jumps. Consider the equation

$$\begin{aligned} Y(t) = \xi(t) + \int_t^T g\Big(t,s,Y(s),Z(t,s),Z(s,t),K(t,s,\cdot),K(s,t,\cdot)\Big)\mathrm{d}s \\ - \int_t^T Z(t,s)\mathrm{d}B(s) - \int_t^T\int_{\mathbb{R}^*} K(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e), \quad t\in[0,T], \end{aligned} \tag{2.1}$$

where $\xi(\cdot)\in L^2_{\mathcal{F}_T}(0,T;\mathbb{R}^m)$ and $g:\Omega\times\triangle\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^{m\times d}\times L^2_\nu(\mathbb{R}^m)\times L^2_\nu(\mathbb{R}^m)\to\mathbb{R}^m$ with $s\to g(t,s,y,z,\varsigma,k,\vartheta)$ being $\mathbb{F}$-progressively measurable for any $(y,z,\varsigma,k,\vartheta)\in\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^{m\times d}\times L^2_\nu(\mathbb{R}^m)\times L^2_\nu(\mathbb{R}^m)$.

Assumption 0

• $E\Big[\int_0^T\Big(\int_t^T\Big|g(t,s,0,0,0,0,0)\Big|\mathrm{d}s\Big)^2\mathrm{d}t\Big]<\infty$.

• There exists a constant $L>0$ such that, for any $y_1,y_2\in\mathbb{R}^m$, $z_1,z_2,\varsigma_1,\varsigma_2\in\mathbb{R}^{m\times d}$, k_1,k_2,ϑ_1, $\vartheta_2\in L^2_\nu(\mathbb{R}^m)$,

$$\begin{aligned} &\Big|g(t,s,y_1,z_1,\varsigma_1,k_1,\vartheta_1)-g(t,s,y_2,z_2,\vartheta_2,,k_2,\varsigma_2)\Big|^2 \\ &\le L\Big(|y_1-y_2|^2+|z_1-z_2|^2+|\varsigma_1-\varsigma_2|^2+\|k_1-k_2\|^2_\nu+\|\vartheta_1-\vartheta_2\|^2_\nu\Big). \end{aligned}$$

Definition 2.1 A trio of processes $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))\in\mathcal{H}^2[0,T]$ is called an adapted solution of BSVIE (2.1) if (2.1) holds in the usual Itô's sense for a Lebesgue measure for almost every $t\in[0,T]$. Moreover, an adapted solution $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))$ is called an adapted M-solution of (2.1) if, for any $S\in[0,T)$, the following relation holds:

$$Y(t)=E[Y(t)|\mathcal{F}_S]+\int_S^t Z(t,s)\mathrm{d}B(s)+\int_S^t\int_{\mathbb{R}^*}K(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e), \qquad \text{a.e. } t\in[S,T]. \tag{2.2}$$

The next proposition can be seen as a special case of Theorem 2 of [16] (or Proposition 3.12 of [21] in the Itô setting) with a Lipschitz generator. By Lemma 5.3 of [21] (or Lemma 3.1 of [37]), we then easily obtain Proposition 2.3, which concerns the simple BSVIE (2.3) with the generator g being independent of Y and Z.

Proposition 2.2 Let Assumption 0 hold. Then, for any $\xi(\cdot)\in L^2_{\mathcal{F}_T}([0,T];\mathbb{R}^m)$, there exists a unique adapted M-solution $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))\in\mathcal{H}^2[0,T]$ to BSVIE (2.1).

Proposition 2.3 Let $g(\cdot)\in L^2_{\mathbb{F}}(\triangle;\mathbb{R}^m)$ and let $\xi(\cdot)\in L^2_{\mathcal{F}_T}([0,T];\mathbb{R}^m)$. Then the BSVIE

$$Y(t)=\xi(t)+\int_t^T g\Big(t,s\Big)\mathrm{d}s-\int_t^T Z(t,s)\mathrm{d}W(s)-\int_t^T\int_{\mathbb{R}^*}K(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e), \quad t\in[0,T] \tag{2.3}$$

admits a unique solution $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))\in\mathcal{H}^2_\triangle[0,T]$. Moreover, the following estimate is true for some constant $\beta>0$:

$$\begin{aligned} &E\Big[\int_0^T\Big(\mathrm{e}^{\beta t}|Y(t)|^2+\int_t^T\mathrm{e}^{\beta s}|Z(t,s)|^2\mathrm{d}s+\int_t^T\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|K(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\Big)\mathrm{d}t\Big] \\ &\le CE\Big[\int_0^T\mathrm{e}^{\beta T}|\xi(t)|^2\mathrm{d}t\Big]+\frac{C}{\beta}E\Big[\int_0^T\int_t^T\mathrm{e}^{\beta s}|g(t,s)|^2\mathrm{d}s\mathrm{d}t\Big]. \end{aligned} \tag{2.4}$$

Under Theorem 3.5 of [1] and Proposition 2.3, in a manner analogous to the argument of Theorem 3.4 used in [35] (or Theorem 4.2 of [28]), we can get Proposition 2.4 (that can also be seen as a special case of Proposition 6.3 of [21] in the Itô setting), so we omit the details of the proof here.

Proposition 2.4 For $i=1,2$, let $g^i:\Omega\times\triangle\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\to\mathbb{R}$ satisfy Assumption 0. For any terminal conditions $\xi^1(\cdot),\xi^2(\cdot)\in L^2_{\mathcal{F}_T}([0,T];\mathbb{R})$, denote by $\Big(Y^i(\cdot),Z^i(\cdot,\cdot),K^i(\cdot,\cdot,\cdot)\Big)$ the solution of

$$
\begin{aligned}
Y^i(t)=\xi^i(t)+&\int_t^T g^i\Big(t,s,Y^i(s),Z^i(t,s),K^i(t,s,\cdot)\Big)\mathrm{d}s\\
&-\int_t^T Z^i(t,s)\mathrm{d}B(s)-\int_t^T\int_{\mathbb{R}^*}K^i(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e),\quad t\in[0,T].
\end{aligned}\tag{2.5}
$$

Suppose that the following conditions hold:

(i) For each $t\in[0,T]$, $\xi^1(t)\le\xi^2(t)$, a.s.;

(ii) For each $(t,s,y,z,k)\in\triangle\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})$,

$$g^1(t,s,y,z,k)\le\overline{g}(t,s,y,z,k)\le g^2(t,s,y,z,k),$$

where the map $\overline{g}:\Omega\times\triangle\times\mathbb{R}\times\mathbb{R}\times L^2_\nu\to\mathbb{R}$ satisfies Assumption 0 with $y\to\overline{g}(t,s,y,z,k)$ being nondecreasing and, in addition, there exists a bounded predictable process $\Lambda(t,s,e)$ such that, for all $y,z\in\mathbb{R},k,k'\in L^2_\nu(\mathbb{R})$,

$$\overline{g}(t,s,y,z,k)-\overline{g}(t,s,y,z,k')\ge\int_{\mathbb{R}^*}\Lambda(t,s,e)\Big(k(e)-k'(e)\Big)\nu(\mathrm{d}e),$$

with $\Lambda(t,s,e)$ satisfying that

$$\Lambda(t,s,e)>-1,\quad |\Lambda(t,s,e)|\le\Psi(e),\qquad \Psi\in L^2_\nu(\mathbb{R}).$$

Then $Y^1(t)\le Y^2(t)$, a.s., $t\in[0,T]$.

3 Well-Posedness of ABSVIEs with Jumps

In this section, the existence and uniqueness of ABSVIE (1.2) will be studied. For simplicity of presentation, we will consider the following equation instead of (1.2):

$$
\begin{cases}
Y(t)=\displaystyle\int_t^T g\Big(t,s,Y(s+\delta(s)),Z(t,s+\gamma(s)),Z(s+\gamma(s),t),K(t,s+\zeta(s),\cdot),K(s+\zeta(s),t,\cdot)\Big)\mathrm{d}s\\
\qquad\qquad -\displaystyle\int_t^T Z(t,s)\mathrm{d}B(s)-\int_t^T\int_{\mathbb{R}^*}K(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e)+\xi(t),\quad t\in[0,T];\\
Y(t)=\xi(t),\quad t\in[T,T+K];\\
Z(t,s)=\eta(t,s),\quad K(t,s,\cdot)=\theta(t,s,\cdot),\quad (t,s)\in[0,T+K]^2\setminus[0,T]^2.
\end{cases}\tag{3.1}
$$

Here $\delta(\cdot)$, $\gamma(\cdot)$ and $\zeta(\cdot)$ are given $\mathbb{R}^+$-valued continuous functions satisfying that

(i) $K\ge0$ is a constant with, for each $s\in[0,T]$,

$$s+\delta(s)\le T+K,\qquad s+\gamma(s)\le T+K,\qquad s+\zeta(s)\le T+K;$$

(ii) there exists a constant $\sigma\ge0$ with, for all $t\in[0,T]$ and for all non-negative integrable functions $f_i(\cdot),i=1,2,3$,

$$\int_t^T f_1(s+\delta(s))\mathrm{d}s\le\sigma\int_t^{T+K}f_1(s)\mathrm{d}s,\qquad \int_t^T f_2(t,s+\gamma(s))\mathrm{d}s\le\sigma\int_t^{T+K}f_2(t,s)\mathrm{d}s,$$

$$\int_t^T f_2(s+\gamma(s),t)\mathrm{d}s \le \sigma\int_t^{T+K} f_2(s,t)\mathrm{d}s, \quad \int_t^T f_3(t,s+\zeta(s))\mathrm{d}s \le \sigma\int_t^{T+K} f_3(t,s)\mathrm{d}s,$$
$$\int_t^T f_3(s+\zeta(s),t)\mathrm{d}s \le \sigma\int_t^{T+K} f_3(s,t)\mathrm{d}s. \tag{3.2}$$

Assumption 1 For all $(t,s)\in\triangle$, $u_i\in[s,T+K]$, $i=1,2,\cdots,5$, $g:\Omega\times\triangle\times L^2_{\mathcal{F}_{u_1}}(\Omega;\mathbb{R}^m)\times L^2_{\mathcal{F}_{u_2}}(\Omega;\mathbb{R}^{m\times d})\times L^2_{\mathcal{F}_{u_3}}(\Omega;\mathbb{R}^{m\times d})\times L^2_{\mathcal{F}_{u_4}}(\Omega;L^2_\nu(\mathbb{R}^m))\times L^2_{\mathcal{F}_{u_5}}(\Omega;L^2_\nu(\mathbb{R}^m))\to L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m)$ satisfies that

• $E\Big[\int_0^T\Big(\int_t^T\Big|g(t,s,0,0,0,0,0)\Big|\mathrm{d}s\Big)^2\mathrm{d}t\Big]<\infty$;

• there exists a constant $L>0$ such that, for all $(t,s)\in\triangle$, $\psi_1(\cdot),\psi_2(\cdot)\in L^2_{\mathbb{F}}([s,T+K];\mathbb{R}^m)$, $\varphi_1(t,\cdot)$, $\varphi_2(t,\cdot),\phi_1(\cdot,t),\phi_2(\cdot,t)\in L^2_{\mathbb{F}}([s,T+K];\mathbb{R}^{m\times d})$, $\zeta_1(t,\cdot,\cdot)$, $\zeta_2(t,\cdot,\cdot)$, $\varrho_1(\cdot,t,\cdot)$, $\varrho_2(\cdot,t,\cdot)\in L^2_N([s,T+K];\mathbb{R}^m)$,

$$\begin{aligned}&\Big|g(t,s,\psi_1(u_1),\varphi_1(t,u_2),\phi_1(u_3,t),\zeta_1(t,u_4,\cdot),\varrho_1(u_5,t,\cdot))\\&\quad-g(t,s,\psi_2(u_1),\varphi_2(t,u_2),\phi_2(u_3,t),\zeta_2(t,u_4,\cdot),\varrho_2(u_5,t,\cdot))\Big|^2\\&\le L\mathrm{e}^{\mathcal{F}_s}\Big[\Big(|\psi_1(u_1)-\psi_2(u_1)|^2+|\varphi_1(t,u_2)-\varphi_2(t,u_2)|^2+|\phi_1(u_3,t)-\phi_2(u_3,t)|^2\Big)\Big]\\&\quad+L\mathrm{e}^{\mathcal{F}_s}\Big[\Big(\|\zeta_1(t,u_4,\cdot)-\zeta_2(t,u_4,\cdot)\|^2_\nu+\|\varrho_1(u_5,t,\cdot)-\varrho_2(u_5,t,\cdot)\|^2_\nu\Big)\Big].\end{aligned}$$

For convenience, we also denote by $\mathcal{M}^2[0,T+K]$ the set of all $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))\in\mathcal{H}^2[0,T+K]$ with the following relation:

$$Y(t)=E[Y(t)]+\int_0^t Z(t,s)\mathrm{d}W(s)+\int_0^t\int_{\mathbb{R}^*}K(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e), \qquad t\in[0,T+K]. \tag{3.3}$$

Note that, for each $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))\in\mathcal{H}^2[0,T+k]$, we have that

$$\begin{aligned}&E\Big[\int_0^{T+K}\Big(\mathrm{e}^{\beta t}|Y(t)|^2+\int_t^{T+K}\mathrm{e}^{\beta s}|Z(t,s)|^2\mathrm{d}s+\int_t^{T+K}\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|K(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\Big)\mathrm{d}t\Big]\\&\le E\Big[\int_0^{T+K}\Big(\mathrm{e}^{\beta t}|Y(t)|^2+\int_0^{T+K}\mathrm{e}^{\beta s}|Z(t,s)|^2\mathrm{d}s+\int_0^{T+K}\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|K(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\Big)\mathrm{d}t\Big]\\&\le 2E\Big[\int_0^{T+K}\Big(\mathrm{e}^{\beta t}|Y(t)|^2+\int_t^{T+K}\mathrm{e}^{\beta s}|Z(t,s)|^2\mathrm{d}s+\int_t^{T+K}\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|K(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\Big)\mathrm{d}t\Big].\end{aligned}$$

Hence, we can use an equivalent norm in $\mathcal{M}^2[0,T+K]$ defined by

$$E\Big[\int_0^{T+K}\Big(\mathrm{e}^{\beta t}|Y(t)|^2+\int_t^{T+K}\mathrm{e}^{\beta s}|Z(t,s)|^2\mathrm{d}s+\int_t^{T+K}\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|K(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\Big)\mathrm{d}t\Big]. \tag{3.4}$$

We now prove the existence and uniqueness of solutions to ABDSVIE (3.1).

Theorem 3.1 Let Assumption 1 hold. Then, for any $(\xi(\cdot),\eta(\cdot,\cdot),\theta(\cdot,\cdot,\cdot))\in\mathcal{M}^2[0,T+K]$, there exists a unique M-solution $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))\in\mathcal{M}^2[0,T+K]$ to equation (3.1).

Proof For any $(\xi(\cdot),\eta(\cdot,\cdot),\theta(\cdot,\cdot,\cdot))\in\mathcal{M}^2[0,T+K]$, we consider the equation

$$\begin{cases}Y(t)=\xi(t)+\displaystyle\int_t^T\overline{g}(t,s)\mathrm{d}s-\int_t^T Z(t,s)\mathrm{d}B(s)-\int_t^T\int_{\mathbb{R}^*}K(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e), \quad t\in[0,T];\\ Y(t)=\xi(t), \quad t\in[T,T+K];\\ Z(t,s)=\eta(t,s), \quad K(t,s,\cdot)=\theta(t,s,\cdot), \quad (t,s)\in[0,T+K]^2\setminus[0,T]^2,\end{cases} \tag{3.5}$$

where $\overline{g}(t,s) = g\Big(t,s,Y(s+\delta(s)),Z(t,s+\gamma(s)),Z(s+\gamma(s),t),K(t,s+\zeta(s),\cdot),K(s+\zeta(s),t,\cdot)\Big)$. From Propositions 2.2 and 2.3, it is clearly seen that ABSVIE (3.5) admits a unique solution $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot)) \in \mathcal{H}^2_{\triangle}[0,T]$. Now we define $Z(\cdot,\cdot)$ and $K(\cdot,\cdot,\cdot)$ on $\triangle^c$ satisfying the relation (2.2). For any $(\xi(\cdot),\eta(\cdot,\cdot),\theta(\cdot,\cdot,\cdot)) \in \mathcal{M}^2[0,T+K]$, we notice that

$$Y(t)=\xi(t),\ t\in[T,T+K];\quad Z(t,s)=\eta(t,s),\ K(t,s,\cdot)=\theta(t,s,\cdot),\ (t,s)\in[0,T+K]^2\setminus[0,T]^2.$$

Then we obtain that $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot)) \in \mathcal{M}^2[0,T+K]$ is an M-solution to (3.5). Therefore, we can define the map $\Phi:\mathcal{M}^2[0,T+K]\to\mathcal{M}^2[0,T+K]$ by

$$\Phi(y(\cdot),z(\cdot,\cdot),k(\cdot,\cdot,\cdot)) := (Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot)),\quad (y(\cdot),z(\cdot,\cdot),k(\cdot,\cdot,\cdot))\in M^2[0,T+K].$$

Next, we show that $\Phi(\cdot,\cdot)$ is a contractive map under the norm $\|\cdot\|_{\mathcal{M}^2[0,T+K]}$.

For any $(y_1(\cdot),z_1(\cdot,\cdot),k_1(\cdot,\cdot,\cdot)),(y_2(\cdot),z_2(\cdot,\cdot),,k_1(\cdot,\cdot,\cdot))\in\mathcal{M}^2[0,T+K]$, we set

$$\begin{aligned}(Y_1(\cdot),Z_1(\cdot,\cdot),K_1(\cdot,\cdot,\cdot)) &= \Phi(y_1(\cdot),z_1(\cdot,\cdot),k_1(\cdot,\cdot,\cdot)),\\ (Y_2(\cdot),Z_2(\cdot,\cdot),K_2(\cdot,\cdot,\cdot)) &= \Phi(y_2(\cdot),z_2(\cdot,\cdot),k_2(\cdot,\cdot,\cdot)).\end{aligned}$$

Moreover, we denote their differences by

$$\begin{aligned}(\widetilde{y}(\cdot),\widetilde{z}(\cdot,\cdot),\widetilde{k}(\cdot,\cdot,\cdot)) &= \big(y_1(\cdot)-y_2(\cdot),z_1(\cdot,\cdot)-z_2(\cdot,\cdot),k_1(\cdot,\cdot,\cdot)-k_2(\cdot,\cdot,\cdot)\big),\\ (\widetilde{Y}(\cdot),\widetilde{Z}(\cdot,\cdot),\widetilde{K}(\cdot,\cdot,\cdot)) &= \big(Y_1(\cdot)-Y_2(\cdot),Z_1(\cdot,\cdot)-Z_2(\cdot,\cdot),K_1(\cdot,\cdot,\cdot)-K_2(\cdot,\cdot,\cdot)\big).\end{aligned}$$

Note that, for any $(y(\cdot),z(\cdot,\cdot),k(\cdot,\cdot,\cdot))\in\mathcal{M}^2[0,T+K]$, we have that

$$y(t)-E[y(t)]=\int_0^t z(t,s)\mathrm{d}W(s)+\int_0^t\int_{\mathbb{R}^*}k(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e),\quad t\in[0,T+K]. \tag{3.6}$$

Taking the square at both sides of (3.6),

$$\begin{aligned}&\Big|\int_0^t z(t,s)\mathrm{d}W(s)\Big|^2+\Big|\int_0^t\int_{\mathbb{R}^*}k(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e)\Big|^2\\ &\quad+2\int_0^t z(t,s)\mathrm{d}W(s)\int_0^t\int_{\mathbb{R}^*}k(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e)\\ &=\Big|y(t)\Big|^2+\Big|E[y(t)]\Big|^2-2y(t)E[y(t)].\end{aligned} \tag{3.7}$$

Hence,

$$E\Big[\int_0^t\Big|z(t,s)\Big|^2\mathrm{d}s\Big]+E\Big[\int_0^t\int_{\mathbb{R}^*}|k(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\Big]\le E\Big[\Big|y(t)\Big|^2\Big], \tag{3.8}$$

which implies that

$$\begin{aligned}&E\Big[\int_0^{T+K}\int_0^t \mathrm{e}^{\beta s}|z(t,s)|^2\mathrm{d}s\mathrm{d}t\Big]+E\Big[\int_0^{T+K}\int_0^t\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|k(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\mathrm{d}t\Big]\\ &\le E\Big[\int_0^{T+K}\mathrm{e}^{\beta t}|y(t)|^2\mathrm{d}t\Big].\end{aligned} \tag{3.9}$$

Applying the estimate (2.4), Assumption 1, (3.2) and (3.9), we have that

$$\begin{aligned}&E\Big[\int_0^T\Big(\mathrm{e}^{\beta t}|\widetilde{Y}(t)|^2+\int_t^T\mathrm{e}^{\beta s}|\widetilde{Z}(t,s)|^2\mathrm{d}s+\int_t^T\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|K(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\Big)\mathrm{d}t\Big]\\ &\le\frac{C}{\beta}E\Big[\int_0^T\int_t^T\mathrm{e}^{\beta s}\Big|g(t,s,y_1(s+\delta(s)),z_1(t,s+\gamma(s)),z_1(s+\gamma(s),t),\\ &\quad k_1(t,s+\zeta(s),\cdot),k_1(s+\zeta(s),t,\cdot))-g(t,s,y_2(s+\delta(s)),z_2(t,s+\gamma(s)),\end{aligned}$$

$$
\begin{aligned}
&\quad z_2(s+\gamma(s),t),k_2(t,s+\zeta(s),\cdot),k_2(s+\zeta(s),t,\cdot))\Big|^2\mathrm{d}s\mathrm{d}t\Big]\\
&\le \frac{CL}{\beta}E\Big[\int_0^T\int_t^T \mathrm{e}^{\beta s}\Big(|\widetilde{y}(s+\delta(s))|^2+|\widetilde{z}(t,s+\gamma(s))|^2+|\widetilde{z}(s+\gamma(s),t)|^2\Big)\mathrm{d}s\mathrm{d}t\Big]\\
&\quad+\frac{CL}{\beta}E\Big[\int_0^T\int_t^T \mathrm{e}^{\beta s}\Big(\|\widetilde{k}(t,s+\zeta(s))\|_\nu^2+\|\widetilde{k}(s+\zeta(s),t)\|_\nu^2\Big)\mathrm{d}s\mathrm{d}t\Big]\\
&\le \frac{C\sigma LT}{\beta}E\Big[\int_0^{T+K}\mathrm{e}^{\beta s}|\widetilde{y}(s)|^2\mathrm{d}s\Big]+\frac{C\sigma L}{\beta}E\Big[\int_0^{T+K}\int_t^{T+K}\mathrm{e}^{\beta s}\big(|\widetilde{z}(t,s)|^2+|\widetilde{z}(s,t)|^2\big)\mathrm{d}s\mathrm{d}t\Big]\\
&\quad+\frac{C\sigma L}{\beta}E\Big[\int_0^{T+K}\int_t^{T+K}\mathrm{e}^{\beta s}\Big(\|\widetilde{k}(t,s)\|_\nu^2+\|\widetilde{k}(s,t)\|_\nu^2\Big)\mathrm{d}s\mathrm{d}t\Big]\\
&= \frac{C\sigma LT}{\beta}E\Big[\int_0^{T+K}\mathrm{e}^{\beta s}|\widetilde{y}(s)|^2\mathrm{d}s\Big]+\frac{C\sigma L}{\beta}E\Big[\int_0^{T+K}\int_t^{T+K}\mathrm{e}^{\beta s}|\widetilde{z}(t,s)|^2\mathrm{d}s\mathrm{d}t\Big]\\
&\quad+\frac{C\sigma L}{\beta}E\Big[\int_0^{T+K}\int_t^{T+K}\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|\widetilde{k}(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\mathrm{d}t+\int_0^{T+K}\int_0^t\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|\widetilde{z}(t,s)|^2\mathrm{d}s\mathrm{d}t\Big]\\
&\quad+\frac{C\sigma L}{\beta}E\Big[\int_0^{T+K}\int_0^t\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|\widetilde{k}(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\mathrm{d}t\Big]\\
&\le \frac{C\sigma L(T+1)}{\beta}E\Big[\int_0^{T+K}\mathrm{e}^{\beta t}|\widetilde{y}(t)|^2\mathrm{d}t\Big]+\frac{C\sigma L}{\beta}E\Big[\int_0^{T+K}\int_t^{T+K}\mathrm{e}^{\beta s}|\widetilde{z}(t,s)|^2\mathrm{d}s\mathrm{d}t\Big]\\
&\quad+\frac{C\sigma L}{\beta}E\Big[\int_0^{T+K}\int_t^{T+K}\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|\widetilde{k}(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\mathrm{d}t\Big].
\end{aligned}\tag{3.10}
$$

Thus, we have that

$$
\begin{aligned}
&E\Big[\int_0^T\Big(\mathrm{e}^{\beta t}|\widetilde{Y}(t)|^2+\int_t^T\mathrm{e}^{\beta s}|\widetilde{Z}(t,s)|^2\mathrm{d}s+\int_t^T\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|K(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\Big)\mathrm{d}t\Big]\\
&\le \frac{\sigma L(T+1)C}{\beta}E\Big[\int_0^{T+K}\Big(\mathrm{e}^{\beta t}|\widetilde{y}(t)|^2+\int_t^{T+K}\mathrm{e}^{\beta s}|\widetilde{z}(t,s)|^2\mathrm{d}s\\
&\quad+\int_t^{T+K}\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|k(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\Big)\mathrm{d}t\Big].
\end{aligned}\tag{3.11}
$$

Note that

$$
\widetilde{Y}(t)=0,\quad t\in[T,T+K];\qquad \widetilde{Z}(t,s)=0,\quad \widetilde{K}(t,s,\cdot)=0,\quad (t,s)\in[0,T+K]^2\setminus[0,T]^2.
$$

Therefore, we have that

$$
\begin{aligned}
&E\Big[\int_0^{T+K}\Big(\mathrm{e}^{\beta t}|\widetilde{Y}(t)|^2+\int_t^{T+K}\mathrm{e}^{\beta s}|\widetilde{Z}(t,s)|^2\mathrm{d}s+\int_t^{T+K}\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|K(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\Big)\mathrm{d}t\Big]\\
&\le \varepsilon E\Big[\int_0^{T+K}\Big(\mathrm{e}^{\beta t}|\widetilde{y}(t)|^2+\int_t^{T+K}\mathrm{e}^{\beta s}|\widetilde{z}(t,s)|^2\mathrm{d}s+\int_t^{T+K}\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}|k(t,s,e)|^2\nu(\mathrm{d}e)\mathrm{d}s\Big)\mathrm{d}t\Big],
\end{aligned}\tag{3.12}
$$

where $\varepsilon=\frac{\sigma L(T+1)C}{\beta}$. By choosing $\beta=2\sigma L(T+1)C+1$, the mapping Φ is a contractive map on $\mathcal{M}^2[0,T+k]$. Thus, we can see that there exists a unique M-solution $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))\in\mathcal{M}^2[0,T+K]$ of (3.1). □

Next, for the general ABSVIEs with jumps (1.2), we need the following assumptions:

Assumption 2 For all $(t,s)\in\triangle$, $u_i\in[s,T+K], i=1,2,\cdots,5$, $g:\Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^{m\times d}\times L_\nu^2(\mathbb{R}^m)\times L_\nu^2(\mathbb{R}^m)\times L^2_{\mathcal{F}_{u_1}}(\Omega;\mathbb{R}^m)\times L^2_{\mathcal{F}_{u_2}}(\Omega;\mathbb{R}^{m\times d})\times L^2_{\mathcal{F}_{u_3}}(\Omega;\mathbb{R}^{m\times d})\times L^2_{\mathcal{F}_{u_4}}(\Omega;L_\nu^2(\mathbb{R}^m))\times L^2_{\mathcal{F}_{u_5}}(\Omega;L_\nu^2(\mathbb{R}^m))\to L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m)$ satisfies that

- $E\Big[\int_0^T\Big(\int_t^T\Big|g(t,s,0,0,0,0,0,0,0,0,0,0)\Big|\mathrm{d}s\Big)^2\mathrm{d}t\Big]<\infty$;
- there exists a constant $L>0$ such that, for all $(t,s)\in\triangle$, $y_1,y_2\in\mathbb{R}^m$, $z_1,z_2,\varsigma_1,\varsigma_2\in\mathbb{R}^{m\times d}$, k_1,k_2, ϑ_1, $\vartheta_2\in L^2_\nu(\mathbb{R}^m)$, $\psi_1(\cdot),\psi_2(\cdot)\in L^2_{\mathbb{F}}([s,T+K];\mathbb{R}^m)$, $\varphi_1(t,\cdot)$, $\varphi_2(t,\cdot)$, $\phi_1(\cdot,t)$, $\phi_2(\cdot,t)\in L^2_{\mathbb{F}}([s,T+K];\mathbb{R}^{m\times d})$, $\zeta_1(t,\cdot,\cdot)$, $\zeta_2(t,\cdot,\cdot)$, $\varrho_1(\cdot,t,\cdot)$, $\varrho_2(\cdot,t,\cdot)\in L^2_N([s,T+K];\mathbb{R}^m)$,

$$\begin{aligned}&\Big|g(t,s,y_1,z_1,\varsigma_1,k_1,\vartheta_1,\psi_1(u_1),\varphi_1(t,u_2),\phi_1(u_3,t),\zeta_1(t,u_4,\cdot),\varrho_1(u_5,t,\cdot))\\&\quad-g(t,s,y_2,z_2,\varsigma_2,k_2,\vartheta_2,\psi_2(u_1),\varphi_2(t,u_2),\phi_2(u_3,t),\zeta_2(t,u_4,\cdot),\varrho_2(u_5,t,\cdot))\Big|^2\\&\le L\Big(|y_1-y_2|^2+|z_1-z_2|^2+|\varsigma_1-\varsigma_2|^2+\|k_1-k_2\|^2_\nu+\|\vartheta_1-\vartheta_2\|^2_\nu\Big)\\&\quad+L\mathrm{e}^{\mathcal{F}_s}\Big[\Big(|\psi_1(u_1)-\psi_2(u_1)|^2+|\varphi_1(t,u_2)-\varphi_2(t,u_2)|^2+|\phi_1(u_3,t)-\phi_2(u_3,t)|^2\Big)\Big]\\&\quad+L\mathrm{e}^{\mathcal{F}_s}\Big[\Big(\|\zeta_1(t,u_4,\cdot)-\zeta_2(t,u_4,\cdot)\|^2_\nu+\|\varrho_1(u_5,t,\cdot)-\varrho_2(u_5,t,\cdot)\|^2_\nu\Big)\Big].\end{aligned}$$

For the proof of the existence and uniqueness of ABSVIE (1.2), by the proof of Theorem 3.1, we easily obtain the following result:

Theorem 3.2 Let Assumption 2 hold. Then, for any $(\xi(\cdot),\eta(\cdot,\cdot),\theta(\cdot,\cdot,\cdot))\in\mathcal{M}^2[0,T+K]$, there exists a unique M-solution $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))\in\mathcal{M}^2[0,T+K]$ to ABSVIE (1.2).

Example 3.3 Consider the following linear ABSVIE:

$$\begin{cases}Y(t)=\xi(t)+\displaystyle\int_t^T Y(s)+Z(t,s)+Z(s,t)+\mathrm{e}^{\mathcal{F}_s}\big[Y(s+\delta(s))+Z(t,s+\gamma(s))\\\qquad+Z(s+\gamma(s),t)+K(t,s+\zeta(s),\cdot)+K(s+\zeta(s),t,\cdot)\big]\mathrm{d}s-\displaystyle\int_t^T Z(t,s)\mathrm{d}B(s)\\\qquad-\displaystyle\int_t^T\int_{\mathbb{R}^*}K(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e),\quad t\in[0,T];\\Y(t)=\xi(t),\quad t\in[T,T+K];\\Z(t,s)=\eta(t,s),\quad K(t,s,\cdot)=\theta(t,s,\cdot),\quad (t,s)\in[0,T+K]^2\setminus[0,T]^2.\end{cases}\tag{3.13}$$

From Theorem 3.2 there exists a unique M-solution $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))\in\mathcal{M}^2[0,T+K]$ solving the above equation for any $(\xi(\cdot),\eta(\cdot,\cdot),\theta(\cdot,\cdot,\cdot))\in\mathcal{M}^2[0,T+K]$.

Note that if the generator $g(\cdot)$ of (1.2) is independent of $Z(s,t),Z(s+\gamma(s),t),K(s,t,\cdot)$ and $K(s+\zeta(s),t,\cdot)$, then, for the related equation, we only need the values $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))$ for $0\le t\le s\le T+K$. Thus the M-solution to the related equation is not necessary. From Theorem 3.2, we have the following corollary:

Corollary 3.4 Let Assumption 2 hold. Then, for any $(\xi(\cdot),\eta(\cdot,\cdot),\theta(\cdot,\cdot,\cdot))\in\mathcal{H}^2_\triangle[0,T+K]$, there exists a unique solution $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))\in\mathcal{H}^2_\triangle[0,T+K]$ to the following equation:

$$\begin{cases}Y(t)=\xi(t)+\displaystyle\int_t^T g\Big(t,s,Y(s),Z(t,s),K(t,s,\cdot),Y(s+\delta(s)),Z(t,s+\gamma(s)),K(t,s+\zeta(s),\cdot)\Big)\mathrm{d}s\\\qquad-\displaystyle\int_t^T Z(t,s)\mathrm{d}B(s)-\int_t^T\int_{\mathbb{R}^*}K(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e),\quad t\in[0,T];\\Y(t)=\xi(t),\quad t\in[T,T+K];\\Z(t,s)=\eta(t,s),\quad K(t,s,\cdot)=\theta(t,s,\cdot),\quad (t,s)\in[0,T+K]^2\setminus[0,T]^2,\quad t\le s.\end{cases}\tag{3.14}$$

Remark 3.5 The results of Theorems 3.1, 3.2 and Corollary 3.4 remain true when $\xi(t)$, $t\in[0,T+K]$ is $\mathcal{F}_{T\vee t}$-measurable. Under non-Lipschitz coefficients in [24, 37], we can obtain a similar result to that of Corollary 3.4.

4 Comparison Theorem for ABSVIEs with Jumps

In this section, inspired by [28, 35], we investigate a comparison theorem for a class of ABSVIE for the one-dimensional case ($m=1$). For notational convenience, we also suppose that $d=l=1$. Consider the following ABSVIEs for $i=0,1$:

$$\begin{cases} Y^i(t)=\xi^i(t)+\displaystyle\int_t^T g^i\Big(t,s,Y^i(s),Z^i(t,s),K^i(t,s,\cdot),Y^i(s+\delta(s))\Big)\mathrm{d}s \\ \qquad\qquad -\displaystyle\int_t^T Z^i(t,s)\mathrm{d}B(s)-\int_t^T\int_{\mathbb{R}^*}K^i(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e),\quad t\in[0,T]; \\ Y(t)=\xi^i(t),\quad t\in[T,T+K]. \end{cases} \tag{4.1}$$

Theorem 4.1 For $i=1,2$, let $g^i(\cdot)$ satisfy Assumption 2 and let $\xi^i(\cdot)\in L^2_{\mathcal{F}_T}([0,T+K];\mathbb{R})$. Denote by $\Big(Y^i(\cdot),Z^i(\cdot,\cdot),K(\cdot,\cdot,\cdot)\Big)$ the solution of (4.1). Suppose that the following conditions hold:

(i) For all $t\in[0,T+K]$, $\xi^1(t)\le\xi^2(t)$, a.s.;

(ii) For all $(t,s,y,z,k,\psi)\in\triangle\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times L^2_{\mathcal{F}_u}(\Omega;\mathbb{R})$ and for any $u\in[s,T+K]$,

$$g^1(t,s,y,z,k,\psi)\le\overline{g}(t,s,y,z,k,\psi)\le g^2(t,s,y,z,\psi),$$

where the map $\overline{g}:\Omega\times\triangle\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times L^2_{\mathcal{F}_u}(\Omega;\mathbb{R})\to\mathbb{R}$ satisfies Assumption 2 such that $y\to\overline{g}(t,s,y,z,k,\psi)$ is nondecreasing, and $\overline{g}(t,s,y,z,k,\psi)$ is increasing with respect to ψ, i.e., $\overline{g}(t,s,y,z,k,\psi_1(u))<\overline{g}(t,s,y,z,k,\psi_2(u))$ if $\psi_1(u)<\psi_2(u)$ with $\psi_1(u),\psi_2(u)\in L^2_{\mathbb{F}}([s,T+K];\mathbb{R})$ and, in addition, there exists a bounded predictable process $\Lambda(t,s,e)$ such that, for any $y,z\in\mathbb{R},k,k'\in L^2_\nu(\mathbb{R}),\psi\in L^2_{\mathcal{F}_u}(\Omega;\mathbb{R})$,

$$\overline{g}(t,s,y,z,k,\psi)-\overline{g}(t,s,y,z,k',\psi)\ge\int_{\mathbb{R}^*}\Lambda(t,s,e)\Big(k(e)-k'(e)\Big)\nu(\mathrm{d}e), \tag{4.2}$$

with $\Lambda(t,s,e)$ satisfying that

$$\Lambda(t,s,e)>-1,\quad |\Lambda(t,s,e)|\le\Psi(e),\qquad \Psi\in L^2_\nu(\mathbb{R}).$$

Then we have that

$$Y^1(t)\le Y^2(t),\qquad \text{a.s.},\ \ t\in[0,T+K].$$

Proof Let $\overline{\xi}(\cdot)\in L^2_{\mathcal{F}_T}([0,T+K];\mathbb{R})$ such that

$$\xi^1(t)\le\overline{\xi}(t)\le\xi^2(t),\qquad \text{a.s.},\ \ t\in[0,T+K].$$

Clearly, there exists a unique solution in $L^2_{\mathbb{F}}([0,T+K];\mathbb{R})\times L^2_{\mathbb{F}}(\triangle;\mathbb{R})\times L^2_N(\triangle;\mathbb{R})$, denoted by $(\overline{Y}(\cdot),\overline{Z}(\cdot,\cdot),\overline{K}(\cdot,\cdot,\cdot))$, solving the equation

$$\begin{cases} \overline{Y}(t)=\overline{\xi}(t)+\displaystyle\int_t^T \overline{g}\Big(t,s,\overline{Y}(s),\overline{Z}(t,s),\overline{K}(t,s,\cdot),\overline{Y}(s+\delta(s))\Big)\mathrm{d}s \\ \qquad\qquad -\displaystyle\int_t^T \overline{Z}(t,s)\mathrm{d}B(s)-\int_t^T\int_{\mathbb{R}^*}\overline{K}(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e),\quad t\in[0,T]; \\ Y(t)=\overline{\xi}(t),\quad t\in[T,T+K]. \end{cases}$$

Set $\widetilde{Y}_0(\cdot) = Y^2(\cdot)$ and let $\Big(\widetilde{Y}_1(\cdot), \widetilde{Z}_1(\cdot,\cdot), \widetilde{K}_1(\cdot,\cdot,\cdot)\Big) \in L^2_{\mathbb{F}}([0,T+K];\mathbb{R}) \times L^2_{\mathbb{F}}(\triangle;\mathbb{R}) \times L^2_N(\triangle;\mathbb{R})$ be the unique solution of the equation

$$\begin{cases} \widetilde{Y}_1(t) = \overline{\xi}(t) + \displaystyle\int_t^T \overline{g}\Big(t,s,\widetilde{Y}_1(s),\widetilde{Z}_1(t,s),\widetilde{K}_1(t,s,\cdot),\widetilde{Y}_0(s+\delta(s))\Big)\mathrm{d}s \\ \qquad\qquad - \displaystyle\int_t^T \widetilde{Z}_1(t,s)\mathrm{d}B(s) - \int_t^T\int_{\mathbb{R}^*} \widetilde{K}_1(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e), \quad t\in[0,T]; \\ \widetilde{Y}_1(t) = \overline{\xi}(t), \quad t\in[T,T+K]. \end{cases}$$

Note that, for all $(t,s,y,z,k,\psi) \in \triangle \times \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}) \times L^2_{\mathcal{F}_u}(\Omega;\mathbb{R})$,

$$\overline{g}(t,s,y,z,k,\widetilde{Y}_0(s+\delta(s))) \le g^2(t,s,y,z,k,\widetilde{Y}_0(s+\delta(s))),$$

and $\overline{\xi}(t) \le \xi^2(t)$, a.s., $t\in[0,T+K]$.

By Proposition 2.4, for any $t\in[0,T+K]$, there exists a measurable set Ω_t^1 with $\mathrm{P}(\Omega_t^1)=0$ such that

$$\widetilde{Y}_1(t) \le \widetilde{Y}_0(t) = Y^2(t), \qquad \omega\in\Omega\backslash\Omega_t^1,\ t\in[0,T+K].$$

Next, we consider the equation

$$\begin{cases} \widetilde{Y}_2(t) = \overline{\xi}(t) + \displaystyle\int_t^T \overline{g}\Big(t,s,\widetilde{Y}_2(s),\widetilde{Z}_2(t,s),\widetilde{K}_2(t,s,\cdot),\widetilde{Y}_1(s+\delta(s))\Big)\mathrm{d}s \\ \qquad\qquad - \displaystyle\int_t^T \widetilde{Z}_2(t,s)\mathrm{d}B(s) - \int_t^T\int_{\mathbb{R}^*} \widetilde{K}_2(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e), \quad t\in[0,T]; \\ \widetilde{Y}_2(t) = \overline{\xi}(t), \quad t\in[T,T+K]. \end{cases}$$

Let $\Big(\widetilde{Y}_2(\cdot), \widetilde{Z}_2(\cdot,\cdot), \widetilde{K}_2(\cdot,\cdot,\cdot)\Big) \in L^2_{\mathbb{F}}([0,T+K];\mathbb{R}) \times L^2_{\mathbb{F}}(\triangle;\mathbb{R}) \times L^2_N(\triangle;\mathbb{R})$ be the unique solution of the above equation. Noting that $\overline{g}(t,s,y,z,k,\psi)$ is increasing in ψ, we obtain, for all $(t,s,y,z,k,\psi) \in \triangle \times \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}) \times L^2_{\mathcal{F}_u}(\Omega;\mathbb{R})$, that

$$\overline{g}(t,s,y,z,k,\widetilde{Y}_1(s+\delta(s))) \le \overline{g}(t,s,y,z,k,\widetilde{Y}_0(s+\delta(s))).$$

Hence, in a fashion similar to the above discussion, for any $t\in[0,T+K]$, there also exists a measurable set Ω_t^2 with $\mathrm{P}(\Omega_t^2)=0$ such that

$$\widetilde{Y}_2(t) \le \widetilde{Y}_1(t), \qquad \omega\in\Omega\backslash\Omega_t^2,\ t\in[0,T+K].$$

By induction, we can construct a sequence $\Big\{\Big(\widetilde{Y}_n(\cdot), \widetilde{Z}_n(\cdot,\cdot), \widetilde{K}_n(\cdot,\cdot,\cdot)\Big)\Big\}_{n\ge1} \in L^2_{\mathbb{F}}([0,T+K];\mathbb{R}) \times L^2_{\mathbb{F}}(\triangle;\mathbb{R}) \times L^2_N(\triangle;\mathbb{R})$ and Ω_t^n with $\mathrm{P}(\Omega_t^n)=0$ such that

$$\begin{cases} \widetilde{Y}_n(t) = \overline{\xi}(t) + \displaystyle\int_t^T \overline{g}\Big(t,s,\widetilde{Y}_n(s),\widetilde{Z}_n(t,s),\widetilde{K}_n(t,s,\cdot),\widetilde{Y}_{n-1}(s+\delta(s))\Big)\mathrm{d}s \\ \qquad\qquad - \displaystyle\int_t^T \widetilde{Z}_n(t,s)\mathrm{d}B(s) - \int_t^T\int_{\mathbb{R}^*} \widetilde{K}_n(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e), \quad t\in[0,T]; \\ \widetilde{Y}_n(t) = \overline{\xi}(t), \quad t\in[T,T+K], \end{cases}$$

and

$$Y^2(t) = \widetilde{Y}_0(t) \ge \widetilde{Y}_1(t) \ge \widetilde{Y}_2(t) \ge \cdots, \qquad \omega\in\Omega\backslash\Big(\bigcup_{n\ge1}\Omega_t^n\Big),\ t\in[0,T+K].$$

Clearly, we see that $\mathrm{P}\Big(\bigcup\limits_{n\ge1}\Omega_t^n\Big)=0$.

Now, we show that the sequence $\Big\{\big(\widetilde{Y}_n(\cdot),\widetilde{Z}_n(\cdot,\cdot),\widetilde{K}_n(\cdot,\cdot,\cdot)\big)\Big\}_{n\geq 1}$ is a Cauchy sequence. Using the estimate (2.4), we obtain that

$$\begin{aligned}
&E\Big[\int_0^T \mathrm{e}^{\beta t}\Big|\widetilde{Y}_n(t)-\widetilde{Y}_{n-1}(t)\Big|^2\mathrm{d}t+\int_0^T\int_t^T e^{\beta s}\Big|\widetilde{Z}_n(t,s)-\widetilde{Z}_{n-1}(t,s)\Big|^2\mathrm{d}s\mathrm{d}t\\
&\quad+\int_0^T\int_t^T\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}\Big|\widetilde{K}_n(t,s,e)-\widetilde{K}_{n-1}(t,s,e)\Big|^2\nu(\mathrm{d}e)\mathrm{d}s\mathrm{d}t\Big]\\
&\leq\frac{C}{\beta}E\Big[\int_0^T\int_t^T\mathrm{e}^{\beta s}\Big|\overline{g}\big(t,s,\widetilde{Y}_n(s),\widetilde{Z}_n(t,s),\widetilde{K}_n(t,s,\cdot),\widetilde{Y}_{n-1}(s+\delta(s))\big)\\
&\quad-\overline{g}\big(t,s,\widetilde{Y}_{n-1}(s),\widetilde{Z}_{n-1}(t,s),\widetilde{K}_{n-1}(t,s,\cdot),\widetilde{Y}_{n-2}(s+\delta(s))\big)\Big|^2\mathrm{d}s\mathrm{d}t\Big]\\
&\leq\frac{C\sigma L(T+1)}{\beta}E\Big[\int_0^T\mathrm{e}^{\beta t}\Big|\widetilde{Y}_n(t)-\widetilde{Y}_{n-1}(t)\Big|^2\mathrm{d}t+\int_0^T\int_t^T e^{\beta s}\Big|\widetilde{Z}_n(t,s)-\widetilde{Z}_{n-1}(t,s)\Big|^2\mathrm{d}s\mathrm{d}t\\
&\quad+\int_0^T\int_t^T\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}\Big|\widetilde{K}_n(t,s,e)-\widetilde{K}_{n-1}(t,s,e)\Big|^2\nu(\mathrm{d}e)\mathrm{d}s\mathrm{d}t\Big]\\
&\quad+\frac{C\sigma L(T+1)}{\beta}E\Big[\int_0^{T+K}\mathrm{e}^{\beta t}\Big|\widetilde{Y}_{n-1}(t)-\widetilde{Y}_{n-2}(t)\Big|^2\mathrm{d}t\Big].
\end{aligned}$$

Note that

$$\widetilde{Y}_n(t)-\widetilde{Y}_{n-1}(t)=0,\quad t\in[T,T+K].$$

By choosing $\beta=3C\sigma L(T+1)$, we have that

$$\begin{aligned}
&E\Big[\int_0^{T+K}\mathrm{e}^{\beta t}\Big|\widetilde{Y}_n(t)-\widetilde{Y}_{n-1}(t)\Big|^2\mathrm{d}t+\int_0^T\int_t^T e^{\beta s}\Big|\widetilde{Z}_n(t,s)-\widetilde{Z}_{n-1}(t,s)\Big|^2\mathrm{d}s\mathrm{d}t\\
&\quad+\int_0^T\int_t^T\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}\Big|\widetilde{K}_n(t,s,e)-\widetilde{K}_{n-1}(t,s,e)\Big|^2\nu(\mathrm{d}e)\mathrm{d}s\mathrm{d}t\Big]\\
&\leq\frac12E\Big[\int_0^{T+K}\mathrm{e}^{\beta t}\Big|\widetilde{Y}_{n-1}(t)-\widetilde{Y}_{n-2}(t)\Big|^2\mathrm{d}t\Big]\\
&\leq\frac12E\Big[\int_0^{T+K}\mathrm{e}^{\beta t}\Big|\widetilde{Y}_{n-1}(t)-\widetilde{Y}_{n-2}(t)\Big|^2\mathrm{d}t+\int_0^T\int_t^T e^{\beta s}\Big|\widetilde{Z}_{n-1}(t,s)-\widetilde{Z}_{n-2}(t,s)\Big|^2\mathrm{d}s\mathrm{d}t\\
&\quad+\int_0^T\int_t^T\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}\Big|\widetilde{K}_{n-1}(t,s,e)-\widetilde{K}_{n-2}(t,s,e)\Big|^2\nu(\mathrm{d}e)\mathrm{d}s\mathrm{d}t\Big]\\
&\leq\Big(\frac12\Big)^{n-2}E\Big[\int_0^{T+K}\mathrm{e}^{\beta t}\Big|\widetilde{Y}_2(t)-\widetilde{Y}_1(t)\Big|^2\mathrm{d}t+\int_0^T\int_t^T e^{\beta s}\Big|\widetilde{Z}_2(t,s)-\widetilde{Z}_1(t,s)\Big|^2\mathrm{d}s\mathrm{d}t\\
&\quad+\int_0^T\int_t^T\int_{\mathbb{R}^*}\mathrm{e}^{\beta s}\Big|\widetilde{K}_2(t,s,e)-\widetilde{K}_1(t,s,e)\Big|^2\nu(\mathrm{d}e)\mathrm{d}s\mathrm{d}t\Big].
\end{aligned}$$

Thus, we obtain that $\Big\{\big(\widetilde{Y}_n(\cdot),\widetilde{Z}_n(\cdot,\cdot),\widetilde{K}_n(\cdot,\cdot,\cdot)\big)\Big\}_{n\geq 2}$ is a Cauchy sequence. We denote the limits by $\big(\widetilde{Y}(\cdot),\widetilde{Z}(\cdot,\cdot),\widetilde{Z}(\cdot,\cdot,\cdot)\big)$, and then $\big(\widetilde{Y}(\cdot),\widetilde{Z}(\cdot,\cdot),\widetilde{K}(\cdot,\cdot,\cdot)\big)\in L^2_{\mathbb{F}}([0,T+K];\mathbb{R})\times L^2_{\mathbb{F}}(\triangle;\mathbb{R})\times L^2_N(\triangle;\mathbb{R})$ such that

$$\begin{aligned}
&\lim_{n\to\infty}E\Big[\int_0^{T+K}\mathrm{e}^{\beta t}\Big|\widetilde{Y}_n(t)-\widetilde{Y}(t)\Big|^2\mathrm{d}t+\int_0^T\int_t^T e^{\beta s}\Big|\widetilde{Z}_n(t,s)-\widetilde{Z}(t,s)\Big|^2\mathrm{d}s\mathrm{d}t\\
&\quad+\int_0^T\int_t^T\int_{\mathbb{R}^*}e^{\beta s}\Big|\widetilde{K}_n(t,s,e)-\widetilde{K}(t,s,e)\Big|^2\nu(\mathrm{d}e)\mathrm{d}s\mathrm{d}t\Big]=0.
\end{aligned}$$

Furthermore, we have that

$$\begin{cases} \widetilde{Y}(t) = \overline{\xi}(t) + \displaystyle\int_t^T \overline{g}\Big(t, s, \widetilde{Y}(s), \widetilde{Z}(t,s), \widetilde{K}(t,s,\cdot), \widetilde{Y}(s+\delta(s))\Big)\mathrm{d}s \\ \qquad\qquad - \displaystyle\int_t^T \widetilde{Z}(t,s)\mathrm{d}B(s) - \int_t^T \int_{\mathbb{R}^*} \widetilde{K}(t,s,e)\widetilde{N}(\mathrm{d}s, \mathrm{d}e), \quad t \in [0,T]; \\ \widetilde{Y}_2(t) = \overline{\xi}(t), \quad t \in [T, T+K]. \end{cases}$$

By the uniqueness of the solutions to the ABSVIE, we obtain that

$$\overline{Y}(t) = \widetilde{Y}(t) \le \widetilde{Y}_0(t) = Y^2(t), \qquad \text{a.s., } t \in [0, T+K].$$

Similarly, we can show that $Y^1(t) \le \overline{Y}(t)$, a.s., $t \in [0, T+K]$. Thus, we have completed the proof. □

Remark 4.2 Due to technical problems, the comparison theorems for BSVIEs with generators g depending on the terms $Z(s,t)$ and $K(s,t,\cdot)$ only hold for some special cases, and may not be obtained for the general case (see [35]), so we prefer not to discuss this situation here.

5 Dynamic Risk Measures

In this section, as an application of ABSVIEs with jumps, under Assumption 2, we construct dynamic risk measures by means of ABSVIEs. For the reader's convenience, we introduce the concept of related dynamic risk measures induced by BSVIEs from [41].

Definition 5.1 A map $\rho : [0, T+K] \times L^2_{\mathcal{F}_T}([0, T+K]; \mathbb{R}) \to L^2_{\mathbb{F}}([0, T+K]; \mathbb{R})$ is called a dynamic risk measure if the following hold:

• (Past independence) For any $\xi(\cdot), \eta(\cdot) \in L^2_{\mathcal{F}_T}([0, T+K]; \mathbb{R})$, if

$$\xi(s) = \eta(s), \qquad \text{a.s., } s \in [t, T+K]$$

for some $t \in [0, T+K)$, then $\rho(t; \xi(\cdot)) = \rho(t; \eta(\cdot))$, a.s..

• (Monotonicity) For any $\xi(\cdot), \eta(\cdot) \in L^2_{\mathcal{F}_T}([0, T+K]; \mathbb{R})$, if

$$\xi(s) \le \eta(s), \qquad \text{a.s., } s \in [t, T+K]$$

for some $t \in [0, T+K)$, then $\rho(s; \xi(\cdot)) \ge \rho(s; \eta(\cdot))$, a.s., $s \in [t, T+K]$.

Definition 5.2 A dynamic risk measure $\rho : [0, T+K] \times L^2_{\mathcal{F}_T}([0, T+K]; \mathbb{R}) \to L^2_{\mathbb{F}}([0, T+K]; \mathbb{R})$ is called a dynamic coherent risk measure if the following hold:

• (Cash invariance) There exists a deterministic integrable function $r(\cdot)$ such that, for any $\xi(\cdot) \in L^2_{\mathcal{F}_T}([0, T+K]; \mathbb{R})$ and any constant c,

$$\rho(t; \xi(\cdot) + c) = \rho(t; \xi(\cdot)) - c\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}, \qquad \text{a.s., } t \in [0, T+K].$$

• (Subadditivity) $\rho(t; \xi(\cdot) + \eta(\cdot)) \le \rho(t; \xi(\cdot)) + \rho(t; \eta(\cdot))$, a.s., $t \in [0, T+K]$.

• (Positive homogeneity) For any $\xi(\cdot) \in L^2_{\mathcal{F}_T}([0, T+K]; \mathbb{R})$ and $\lambda > 0$,

$$\rho(t; \lambda\xi(\cdot)) = \lambda\rho(t; \xi(\cdot)), \qquad \text{a.s., } t \in [0, T+K].$$

Definition 5.3 A dynamic risk measure $\rho : [0, T+K] \times L^2_{\mathcal{F}_T}([0, T+K]; \mathbb{R}) \to L^2_{\mathbb{F}}([0, T+K]; \mathbb{R})$ is called a dynamic convex risk measure if the following hold:

• (Cash invariance) There exists a deterministic integrable function $r(\cdot)$ such that, for any $\xi(\cdot)\in L^2_{\mathcal{F}_T}([0,T+K];\mathbb{R})$ and any constant c,

$$\rho(t;\xi(\cdot)+c)=\rho(t;\xi(\cdot))-c\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s},\qquad \text{a.s., } t\in[0,T+K].$$

• (Convexity) For any $\xi(\cdot),\eta(\cdot)\in L^2_{\mathcal{F}_T}([0,T+K];\mathbb{R})$ and $\lambda\in[0,1]$,

$$\rho\big(t;\lambda\xi(\cdot)+(1-\lambda)\eta(\cdot)\big)\le\lambda\rho(t;\xi(\cdot))+(1-\lambda)\rho(t;\eta(\cdot)),\qquad \text{a.s., } t\in[0,T+K].$$

We denote that

$$\rho(t;\xi(\cdot))=Y(t),\qquad t\in[0,T+K],\tag{5.1}$$

where $Y(\cdot)$ is the first component of the solution $(Y(\cdot),Z(\cdot,\cdot),K(\cdot,\cdot,\cdot))$ of the following class of ABSVIEs:

$$\begin{cases}Y(t)=-\xi(t)+\displaystyle\int_t^T g\big(t,s,Y(s),Z(t,s),K(t,s,\cdot),Y(s+\delta(s))\big)\mathrm{d}s\\ \qquad\qquad -\displaystyle\int_t^T Z(t,s)\mathrm{d}B(s)-\int_t^T\int_{\mathbb{R}^*}K(t,s,e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e),\quad t\in[0,T];\\ Y(t)=-\xi(t),\quad t\in[T,T+K].\end{cases}\tag{5.2}$$

Here $\xi(\cdot)\in L^2_{\mathcal{F}_T}([0,T+K];\mathbb{R})$ and the generator g satisfies Assumption 2.

Let the generator be given by

$$g(t,s,y,z,k,\psi)=r(s)(y+\psi)+g_0(t,s,z,k)\tag{5.3}$$

for all $(t,s,y,z,k,\psi)\in\triangle\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times L^2_{\mathcal{F}_u}(\Omega;\mathbb{R})$, $u\in[s,T+K]$, where $r(\cdot)$ is a non-negative deterministic function.

In light of the previous sections, and in a manner analogous to the argument regarding dynamic risk measures in [41], we can get the results (since the proofs of Proposition 5.4 and Theorem 5.5 are almost the same as in [41], we omit the proofs here):

Proposition 5.4 Assume that g satisfies Assumption 2 and let ρ be defined as (5.1). Then the following hold:

(i) (Convexity) Assume that g satisfies (4.2) such that $y\to g(t,s,y,z,k,\psi)$ is nondecreasing and that $g(t,s,y,z,k,\psi)$ is increasing with respect to ψ. If $g(t,s,y,z,k,\psi)$ is convex in (y,z,k,ψ), i.e., for any $(t,s)\in\triangle,(y_1,z_1,k_1,\psi_1),(y_2,z_2,k_2,\psi_2)\in\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times L^2_{\mathcal{F}_u}(\Omega;\mathbb{R})$, $\lambda\in[0,1]$,

$$\begin{aligned}&g\Big(t,s,\lambda y_1+(1-\lambda)y_2,\lambda z_1+(1-\lambda)z_2,\lambda k_1+(1-\lambda)k_2,\lambda\psi_1+(1-\lambda)\psi_2\Big)\\ &\le\lambda g\Big(t,s,y_1,k_1,z_1,\psi_1\Big)+(1-\lambda)g\Big(t,s,y_2,k_2,z_2,\psi_2\Big),\end{aligned}$$

then ρ is convex.

(ii) (Cash invariance) Let g be given by (5.3). Then ρ is cash invariant. In particular, if g is independent of y and ψ, then, for any $\xi(\cdot)\in L^2_{\mathcal{F}_T}([0,T+K];\mathbb{R})$ and any constant c,

$$\rho(t;\xi(\cdot)+c)=\rho(t;\xi(\cdot))-c,\qquad \text{a.s., } t\in[0,T+K].$$

(iii) (Past independence) ρ is past independent.

(iv) (Monotonicity) Assume that g satisfies (4.2) such that $y\to g(t,s,y,z,k,\psi)$ is nondecreasing and that $g(t,s,y,z,k,\psi)$ is increasing with respect to ψ. Then ρ is monotonic.

(v) (Subadditivity) Assume that g satisfies (4.2) such that $y \to g(t,s,y,z,k,\psi)$ is nondecreasing and that $g(t,s,y,z,k,\psi)$ is increasing with respect to ψ. If $g(t,s,y,z,k,\psi)$ is subadditive in (y,z,k,ψ), i.e., for all $(t,s) \in \triangle, (y_1,z_1,k_1,\psi_1), (y_2,z_2,k_2,\psi_2) \in \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}) \times L^2_{\mathcal{F}_u}(\Omega;\mathbb{R})$,

$$g(t,s,y_1+y_2,z_1+z_2,k_1+k_2,\psi_1+\psi_2) \leq g(t,s,y_1,z_1,k_1,\psi_1) + g(t,s,y_2,z_2,k_2,\psi_2),$$

then ρ is subadditive.

(vi) (Positive homogeneity) Assume that g is positively homogeneous in (y,z,k,ψ), i.e., for any $(t,s) \in \triangle, (y,z,k,\psi) \in \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}) \times L^2_{\mathcal{F}_u}(\Omega;\mathbb{R}), \alpha > 0$,

$$g\big(t,s,\alpha y,\alpha z,\alpha k,\alpha\psi\big) = \alpha g\big(t,s,y,z,k,\psi\big),$$

then ρ is positively homogeneous.

From the definitions of the dynamic risk measures induced by the BSVIEs and Proposition 5.4, we have the following result:

Theorem 5.5 Assume that the generator g given by (5.3) satisfies Assumption 2 and (4.2) such that $y \to g(t,s,y,z,k,\psi)$ is nondecreasing and that $g(t,s,y,z,k,\psi)$ is increasing with respect to ψ. Then ρ defined by (5.1) is a dynamic convex risk measure if g is convex. Moreover, ρ defined by (5.1) is a dynamic coherent risk measure if g is positively homogeneous and subadditive.

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