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ON THE RIGOROUS MATHEMATICAL DERIVATION FOR THE VISCOUS PRIMITIVE EQUATIONS WITH DENSITY STRATIFICATION*

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Abstract In this paper, we rigorously derive the governing equations describing the motion of a stable stratified fluid, from the mathematical point of view. In particular, we prove that the scaled Boussinesq equations strongly converge to the viscous primitive equations with density stratification as the aspect ratio goes to zero, and the rate of convergence is of the same order as the aspect ratio. Moreover, in order to obtain this convergence result, we also establish the global well-posedness of strong solutions to the viscous primitive equations with density stratification.

Key words Boussinesq equations; primitive equations; density stratification; hydrostatic approximation; strong convergence

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1 Introduction

The primitive equations are considered to be a fundamental model in geophysical flows ([32, 33, 37, 40, 41]). For large-scale oceanic dynamics, an important feature is that the vertical scale of the ocean is much smaller than the horizontal scale, which means that we can use hydrostatic approximation to simulate the motion of the ocean in the vertical direction. Owing to this fact, and the high accuracy of hydrostatic approximation, the primitive equations for oceanic dynamics can be formally derived from the Boussinesq equations (see [10, 26]).

The small aspect ratio limit from the Navier-Stokes equations to the primitive equations was first studied by Azérad-Guillén [1] in a weak sense, then by Li-Titi [28] in a strong sense with error estimates, and finally by Furukawa *et al.* [14] in a strong sense but under a relaxed regularity on the initial condition. Subsequently, the strong convergence of solutions of the scaled Navier-Stokes equations to the corresponding ones of the primitive equations with only horizontal viscosity was obtained by Li-Titi-Yuan [30]. In addition, the rigorous justification

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of the hydrostatic approximation from the scaled Boussinesq equations with rotation to the primitive equations with full viscosity and diffusivity is due to the work of the authors in [34].

Fluid flow is strongly influenced by the effect of stratification from a physical point of view ([32, 33, 40]). An important observation for the effect of stratification is that the density of a fluid changes with depth. Furthermore, the density stratification term plays an important role in mathematical studies of the primitive equations with partial dissipation ([5–8, 11]). These two facts show that the density stratification term is of great significance both physically and mathematically. Therefore, the aim of this paper is to derive, rigorously, the governing equations describing the motion of a stable stratified fluid, i.e., the viscous primitive equations with density stratification, from the mathematical point of view.

Let $\Omega_\tau = M \times (-\tau, \tau)$ be a τ-dependent domain, where $M = (0,1) \times (0,1)$. Here, $\tau = H/L$ is called the aspect ratio, which measures the ratio of the vertical scale H to the horizontal scale L of the ocean. For large-scale ocean circulation, the aspect ratio τ is close to 10^{-3}, so it is much less than 1.

Denote by $\nabla_h = (\partial_x, \partial_y)$ the horizontal gradient operator. Then the horizontal Laplacian operator Δ_h is given by

$$\Delta_h = \nabla_h \cdot \nabla_h = \partial_{xx} + \partial_{yy}.$$

Let us consider the anisotropic Boussinesq equations defined on Ω_τ,

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla \pi + \dfrac{g\varrho}{\rho_b} \vec{k} = \mu_h \Delta_h u + \mu_z \partial_{zz} u, \\ \partial_t \varrho + u \cdot \nabla \varrho = \kappa_h \Delta_h \varrho + \kappa_z \partial_{zz} \varrho, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where the three dimensional velocity field $u = (v, w) = (v_1, v_2, w)$, the pressure π and the density ϱ are the unknowns. g is the gravitational acceleration and ρ_b is the reference constant density. $\vec{k} = (0,0,1)$ is unit vector pointing to the z-direction. μ_h and μ_z represent the horizontal and vertical viscosity coefficients, respectively, while κ_h and κ_z represent the horizontal and vertical heat conduction coefficients, respectively.

For simplicity, the reference constant density ρ_b is set to be $\rho_b = 1$. In fact, the anisotropic Boussinesq equations (1.1) have an elementary exact solution $(u, \pi, \varrho) = (0, \bar{p}(z), \bar{\varrho}(z))$ satisfying the hydrostatic approximation

$$\frac{\mathrm{d}\bar{p}(z)}{\mathrm{d}z} + g\bar{\varrho}(z) = 0.$$

Assume that

$$p(x,y,z,t) = \pi(x,y,z,t) - \bar{p}(z), \quad \rho(x,y,z,t) = \varrho(x,y,z,t) - \bar{\varrho}(z).$$

Then the anisotropic Boussinesq equations (1.1) become

$$\begin{cases} \partial_t v + (v \cdot \nabla_h) v + w \partial_z v + \nabla_h p = \mu_h \Delta_h v + \mu_z \partial_{zz} v, \\ \partial_t w + v \cdot \nabla_h w + w \partial_z w + \partial_z p + g\rho = \mu_h \Delta_h w + \mu_z \partial_{zz} w, \\ \partial_t \rho + v \cdot \nabla_h \rho + w \partial_z \rho + \left(\dfrac{\mathrm{d}\bar{\varrho}(z)}{\mathrm{d}z} \right) w = \kappa_h \Delta_h \rho + \kappa_z \partial_{zz} \rho + \kappa_z \dfrac{\mathrm{d}^2 \bar{\varrho}(z)}{\mathrm{d}z^2}, \\ \nabla_h \cdot v + \partial_z w = 0. \end{cases} \tag{1.2}$$

Let $N=\left(-g\frac{\mathrm{d}\bar{\varrho}(z)}{\mathrm{d}z}\right)^{1/2}$. If $N>0$, then N is called the buoyancy or Brunt-Väisälä frequency. When $\frac{\mathrm{d}\bar{\varrho}(z)}{\mathrm{d}z}<0$, the density decreases with height, and the lighter fluid is above the heavier fluid, which is a situation referred to as stable stratification.

First, we transform the anisotropic Boussinesq equations (1.2), defined on the τ-dependent domain Ω_τ, to the scaled Boussinesq equations defined on a fixed domain. To this end, we introduce new unknowns with the subscript τ,

$$\begin{aligned}
&u_\tau=(v_\tau,w_\tau), v_\tau(x,y,z,t)=v(x,y,\tau z,t),\\
&w_\tau(x,y,z,t)=\frac{1}{\tau}w(x,y,\tau z,t), p_\tau(x,y,z,t)=p(x,y,\tau z,t),\\
&\rho_\tau(x,y,z,t)=(g\tau)\rho(x,y,\tau z,t), \bar{p}_\tau(z)=\bar{p}(\tau z), \bar{\varrho}_\tau(z)=(g\tau)\bar{\varrho}(\tau z),
\end{aligned}$$

for any $(x,y,z)\in\Omega=:M\times(-1,1)$ and for any $t\in(0,\infty)$. Then the last two scalings allow us to write the pressure and the density non-dimensionally as

$$\bar{p}_\tau(z)+p_\tau(x,y,z,t)=\bar{p}(\tau z)+p(x,y,\tau z,t)=\pi(x,y,\tau z,t)$$

and

$$\bar{\varrho}_\tau(z)+\rho_\tau(x,y,z,t)=(g\tau)(\bar{\varrho}(\tau z)+\rho(x,y,\tau z,t))=(g\tau)\varrho(x,y,\tau z,t),$$

respectively.

Suppose that $\mu_h=\kappa_h=1$ and that $\mu_z=\kappa_z=\tau^2$. Under these scalings, the anisotropic Boussinesq equations (1.2) defined on Ω_τ can be written as the scaled Boussinesq equations

$$\begin{cases}
\partial_t v_\tau+(v_\tau\cdot\nabla_h)v_\tau+w_\tau\partial_z v_\tau+\nabla_h p_\tau=\Delta_h v_\tau+\partial_{zz}v_\tau,\\
\tau(\partial_t w_\tau+v_\tau\cdot\nabla_h w_\tau+w_\tau\partial_z w_\tau)+\dfrac{1}{\tau}(\partial_z p_\tau+\rho_\tau)=\tau\Delta_h w_\tau+\tau\partial_{zz}w_\tau,\\
\partial_t\rho_\tau+v_\tau\cdot\nabla_h\rho_\tau+w_\tau\partial_z\rho_\tau+w_\tau\dfrac{\mathrm{d}\bar{\varrho}_\tau}{\mathrm{d}z}=\Delta_h\rho_\tau+\partial_{zz}\rho_\tau+\dfrac{\mathrm{d}^2\bar{\varrho}_\tau}{\mathrm{d}z^2},\\
\nabla_h\cdot v_\tau+\partial_z w_\tau=0,
\end{cases}\tag{1.3}$$

defined on the fixed domain Ω.

When the fluid is steadily stratified, we can assume, for simplicity, that $\bar{\varrho}(z)=1-(1/g)N^2z$ for some positive constant N^2, where N represents the strength of the stable stratification. This assumption leads to $\bar{\varrho}_\tau(z)=(g\tau)\bar{\varrho}(\tau z)=g\tau-\tau^2N^2z$, and hence the third equation of the scaled Boussinesq equations (1.3) becomes

$$\partial_t\rho_\tau+v_\tau\cdot\nabla_h\rho_\tau+w_\tau\partial_z\rho_\tau-\tau^2N^2w_\tau=\Delta_h\rho_\tau+\partial_{zz}\rho_\tau.$$

Set $\tau^2\cdot N^2=1$, i.e., $N\sim 1/\tau$, which means that the stratification effect is very strong. In such a case, the scaled Boussinesq equations (1.3) can be rewritten as

$$\begin{cases}
\partial_t v_\tau-\Delta v_\tau+(v_\tau\cdot\nabla_h)v_\tau+w_\tau\partial_z v_\tau+\nabla_h p_\tau=0,\\
\tau^2\left(\partial_t w_\tau-\Delta w_\tau+v_\tau\cdot\nabla_h w_\tau+w_\tau\partial_z w_\tau\right)+\partial_z p_\tau+\rho_\tau=0,\\
\partial_t\rho_\tau-\Delta\rho_\tau+v_\tau\cdot\nabla_h\rho_\tau+w_\tau\partial_z\rho_\tau-w_\tau=0,\\
\nabla_h\cdot v_\tau+\partial_z w_\tau=0.
\end{cases}\tag{1.4}$$

Next, we supply the scaled Boussinesq equations (1.4) with the following boundary and initial conditions:

$$v_\tau, w_\tau, p_\tau \text{ and } \rho_\tau \text{ are periodic in } x,y,z,\tag{1.5}$$

$$(v_\tau, w_\tau, \rho_\tau)|_{t=0} = (v_0, w_0, \rho_0). \tag{1.6}$$

Here (v_0, w_0, ρ_0) is given. Moreover, we also equip system (1.4) with the following symmetry condition:

$$v_\tau, w_\tau, p_\tau \text{ and } \rho_\tau \text{ are even, odd, even and odd with respect to } z, \text{ respectively.} \tag{1.7}$$

Note that the above symmetry condition is preserved by the scaled Boussinesq equations (1.4), i.e., that it holds provided that the initial data satisfy this symmetry condition. Due to this fact, throughout this paper, we always suppose that the initial data satisfy that

$$v_0, w_0 \text{ and } \rho_0 \text{ are periodic in } x, y, z, \text{ and are even, odd and odd in } z, \text{ respectively.} \tag{1.8}$$

In this paper, we will use the same notation, $L^p(\Omega)$ or $H^m(\Omega)$, to denote both a space itself and its finite product spaces. For convenience, we denote by notations $\|\cdot\|_p$ and $\|\cdot\|_{p,M}$ the $L^p(\Omega)$ norm and the $L^p(M)$ norm, respectively. Moreover, since the scaled Boussinesq equations (1.4) satisfy the symmetry condition (1.7), it follows from the divergence-free condition that w_0 is uniquely determined as

$$w_0(x, y, z) = -\int_0^z \nabla_h \cdot v_0(x, y, \xi)\mathrm{d}\xi \tag{1.9}$$

for any $(x, y) \in M$ and $z \in (-1, 1)$. Hence only the initial condition of (v_τ, ρ_τ) is given throughout the paper.

For the proof of the global existence of weak solutions to the scaled Boussinesq equations (1.4), we refer to the work of Lions-Temam-Wang [26, Part IV]. Specifically, for any initial data $(u_0, \rho_0) = (v_0, w_0, \rho_0) \in L^2(\Omega)$, with $\nabla \cdot u_0 = 0$, we can prove that there exists a global weak solution $(v_\tau, w_\tau, \rho_\tau)$ of the scaled Boussinesq equations (1.4), subject to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7). Moreover, by an argument similar to that of Lions-Temam-Wang [26, Part IV], we can also show that we have a unique local strong solution $(v_\tau, w_\tau, \rho_\tau)$ for initial data $(u_0, \rho_0) = (v_0, w_0, \rho_0) \in H^1(\Omega)$, with $\nabla \cdot u_0 = 0$. The weak solutions of the scaled Boussinesq equations (1.4) are defined as follows:

Definition 1.1 Given $(u_0, \rho_0) = (v_0, w_0, \rho_0) \in L^2(\Omega)$, with $\nabla \cdot u_0 = 0$, we say that a space periodic function $(v_\tau, w_\tau, \rho_\tau)$ is a weak solution of system (1.4), subject to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7), if

(i) $(v_\tau, w_\tau, \rho_\tau) \in C_w([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ for any $T > 0$, where C_w is the space of weakly continuous function in time;

(ii) $(v_\tau, w_\tau, \rho_\tau)$ satisfies the integral equality

$$\begin{aligned}
&\int_0^\infty \int_\Omega \Big\{(-v_\tau \cdot \partial_t \varphi_h - \tau^2 w_\tau \partial_t \varphi_3 - \rho_\tau \partial_t \psi + \rho_\tau \varphi_3 - w_\tau \psi) \\
&\quad + \left[\nabla v_\tau : \nabla \varphi_h + \tau^2 \nabla w_\tau \cdot \nabla \varphi_3 + \nabla \rho_\tau \cdot \nabla \psi\right] \\
&\quad + \left[(u_\tau \cdot \nabla) v_\tau \cdot \varphi_h + \tau^2 (u_\tau \cdot \nabla w_\tau)\varphi_3 + (u_\tau \cdot \nabla \rho_\tau)\psi\right] \Big\}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t \\
= &\int_\Omega \left(v_0 \cdot \varphi_h(0) + \tau^2 w_0 \varphi_3(0) + \rho_0 \psi(0)\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z
\end{aligned}$$

for any spatially periodic function $(\varphi, \psi) = (\varphi_h, \varphi_3, \psi)$, with $\varphi_h = (\varphi_1, \varphi_2)$, such that $\nabla \cdot \varphi = 0$ and $(\varphi, \psi) \in C_c^\infty(\overline{\Omega} \times [0, \infty))$.

Remark 1.2 Similar to the theory of three-dimensional Navier-Stokes equations (see Temam [39, Ch.III, Remark 4.1] and Robinson *et al.* [36, Theorem 4.6]), we can prove that $(v_\tau, w_\tau, \rho_\tau)$ satisfies the energy inequality

$$\begin{aligned}&\frac{1}{2}\left(\|v_\tau(t)\|_2^2+\tau^2\|w_\tau(t)\|_2^2+\|\rho_\tau(t)\|_2^2\right)+\int_0^t\left(\|\nabla v_\tau\|_2^2+\tau^2\|\nabla w_\tau\|_2^2+\|\nabla\rho_\tau\|_2^2\right)\mathrm{d}s\\&\le\frac{1}{2}\left(\|v_0\|_2^2+\tau^2\|w_0\|_2^2+\|\rho_0\|_2^2\right)\end{aligned}\tag{1.10}$$

for a.e. $t \in [0, \infty)$, as long as the weak solution $(v_\tau, w_\tau, \rho_\tau)$ is obtained by the Galerkin method.

In consequence, this paper aims to study the small aspect ratio limit for system (1.4). In other words, when the aspect ratio τ goes to zero, we are going to prove that the scaled Boussinesq equations (1.4) converge to the viscous primitive equations with density stratification

$$\begin{cases}\partial_t v-\Delta v+(v\cdot\nabla_h)v+w\partial_z v+\nabla_h p=0,\\ \partial_z p+\rho=0,\\ \partial_t\rho-\Delta\rho+v\cdot\nabla_h\rho+w\partial_z\rho-w=0,\\ \nabla_h\cdot v+\partial_z w=0,\end{cases}\tag{1.11}$$

in a suitable sense, where the density stratification term w in the third equation of system (1.11) provides additional dissipation for this system. Moreover, the resulting system (1.11) satisfies the same boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7) as system (1.4).

Next, we want to recall some results concerning the primitive equations. The global existence of weak solutions of the full primitive equations was first given by Lions-Temam-Wang [25–27], but the question of uniqueness to this mathematical model is still unknown, except for some special cases [3, 20, 22, 29, 38]. The existence and uniqueness of strong solutions to the primitive equations with full dissipation in different settings is due to the work of Cao-Titi [10], Kobelkov [21], Kukavica-Ziane [23, 24], Hieber-Kashiwabara [18], and Hieber *et al.* [17], as well as Giga *et al.* [15]. The study of the global strong solutions to the primitive equations is naturally carried out in the cases of partial dissipation. More details on these cases can be found in the work of Cao-Titi [11], Fang-Han [13], Li-Yuan [31], and Cao-Li-Titi [5–9]. However, the inviscid primitive equations with or without rotation are known to be ill-posed in Sobolev spaces, and the smooth solutions may develop singularity in finite time, see Renardy [35], Han-Kwan and Nguyen [16], Ibrahim-Lin-Titi [19], Wong [42], and Cao *et al.* [4].

The rest of this paper is organized as follows: some auxiliary lemmas frequently used in the proof are collected in Section 2; the main results of the paper are stated in Section 3; the global well-posedness of strong solutions to the viscous primitive equations with density stratification (1.11) is established in Section 4; the proofs of Theorems 3.2 and 3.3 are presented in Sections 5 and 6, respectively.

2 Preliminaries

In this section, we present some Ladyzhenskaya-type inequalities in three dimensions for a class of integrals which are frequently used throughout the paper.

Lemma 2.1 ([12]) We have the inequalities

$$\begin{aligned}
&\int_M \left(\int_{-1}^1 \varphi(x,y,z)\mathrm{d}z\right)\left(\int_{-1}^1 \psi(x,y,z)\phi(x,y,z)\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y\\
\le\ & C\left\|\varphi\right\|_2^{1/2}\left(\left\|\varphi\right\|_2^{1/2}+\left\|\nabla_h\varphi\right\|_2^{1/2}\right)\left\|\psi\right\|_2^{1/2}\left(\left\|\psi\right\|_2^{1/2}+\left\|\nabla_h\psi\right\|_2^{1/2}\right)\left\|\phi\right\|_2,\\
&\int_M \left(\int_{-1}^1 \varphi(x,y,z)\mathrm{d}z\right)\left(\int_{-1}^1 \psi(x,y,z)\phi(x,y,z)\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y\\
\le\ & C\left\|\psi\right\|_2^{1/2}\left(\left\|\psi\right\|_2^{1/2}+\left\|\nabla_h\psi\right\|_2^{1/2}\right)\left\|\phi\right\|_2^{1/2}\left(\left\|\phi\right\|_2^{1/2}+\left\|\nabla_h\phi\right\|_2^{1/2}\right)\left\|\varphi\right\|_2
\end{aligned}$$

for every φ,ψ,ϕ such that the quantities on the right-hand side are finite, where C is a positive constant.

Lemma 2.2 ([28]) Let $\varphi=(\varphi_1,\varphi_2,\varphi_3)$, and let ψ and ϕ be periodic functions in Ω. Denote by $\varphi_h=(\varphi_1,\varphi_2)$ the horizontal components of the function φ. There exists a positive constant C such that it holds that

$$\left|\int_\Omega (\varphi\cdot\nabla\psi)\,\phi\mathrm{d}x\mathrm{d}y\mathrm{d}z\right|\le C\left\|\nabla\varphi_h\right\|_2^{1/2}\left\|\Delta\varphi_h\right\|_2^{1/2}\left\|\nabla\psi\right\|_2^{1/2}\left\|\Delta\psi\right\|_2^{1/2}\left\|\phi\right\|_2,$$

provided that $\varphi\in H^1(\Omega)$, with $\nabla\cdot\varphi=0$ in Ω, $\int_\Omega\varphi\mathrm{d}x\mathrm{d}y\mathrm{d}z=0$, and $\varphi_3|_{z=0}=0$, $\nabla\psi\in H^1(\Omega)$ and $\phi\in L^2(\Omega)$.

3 Main Results

Now we state the main results of this paper. In order to obtain the strong convergence results, i.e., Theorems 3.2 and 3.3, we first establish the global well-posedness of strong solutions to the viscous primitive equations with density stratification (1.11) for initial data $(v_0,\rho_0)\in H^1(\Omega)$.

Theorem 3.1 Suppose that we have a periodic function pair $(v_0,\rho_0)\in H^1(\Omega)$, with

$$\int_{-1}^1\nabla_h\cdot v_0(x,y,z)\mathrm{d}z=0,\ \int_\Omega v_0(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=0,\ \text{and}\ \int_\Omega\rho_0(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=0.$$

Then, for any $T>0$, there exists a unique strong solution (v,ρ) depending continuously on the initial data to system (1.11) on the time interval $[0,T]$, subject to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7), such that $(v,\rho)\in C([0,T];H^1(\Omega))\cap L^2(0,T;H^2(\Omega))$ and $(\partial_t v,\partial_t\rho)\in L^2(0,T;L^2(\Omega))$.

The global existence of weak solutions to the scaled Boussinesq equations (1.4) basically follows the proof in Lions-Temam-Wang [26, Part IV]. For initial data $(v_0,\rho_0)\in H^1(\Omega)$, it can be deduced from (1.9) that $(v_0,w_0,\rho_0)\in L^2(\Omega)$, which implies that system (1.4) has a global weak solution $(v_\tau,w_\tau,\rho_\tau)$. For this case, we have the following strong convergence theorem.

Theorem 3.2 Take a periodic function pair $(v_0,\rho_0)\in H^1(\Omega)$ such that

$$\int_{-1}^1\nabla_h\cdot v_0(x,y,z)\mathrm{d}z=0,\ \int_\Omega v_0(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=0,\ \text{and}\ \int_\Omega\rho_0(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=0.$$

Suppose that $(v_\tau,w_\tau,\rho_\tau)$ is a global weak solution of system (1.4), satisfying the energy inequality (1.10), and that (v,ρ) is the unique global strong solution of system (1.11), with the same boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7). Let

$$(V_\tau,W_\tau,\Gamma_\tau)=(v_\tau-v,w_\tau-w,\rho_\tau-\rho).$$

Then, for any $T>0$, it holds that

$$\sup_{0\le t\le T}\left(\|(V_\tau,\tau W_\tau,\Gamma_\tau)\|_2^2\right)(t)+\int_0^T\|\nabla(V_\tau,\tau W_\tau,\Gamma_\tau)\|_2^2\mathrm{d}t\le\tau^2\widetilde{\mathcal{K}_1}(T),$$

where $\widetilde{\mathcal{K}_1}(t)$ is a nonnegative continuously increasing function that does not depend on τ. As a result, we have the strong convergences

$$\begin{aligned}&(v_\tau,\tau w_\tau,\rho_\tau)\to(v,0,\rho)\quad\text{in } L^\infty\left(0,T;L^2(\Omega)\right),\\&(\nabla v_\tau,\tau\nabla w_\tau,\nabla\rho_\tau,w_\tau)\to(\nabla v,0,\nabla\rho,w)\quad\text{in } L^2\left(0,T;L^2(\Omega)\right),\end{aligned}$$

and the rate of convergence is of the order $O(\tau)$.

Next we assume that initial data (v_0,ρ_0) belongs to $H^2(\Omega)$. Then, from (1.9), it follows that (v_0,w_0,ρ_0) belongs to $H^1(\Omega)$. By an argument similar to that of Lions-Temam-Wang [26, Part IV], there exists a unique local strong solution $(v_\tau,w_\tau,\rho_\tau)$ to system (1.4), subject to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7). In this case, we also have the following strong convergence theorem.

Theorem 3.3 Take a periodic function pair $(v_0,\rho_0)\in H^2(\Omega)$ such that

$$\int_{-1}^1\nabla_h\cdot v_0(x,y,z)\mathrm{d}z=0,\ \int_\Omega v_0(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=0,\ \text{and}\ \int_\Omega\rho_0(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=0.$$

Suppose that $(v_\tau,w_\tau,\rho_\tau)$ is the unique local strong solution of system (1.4), and that (v,ρ) is the unique global strong solution of system (1.11), with the same boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7). Let

$$(V_\tau,W_\tau,\Gamma_\tau)=(v_\tau-v,w_\tau-w,\rho_\tau-\rho).$$

Then, for any $T>0$, there is a small positive constant $\tau(T)=\frac{3\beta_0}{4\sqrt{\widetilde{\mathcal{K}_2}(T)}}$ such that system (1.4) exists a unique strong solution $(v_\tau,w_\tau,\rho_\tau)$ on the time interval $[0,T]$, and that system (6.1)–(6.4) (see Section 6, below) has the estimate

$$\sup_{0\le t\le T}\left(\|(V_\tau,\tau W_\tau,\Gamma_\tau)\|_{H^1}^2\right)(t)+\int_0^T\|\nabla(V_\tau,\tau W_\tau,\Gamma_\tau)\|_{H^1}^2\mathrm{d}t\le\tau^2\widetilde{\mathcal{K}_3}(T),$$

provided that $\tau\in(0,\tau(T))$, where both $\widetilde{\mathcal{K}_2}(t)$ and $\widetilde{\mathcal{K}_3}(t)$ are the nonnegative continuously increasing functions that do not depend on τ. As a result, we have the strong convergences

$$\begin{aligned}&(v_\tau,\tau w_\tau,\rho_\tau)\to(v,0,\rho)\ \text{ in } L^\infty\left(0,T;H^1(\Omega)\right),\\&(\nabla v_\tau,\tau\nabla w_\tau,\nabla\rho_\tau,w_\tau)\to(\nabla v,0,\nabla\rho,w)\ \text{ in } L^2\left(0,T;H^1(\Omega)\right),\\&w_\tau\to w\ \text{ in } L^\infty\left(0,T;L^2(\Omega)\right),\end{aligned}$$

and the rate of convergence is of the order $O(\tau)$.

Remark 3.4 It should be pointed out that the case where $\bar{\varrho}(z)=$ Constant has been studied by the authors (see [34]). Compared with [34], the third equation of the resulting limit system here contains the density stratification term w. In order to establish the global H^1 theory for system (1.11), we carry out the a priori estimates on v and ρ, simultaneously. In this way, the global well-posedness of strong solutions to the viscous primitive equations with density stratification (1.11) is obtained, and the first order energy estimate on strong solutions will be used in the proof of Theorem 3.2. Moreover, Theorem 3.3 is proven by establishing the second order energy estimate on strong solutions of system (1.11).

4 Global Well-Posedness of the Primitive Equations

In this section, we establish the global well-posedness of strong solutions to the viscous primitive equations with density stratification (1.11), subject to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7).

Before this, we use the symmetry condition (1.7) to reformulate system (1.11). This symmetry condition indicates that $w|_{z=0} = 0$. Integrating the last equation of system (1.11) with respect to z yields

$$w(x,y,z,t) = -\int_0^z \nabla_h \cdot v(x,y,\xi,t)\mathrm{d}\xi.$$

We integrate the second equation of system (1.11) with respect to z to obtain that

$$p(x,y,z,t) = p_\gamma(x,y,t) - \int_0^z \rho(x,y,\xi,t)\mathrm{d}\xi,$$

in which $p_\gamma(x,y,t)$ represents the unknown surface pressure as $z = 0$. Based on the above relations, we can recast system (1.11) as

$$\begin{aligned} &\partial_t v - \Delta v + (v\cdot\nabla_h)v - \left(\int_0^z \nabla_h \cdot v(x,y,\xi,t)\mathrm{d}\xi\right)\partial_z v + \nabla_h p_\gamma(x,y,t) \\ &-\int_0^z \nabla_h \rho(x,y,\xi,t)\mathrm{d}\xi = 0, \end{aligned} \tag{4.1}$$

$$\partial_t \rho - \Delta\rho + v\cdot\nabla_h\rho - \left(\int_0^z \nabla_h\cdot v(x,y,\xi,t)\mathrm{d}\xi\right)\partial_z\rho + \int_0^z \nabla_h \cdot v(x,y,\xi,t)\mathrm{d}\xi = 0, \tag{4.2}$$

satisfying the boundary and initial conditions

$$v \text{ and } \rho \text{ are periodic in } x,y,z,$$

$$(v,\rho)|_{t=0} = (v_0,\rho_0),$$

and the symmetry condition that

$$v \text{ and } \rho \text{ are even and odd with respect to } z, \text{ respectively.}$$

4.1 L^2 Estimates on v and ρ

Taking the $L^2(\Omega)$ inner product of (4.1) and (4.2) with v and ρ, respectively, and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|_2^2 + \|\rho\|_2^2\right) + \|\nabla v\|_2^2 + \|\nabla\rho\|_2^2 \\ =& \int_\Omega \left(\int_0^z \nabla_h\rho(x,y,\xi,t)\mathrm{d}\xi\right)\cdot v\mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_\Omega\left(\int_0^z \nabla_h\cdot v(x,y,\xi,t)\mathrm{d}\xi\right)\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0, \end{aligned}$$

where we have used the facts that

$$\int_\Omega \left[(v\cdot\nabla_h)\, v - \left(\int_0^z \nabla_h\cdot v(x,y,\xi,t)\mathrm{d}\xi\right)\partial_z v\right]\cdot v\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0,$$

$$\int_\Omega \left[v\cdot\nabla_h\rho - \left(\int_0^z \nabla_h\cdot v(x,y,\xi,t)\mathrm{d}\xi\right)\partial_z\rho\right]\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$

and

$$\int_\Omega \nabla_h p_\gamma(x,y,t)\cdot v\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0.$$

Integrating the differential equation above in time between 0 to t, we have that

$$\left(\|v\|_2^2+\|\rho\|_2^2\right)(t)+\int_0^t\left(\|\nabla v\|_2^2+\|\nabla\rho\|_2^2\right)\mathrm{d}s\le\eta_1, \tag{4.3}$$

where

$$\eta_1=C\left(\|v_0\|_{H^1}^2+\|\rho_0\|_{H^1}^2\right).$$

4.2 L^4 Estimates on v and ρ

Multiplying (4.1) and (4.2) by $|v|^2v$ and $|\rho|^2\rho$, respectively, and integrating over Ω, then it follows from integration by parts that

$$\begin{aligned}
&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_4^4+\int_\Omega|v|^2\left(|\nabla v|^2+2\left|\nabla|v|\right|^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_4^4+\int_\Omega|\rho|^2\left(|\nabla\rho|^2+2\left|\nabla|\rho|\right|^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
=&\int_\Omega\left(\int_0^z v(x,y,\xi,t)\mathrm{d}\xi\right)\cdot\left[\nabla_h\left(|\rho|^2\rho\right)\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&-\int_\Omega\left(\int_0^z\rho(x,y,\xi,t)\mathrm{d}\xi\right)(\nabla_h\cdot|v|^2v)\mathrm{d}x\mathrm{d}y\mathrm{d}z-\int_\Omega\nabla_h p_\gamma(x,y,t)\cdot|v|^2v\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
=:&\,D_1+D_2+D_3,
\end{aligned}\tag{4.4}$$

in which we have used the facts that

$$\begin{aligned}
&\int_\Omega\left[(v\cdot\nabla_h)v-\left(\int_0^z\nabla_h\cdot v(x,y,\xi,t)\mathrm{d}\xi\right)\partial_z v\right]\cdot|v|^2v\mathrm{d}x\mathrm{d}y\mathrm{d}z=0,\\
&\int_\Omega\left[v\cdot\nabla_h\rho-\left(\int_0^z\nabla_h\cdot v(x,y,\xi,t)\mathrm{d}\xi\right)\partial_z\rho\right]|\rho|^2\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z=0.
\end{aligned}$$

We now estimate the first integral term D_1 on the right-hand side of (4.4). Using Hölder's inequality yields that

$$\begin{aligned}
D_1:=&\int_\Omega\left(\int_0^z v(x,y,\xi,t)\mathrm{d}\xi\right)\cdot\left[\nabla_h\left(|\rho|^2\rho\right)\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
\le&\, C\int_M\left(\int_{-1}^1|v|\mathrm{d}z\right)\left(\int_{-1}^1|\rho|^2|\nabla_h\rho|\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y\\
\le&\, C\int_M\left(\int_{-1}^1|v|\mathrm{d}z\right)\left(\int_{-1}^1|\rho|^2\mathrm{d}z\right)^{1/2}\left(\int_{-1}^1|\rho|^2|\nabla_h\rho|^2\mathrm{d}z\right)^{1/2}\mathrm{d}x\mathrm{d}y\\
\le&\, C\left(\int_\Omega|v|^4\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{1/4}\left(\int_\Omega|\rho|^4\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{1/4}\left(\int_\Omega|\rho|^2|\nabla_h\rho|^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{1/2}\\
\le&\, C\|v\|_4\|\rho\|_4\left(\int_\Omega|\rho|^2|\nabla\rho|^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{1/2}.
\end{aligned}$$

Due to Young's inequality, we have that

$$\begin{aligned}
D_1\le&\, C\|v\|_4^2\|\rho\|_4^2+\frac{3}{8}\int_\Omega|\rho|^2|\nabla\rho|^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
\le&\, C\left(\|v\|_4^4+\|\rho\|_4^4\right)+\frac{3}{8}\int_\Omega|\rho|^2|\nabla\rho|^2\mathrm{d}x\mathrm{d}y\mathrm{d}z.
\end{aligned}\tag{4.5}$$

A similar argument as that for the integral term D_1 gives that

$$D_2:=\int_\Omega\left(-\int_0^z\rho(x,y,\xi,t)\mathrm{d}\xi\right)(\nabla_h\cdot|v|^2v)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\leq C\left(\|v\|_4^4+\|\rho\|_4^4\right)+\frac{3}{8}\int_\Omega |v|^2|\nabla v|^2\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{4.6}$$

For the last integral term D_3 on the right-hand side of (4.4), we use Lemma 2.1 and Poincaré's inequality to get that

$$\begin{aligned}
D_3 := &-\int_\Omega \nabla_h p_\gamma(x,y,t)\cdot |v|^2 v\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&\leq \int_M |\nabla_h p_\gamma(x,y,t)|\left(\int_{-1}^1 |v||v|^2\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y\\
&\leq C\|v\|_2^{1/2}\|\nabla v\|_2^{1/2}\left(\|v\|_4^2+\|v\|_4\||v|\nabla v\|_2^{1/2}\right)\|\nabla_h p_\gamma\|_{2,M}.
\end{aligned}\tag{4.7}$$

Applying the operator div_h to equation (4.1) and integrating the resulting equation with respect to z from -1 to 1, we can see that $p_\gamma(x,y,t)$ satisfies the system

$$\begin{cases}
-\Delta_h p_\gamma = \dfrac{1}{2}\displaystyle\int_{-1}^1 \nabla_h\cdot\left(\nabla_h\cdot(v\otimes v)-\int_0^z \nabla_h\rho\mathrm{d}\xi\right)\mathrm{d}z,\\
\displaystyle\int_M p_\gamma(x,y,t)\mathrm{d}x\mathrm{d}y=0, p_\gamma \ \text{ is periodic in } x,y,
\end{cases}$$

where the condition $\int_M p_\gamma(x,y,t)\mathrm{d}x\mathrm{d}y=0$ is imposed to guarantee the uniqueness of $p_\gamma(x,y,t)$. By virtue of the elliptic estimates and Poincaré's inequality, we obtain that

$$\begin{aligned}
\|\nabla_h p_\gamma\|_{2,M} &\leq C\left\|\int_{-1}^1\left(\nabla_h\cdot(v\otimes v)-\int_0^z\nabla_h\rho\mathrm{d}\xi\right)\mathrm{d}z\right\|_{2,M}\\
&\leq C\left(\|\nabla_h\cdot(v\otimes v)\|_2+\|\nabla_h\rho\|_2\right)\\
&\leq C\left(\||v|\nabla v\|_2+\|\nabla\rho\|_2\right).
\end{aligned}\tag{4.8}$$

Substituting (4.8) into (4.7) and then using Young's inequality yields that

$$\begin{aligned}
D_3 &\leq C\|v\|_2^{1/2}\|\nabla v\|_2^{1/2}\left(\|v\|_4^2+\|v\|_4\||v|\nabla v\|_2^{1/2}\right)\left(\|\nabla\rho\|_2+\||v|\nabla v\|_2\right)\\
&\leq C\|v\|_2^{1/2}\|\nabla v\|_2^{1/2}\left(\|\nabla\rho\|_2\|v\|_4^2+\|\nabla\rho\|_2\|v\|_4\||v|\nabla v\|_2^{1/2}\right)\\
&\quad+C\|v\|_2^{1/2}\|\nabla v\|_2^{1/2}\left(\|v\|_4^2\||v|\nabla v\|_2+\|v\|_4\||v|\nabla v\|_2^{3/2}\right)\\
&\leq C\left(\|v\|_2\|\nabla v\|_2+\|v\|_2^2\|\nabla v\|_2^2+\|\nabla\rho\|_2^2\right)\left(\|v\|_4^4+\|\rho\|_4^4\right)\\
&\quad+\frac{1}{4}\left(\|v\|_2\|\nabla v\|_2+\|\nabla\rho\|_2^2\right)+\frac{1}{4}\||v|\nabla v\|_2^2.
\end{aligned}\tag{4.9}$$

Adding (4.5), (4.6) and (4.9) leads to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|_4^4+\|\rho\|_4^4\right)+\int_\Omega\left(|v|^2|\nabla v|^2+|\rho|^2|\nabla\rho|^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&\leq C\left(\|v\|_2\|\nabla v\|_2+\|v\|_2^2\|\nabla v\|_2^2+\|\nabla\rho\|_2^2+1\right)\left(\|v\|_4^4+\|\rho\|_4^4\right)\\
&\quad+\left(\|v\|_2\|\nabla v\|_2+\|\nabla\rho\|_2^2\right).
\end{aligned}$$

Owing to Grönwall's inequality, we get from (4.3) that

$$\begin{aligned}
&\left(\|v\|_4^4+\|\rho\|_4^4\right)(t)+\int_0^t\int_\Omega\left(|v|^2|\nabla v|^2+|\rho|^2|\nabla\rho|^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}s\\
&\leq \exp\left\{C\int_0^t\left(\|v\|_2\|\nabla v\|_2+\|v\|_2^2\|\nabla v\|_2^2+\|\nabla\rho\|_2^2+1\right)\mathrm{d}s\right\}
\end{aligned}$$

$$\times\left[\|v_0\|_4^4+\|\rho_0\|_4^4+\int_0^t\left(\|v\|_2\|\nabla v\|_2+\|\nabla\rho\|_2^2\right)\mathrm{d}s\right]\le\eta_2(t),\tag{4.10}$$

where

$$\eta_2(t)=(t+2)\mathrm{e}^{C(t+2)\left(\eta_1^2+\eta_1+1\right)}\left[\|v_0\|_{H^1}^4+\|\rho_0\|_{H^1}^4+\eta_1\right].$$

4.3 L^2 Estimates on $\partial_z v$ and $\partial_z\rho$

Taking the $L^2(\Omega)$ inner product of (4.1) and (4.2) with $-\partial_{zz}v$ and $-\partial_{zz}\rho$, respectively, we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial_z v\|_2^2+\|\partial_z\rho\|_2^2\right)+\|\nabla\partial_z v\|_2^2+\|\nabla\partial_z\rho\|_2^2\\
=&\int_\Omega\left[\left(\int_0^z\nabla_h\cdot v(x,y,\xi,t)\mathrm{d}\xi\right)\partial_{zz}\rho-\left(\int_0^z\nabla_h\rho(x,y,\xi,t)\mathrm{d}\xi\right)\cdot\partial_{zz}v\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&+\int_\Omega\left[v\cdot\nabla_h\rho-\left(\int_0^z\nabla_h\cdot v(x,y,\xi,t)\mathrm{d}\xi\right)\partial_z\rho\right]\partial_{zz}\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&+\int_\Omega\left[(v\cdot\nabla_h)\,v-\left(\int_0^z\nabla_h\cdot v(x,y,\xi,t)\mathrm{d}\xi\right)\partial_z v\right]\cdot\partial_{zz}v\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
=:&D_1+D_2+D_3.\end{aligned}\tag{4.11}$$

For the first integral term D_1 on the right-hand side of (4.11), we use integration by parts, Hölder's inequality and Young's inequality to get that

$$\begin{aligned}D_1:=&\int_\Omega\left[\left(\int_0^z\nabla_h\cdot v\mathrm{d}\xi\right)\partial_{zz}\rho-\left(\int_0^z\nabla_h\rho\mathrm{d}\xi\right)\cdot\partial_{zz}v\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
=&\int_\Omega(v\cdot\nabla_h\partial_z\rho-\rho\nabla_h\cdot\partial_z v)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
\le&\|v\|_2\|\nabla_h\partial_z\rho\|_2+\|\rho\|_2\|\nabla_h\partial_z v\|_2\\
\le&C\left(\|v\|_2^2+\|\rho\|_2^2\right)+\frac{1}{6}\left(\|\nabla\partial_z v\|_2^2+\|\nabla\partial_z\rho\|_2^2\right).\end{aligned}$$

Next, we estimate the second integral term D_2 on the right-hand side of (4.11). Using integration by parts, Hölder's inequality, the Lebesgue interpolation inequality, the Sobolev embedding theorem, and Poincaré's inequality gives that

$$\begin{aligned}D_2:=&\int_\Omega\left[v\cdot\nabla_h\rho-\left(\int_0^z\nabla_h\cdot v(x,y,\xi,t)\mathrm{d}\xi\right)\partial_z\rho\right]\partial_{zz}\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
=&\int_\Omega\left[(\nabla_h\cdot\partial_z v)\,\rho\partial_z\rho+(\partial_z v\cdot\nabla_h\partial_z\rho)\,\rho-2\,(v\cdot\nabla_h\partial_z\rho)\,\partial_z\rho\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
\le&\|\rho\|_4\|\partial_z\rho\|_4\|\nabla_h\partial_z v\|_2+\left(\|\rho\|_4\|\partial_z v\|_4+\|v\|_4\|\partial_z\rho\|_4\right)\|\nabla_h\partial_z\rho\|_2\\
\le&C\|\rho\|_4\|\partial_z\rho\|_2^{1/4}\|\nabla\partial_z\rho\|_2^{3/4}\|\nabla\partial_z v\|_2+C\|v\|_4\|\partial_z\rho\|_2^{1/4}\|\nabla\partial_z\rho\|_2^{7/4}\\
&+C\|\rho\|_4\|\partial_z v\|_2^{1/4}\|\nabla\partial_z v\|_2^{3/4}\|\nabla\partial_z\rho\|_2.\end{aligned}$$

By virtue of Young's inequality, we have that

$$D_2\le C\left(\|v\|_4^8+\|\rho\|_4^8\right)\left(\|\partial_z v\|_2^2+\|\partial_z\rho\|_2^2\right)+\frac{1}{6}\left(\|\nabla\partial_z v\|_2^2+\|\nabla\partial_z\rho\|_2^2\right).$$

With an argument similar to that for the second integral term D_2 on the right-hand side of (4.11), the last integral term D_3 can be estimated as

$$D_3:=\int_\Omega\left[(v\cdot\nabla_h)\,v-\left(\int_0^z\nabla_h\cdot v(x,y,\xi,t)\mathrm{d}\xi\right)\partial_z v\right]\cdot\partial_{zz}v\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$
\begin{aligned}
&= \int_\Omega \left[(\nabla_h \cdot \partial_z v)\, v \cdot \partial_z v + (\partial_z v \cdot \nabla_h)\, \partial_z v \cdot v - 2\,(v \cdot \nabla_h)\, \partial_z v \cdot \partial_z v \right] \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\le C \left\|v\right\|_4^8 \left(\left\|\partial_z v\right\|_2^2 + \left\|\partial_z \rho\right\|_2^2 \right) + \frac{1}{6} \left\|\nabla \partial_z v\right\|_2^2 .
\end{aligned}
$$

Combining the estimates for D_1, D_2 and D_3 gives that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left(\left\|\partial_z v\right\|_2^2 + \left\|\partial_z \rho\right\|_2^2 \right) + \left\|\nabla \partial_z v\right\|_2^2 + \left\|\nabla \partial_z \rho\right\|_2^2 \\
&\le C \left(\left\|v\right\|_4^8 + \left\|\rho\right\|_4^8 \right) \left(\left\|\partial_z v\right\|_2^2 + \left\|\partial_z \rho\right\|_2^2 \right) + C \left(\left\|v\right\|_2^2 + \left\|\rho\right\|_2^2 \right).
\end{aligned}
$$

Using Grönwall's inequality, it follows from (4.3) and (4.10) that

$$
\begin{aligned}
&\left(\left\|\partial_z v\right\|_2^2 + \left\|\partial_z \rho\right\|_2^2 \right)(t) + \int_0^t \left(\left\|\nabla \partial_z v\right\|_2^2 + \left\|\nabla \partial_z \rho\right\|_2^2 \right) \mathrm{d}s \\
&\le \exp\left\{ C \int_0^t \left(\left\|v\right\|_4^8 + \left\|\rho\right\|_4^8 \right) \mathrm{d}s \right\} \left[\left\|\partial_z v_0\right\|_2^2 + \left\|\partial_z \rho_0\right\|_2^2 + C \int_0^t \left(\left\|v\right\|_2^2 + \left\|\rho\right\|_2^2 \right) \mathrm{d}s \right] \\
&\le \eta_3(t), \qquad (4.12)
\end{aligned}
$$

where

$$
\eta_3(t) = C(t+1)\mathrm{e}^{Ct\eta_2^2(t)} \left[\left\|v_0\right\|_{H^1}^2 + \left\|\rho_0\right\|_{H^1}^2 + \eta_1 \right].
$$

4.4 L^2 Estimates on ∇v and $\nabla \rho$

Multiplying (4.1) and (4.2) by $\partial_t v - \Delta v$ and $\partial_t \rho - \Delta \rho$, respectively, integrating over Ω, and integrating by parts, we get that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left(\left\|\nabla v\right\|_2^2 + \left\|\nabla \rho\right\|_2^2 \right) + \left\|\partial_t v\right\|_2^2 + \left\|\Delta v\right\|_2^2 + \left\|\partial_t \rho\right\|_2^2 + \left\|\Delta \rho\right\|_2^2 \\
&= \int_\Omega \left[\left(\int_0^z \nabla_h \cdot v(x,y,\xi,t)\mathrm{d}\xi \right) (\Delta\rho - \partial_t \rho) - \left(\int_0^z \nabla_h \rho(x,y,\xi,t)\mathrm{d}\xi \right) \cdot (\Delta v - \partial_t v) \right] \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\quad + \int_\Omega \left[v \cdot \nabla_h \rho\, (\Delta \rho - \partial_t \rho) + (v \cdot \nabla_h)\, v \cdot (\Delta v - \partial_t v) \right] \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\quad + \int_\Omega \left(\int_0^z \nabla_h \cdot v(x,y,\xi,t)\mathrm{d}\xi \right) \left[\partial_z \rho\, (\partial_t \rho - \Delta \rho) + \partial_z v \cdot (\partial_t v - \Delta v) \right] \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&=: D_1 + D_2 + D_3. \qquad (4.13)
\end{aligned}
$$

Due to Hölder's inequality and Young's inequality, the first integral term D_1 on the right-hand side of (4.13) can be bounded as

$$
\begin{aligned}
D_1 :&= \int_\Omega \Bigg[\left(\int_0^z \nabla_h \cdot v(x,y,\xi,t)\mathrm{d}\xi \right) (\Delta \rho - \partial_t \rho) \\
&\qquad - \left(\int_0^z \nabla_h \rho(x,y,\xi,t)\mathrm{d}\xi \right) \cdot (\Delta v - \partial_t v) \Bigg] \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\le \int_M \Bigg[\left(\int_{-1}^1 |\nabla_h v|\mathrm{d}z \right) \left(\int_{-1}^1 |\partial_t \rho| + |\Delta \rho|)\mathrm{d}z \right) \\
&\qquad + \left(\int_{-1}^1 |\nabla_h \rho|\mathrm{d}z \right) \left(\int_{-1}^1 |\partial_t v| + |\Delta v|)\mathrm{d}z \right) \Bigg] \mathrm{d}x\mathrm{d}y \\
&\le C \left\|\nabla_h v\right\|_2 \left(\left\|\partial_t \rho\right\|_2 + \left\|\Delta \rho\right\|_2 \right) + C \left\|\nabla_h \rho\right\|_2 \left(\left\|\partial_t v\right\|_2 + \left\|\Delta v\right\|_2 \right) \\
&\le C \left(\left\|\nabla v\right\|_2^2 + \left\|\nabla \rho\right\|_2^2 \right) + \frac{1}{6} \left(\left\|\partial_t v\right\|_2^2 + \left\|\Delta v\right\|_2^2 + \left\|\partial_t \rho\right\|_2^2 + \left\|\Delta \rho\right\|_2^2 \right). \qquad (4.14)
\end{aligned}
$$

For the second integral term D_2 on the right-hand side of (4.13), we use Lemma 2.1, Poincaré's inequality and Young's inequality to obtain that

$$\begin{aligned}
D_2 :&= \int_\Omega [v\cdot\nabla_h\rho\,(\Delta\rho-\partial_t\rho)+(v\cdot\nabla_h)\,v\cdot(\Delta v-\partial_t v)]\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&\le \int_M \left(\int_{-1}^1 (|v|+|\partial_z v|)\,\mathrm{d}z\right)\left(\int_{-1}^1 |\nabla_h\rho|\,(|\partial_t\rho|+|\Delta\rho|)\,\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y\\
&\quad+\int_M \left(\int_{-1}^1 (|v|+|\partial_z v|)\,\mathrm{d}z\right)\left(\int_{-1}^1 |\nabla_h v|\,(|\partial_t v|+|\Delta v|)\,\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y\\
&\le C\left(\|v\|_2^{1/2}\|\nabla v\|_2^{1/2}+\|\partial_z v\|_2^{1/2}\|\nabla\partial_z v\|_2^{1/2}\right)\|\nabla\rho\|_2^{1/2}\|\Delta\rho\|_2^{1/2}\,(\|\partial_t\rho\|_2+\|\Delta\rho\|_2)\\
&\quad+C\left(\|v\|_2^{1/2}\|\nabla v\|_2^{1/2}+\|\partial_z v\|_2^{1/2}\|\nabla\partial_z v\|_2^{1/2}\right)\|\nabla v\|_2^{1/2}\|\Delta v\|_2^{1/2}\,(\|\partial_t v\|_2+\|\Delta v\|_2)\\
&\le C\left(\|v\|_2^2\|\nabla v\|_2^2+\|\partial_z v\|_2^2\|\nabla\partial_z v\|_2^2\right)\left(\|\nabla v\|_2^2+\|\nabla\rho\|_2^2\right)\\
&\quad+\frac{1}{6}\left(\|\partial_t v\|_2^2+\|\Delta v\|_2^2+\|\partial_t\rho\|_2^2+\|\Delta\rho\|_2^2\right).
\end{aligned}\tag{4.15}$$

Finally, it remains to estimate the last integral term D_3 on the right-hand side of (4.13). A similar argument as that for D_2 yields that

$$\begin{aligned}
D_3 :&= \int_\Omega \left(\int_0^z \nabla_h\cdot v(x,y,\xi,t)\mathrm{d}\xi\right)[\partial_z\rho\,(\partial_t\rho-\Delta\rho)+\partial_z v\cdot(\partial_t v-\Delta v)]\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&\le C\left(\|\partial_z v\|_2^2\|\nabla\partial_z v\|_2^2+\|\partial_z\rho\|_2^2\|\nabla\partial_z\rho\|_2^2\right)\left(\|\nabla v\|_2^2+\|\nabla\rho\|_2^2\right)\\
&\quad+\frac{1}{6}\left(\|\partial_t v\|_2^2+\|\Delta v\|_2^2+\|\partial_t\rho\|_2^2+\|\Delta\rho\|_2^2\right).
\end{aligned}\tag{4.16}$$

Summing (4.14), (4.15) and (4.16) gives that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v\|_2^2+\|\nabla\rho\|_2^2\right)+\frac{1}{2}\left(\|\partial_t v\|_2^2+\|\Delta v\|_2^2+\|\partial_t\rho\|_2^2+\|\Delta\rho\|_2^2\right)\\
&\le C\left(\|v\|_2^2\|\nabla v\|_2^2+\|\partial_z v\|_2^2\|\nabla\partial_z v\|_2^2+\|\partial_z\rho\|_2^2\|\nabla\partial_z\rho\|_2^2+1\right)\left(\|\nabla v\|_2^2+\|\nabla\rho\|_2^2\right).
\end{aligned}\tag{4.17}$$

Applying Grönwall's inequality to the above inequality, it follows from (4.3) and (4.12) that

$$\begin{aligned}
&\left(\|\nabla v\|_2^2+\|\nabla\rho\|_2^2\right)(t)+\int_0^t\left(\|\partial_t v\|_2^2+\|\Delta v\|_2^2+\|\partial_t\rho\|_2^2+\|\Delta\rho\|_2^2\right)\mathrm{d}s\\
&\le \exp\left\{C\int_0^t\left(\|v\|_2^2\|\nabla v\|_2^2+\|\partial_z v\|_2^2\|\nabla\partial_z v\|_2^2+\|\partial_z\rho\|_2^2\|\nabla\partial_z\rho\|_2^2+1\right)\mathrm{d}s\right\}\left(\|\nabla v_0\|_2^2+\|\nabla\rho_0\|_2^2\right)\\
&\le \eta_4(t),
\end{aligned}\tag{4.18}$$

where

$$\eta_4(t)=\mathrm{e}^{C(t+2)\left(\eta_1^2+\eta_3^2(t)+1\right)}\left(\|v_0\|_{H^1}^2+\|\rho_0\|_{H^1}^2\right).$$

Based on the above energy estimates, we now give the proof of Theorem 3.1.

Proof of Theorem 3.1 Adding (4.3) and (4.18), we have that

$$\sup_{0\le s\le t}\left(\|v\|_{H^1}^2+\|\rho\|_{H^1}^2\right)(s)+\int_0^t\left(\|\partial_t v\|_2^2+\|\nabla v\|_{H^1}^2+\|\partial_t\rho\|_2^2+\|\nabla\rho\|_{H^1}^2\right)\mathrm{d}s\le\eta_1+\eta_4(t),$$

where $\eta_4(t)$ is a nonnegative continuously increasing function defined on $[0,\infty)$. For any $T>0$, the following estimate holds

$$\sup_{0\le t\le T}\left(\|v\|_{H^1}^2+\|\rho\|_{H^1}^2\right)(t)+\int_0^T\left(\|\partial_t v\|_2^2+\|\nabla v\|_{H^1}^2+\|\partial_t\rho\|_2^2+\|\nabla\rho\|_{H^1}^2\right)\mathrm{d}t\le\eta_1+\eta_4(T).$$

As a consequence, the strong solution (v,ρ) exists globally in time, with $(v,\rho)\in C([0,T];H^1(\Omega))\cap L^2(0,T;H^2(\Omega))$ and $(\partial_t v,\partial_t\rho)\in L^2(0,T;L^2(\Omega))$.

Moreover, the proofs of continuous dependence on the initial data and the uniqueness of strong solutions are standard (see, e.g., [10]), so we omit these proofs here. $\square$

5 Strong Convergence for H^1 Initial Data

In this section, assume that initial data (v_0,ρ_0) belongs to $H^1(\Omega)$ with

$$\int_{-1}^{1}\nabla_h\cdot v_0(x,y,z)\mathrm{d}z=0,\quad \text{for all } (x,y)\in M.$$

We prove that the scaled Boussinesq equations (1.4) strongly converge to the viscous primitive equations with density stratification (1.11) as the aspect ratio τ goes to zero.

The following proposition is formally obtained by testing the scaled Boussinesq equations (1.4) with (v,w,ρ). As for the rigorous justification for this proposition, we refer to the work of Li-Titi [28] and Bardos *et al.* [2].

Proposition 5.1 Given a periodic function pair $(v_0,\rho_0)\in H^1(\Omega)$ with

$$\int_{-1}^{1}\nabla_h\cdot v_0\mathrm{d}z=0\ \text{ and }\ w_0(x,y,z)=-\int_0^z\nabla_h\cdot v_0(x,y,\xi)\mathrm{d}\xi,$$

suppose that $(v_\tau,w_\tau,\rho_\tau)$ is a global weak solution of system (1.4), satisfying the energy inequality (1.10), and that (v,ρ) is the unique global strong solution of system (1.11). Then it holds that

$$\begin{aligned}
&\left(\int_\Omega\left(v_\tau\cdot v+\tau^2w_\tau w+\rho_\tau\rho\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)(r)-\frac{\tau^2}{2}\|w(r)\|_2^2\\
&+\int_0^r\int_\Omega\left(\nabla v_\tau:\nabla v+\tau^2\nabla w_\tau\cdot\nabla w+\nabla\rho_\tau\cdot\nabla\rho\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\
=\ &\|v_0\|_2^2+\int_0^r\int_\Omega\left[-(u_\tau\cdot\nabla)v_\tau\cdot v-\tau^2(u_\tau\cdot\nabla w_\tau)w-(u_\tau\cdot\nabla\rho_\tau)\rho\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\
&+\frac{\tau^2}{2}\|w_0\|_2^2+\tau^2\int_0^r\int_\Omega\left(\int_0^z\partial_t v(x,y,\xi,t)\mathrm{d}\xi\right)\cdot\nabla_hW_\tau\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\
&+\|\rho_0\|_2^2+\int_0^r\int_\Omega\left(v_\tau\cdot\partial_tv+\rho_\tau\partial_t\rho+w_\tau\rho-\rho_\tau w\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t
\end{aligned}\tag{5.1}$$

for any $r\in[0,\infty)$.

With the help of this proposition, we can now estimate the difference function $(V_\tau,W_\tau,\Gamma_\tau)$.

Proposition 5.2 Let $(V_\tau,W_\tau,\Gamma_\tau)=(v_\tau-v,w_\tau-w,\rho_\tau-\rho)$. Under the same assumptions as in Proposition 5.1, it holds that

$$\sup_{0\le s\le t}\left(\|(V_\tau,\tau W_\tau,\Gamma_\tau)\|_2^2\right)(s)+\int_0^t\|\nabla(V_\tau,\tau W_\tau,\Gamma_\tau)\|_2^2\,\mathrm{d}s\le\tau^2\mathcal{K}_1(t)$$

for any $t\in[0,\infty)$, where

$$\mathcal{K}_1(t)=C\mathrm{e}^{C\eta_4^2(t)}\left[\eta_4(t)+\eta_4^2(t)+\left(\|v_0\|_2^2+\tau^2\|w_0\|_2^2+\|\rho_0\|_2^2\right)^2\right].$$

Here C is a positive constant that does not depend on τ.

Proof Multiplying the first three equations in system (1.11) by v_τ, w_τ and ρ_τ, respectively, and integrating over $\Omega\times(0,r)$, it follows from integration by parts that

$$\begin{aligned}&\int_0^r\int_\Omega(v_\tau\cdot\partial_t v+\rho_\tau\partial_t\rho+\nabla v_\tau:\nabla v+\nabla\rho_\tau\cdot\nabla\rho)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\ =&\int_0^r\int_\Omega[w\rho_\tau-\rho w_\tau-(u\cdot\nabla)v\cdot v_\tau-(u\cdot\nabla\rho)\rho_\tau]\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t.\end{aligned}\tag{5.2}$$

Replacing $(v_\tau,w_\tau,\rho_\tau)$ with (v,w,ρ), a similar argument gives that

$$\frac12\left(\|v(r)\|_2^2+\|\rho(r)\|_2^2\right)+\int_0^r\left(\|\nabla v\|_2^2+\|\nabla\rho\|_2^2\right)\mathrm{d}t=\frac12\left(\|v_0\|_2^2+\|\rho_0\|_2^2\right).\tag{5.3}$$

Thanks to Remark 1.2, the weak solution $(v_\tau,w_\tau,\rho_\tau)$ of system (1.4) satisfies the energy inequality

$$\begin{aligned}&\frac12\left(\|v_\tau(r)\|_2^2+\tau^2\|w_\tau(r)\|_2^2+\|\rho_\tau(r)\|_2^2\right)+\int_0^r\left(\|\nabla v_\tau\|_2^2+\tau^2\|\nabla w_\tau\|_2^2+\|\nabla\rho_\tau\|_2^2\right)\mathrm{d}t\\ \le&\frac12\left(\|v_0\|_2^2+\tau^2\|w_0\|_2^2+\|\rho_0\|_2^2\right).\end{aligned}\tag{5.4}$$

Subtracting the sum of (5.1) and (5.2) from the sum of (5.3) and (5.4), we have that

$$\begin{aligned}&\frac12\left(\|V_\tau(r)\|_2^2+\tau^2\|W_\tau(r)\|_2^2+\|\Gamma_\tau(r)\|_2^2\right)+\int_0^r\left(\|\nabla V_\tau\|_2^2+\tau^2\|\nabla W_\tau\|_2^2+\|\nabla\Gamma_\tau\|_2^2\right)\mathrm{d}t\\ \le&\int_0^r\int_\Omega[(u_\tau\cdot\nabla\rho_\tau)\rho+(u\cdot\nabla\rho)\rho_\tau]\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\ &+\tau^2\int_0^r\int_\Omega\left[-\left(\int_0^z\partial_t v(x,y,\xi,t)\mathrm{d}\xi\right)\cdot\nabla_h W_\tau-\nabla w\cdot\nabla W_\tau\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\ &+\int_0^r\int_\Omega[(u_\tau\cdot\nabla)v_\tau\cdot v+(u\cdot\nabla)v\cdot v_\tau]\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t+\tau^2\int_0^r\int_\Omega(u_\tau\cdot\nabla w_\tau)w\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\ =:&J_1+J_2+J_3+J_4.\end{aligned}\tag{5.5}$$

First, we estimate the integral term J_1 on the right-hand side of (5.5). Using Hölder's inequality, Lemma 2.1 and Young's inequality gives that

$$\begin{aligned}J_1:=&\int_0^r\int_\Omega[(u_\tau\cdot\nabla\rho_\tau)\rho+(u\cdot\nabla\rho)\rho_\tau]\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\ =&\int_0^r\int_\Omega[(V_\tau\cdot\nabla_h\Gamma_\tau)\rho-(\partial_z W_\tau)\Gamma_\tau\rho-W_\tau\Gamma_\tau\partial_z\rho]\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\ =&\int_0^r\int_\Omega[(V_\tau\cdot\nabla_h\Gamma_\tau)\rho+(\nabla_h\cdot V_\tau)\Gamma_\tau\rho]\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\ &+\int_0^r\int_\Omega\Gamma_\tau(\partial_z\rho)\left(\int_0^z(\nabla_h\cdot V_\tau)\mathrm{d}\xi\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\ \le&\int_0^r\int_\Omega(|V_\tau||\nabla_h\Gamma_\tau||\rho|+|\nabla_h V_\tau||\Gamma_\tau||\rho|)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\ &+\int_M\left(\int_{-1}^1|\nabla_h V_\tau|\mathrm{d}z\right)\left(\int_{-1}^1|\Gamma_\tau||\partial_z\rho|\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y\\ \le&C\int_0^r\|\nabla\rho\|_2^2\|\Delta\rho\|_2^2\left(\|V_\tau\|_2^2+\|\Gamma_\tau\|_2^2\right)\mathrm{d}t+\frac18\int_0^r\left(\|\nabla V_\tau\|_2^2+\|\nabla\Gamma_\tau\|_2^2\right)\mathrm{d}t.\end{aligned}\tag{5.6}$$

Note that the divergence-free condition, the Sobolev embedding $H^1\subset L^6$ and Poincaré's inequality are used in the above estimate. By virtue of Hölder's inequality and Young's inequality, we obtain that

$$
\begin{aligned}
J_2 :=& \tau^2 \int_0^r \int_\Omega \left[-\left(\int_0^z \partial_t v(x,y,\xi,t) \mathrm{d}\xi \right) \cdot \nabla_h W_\tau - \nabla w \cdot \nabla W_\tau \right] \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t \\
\le& C\tau^2 \int_0^r \left(\|\partial_t v\|_2^2 + \|\nabla_h w\|_2^2 + \|\partial_z w\|_2^2 \right) \mathrm{d}t + \frac{1}{8} \int_0^r \tau^2 \|\nabla W_\tau\|_2^2 \mathrm{d}t \\
\le& C\tau^2 \int_0^r \left[\|\partial_t v\|_2^2 + \int_\Omega \left(\int_0^z \nabla_h (\nabla_h \cdot v) \mathrm{d}\xi \right)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z \right] \mathrm{d}t \\
&+ C\tau^2 \int_0^r \|\nabla_h v\|_2^2 \mathrm{d}t + \frac{1}{8} \int_0^r \tau^2 \|\nabla W_\tau\|_2^2 \mathrm{d}t \\
\le& C\tau^2 \int_0^r \left(\|\partial_t v\|_2^2 + \|\Delta v\|_2^2 \right) \mathrm{d}t + \frac{1}{8} \int_0^r \tau^2 \|\nabla W_\tau\|_2^2 \mathrm{d}t.
\end{aligned} \tag{5.7}
$$

The bounds for J_3 and J_4 on the right-hand side of (5.5) can be found in Li-Titi [28, Proposition 4.2]. These two integral terms can be estimated as

$$
\begin{aligned}
J_3 :=& \int_0^r \int_\Omega [(u_\tau \cdot \nabla) v_\tau \cdot v + (u \cdot \nabla) v \cdot v_\tau] \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t \\
\le& C \int_0^r \|\nabla v\|_2^2 \|\Delta v\|_2^2 \|V_\tau\|_2^2 \mathrm{d}t + \frac{1}{8} \int_0^r \|\nabla V_\tau\|_2^2 \mathrm{d}t
\end{aligned} \tag{5.8}
$$

and

$$
\begin{aligned}
J_4 :=& \tau^2 \int_0^r \int_\Omega (u_\tau \cdot \nabla w_\tau) w \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t \\
\le& C\tau^2 \int_0^r \left(\|v_\tau\|_2^2 \|\nabla v_\tau\|_2^2 + \|\nabla v\|_2^2 \|\Delta v\|_2^2 + \tau^4 \|w_\tau\|_2^2 \|\nabla w_\tau\|_2^2 \right) \mathrm{d}t \\
&+ \frac{1}{8} \int_0^r \left(\|\nabla V_\tau\|_2^2 + \tau^2 \|\nabla W_\tau\|_2^2 \right) \mathrm{d}t,
\end{aligned} \tag{5.9}
$$

respectively.

Adding (5.6), (5.7), (5.8) and (5.9) yields that

$$
\begin{aligned}
h(t) :=& \left(\|(V_\tau, \Gamma_\tau)\|_2^2 + \tau^2 \|W_\tau\|_2^2 \right)(t) + \int_0^t \left(\|\nabla (V_\tau, \Gamma_\tau)\|_2^2 + \tau^2 \|\nabla W_\tau\|_2^2 \right) \mathrm{d}s \\
\le& C \int_0^t \|\nabla v\|_2^2 \|\Delta v\|_2^2 \|V_\tau\|_2^2 \mathrm{d}s + C\tau^2 \int_0^t \left(\|\partial_t v\|_2^2 + \|\Delta v\|_2^2 \right) \mathrm{d}s \\
&+ C\tau^2 \int_0^t \left(\|v_\tau\|_2^2 \|\nabla v_\tau\|_2^2 + \|\nabla v\|_2^2 \|\Delta v\|_2^2 + \tau^4 \|w_\tau\|_2^2 \|\nabla w_\tau\|_2^2 \right) \mathrm{d}s \\
&+ C \int_0^t \|\nabla \rho\|_2^2 \|\Delta \rho\|_2^2 \left(\|V_\tau\|_2^2 + \|\Gamma_\tau\|_2^2 \right) \mathrm{d}s =: H(t)
\end{aligned}
$$

for a.e. $t \in [0, \infty)$. Taking the derivative of $H(t)$ with respect to t, we obtain that

$$
\begin{aligned}
H'(t) \le& C\tau^2 \left(\|v_\tau\|_2^2 \|\nabla v_\tau\|_2^2 + \|\nabla v\|_2^2 \|\Delta v\|_2^2 + \tau^4 \|w_\tau\|_2^2 \|\nabla w_\tau\|_2^2 \right) \\
&+ C \left(\|\nabla v\|_2^2 \|\Delta v\|_2^2 + \|\nabla \rho\|_2^2 \|\Delta \rho\|_2^2 \right) \left(\|V_\tau\|_2^2 + \|\Gamma_\tau\|_2^2 \right) + C\tau^2 \left(\|\partial_t v\|_2^2 + \|\Delta v\|_2^2 \right) \\
\le& C \left(\|\nabla v\|_2^2 \|\Delta v\|_2^2 + \|\nabla \rho\|_2^2 \|\Delta \rho\|_2^2 \right) H(t) + C\tau^2 \left(\|\partial_t v\|_2^2 + \|\Delta v\|_2^2 \right) \\
&+ C\tau^2 \left(\|v_\tau\|_2^2 \|\nabla v_\tau\|_2^2 + \|\nabla v\|_2^2 \|\Delta v\|_2^2 + \tau^4 \|w_\tau\|_2^2 \|\nabla w_\tau\|_2^2 \right).
\end{aligned}
$$

Note that the fact that $H(0)=0$. Applying Grönwall's inequality to the above inequality, it follows from (4.18) and (5.4) that

$$\begin{aligned}
h(t) :=& \left(\|(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|W_\tau\|_2^2\right)(t)+\int_0^t\left(\|\nabla(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\nabla W_\tau\|_2^2\right)\mathrm{d}s\\
\le& C\tau^2\exp\left\{C\int_0^t\left(\|\nabla v\|_2^2\|\Delta v\|_2^2+\|\nabla\rho\|_2^2\|\Delta\rho\|_2^2\right)\mathrm{d}s\right\}\\
&\times\int_0^t\left(\|\partial_t v\|_2^2+\|\Delta v\|_2^2+\|v_\tau\|_2^2\|\nabla v_\tau\|_2^2+\|\nabla v\|_2^2\|\Delta v\|_2^2+\tau^4\|w_\tau\|_2^2\|\nabla w_\tau\|_2^2\right)\mathrm{d}s\\
\le& C\tau^2\mathrm{e}^{C\eta_4^2(t)}\left[\eta_4(t)+\eta_4^2(t)+\left(\|v_0\|_2^2+\tau^2\|w_0\|_2^2+\|\rho_0\|_2^2\right)^2\right].
\end{aligned}$$

This completes the proof. □

Based on Proposition 5.2, the proof of Theorem 3.2 is as follows:

Proof of Theorem 3.2 For any $T>0$, according to Proposition 5.2, the following estimate holds:

$$\sup_{0\le t\le T}\left(\|(V_\tau,\tau W_\tau,\Gamma_\tau)\|_2^2\right)(t)+\int_0^T\|\nabla(V_\tau,\tau W_\tau,\Gamma_\tau)\|_2^2\mathrm{d}t\le\tau^2\widetilde{\mathcal{K}_1}(T).$$

Here

$$\widetilde{\mathcal{K}_1}(T)=C\mathrm{e}^{C\eta_4^2(T)}\left[\eta_4(T)+\eta_4^2(T)+\left(\|v_0\|_2^2+\|w_0\|_2^2+\|\rho_0\|_2^2\right)^2\right],$$

and C is a positive constant that does not depend on τ. It can be deduced from the above estimate that

$$\begin{aligned}
&(v_\tau,\tau w_\tau,\rho_\tau)\to(v,0,\rho)\quad\text{in } L^\infty\left(0,T;L^2(\Omega)\right),\\
&(\nabla v_\tau,\tau\nabla w_\tau,\nabla\rho_\tau,w_\tau)\to(\nabla v,0,\nabla\rho,w)\quad\text{in } L^2\left(0,T;L^2(\Omega)\right).
\end{aligned}$$

Obviously, the rate of convergence is of the order $O(\tau)$. The theorem is proven. □

6 Strong Convergence for H^2 Initial Data

In this section, assume that initial data (v_0,ρ_0) lies in $H^2(\Omega)$, where initial velocity v_0 satisfies that

$$\int_{-1}^1\nabla_h\cdot v_0(x,y,z)\mathrm{d}z=0,\quad\text{for all } (x,y)\in M.$$

We prove that the scaled Boussinesq equations (1.4) strongly converge to the viscous primitive equations with density stratification (1.11) as the aspect ratio τ goes to zero. In this case, there is a unique local strong solution $(v_\tau,w_\tau,\rho_\tau)$ to system (1.4), subject to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7). Denote by T_τ^* the maximal existence time of this local strong solution.

Let

$$\begin{aligned}
&(U_\tau,\Gamma_\tau,P_\tau)=(V_\tau,W_\tau,\Gamma_\tau,P_\tau),\\
&(V_\tau,W_\tau,\Gamma_\tau,P_\tau)=(v_\tau-v,w_\tau-w,\rho_\tau-\rho,p_\tau-p).
\end{aligned}$$

We subtract system (1.11) from system (1.4) to obtain system

$$\partial_tV_\tau-\Delta V_\tau+(U_\tau\cdot\nabla)V_\tau+(u\cdot\nabla)V_\tau+(U_\tau\cdot\nabla)v+\nabla_hP_\tau=0,\tag{6.1}$$

$$\tau^2(\partial_t W_\tau - \Delta W_\tau + U_\tau \cdot \nabla W_\tau + U_\tau \cdot \nabla w + u \cdot \nabla W_\tau) + \partial_z P_\tau$$
$$+ \Gamma_\tau + \tau^2(\partial_t w - \Delta w + u \cdot \nabla w) = 0, \tag{6.2}$$
$$\partial_t \Gamma_\tau - \Delta \Gamma_\tau + U_\tau \cdot \nabla \Gamma_\tau + U_\tau \cdot \nabla \rho + u \cdot \nabla \Gamma_\tau - W_\tau = 0, \tag{6.3}$$
$$\nabla_h \cdot V_\tau + \partial_z W_\tau = 0, \tag{6.4}$$

defined on $\Omega \times (0, T_\tau^*)$.

Proposition 6.1 Suppose that $(v_0, \rho_0) \in H^2(\Omega)$ with $\int_{-1}^1 \nabla_h \cdot v_0 \mathrm{d}z = 0$. Then system (6.1)–(6.4) has the basic energy estimate

$$\sup_{0 \le s \le t} \left(\|(V_\tau, \tau W_\tau, \Gamma_\tau)\|_2^2 \right)(s) + \int_0^t \|\nabla (V_\tau, \tau W_\tau, \Gamma_\tau)\|_2^2 \,\mathrm{d}s \le \tau^2 \mathcal{K}_1(t)$$

for any $t \in [0, T_\tau^*)$, where

$$\mathcal{K}_1(t) = C\mathrm{e}^{C\eta_4^2(t)} \left[\eta_4(t) + \eta_4^2(t) + \left(\|v_0\|_2^2 + \tau^2 \|w_0\|_2^2 + \|\rho_0\|_2^2 \right)^2 \right].$$

Here C is a positive constant that does not depend on τ.

It is important to note that the Proposition 6.1 is a direct consequence of Proposition 5.2. Moreover, the basic energy estimate on system (6.1)–(6.4) can also be obtained by the energy method. The strong solution $(v_\tau, w_\tau, \rho_\tau)$ to system (1.4) is local, so is the basic energy estimate. In order to obtain the first order energy estimate for system (6.1)–(6.4), we need to perform the second order energy estimate on system (1.11).

Proposition 6.2 Suppose that $(v_0, \rho_0) \in H^2(\Omega)$ with $\int_{-1}^1 \nabla_h \cdot v_0 \mathrm{d}z = 0$. Then system (1.11) has the second order energy estimate

$$\sup_{0 \le s \le t} \left(\|\Delta v\|_2^2 + \|\Delta \rho\|_2^2 \right)(s) + \int_0^t \left(\|\nabla \partial_t v\|_2^2 + \|\nabla \Delta v\|_2^2 + \|\nabla \partial_t \rho\|_2^2 + \|\nabla \Delta \rho\|_2^2 \right) \mathrm{d}s \le \eta_5(t)$$

for any $t \in [0, \infty)$, where

$$\eta_5(t) = \mathrm{e}^{C(t+2)\left(\eta_4^2(t)+1\right)} \left[\|v_0\|_{H^2}^2 + \|\rho_0\|_{H^2}^2 \right].$$

Proof Taking the $L^2(\Omega)$ inner product of (4.1) and (4.2) with $\Delta(\Delta v - \partial_t v)$ and $\Delta(\Delta \rho - \partial_t \rho)$, respectively, we deduce from integration by parts that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left(\|\Delta v\|_2^2 + \|\Delta \rho\|_2^2 \right) + \|\nabla \partial_t v\|_2^2 + \|\nabla \Delta v\|_2^2 + \|\nabla \partial_t \rho\|_2^2 + \|\nabla \Delta \rho\|_2^2 \\
={}& \int_\Omega \nabla \left(\int_0^z \nabla_h \cdot v(x,y,\xi,t) \mathrm{d}\xi \right) \cdot \nabla (\Delta \rho - \partial_t \rho) \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&+ \int_\Omega \nabla \left(\int_0^z \nabla_h \rho(x,y,\xi,t) \mathrm{d}\xi \right) : \nabla (\partial_t v - \Delta v) \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&+ \int_\Omega \nabla \left[v \cdot \nabla_h \rho - \left(\int_0^z \nabla_h \cdot v(x,y,\xi,t) \mathrm{d}\xi \right) \partial_z \rho \right] \cdot \nabla (\Delta \rho - \partial_t \rho) \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&+ \int_\Omega \nabla \left[(v \cdot \nabla_h) v - \left(\int_0^z \nabla_h \cdot v(x,y,\xi,t) \mathrm{d}\xi \right) \partial_z v \right] : \nabla (\Delta v - \partial_t v) \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
=:{}& G_1 + G_2 + G_3 + G_4.
\end{aligned} \tag{6.5}$$

For the first integral term G_1 on the right-hand side of (6.5), we use Hölder's inequality and Young's inequality to get that

$$G_1 := \int_\Omega \nabla \left(\int_0^z \nabla_h \cdot v(x,y,\xi,t) \mathrm{d}\xi \right) \cdot \nabla (\Delta \rho - \partial_t \rho) \,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$
\begin{aligned}
&= \int_\Omega \nabla_h \left(\int_0^z \nabla_h \cdot v(x,y,\xi,t)\mathrm{d}\xi \right) \cdot \nabla_h \left(\Delta\rho - \partial_t \rho \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\quad + \int_\Omega (\nabla_h \cdot v) \left(\partial_z \Delta\rho - \partial_z \partial_t \rho \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_\Omega \left[\int_0^z \nabla_h \cdot \partial_i v(x,y,\xi,t)\mathrm{d}\xi \right] \left(\partial_i \Delta\rho - \partial_i \partial_t \rho \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\quad + \int_\Omega (\nabla_h \cdot v) \left(\partial_z \Delta\rho - \partial_z \partial_t \rho \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\le \int_M \left(\int_{-1}^1 |\partial_i \nabla_h v| \mathrm{d}z \right) \left(\int_{-1}^1 \left(|\partial_i \partial_t \rho| + |\partial_i \Delta\rho| \right) \mathrm{d}z \right) \mathrm{d}x\mathrm{d}y \\
&\quad + \int_\Omega |\nabla_h v| \left(|\partial_z \partial_t \rho| + |\partial_z \Delta\rho| \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\le C \left(\|\nabla v\|_2^2 + \|\Delta v\|_2^2 \right) + \frac{1}{8} \left(\|\nabla \partial_t \rho\|_2^2 + \|\nabla \Delta\rho\|_2^2 \right). \qquad (6.6)
\end{aligned}
$$

Next, we estimate the second integral term G_2 on the right-hand side of (6.5). Using the same method as for the integral term G_1 gives that

$$
\begin{aligned}
G_2 :&= \int_\Omega \nabla \left(\int_0^z \nabla_h \rho(x,y,\xi,t)\mathrm{d}\xi \right) : \nabla \left(\partial_t v - \Delta v \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\le C \left(\|\nabla \rho\|_2^2 + \|\Delta \rho\|_2^2 \right) + \frac{1}{8} \left(\|\nabla \partial_t v\|_2^2 + \|\nabla \Delta v\|_2^2 \right). \qquad (6.7)
\end{aligned}
$$

Due to Lemma 2.2 and Young's inequality, the third integral term G_3 on the right-hand side of (6.5) can be estimated as

$$
\begin{aligned}
G_3 :&= \int_\Omega \nabla \left[v \cdot \nabla_h \rho - \left(\int_0^z \nabla_h \cdot v(x,y,\xi,t)\mathrm{d}\xi \right) \partial_z \rho \right] \cdot \nabla \left(\Delta\rho - \partial_t \rho \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_\Omega \nabla \left(u \cdot \nabla \rho \right) \cdot \nabla \left(\Delta\rho - \partial_t \rho \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_\Omega \left(\partial_i u \cdot \nabla \rho + u \cdot \partial_i \nabla \rho \right) \left(\partial_i \Delta\rho - \partial_i \partial_t \rho \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\le C \|\partial_i \nabla v\|_2^{1/2} \|\partial_i \Delta v\|_2^{1/2} \|\nabla \rho\|_2^{1/2} \|\Delta \rho\|_2^{1/2} \left(\|\partial_i \partial_t \rho\|_2 + \|\partial_i \Delta\rho\|_2 \right) \\
&\quad + C \|\nabla v\|_2^{1/2} \|\Delta v\|_2^{1/2} \|\partial_i \nabla \rho\|_2^{1/2} \|\partial_i \Delta \rho\|_2^{1/2} \left(\|\partial_i \partial_t \rho\|_2 + \|\partial_i \Delta\rho\|_2 \right) \\
&\le C \left(\|\nabla v\|_2^2 \|\Delta v\|_2^2 + \|\nabla \rho\|_2^2 \|\Delta \rho\|_2^2 \right) \left(\|\Delta v\|_2^2 + \|\Delta \rho\|_2^2 \right) \\
&\quad + \frac{1}{8} \left(\|\nabla \Delta v\|_2^2 + \|\nabla \partial_t \rho\|_2^2 + \|\nabla \Delta \rho\|_2^2 \right). \qquad (6.8)
\end{aligned}
$$

For the last integral term G_4 on the right-hand side of (6.5), a similar argument as to that for the integral term G_3 gives that

$$
\begin{aligned}
G_4 :&= \int_\Omega \nabla \left[(v \cdot \nabla_h) v - \left(\int_0^z \nabla_h \cdot v(x,y,\xi,t)\mathrm{d}\xi \right) \partial_z v \right] : \nabla \left(\Delta v - \partial_t v \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_\Omega \nabla \left[(u \cdot \nabla) v \right] : \nabla \left(\Delta v - \partial_t v \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\le C \|\nabla v\|_2^2 \|\Delta v\|_2^2 \|\Delta v\|_2^2 + \frac{1}{8} \left(\|\nabla \partial_t v\|_2^2 + \|\nabla \Delta v\|_2^2 \right). \qquad (6.9)
\end{aligned}
$$

Substituting (6.6)–(6.9) into (6.5), we have that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \left(\|\Delta v\|_2^2 + \|\Delta \rho\|_2^2 \right) + \frac{1}{2} \left(\|\nabla \partial_t v\|_2^2 + \|\nabla \Delta v\|_2^2 + \|\nabla \partial_t \rho\|_2^2 + \|\nabla \Delta \rho\|_2^2 \right)
$$

$$\leq C\left(\|\nabla v\|_2^2\|\Delta v\|_2^2+\|\nabla \rho\|_2^2\|\Delta \rho\|_2^2+1\right)\left(\|\Delta v\|_2^2+\|\Delta \rho\|_2^2\right).$$

Thanks to Grönwall's inequality, it follows from (4.18) that

$$\begin{aligned}&\left(\|\Delta v\|_2^2+\|\Delta \rho\|_2^2\right)(t)+\int_0^t\left(\|\nabla\partial_t v\|_2^2+\|\nabla\Delta v\|_2^2+\|\nabla\partial_t\rho\|_2^2+\|\nabla\Delta\rho\|_2^2\right)\mathrm{d}s\\ &\leq \exp\left\{C\int_0^t\left(\|\nabla v\|_2^2\|\Delta v\|_2^2+\|\nabla \rho\|_2^2\|\Delta \rho\|_2^2+1\right)\mathrm{d}s\right\}\left(\|\Delta v_0\|_2^2+\|\Delta \rho_0\|_2^2\right)\\ &\leq \mathrm{e}^{C(t+2)\left(\eta_4^2(t)+1\right)}\left[\|v_0\|_{H^2}^2+\|\rho_0\|_{H^2}^2\right].\end{aligned}$$

The proof is completed. □

With the help of Proposition 6.2, we can perform the first order energy estimate on system (6.1)–(6.4).

Proposition 6.3 Suppose that $(v_0,\rho_0)\in H^2(\Omega)$ with $\int_{-1}^1\nabla_h\cdot v_0\mathrm{d}z=0$. Then there exists a small positive constant β_0 such that system (6.1)–(6.4) has the first order energy estimate

$$\sup_{0\leq s\leq t}\left(\|\nabla(V_\tau,\tau W_\tau,\Gamma_\tau)\|_2^2\right)(s)+\int_0^t\|\Delta(V_\tau,\tau W_\tau,\Gamma_\tau)\|_2^2\,\mathrm{d}s\leq \tau^2\mathcal{K}_2(t)$$

for any $t\in[0,T_\tau^*)$, provided that

$$\sup_{0\leq s\leq t}\left(\|\nabla(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\nabla W_\tau\|_2^2\right)(s)\leq \beta_0^2,$$

where

$$\mathcal{K}_2(t)=C\mathrm{e}^{C(1+\tau^4)\eta_5^2(t)}\left[\eta_5(t)+\eta_5^2(t)\right].$$

Here C is a positive constant that does not depend on τ.

Proof Multiplying the first three equations in system (6.1)–(6.4) by $-\Delta V_\tau$, $-\Delta W_\tau$ and $-\Delta\Gamma_\tau$, respectively, then integrating over Ω and finally integrating by parts gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\nabla W_\tau\|_2^2\right)+\|\Delta(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\Delta W_\tau\|_2^2\\ &=\int_\Omega(U_\tau\cdot\nabla\Gamma_\tau+u\cdot\nabla\Gamma_\tau+U_\tau\cdot\nabla\rho)\Delta\Gamma_\tau\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &\quad+\tau^2\int_\Omega\left(\partial_t w-\Delta w+u\cdot\nabla w\right)\Delta W_\tau\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &\quad+\tau^2\int_\Omega\left(U_\tau\cdot\nabla W_\tau+u\cdot\nabla W_\tau+U_\tau\cdot\nabla w\right)\Delta W_\tau\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &\quad+\int_\Omega\left[(U_\tau\cdot\nabla)V_\tau+(u\cdot\nabla)V_\tau+(U_\tau\cdot\nabla)v\right]\cdot\Delta V_\tau\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &=:I_1+I_2+I_3+I_4.\end{aligned}\tag{6.10}$$

For the first integral term I_1 on the right-hand side of (6.10), we apply Lemma 2.2 and Poincaré's inequality and Young's inequality to obtain that

$$\begin{aligned}I_1:=&\int_\Omega(U_\tau\cdot\nabla\Gamma_\tau+u\cdot\nabla\Gamma_\tau+U_\tau\cdot\nabla\rho)\Delta\Gamma_\tau\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ \leq& C\|\nabla V_\tau\|_2^{1/2}\|\Delta V_\tau\|_2^{1/2}\|\nabla\Gamma_\tau\|_2^{1/2}\|\Delta\Gamma_\tau\|_2^{1/2}\|\Delta\Gamma_\tau\|_2\\ &+C\|\nabla v\|_2^{1/2}\|\Delta v\|_2^{1/2}\|\nabla\Gamma_\tau\|_2^{1/2}\|\Delta\Gamma_\tau\|_2^{3/2}\\ &+C\|\nabla V_\tau\|_2^{1/2}\|\Delta V_\tau\|_2^{1/2}\|\nabla\rho\|_2^{1/2}\|\Delta\rho\|_2^{1/2}\|\Delta\Gamma_\tau\|_2\end{aligned}$$

$$\begin{aligned}&\le \frac{5}{64}\left(\|\Delta V_\tau\|_2^2+\|\Delta\Gamma_\tau\|_2^2\right)+C\left(\|\nabla V_\tau\|_2^2+\|\nabla\Gamma_\tau\|_2^2\right)\\&\quad\times\left[\left(\|\Delta V_\tau\|_2^2+\|\Delta\Gamma_\tau\|_2^2\right)+\|\Delta\rho\|_2^2\|\nabla\Delta\rho\|_2^2+\|\Delta v\|_2^2\|\nabla\Delta v\|_2^2\right].\end{aligned}\tag{6.11}$$

The integral terms I_2, I_3 and I_4 on the right-hand side of (6.10) can be bounded as

$$\begin{aligned}I_2:&=\tau^2\int_\Omega(\partial_t w-\Delta w+u\cdot\nabla w)\,\Delta W_\tau \mathrm{d}x\mathrm{d}y\mathrm{d}z\\&\le\tau^2C\left(\|\Delta v\|_2^2\|\nabla\Delta v\|_2^2+\|\nabla\partial_t v\|_2^2+\|\nabla\Delta v\|_2^2\right)+\frac{5}{64}\tau^2\|\Delta W_\tau\|_2^2,\end{aligned}\tag{6.12}$$

$$\begin{aligned}I_3:&=\tau^2\int_\Omega(U_\tau\cdot\nabla W_\tau+u\cdot\nabla W_\tau+U_\tau\cdot\nabla w)\,\Delta W_\tau \mathrm{d}x\mathrm{d}y\mathrm{d}z\\&\le\frac{5}{64}\left(\|\Delta V_\tau\|_2^2+\tau^2\|\Delta W_\tau\|_2^2\right)+C\left(\|\nabla V_\tau\|_2^2+\tau^2\|\nabla W_\tau\|_2^2\right)\\&\quad\times\left[\left(\|\Delta V_\tau\|_2^2+\tau^2\|\Delta W_\tau\|_2^2\right)+(1+\tau^4)\|\Delta v\|_2^2\|\nabla\Delta v\|_2^2\right]\end{aligned}\tag{6.13}$$

and

$$\begin{aligned}I_4:&=\int_\Omega[(U_\tau\cdot\nabla)V_\tau+(u\cdot\nabla)V_\tau+(U_\tau\cdot\nabla)v]\cdot\Delta V_\tau \mathrm{d}x\mathrm{d}y\mathrm{d}z\\&\le C\left(\|\nabla V_\tau\|_2^2\|\Delta V_\tau\|_2^2+\|\Delta v\|_2^2\|\nabla\Delta v\|_2^2\|\nabla V_\tau\|_2^2\right)+\frac{5}{64}\|\Delta V_\tau\|_2^2\\&\le C\|\nabla V_\tau\|_2^2\left(\|\Delta V_\tau\|_2^2+\|\Delta v\|_2^2\|\nabla\Delta v\|_2^2\right)+\frac{5}{64}\|\Delta V_\tau\|_2^2,\end{aligned}\tag{6.14}$$

respectively. The details of the above calculations can be found in [28, Proposition 5.2].

Combining the estimates for (6.11), (6.12), (6.13) and (6.14), we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\nabla W_\tau\|_2^2\right)+\frac{11}{16}\left(\|\Delta(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\Delta W_\tau\|_2^2\right)\\&\le C_\delta\left(\|\nabla(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\nabla W_\tau\|_2^2\right)\Big[\left(\|\Delta(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\Delta W_\tau\|_2^2\right)\\&\quad+\|\Delta\rho\|_2^2\|\nabla\Delta\rho\|_2^2+(1+\tau^4)\|\Delta v\|_2^2\|\nabla\Delta v\|_2^2\Big]\\&\quad+\tau^2C_\delta\left(\|\Delta v\|_2^2\|\nabla\Delta v\|_2^2+\|\nabla\partial_t v\|_2^2+\|\nabla\Delta v\|_2^2\right).\end{aligned}\tag{6.15}$$

Using the assumption given by the proposition,

$$\sup_{0\le s\le t}\left(\|\nabla(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\nabla W_\tau\|_2^2\right)(s)\le\beta_0^2,$$

and choosing $\beta_0=\sqrt{\frac{3}{16C_\delta}}$, it can be deduced from inequality (6.15) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\nabla W_\tau\|_2^2\right)+\left(\|\Delta(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\Delta W_\tau\|_2^2\right)\\&\le C_\delta\left[\|\Delta\rho\|_2^2\|\nabla\Delta\rho\|_2^2+(1+\tau^4)\|\Delta v\|_2^2\|\nabla\Delta v\|_2^2\right]\left(\|\nabla(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\nabla W_\tau\|_2^2\right)\\&\quad+\tau^2C_\delta\left(\|\Delta v\|_2^2\|\nabla\Delta v\|_2^2+\|\nabla\partial_t v\|_2^2+\|\nabla\Delta v\|_2^2\right).\end{aligned}$$

Note that the fact that $(V_\tau,W_\tau,\Gamma_\tau)|_{t=0}=0$. Applying Grönwall's inequality to the above inequality, it follows from Proposition 6.2 that

$$\begin{aligned}&\left(\|\nabla(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\nabla W_\tau\|_2^2\right)(t)+\int_0^t\left(\|\Delta(V_\tau,\Gamma_\tau)\|_2^2+\tau^2\|\Delta W_\tau\|_2^2\right)\mathrm{d}s\\&\le\tau^2C_\delta\exp\left\{C_\delta\int_0^t\left[\|\Delta\rho\|_2^2\|\nabla\Delta\rho\|_2^2+(1+\tau^4)\|\Delta v\|_2^2\|\nabla\Delta v\|_2^2\right]\mathrm{d}s\right\}\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^t \left(\|\Delta v\|_2^2 \|\nabla \Delta v\|_2^2 + \|\nabla \partial_t v\|_2^2 + \|\nabla \Delta v\|_2^2 \right) \mathrm{d}s \\
&\le \tau^2 C_\delta \mathrm{e}^{C_\delta (1+\tau^4)\eta_5^2(t)} \left[\eta_5(t) + \eta_5^2(t) \right].
\end{aligned}
$$

This completes the proof. □

Proposition 6.4 Let T_τ^* be the maximal existence time of strong solution $(v_\tau, w_\tau, \rho_\tau)$ to system (1.4), corresponding to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7). Then, for any $T > 0$, there exists a small positive constant $\tau(T) = \frac{3\beta_0}{4\sqrt{\widetilde{\mathcal{K}_2}(T)}}$ such that $T_\tau^* > T$, provided that $\tau \in (0, \tau(T))$. Furthermore, system (6.1)–(6.4) has the energy estimate

$$
\sup_{0 \le t \le T} \left(\|(V_\tau, \tau W_\tau, \Gamma_\tau)\|_{H^1}^2 \right)(t) + \int_0^T \|\nabla (V_\tau, \tau W_\tau, \Gamma_\tau)\|_{H^1}^2 \mathrm{d}t \le \tau^2 \left(\widetilde{\mathcal{K}_1}(T) + \widetilde{\mathcal{K}_2}(T) \right),
$$

where

$$
\widetilde{\mathcal{K}_1}(T) = C\mathrm{e}^{C\eta_4^2(T)} \left[\eta_4(T) + \eta_4^2(T) + \left(\|v_0\|_2^2 + \|w_0\|_2^2 + \|\rho_0\|_2^2 \right)^2 \right],
$$

and

$$
\widetilde{\mathcal{K}_2}(T) = C' \mathrm{e}^{C'\eta_5^2(T)} \left[\eta_5(T) + \eta_5^2(T) \right].
$$

Here both C and C' are positive constants that do not depend on τ.

The proof of Proposition 6.4 is similar to that given by Pu-Zhou [34, Proposition 4.1], and so is omitted here. Based on Proposition 6.4, we give the proof of Theorem 3.3.

Proof of Theorem 3.3 For any $T > 0$, thanks to Proposition 6.4, there exists a small positive constant $\tau(T) = \frac{3\beta_0}{4\sqrt{\widetilde{\mathcal{K}_2}(T)}}$ such that $T_\tau^* > T$, provided that $\tau \in (0, \tau(T))$, which implies that system (1.4), corresponding to the boundary and initial conditions (1.5)–(1.6) and the symmetry condition (1.7), has a unique strong solution $(v_\tau, w_\tau, \rho_\tau)$ on the time interval $[0, T]$ for all $\tau \in (0, \tau(T))$. Moreover, the following estimate holds

$$
\begin{aligned}
&\sup_{0 \le t \le T} \left(\|(V_\tau, \tau W_\tau, \Gamma_\tau)\|_{H^1}^2 \right)(t) + \int_0^T \|\nabla (V_\tau, \tau W_\tau, \Gamma_\tau)\|_{H^1}^2 \mathrm{d}t \\
&\le \tau^2 \left(\widetilde{\mathcal{K}_1}(T) + \widetilde{\mathcal{K}_2}(T) \right) =: \tau^2 \widetilde{\mathcal{K}_3}(T).
\end{aligned}
$$

Here $\widetilde{\mathcal{K}_3}(t)$ is a nonnegative continuously increasing function that does not depend on τ. Finally, it is clear that the strong convergences stated in Theorem 3.3 are the direct consequences of the above estimate. The theorem is thus proven. □

References

[1] Azérad P, Guillén F. Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics. SIAM J Math Anal, 2001, **33**(4): 847–859

[2] Bardos C, Lopes Filho M C, Niu D, et al. Stability of two-dimensional viscous incompressible flows under three-dimensional perturbations and inviscid symmetry breaking. SIAM J Math Anal, 2013, **45**(3): 1871–1885

[3] Bresch D, Guillén-González F, Masmoudi N, Rodríguez-Bellido M A. On the uniqueness of weak solutions of the two-dimensional primitive equations. Differ Integral Equ, 2003, **16**(1): 77–94

[4] Cao C, Ibrahim S, Nakanishi K, Titi E S. Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics. Commun Math Phys, 2015, **337**(2): 473–482

[5] Cao C, Li J, Titi E S. Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity. Arch Ration Mech Anal, 2014, **214**(1): 35–76

[6] Cao C, Li J, Titi E S. Global well-posedness of strong solutions to the 3D primitive equations with horizontal eddy diffusivity. J Differ Equ, 2014, **257**(11): 4108–4132

[7] Cao C, Li J, Titi E S. Global well-posedness of the three-dimensional primitive equations with only horizontal viscosity and diffusion. Commun Pure Appl Math, 2016, **69**(8): 1492–1531

[8] Cao C, Li J, Titi E S. Strong solutions to the 3D primitive equations with horizontal dissipation: near H^1 initial data. J Funct Anal, 2017, **272**(11): 4606–4641

[9] Cao C, Li J, Titi E S. Global well-posedness of the 3D primitive equations with horizontal viscosity and vertical diffusivity. Phys D, 2020, **412**: 132606

[10] Cao C, Titi E S. Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics. Ann Math, 2007, **166**(1): 245–267

[11] Cao C, Titi E S. Global well-posedness of the 3D primitive equations with partial vertical turbulence mixing heat diffusion. Commun Math Phys, 2012, **310**(2): 537–568

[12] Cao C, Titi E S. Global well-posedness and finite-dimensional global attractor for a 3-D planetary geostrophic viscous model. Commun Pure Appl Math, 2003, **56**(2): 198–233

[13] Fang D, Han B. Global well-posedness for the 3D primitive equations in anisotropic framework. J Math Anal Appl, 2020, **484**(2): 123714

[14] Furukawa K, Giga Y, Hieber M, et al. Rigorous justification of the hydrostatic approximation for the primitive equations by scaled Navier-Stokes equations. Nonlinearity, 2020, **33**(12): 6502–6516

[15] Giga Y, Gries M, Hieber M, et al. The hydrostatic Stokes semigroup and well-posedness of the primitive equations on spaces of bounded functions. J Funct Anal, 2020, **279**(3): 108561

[16] Han-Kwan D, Nguyen T. Ill-posedness of the hydrostatic Euler and singular Vlasov equations. Arch Ration Mech Anal, 2016, **221**(3): 1317–1344

[17] Hieber M, Hussein A, Kashiwabara T. Global strong L^p well-posedness of the 3D primitive equations with heat and salinity diffusion. J Differ Equ, 2016, **261**(12): 6950–6981

[18] Hieber M, Kashiwabara T. Global strong well-posedness of the three dimensional primitive equations in L^p-spaces. Arch Ration Mech Anal, 2016, **221**(3): 1077–1115

[19] Ibrahim S, Lin Q, Titi E S. Finite-time blowup and ill-posedness in Sobolev spaces of the inviscid primitive equations with rotation. J Differ Equ, 2021, **286**: 557–577

[20] Ju N. On H^2 solutions and z-weak solutions of the 3D primitive equations. Indiana Univ Math J, 2017, **66**(3): 973–996

[21] Kobelkov G M. Existence of a solution in the large for the 3D large-scale ocean dynamics equations. C R Math Acad Sci Paris, 2006, **343**(4): 283–286

[22] Kukavica I, Pei Y, Rusin W, Ziane M. Primitive equations with continuous initial data. Nonlinearity, 2014, **27**(6): 1135–1155

[23] Kukavica I, Ziane M. The regularity of solutions of the primitive equations of the ocean in space dimension three. C R Math Acad Sci Paris, 2007, **345**(5): 257–260

[24] Kukavica I, Ziane M. On the regularity of the primitive equations of the ocean. Nonlinearity, 2007, **20**(12): 2739–2753

[25] Lions J L, Temam R, Wang S. New formulations of the primitive equations of atmosphere and applications. Nonlinearity, 1992, **5**(2): 237–288

[26] Lions J L, Temam R, Wang S. On the equations of the large scale ocean. Nonlinearity, 1992, **5**(5): 1007–1053

[27] Lions J L, Temam R, Wang S. Mathematical theory for the coupled atmosphere-ocean models. J Math Pures Appl, 1995, **74**(2): 105–163

[28] Li J, Titi E S. The primitive equations as the small aspect ratio limit of the Navier-Stokes equations: rigorous justification of the hydrostatic approximation. J Math Pures Appl, 2019, **124**: 30–58

[29] Li J, Titi E S. Existence and uniqueness of weak solutions to viscous primitive equations for a certain class of discontinuous initial data. SIAM J Math Anal, 2017, **49**(1): 1–28

[30] Li J, Titi E S, Yuan G. The primitive equations approximation of the anisotropic horizontally viscous Navier-Stokes equations. J Differ Equ, 2022, **306**: 492–524

[31] Li J, Yuan G. Global well-posedness of z-weak solutions to the primitive equations without vertical diffusivity. J Math Phys, 2022, **63**(2): 021501

[32] Majda A. Introduction to PDEs and Waves for the Atmosphere and Ocean. Providence, RI: American Mathematical Society, 2003

[33] Pedlosky J. Geophysical Fluid Dynamics. New York: Springer, 1987

[34] Pu X, Zhou W. Rigorous derivation of the full primitive equations by the scaled Boussinesq equations with rotation. Bull Malays Math Sci Soc, 2023, **46**(3): Art 88

[35] Renardy M. Ill-posedness of the hydrostatic Euler and Navier-Stokes equations. Arch Ration Mech Anal, 2009, **194**(3): 877–886

[36] Robinson J C, Rodrigo J L, Sadowski W. The Three-Dimensional Navier-Stokes Equations: Classical Theory. Cambridge: Cambridge University Press, 2016

[37] Seidov D. An intermediate model for large-scale ocean circulation studies. Dynam Atmos Oceans, 1996, **25**(1): 25–55

[38] Tachim Medjo T. On the uniqueness of z-weak solutions of the three-dimensional primitive equations of the ocean. Nonlinear Anal Real World Appl, 2010, **11**(3): 1413–1421

[39] Temam R. Navier Stokes Equations: Theory and Numerical Analysis. Amsterdam: North-Holland Publishing Company, 1977

[40] Vallis G K. Atmospheric and Oceanic Fluid Dynamics. Cambridge: Cambridge University Press, 2006

[41] Washington W M, Parkinson C L. An Introduction to Three Dimensional Climate Modeling. Oxford: Oxford University Press, 1986

[42] Wong T K. Blowup of solutions of the hydrostatic Euler equations. Proc Amer Math Soc, 2015, **143**(3): 1119–1125