Acta Mathematica Scientia, 2023, **43B**(2): 821–838
https://doi.org/10.1007/s10473-023-0218-0

Mathematica Acta Scientia
数学物理学报
http://actams.apm.ac.cn

SOME RESULTS REGARDING PARTIAL DIFFERENTIAL POLYNOMIALS AND THE UNIQUENESS OF MEROMORPHIC FUNCTIONS IN SEVERAL VARIABLES*

Manli LIU (刘曼莉)
Department of Mathematics, South China Agricultural University, Guangzhou 510642, China
E-mail: *lml6641@163.com*

Lingyun GAO (高凌云)†
Department of Mathematics, Jinan University, Guangzhou 510632, China
E-mail: *tgaoly@jnu.edu.cn*

Shaomei FANG (房少梅)
Department of Mathematics, South China Agricultural University, Guangzhou 510642, China
E-mail: *dz90@scau.edu.cn*

Abstract In this paper, we mainly investigate the value distribution of meromorphic functions in $\mathbb{C}^m$ with its partial differential and uniqueness problem on meromorphic functions in $\mathbb{C}^m$ and with its k-th total derivative sharing small functions. As an application of the value distribution result, we study the defect relation of a nonconstant solution to the partial differential equation. In particular, we give a connection between the Picard type theorem of Milliox-Hayman and the characterization of entire solutions of a partial differential equation.

Key words meromorphic function in several variables; Nevanlinna theory; partial differential equation; total derivative

2010 MR Subject Classification 32H25; 30D35

1 Introduction and Main Results

Picard's theorem asserts that an entire function in the complex plane $\mathbb{C}$, omitting two distinct complex numbers must be constant. This also implies that a meromorphic function in $\mathbb{C}$ omitting three distinct values must be constant. Picard's theorem has played a decisive role in the development of the theory of entire (meromorphic) functions and other applications. It is a significant strengthening of Liouville's Theorem, which states that a bounded entire function must be constant. Recently, many researchers have paid much attention to Picard's theorem

*Received October 6, 2021; revised March 1, 2022. This work was partially supported by the NSFC (11271227, 11271161), the PCSIRT (IRT1264) and the Fundamental Research Funds of Shandong University (2017JC019).
†Corresponding author

and its applications [5, 7, 9]. Of particular interest in the connection/equivalence between Picard's theorem and the characterization of entire solutions of a differential equation, which can be found in [7], and which can be stated as follows:

Theorem A Let $a(z)$ be an entire function and let $L(z)$ be a meromorphic function in $\mathbb{C}$ with at least two distinct zeros. Then an entire solution of the differential equation $f' + a(z)L(f) = 0$ must be constant.

Furthermore, in [7], it also proved the Picard type theorem for a solution of the partial differential equation as follows:

Theorem B Let $a(z)$ be a nonzero entire function in $\mathbb{C}^m$ and let L be a nonzero meromorphic function in $\mathbb{C}$ with at least two distinct zeros. Then an entire solution f in $\mathbb{C}^m$ to the partial differential equation

$$\sum_{|\alpha|=1}^{n} a_\alpha \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_m} z_m} + a(z)L(f) = 0$$

must be constant, where $(\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m$ is a multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_m$, and a_α is a constant.

Recently, a connection between the Picard type theorem of Polya-Saxer-Milliox and the characterization of entire solutions of a differential equation was given in [8]. Motivated by the above works, we extend Theorem B to a more general form and study the defect relation of a nonconstant solution to the partial differential equation.

Specifically, we let $\mathcal{F}$ be the set of entire functions in $\mathbb{C}^m$ such that, for any function $f \in \mathcal{F}$, f is a constant or must depend on all variables $z_1, \cdots, z_m$. We consider the differential polynomial $Q(z, f)$ of f on $\mathbb{C}^m$, which is defined by

$$Q(z, f) = \sum_{i=1}^{n} \alpha_i(z) \prod_{j=0}^{p} \left(\frac{\partial^j f}{\partial z^j} \right)^{S_{ij}}, \tag{1.1}$$

where $S_{ij} (1 \le i \le n, 0 \le j \le p)$ are nonnegative integers, $\alpha_i \not\equiv 0$ $(1 \le i \le n)$ are small meromorphic functions, and $\frac{\partial^k f}{\partial z^k} = \frac{\partial^k f}{\partial z_1^{\alpha_1} \cdots z_m^{\alpha_m}}$, $(\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m : \alpha_1 + \cdots + \alpha_m = k, k \in \mathbb{N}$. Set

$$d(Q) := \min_{1 \le i \le n} \sum_{j=0}^{p} S_{ij} \quad \text{and} \quad \theta(Q) := \max_{1 \le i \le n} \sum_{j=0}^{p} j S_{ij}.$$

We always assume that Q must contain a nonzero partial differential and that its exponent is a positive integer. Our result is given as follows:

Theorem 1.1 Let $Q(z, f)$ be a polynomial of a partial differential as in (1.1), and let R be a nonconstant rational function. Then,

(i) assuming that $d(Q) > \theta(Q) + \deg R + 2$, any solution of the algebraic partial differential equation $Q(z, f) - 1 = R(f)$ must be constant;

(ii) furthermore, if $R(z) = \prod_{i=1}^{q} (z - a_i)^{l_i}$ is a nonconstant polynomial, $a_i \neq a_j$ for all $i \neq j$, and l_i is positive integer $(i = 1, \cdots, q)$. Then, for any nonconstant entire solution of the algebraic partial differential equation $Q(z, f) - 1 = R(f)$, we have that

$$(\theta(Q) + 1)\Theta(0, f) + \sum_{i=1}^{q} \Theta(a_i, f) \le \theta(Q) - d(Q) + q + 1,$$

provided that $\theta(Q) \geq d(Q) - q - 1$.

Note that our method of proving Theorem 1.1 is different with Theorem B. We use the result on a value distribution of the polynomial partial differential (see Theorem 1.9). Next, we show a Picard type theorem of Milliox-Hayman for an entire function in several variables.

Theorem 1.2 Let $n, n_1, \cdots, n_m$ be positive integers, let f be an entire function in $\mathcal{F}$, and let a be a nonzero complex number. If $f \neq 0$ and

$$f^n \left(\frac{\partial f}{\partial z_1}\right)^{n_1} \cdots \left(\frac{\partial f}{\partial z_m}\right)^{n_m} \neq a,$$

then f is a constant.

Note that we do not consider Theorem 1.2 for the class of functions outside of $\mathcal{F}$. If $f(z_1, \cdots, z_m) = g(z_1, \cdots, z_i)$, where $i < m$, $m \geq 2$, then Theorem 1.2 is considered automatically in $\mathbb{C}^i$ instead of $\mathbb{C}^m$.

Remark 1.3 Theorem 1.2 is still true for a meromorphic function in $\mathcal{F}$ via Theorem 1.9.

From Theorem 1.2, we get the following result in $\mathbb{C}$, due to Hayman:

Corollary 1.4 Let n, n_1 be positive integers, let f be an entire function in $\mathbb{C}$, and let a be a nonzero complex number. If $f \neq 0$ and $f^n(f')^{n_1} \neq a$, then f is a constant.

Theorem 1.5 Let $n, n_1, \cdots, n_m$ be positive integers, $\mathfrak{a}(z)$ be a nonzero entire function in $\mathbb{C}^m$ without zero, and let a be a nonzero complex number. Then an entire solution f in $\mathcal{F}$ to the partial differential equation

$$f^n \left(\frac{\partial f}{\partial z_1}\right)^{n_1} \cdots \left(\frac{\partial f}{\partial z_m}\right)^{n_m} - a = \mathfrak{a}(z) f^l$$

must be constant, where $l \in \mathbb{Z}^+$.

From Theorem 1.5, we get the following result in $\mathbb{C}$:

Corollary 1.6 Let n, n_1 be positive integers, let $\mathfrak{a}(z)$ be a nonzero entire function in $\mathbb{C}$ without zero, and let a be a nonzero complex number. Then an entire solution f to the differential equation

$$f^n(f')^{n_1} - a = \mathfrak{a}(z) f^l$$

must be constant, where $l \in \mathbb{Z}^+$.

Now, we show the equivalence between Theorem 1.2 and Theorem 1.5.

Theorem 1.5 $\Rightarrow$ Theorem 1.2. Since $f \neq 0$ is an entire function and

$$f^n \left(\frac{\partial f}{\partial z_1}\right)^{n_1} \cdots \left(\frac{\partial f}{\partial z_m}\right)^{n_m} \neq a,$$

then $\mathfrak{a}(z) = \frac{f^n \left(\frac{\partial f}{\partial z_1}\right)^{n_1} \cdots \left(\frac{\partial f}{\partial z_m}\right)^{n_m} - a}{f^l}$ is an entire function in $\mathbb{C}^m$ without zero. This implies that f is an entire solution of equation $f^n \left(\frac{\partial f}{\partial z_1}\right)^{n_1} \cdots \left(\frac{\partial f}{\partial z_m}\right)^{n_m} - a = \mathfrak{a}(z) f^l$. By Theorem 1.5, f is a constant function.

Theorem 1.2 $\Rightarrow$ Theorem 1.5. Suppose that f is an entire solution of the equation

$$f^n \left(\frac{\partial f}{\partial z_1}\right)^{n_1} \cdots \left(\frac{\partial f}{\partial z_m}\right)^{n_m} - a = \mathfrak{a}(z) f^l.$$

Then $f \neq 0$ and $f^n \left(\frac{\partial f}{\partial z_1}\right)^{n_1} \cdots \left(\frac{\partial f}{\partial z_m}\right)^{n_m} \neq a$. Indeed, all zeroes of

$$f^n \left(\frac{\partial f}{\partial z_1}\right)^{n_1} \cdots \left(\frac{\partial f}{\partial z_m}\right)^{n_m} - a$$

are the zero of f since $\mathfrak{a}$ has no zero in $\mathbb{C}^m$. Therefore, if there exists z_0 such that

$$f^n \left(\frac{\partial f}{\partial z_1}\right)^{n_1} \cdots \left(\frac{\partial f}{\partial z_m}\right)^{n_m} (z_0) = a.$$

This implies that $f(z_0) = 0$. Then $f^n \left(\frac{\partial f}{\partial z_1}\right)^{n_1} \cdots \left(\frac{\partial f}{\partial z_m}\right)^{n_m} (z_0) = 0$, which is a contradiction. Similarly, $f \neq 0$. By Theorem 1.2, f is constant.

For the convenience of the reader, we recall the definition of a total derivative. Letting f be a meromorphic function on $\mathbb{C}^m$, the total derivative Df of f is defined by

$$Df(z) = \sum_{j=1}^{n} z_j f_{z_j}(z),$$

where $z = (z_1, z_2, \cdots, z_m) \in \mathbb{C}^m$, and f_{z_j} is the partial derivative of f with respect to $z_j(j = 1, 2, \cdots, m$. The k-th order total derivative $D^k f$ of f is defined inductively by

$$D^k f = D(D^{k-1} f), k = 1, 2, \cdots,$$

where $D^0 f = f$. If f is a nonconstant meromorphic function, then $Df \not\equiv 0$.

A total differential polynomial $P(z, f)$ of f on $\mathbb{C}^m$ is defined by

$$P(z, f) = \sum_{i=1}^{n} \alpha_i(z) \Pi_{j=0}^{p} \left(D^j f(z)\right)^{S_{ij}},$$

where $S_{ij}(1 \leq i \leq n, 0 \leq j \leq p)$ are the nonnegative integers, and $\alpha_i \not\equiv 0$ $(1 \leq i \leq n)$ are small meromorphic functions. Set

$$d(P) := \min_{1 \leq i \leq n} \sum_{j=0}^{p} S_{ij} \quad \text{and} \quad \theta(P) := \max_{1 \leq i \leq n} \sum_{j=0}^{p} j S_{ij}.$$

We always assume that P must contain a nonzero k-th order total derivative and that its exponent is a positive integer. Our result on the value distribution of $P(z, f)$ is given as follows:

Theorem 1.7 Let $a_1, \cdots, a_q$ be distinct nonzero complex numbers. Let f be a transcendental meromorphic function on $\mathbb{C}^m$ and let $P(z, f)$ be a non-constant total differential polynomial in f with $d(P) \geq 2$. Then

$$T_f(r) \leq \frac{q\theta(P) + 1}{qd(P) - 1} \overline{N}_f (r, 0) + \frac{1}{qd(P) - 1} \sum_{j=1}^{q} \overline{N}_P (r, a_j) + o(T_f(r))$$

for all $r \in [1, +\infty)$, excluding a set of finite Lebesgue measure.

Moreover, in the case where f is a transcendental entire function, we have that

$$T_f(r) \leq \frac{q\theta(P) + 1}{qd(P)} \overline{N}_f (r, 0) + \frac{1}{qd(P)} \sum_{j=1}^{q} \overline{N}_P (r, a_j) + o(T_f(r))$$

for all $r \in [1, +\infty)$, excluding a set of finite Lebesgue measure.

From Theorem 1.7, we get the following Picard-type theorem:

Corollary 1.8 Let f be a transcendental meromorphic function on $\mathbb{C}^m$. Let $n, n_1 \cdots, n_k$, $k \geq 1$ be positive integers. If $f^n(Df)^{n_1} \cdots (D^k f)^{n_k}$ is not a constant function, then assume all finite values infinitely often as $n + \sum\limits_{t=1}^{k} n_t \geq \sum\limits_{t=1}^{k} tn_t + 3$. Furthermore, if f is a transcendental entire function, the conclusion holds for $n + \sum\limits_{t=1}^{k} n_t \geq \sum\limits_{t=1}^{k} tn_t + 2$.

By the same arguments as those in Theorem 1.7, we get the following result:

Theorem 1.9 Let $a_1, \cdots, a_q$ be distinct nonzero complex numbers. Let f be a non-constant meromorphic function on $\mathbb{C}^m$ and let $Q(z, f)$ be a non-constant partial differential polynomial in f with $d(Q) \geq 2$. Then

$$T_f(r) \leq \frac{q\theta(Q)+1}{qd(Q)-1}\overline{N}_f(r,0) + \frac{1}{qd(Q)-1}\sum_{j=1}^{q}\overline{N}_Q(r,a_j) + o(T_f(r))$$

for all $r \in [1, +\infty)$, excluding a set of finite Lebesgue measure.

Moreover, in the case where f is a transcendental entire function, we have that

$$T_f(r) \leq \frac{q\theta(Q)+1}{qd(Q)}\overline{N}_f(r,0) + \frac{1}{qd(Q)}\sum_{j=1}^{q}\overline{N}_Q(r,a_j) + o(T_f(r))$$

for all $r \in [1, +\infty)$, excluding a set of finite Lebesgue measure.

Remark 1.10 Theorem 1.9 is proven similarly as the Theorem 1.7 by using Remark 2.5 instead of Lemma 2.4.

In 2013, F. Lv considered a Picard-type theorem for a meromorphic function on several complex variables, and obtained $Df - af^n$ assumes all finite values infinitely often with $n \geq 5$. With the aid of his idea, we give a Picard-type theorem below.

Theorem 1.11 Let f be a transcendental meromorphic function on $\mathbb{C}^m$ such that $D^k f \not\equiv 0$. Let a be a finite nonzero constant and let $n \geq k+4$ be an integer. Then $D^k f - af^n$ assumes all finite values infinitely often.

One notices that our result actually provides an extension to some main results of F. Lv; if we take that $k = 1$, the theorem obtained by Lv is a special case of 1.11.

Let f be a meromorphic function in the complex domain. For two meromorphic functions f, g, if $f - \alpha$ and $g - \alpha$ have the same zeros, counting multiplicity (ignoring multiplicity), then f and g share the same function α CM (IM). Usually, we say that a is a small function with respect to f if $T_a(r) = o(T_f(r)) = S(r, f)$ as $r \to \infty$ outside of a possible exceptional set of finite measure.

In recent decades, uniqueness problems on meromorphic functions have been studied deeply via Nevanlinna theory; a large number of research works on the uniqueness problem have been undertaken in a complex plane [1, 2, 6, 12–14, 16], etc. As a very active subject, problems on the uniqueness of entire functions sharing values with the derivatives attracted a lot of attention. In particular, Yi [12] proved the following theorem:

Theorem C Let f and g be two nonconstant entire functions on the complex plane, and let k be a positive integer. If f and g share 0 CM, $f^{(k)}$ and $g^{(k)}$ share 1 CM, and $\delta(0, f) > \frac{1}{2}$, then $f^{(k)}g^{(k)} \equiv 1$, unless $f \equiv g$.

In [6], Jin extended Theorem C to $\mathbb{C}^m$; here f and g are both entire functions. However, it is natural to consider the following questions: in what condition can we get a similar result for transcendental meromorphic functions on $\mathbb{C}^m$ and a small function a of f? In this paper, we apply a different method to that above, and obtain the following result, which answers the above question:

Theorem 1.12 Let $k \geq 1$, let f be a transcendental meromorphic function on $\mathbb{C}^m$, and let $a \not\equiv 0, \infty$ be a small meromorphic function of f. Suppose that $D^k f$ is a nonconstant function. If $f - a$ and $D^k f - a$ share the value 0 CM, and $D^k f$ and a do not have some common poles of the same multiplicity, and

$$2\delta(0, f) + (3 + k)\Theta(\infty, f) > \frac{9}{2} + k, \tag{1.2}$$

then $f \equiv D^k f$.

By arguments the same as those in Theorem 1.12, we get the following result:

Theorem 1.13 Let $k \geq 1$, f be a transcendental meromorphic function on $\mathbb{C}^m$, and let $a \not\equiv 0, \infty$ be a small meromorphic function of f. If $f - a$ and $\frac{\partial^k f}{\partial z^k} - a$ share the value 0 CM, and $\frac{\partial^k f}{\partial z^k}$ and a do not have some common poles of same multiplicity, and

$$2\delta(0, f) + (3 + k)\Theta(\infty, f) > \frac{9}{2} + k,$$

then $f \equiv \frac{\partial^k f}{\partial z^k}$.

2 Some Notations and Auxiliary Lemmas of Nevanlinna Theory

Set $\|z\| = \left(|z_1|^2 + \cdots + |z_m|^2\right)^{1/2}$ for $z = (z_1, \ldots, z_m)$ and define that

$$B_m(r) = \{z \in \mathbb{C}^m : \|z\| < r\}, S_m(r) = \{z \in \mathbb{C}^m : \|z\| = r\} \ (r > 0),$$

$$v_m(z) = dd^c\|z\|^2, \quad \sigma_m(z) = d^c \log \|z\|^2 \wedge \left(dd^c \log \|z\|^2\right)^{m-1}$$

for $z \in \mathbb{C}^m \backslash \{0\}$. Then $\sigma_m(z)$ is a positive measure on $S_m(r)$ with the total measure one. Let $a \in P^1$. Let $f : \mathbb{C}^m \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function. For each $a \in \mathbb{P}^1(\mathbb{C})$ with $f^{-1}(a) \neq \mathbb{C}^m$, we denote by Z_a^f the a -divisor of f, and write $Z_a^f(r) = \overline{B}_m(r) \cap Z_a^f$. In addition, we define that

$$n_f(r, a) = r^{2-2m} \int_{Z_a^f(r)} v_m^{m-1}(z).$$

Then the corresponding counting function $N_f(r, a)$ is defined as

$$N_f(r, a) = \int_0^r [n_f(t, a) - n_f(0, a)] \frac{\mathrm{d}t}{t} + n_f(0, a) \log r,$$

where $n_f(0, a)$ is the Lelong number of Z_a^f at the origin. In particular, we define the divisor $\overline{Z}_a^f = \min\left\{1, Z_a^f\right\}, \overline{n}_f(r, a)$ and the reduced counting function $\overline{N}_f(r, a)$. For positive integer k, define the truncated multiplicity functions on $\mathbb{C}^m$ by $Z_a^{f,k}(z) = \min\{Z_a^f(z), k\}$, and the corresponding truncated counting function by $n_\nu(t) = n_k\left(t, \frac{1}{f-a}\right)$ if $\nu = Z_a^{f,k}(z)$, and the truncated valence function by $N_\nu(t) = N_k\left(t, \frac{1}{f-a}\right)$ if $\nu = Z_a^{f,k}(z)$. The proximity function

$m_f(r,a)$ is defined as

$$m_f(r,a)=\begin{cases}\displaystyle\int_{S_m(r)}\log^+\left|\frac{1}{f(z)-a}\right|\sigma_m(z), & \text{if } a\neq\infty\\ \displaystyle\int_{S_m(r)}\log^+|f(z)|\sigma_m(z), & \text{if } a=\infty\end{cases}$$

and the characteristic function $T_f(r)$ is

$$T_f(r)=m_f(r,a)+N_f(r,a).$$

Now, we recall the quantity

$$\delta(a,f)=\liminf_{r\to\infty}\frac{m_f(r,a)}{T_f(r)}=1-\limsup_{r\to\infty}\frac{N_f(r,a)}{T_f(r)}$$

the defect (or deficiency) of a with respect to f. Then, $0\le\delta(a,f)\le 1$. The quantity

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T_f(r)}{\log r}$$

is said to be the order of f.

Here, for brevity, we replace the notations $m_f(r,a)$, $N_f(r,a)$ and $\overline{N}_f(r,a)$ by $m\left(r,\frac{1}{f-a}\right)$ $N\left(r,\frac{1}{f-a}\right)$ and $\overline{N}\left(r,\frac{1}{f-a}\right)$, respectively. If $a=\infty$, we write $m(r,f),N(r,f)$ and $\overline{N}(r,f)$. Moreover, the notation $'\|P'$ means that the assertion P holds for all $r\in[0,\infty)$ outside of a Borel subset E of the interval $[0,\infty)$ with $\int_E \mathrm{d}r<\infty$.

From the the Logarithmic Derivative Lemma for a meromorphic function in several variables [4, 11, 15], we obtain the following:

Lemma 2.1 Let f be a nonconstant meromorphic function on $\mathbb{C}^m$. Then for any positive integer k,

$$m\left(r,\frac{D^kf}{f}\right)=O\left(\log rT_f(r)\right)$$

holds for all large r outside a set with finite Lebesgue measure.

Lemma 2.2 ([9]) Let f be a non-constant meromorphic function on $\mathbb{C}^m$ and let $S=I_f\cap\left(\operatorname{supp}Z^f_\infty\right)_s$, where A_s denotes the set of singular points of an analytic set A. Supposing that $z_0\notin S$ and $Z^f_\infty(z_0)=p\ge 1$, $Z^{Df}_\infty(z_0)\le p+1$.

From Lemma 2.2, we get the following result for higher total derivative:

Lemma 2.3 Let f be a non-constant meromorphic function on $\mathbb{C}^m$ and let $S=I_f\cap\left(\operatorname{supp}Z^f_\infty\right)_s$, where A_s denotes the set of singular points of an analytic set A. Supposing that $z_0\notin S$ and $Z^f_\infty(z_0)=p\ge 1$, $Z^{D^kf}_\infty(z_0)\le p+k$.

Proof First, we see that all poles of D^kf come from the poles of f. If $k=2$, we apply above lemma for Df and note that $D^2f=D(Df)$, so we get that $Z^{D^2f}_\infty(z_0)\le p+2$ for $z_0\notin S$ and $Z^f_\infty(z_0)=p\ge 1$. Similarly, for $k\ge 2$, we can get that

$$Z^{D^kf}_\infty(z_0)\le p+k.$$

From Lemma 2.3, we get that

$$N_{D^kf}(r,\infty))\le N_f(r,\infty)+k\overline{N}_f(r,\infty),$$

where f is a nonconstant meromorphic function such that $D^kf\not\equiv 0$. □

Lemma 2.4 Let f be a non-constant meromorphic function on $\mathbb{C}^n$ such that $D^k f \not\equiv 0$. Then we have that

$$N_f(r,0) - N_{D^k f}(r,0) \le k\overline{N}_f(r,0). \tag{2.1}$$

Proof Suppose that $z^0 = (z_1^0, z_2^0, \cdots, z_n^0) \notin S$ is a zero of f with the multiplicity p. Now, we distinguish the following two cases:

Case I If z^0 is a zero of f with the multiple $p \le k$, then z^0 is a zero of $D^k f$, or not. Thus, z^0 is a zero counted on the left hand side with a multiple of at most p, and z^0 is a zero counted on the right side hand with a multiple k. Since $p \le k$, one observes that (2.1) holds.

Case II If z^0 is a zero of f with the multiple $p > k$, then z^0 is a zero of $D^k f$ with a multiple of at most $p-k$. Indeed, one can deduce that

$$Df = -f^2 D\frac{1}{f}. \tag{2.2}$$

Since z^0 is a zero of f with a multiple p, then z^0 is a pole of $\frac{1}{f}$ with a multiple p, and by applying Lemma 2.2, we can get that z^0 is a pole of $D\frac{1}{f}$ with a multiple $p_1 \le p+1$. By (2.2), we have that z^0 is a zero of Df with a multiple $\widetilde{p} \ge p-1$. Set that $Df = g$. Immediately, we have that

$$Dg = -g^2 D\frac{1}{g} \Longrightarrow D(Df) = -(Df)^2 D\frac{1}{Df}.$$

From the above equation, we have that z^0 is a zero of Df with a multiple $\widetilde{p}$, and that z^0 is a pole of $\frac{1}{Df}$ with a multiple $\widetilde{p}$. Furthermore, by simple computing, we get that z^0 is a zero of $D^2 f$ with a multiple of at least of $\widetilde{p} - 1 \ge p-2$.

By repeating the proof of above, we can obtain that $D^k f = D(D^{k-1} f)$, and that z^0 is a zero of $D^k f$ with a multiple of at least $p-k$. In this case, z^0 is counted in $N_f(r,0) - N_{D^k f}(r,0)$ at most k times.

Combining Case I and Case II, we get that

$$N_f(r,0) - N_{D^k f}(r,0) \le k\overline{N}_f(r,0),$$

so (2.1) is proved. □

Remark 2.5 Let f be a non-constant meromorphic function on $\mathbb{C}^n$ such that $\frac{\partial^k f}{\partial z^k} \not\equiv 0$. Then we have that

$$N_f(r,0) - N_{\frac{\partial^k f}{\partial z^k}}(r,0) \le k\overline{N}_f(r,0)$$

for any multiple index $(\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m : \alpha_1 + \cdots + \alpha_m = k$, where $\partial z^k = \partial z_1^{\alpha_1} \cdots \partial z_m^{\alpha_m}$.

Proof Remark 2.5 is proved similarly as for Lemma 2.4, by using a result similar as of the Lemma 2.2 for the partial differential ([4], page 105). □

By using Lemma 2.1, and computing similarly as for [6], we get the following:

Lemma 2.6 Let f be a non-constant meromorphic function on $\mathbb{C}^m$ such that $D^k f \not\equiv 0$. Then, for any positive integer k,

$$\| N_{D^k f}(r,0) \le T_{D^k f}(r) - T_f(r) + N_f(r,0) + O(\log r T_f(r)).$$

By using the Logarithmic Derivative Lemma for a meromorphic function in several variables [4], we get the following:

Lemma 2.7 Let f be a non-constant meromorphic function on $\mathbb{C}^m$ such that $\frac{\partial^k f}{\partial z^k} \not\equiv 0$. Then, for any positive integer k,

$$\| N_{\frac{\partial^k f}{\partial z^k}}(r,0) \leq T_{\frac{\partial^k f}{\partial z^k}}(r) - T_f(r) + N_f(r,0) + O(\log rT_f(r)).$$

Lemma 2.8 ([4]) Let $f_j(\not\equiv 0), 1, 2, \cdots, n(\geq 2)$ be linearly independent meromorphic functions on $\mathbb{C}^m$ such that $\sum\limits_{j=1}^{n} f_j \equiv 1$. Take multi-indices $\nu_i \in \mathbb{Z}_+^m (i = 1, \cdots, n-1)$ such that

$$0 < |\nu_i| \leq i(i = 1, \cdots, n-1), |\nu_1| \leq |\nu_2| \leq \cdots \leq |\nu_{n-1}| := \omega,$$

and

$$\mathbf{W} = \mathbf{W}_{\nu_1 \cdots \nu_{n-1}}(f_1, \cdots, f_n) \not\equiv 0.$$

Define $l = |\nu_1| + |\nu_2| + \cdots + |\nu_{n-1}|$ and set $B_n = \max\limits_{2 \leq s \leq n} \left\{ \frac{1}{s-1} \sum\limits_{i=1}^{s-1} |\nu_{n-i}| \right\}$. Then, for $j = 1, \cdots, n$, the inequality

$$T_{f_j}(r) < \sum_{i=1}^{n} N_\omega\left(r, \frac{1}{f_i}\right) + B_n \sum_{i \neq j} \overline{N}_{f_i}(r, \infty) + S(r)$$

holds for $r_0 < r < \rho < R$, where $B_n \leq n-1$ and $S(r) = o(\max\limits_{1 \leq j \leq n} \{T_{f_j}(r)\})$.

Lemma 2.9 Let f_1, f_2, f_3 be meromorphic function on $\mathbb{C}^m$ such that

$$\| \sum_{j=1}^{3} \overline{N}_{f_j}(r,0) + \sum_{j=1}^{3} \overline{N}_{f_j}(r, \infty) \leq (\lambda + o(1))T(r),$$

where $T(r) = \max\limits_{1 \leq j \leq 3} \{T_{f_j}(r)\}$ and one has the constant $\lambda < \frac{1}{2}$. Then f_1, f_2, f_3 are linearly dependent.

Proof Assume that $f_j (j = 1, 2, 3)$ are linearly independent. Then, by applying Lemma 2.8, we have that

$$\| T_{f_i}(r) \leq \sum_{j=1}^{3} N_2\left(r, \frac{1}{f_j}\right) + 2\sum_{i \neq j} \overline{N}_{f_j}(r, \infty) \leq \sum_{j=1}^{3} N_2\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} \overline{N}_{f_j}(r, \infty).$$

Noting that $T(r) = \max\{T_{f_j}(r)\} (j = 1, 2, 3)$,

$$\begin{aligned} \| T(r) &\leq \sum_{j=1}^{3} N_2\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} \overline{N}_{f_j}(r, \infty) \leq \sum_{j=1}^{3} 2\overline{N}_{f_j}(r, 0) + 2\sum_{j=1}^{3} \overline{N}_{f_j}(r, \infty) \\ &\leq 2(\lambda + o(1))T(r) < T(r), \end{aligned}$$

which gives a contradiction, and the conclusion obtained. □

Lemma 2.10 Let f_1, f_2 be two non-constant meromorphic functions on $\mathbb{C}^m$, and let c_1, c_2, c_3 be three nonzero constants. If $c_1 f_1 + c_2 f_2 = c_3$ holds, then

$$\| T_{f_1}(r) \leq \overline{N}_{f_1}(r, 0) + \overline{N}_{f_1}(r, \infty) + \overline{N}_{f_2}(r, 0) + S(r, f).$$

Proof By the second main theorem [3, 10], we have that

$$\| T_{f_1}(r) \leq \overline{N}_{f_1}(r, 0) + \overline{N}_{f_1}(r, \infty) + \overline{N}_{f_1}(r, c_3/c_1) + S(r, f).$$

Noting that $\overline{N}_{f_1}(r, c_3/c_1) = \overline{N}_{f_2}(r, 0)$, we have that

$$\| T_{f_1}(r) \leq \overline{N}_{f_1}(r, 0) + \overline{N}_{f_1}(r, \infty) + \overline{N}_{f_2}(r, 0) + S(r, f).$$

Hence, we obtain the conclusion. □

Remark 2.11 Lemma 2.10 is still true when c_1, c_2, c_3 are three nonzero small functions of f_1 (see [3]).

3 The Proof of Theorem 1.2

Proof If f is not constant, then $f^n\left(\frac{\partial f}{\partial z_1}\right)^{n_1}\cdots\left(\frac{\partial f}{\partial z_m}\right)^{n_m}$ is not constant. Indeed, we see that $f^n\left(\frac{\partial f}{\partial z_1}\right)^{n_1}\cdots\left(\frac{\partial f}{\partial z_m}\right)^{n_m}\not\equiv 0$, since $f\in\mathcal{F}$. Suppose that

$$f^n\left(\frac{\partial f}{\partial z_1}\right)^{n_1}\cdots\left(\frac{\partial f}{\partial z_m}\right)^{n_m}\equiv c, \tag{3.1}$$

where $c\neq 0$ is a constant. Then f does not have any zero. From (3.1), we get

$$\left(\frac{\partial f}{\partial z_1}/f\right)^{n_1}\cdots\left(\frac{\partial f}{\partial z_m}/f\right)^{n_m}=\frac{c}{f^{n+n_1+\cdots+n_m}}.$$

By the Logarithmic Derivative Lemma for a meromorphic function in several variables [4], we deduce that $m(r,\frac{1}{f})=S(r,f)$. Since f has no zero, then $N(r,\frac{1}{f})=0$. This implies that

$$T_f(r)=T(r,\frac{1}{f})+O(1)=S(r,f).$$

This is a contradiction, since f is not constant. Therefore, applying Theorem 1.9, we get that $T_f(r)=S(r,f)$. This is again a contradiction . Hence, Theorem 1.2 is proven. □

4 The Proof of Theorem 1.7

Proof of Theorem 1.7 For any z such that $|f(z)|\leq 1$, since $\sum\limits_{j=0}^{p}S_{ij}\geq d(P)(1\leq i\leq n)$, we have that

$$\frac{1}{|f(z)|^{d(P)}}=\frac{1}{|P(z,f)|}\cdot\frac{|P(z,f)|}{|f(z)|^{d(P)}}\leq\frac{1}{|P(z,f)|}\cdot\sum_{i=1}^{n}\left(|\alpha_i(z)|\prod_{j=0}^{p}\left|\frac{D^jf}{f}\right|^{S_{ij}}\right).$$

The above inequality implies that, for all $z\in\mathbb{C}^m$,

$$\log^+\frac{1}{|f(z)|^{d(P)}}\leq\log^+\left(\frac{1}{|P(z,f)|}\cdot\sum_{i=1}^{n}\left(|\alpha_i(z)|\prod_{j=0}^{p}\left|\frac{D^jf}{f}\right|^{S_{ij}}\right)\right).$$

On the one hand, by the lemma on the logarithmic derivative and the first main theorem, we have that

$$d(P)m_f(r,0)\leq m_P(r,0)+o(T_f(r))=T_P(r)-N_P(r,0)+o(T_f(r)).$$

On the other hand, by the second main theorem (used with the $q+1$ different values $0,a_1,\cdots,a_q$), we have that

$$qT_P(r)\leq\overline{N}_P(r,\infty)+\overline{N}_P(r,0)+\sum_{j=1}^{q}\overline{N}_P(r,a_j)+o(T_f(r)).$$

Hence,

$$d(P)m_f(r,0) \leq \frac{1}{q}\left(\overline{N}_P(r,\infty)+\overline{N}_P(r,0)+\sum_{j=1}^{q}\overline{N}_P(r,a_j)\right)-N_P(r,0)+o(T_f(r)).$$

Using the first main theorem again, one deduces that

$$\begin{aligned}d(P)T_f(r) &= d(P)T_f(r,0)+O(1)\\ &= d(P)m_f(r,0)+d(P)N_f(r,0)+O(1)\\ &\leq \frac{1}{q}\left(\overline{N}_P(r,\infty)+\overline{N}_P(r,0)+\sum_{j=1}^{q}\overline{N}_P(r,a_j)\right)\\ &\quad + d(P)N_f(r,0)-N_P(r,0)+o(T_f(r)).\end{aligned} \tag{4.1}$$

We have that

$$\frac{1}{f^{d(P)}}=\frac{1}{P}\sum_{i=1}^{n}\left(\alpha_i f^{(\sum\limits_{j=0}^{p}S_{ij})-d(P)}\prod_{j=0}^{p}\left(\frac{D^j f}{f}\right)^{S_{ij}}\right),$$

and note that $(\sum\limits_{j=0}^{p}S_{ij})-d(P)\geq 0$. Immediately, we get that

$$\begin{aligned}d(P)Z_0^f(z) &\leq Z_0^P(z)+\max_{1\leq i\leq n}\{Z_\infty^{\alpha_i}(z)+\sum_{j=0}^{p}S_{ij}Z_\infty^{\frac{D^j f}{f}}(z)\}\\ &\leq Z_0^P(z)+\sum_{1\leq i\leq n}Z_\infty^{\alpha_i}(z)+\max\left\{\sum_{j=0}^{p}S_{ij}Z_\infty^{\frac{D^j f}{f}}(z)\right\}\\ &\leq Z_0^P(z)+\sum_{1\leq i\leq n}Z_\infty^{\alpha_i}(z)+\max\left\{\sum_{j=0}^{p}S_{ij}\left(Z_0^f(z)-Z_0^{D^j f}(z)\right)\right\},\end{aligned}$$

by the definition $\overline{Z}_a^f := \min\{Z_a^f, 1\}$. From the above inequality, we can see that

$$d(P)Z_0^f(z)-Z_0^P(z)+\frac{1}{q}\overline{Z}_0^P \leq \frac{1}{q}\overline{Z}_0^f+\sum_{1\leq i\leq n}Z_\infty^{\alpha_i}(z)+\max\left\{\sum_{j=0}^{p}S_{ij}\left(Z_0^f(z)-Z_0^{D^j f}(z)\right)\right\}.$$

Furthermore, applying Lemma 2.4 to the above inequality, one observes that

$$\begin{aligned}&d(P)N_f(r,0)-N_P(r,0)+\frac{1}{q}\overline{N}_P(r,0)\\ &\leq \sum_{i=1}^{n}N_{\alpha_i}(r,\infty)+\frac{1}{q}\overline{N}_f(r,0)+\max\left\{\sum_{j=0}^{p}jS_{ij}\right\}\overline{N}_f(r,0)+S(r,f)\\ &\leq \sum_{i=1}^{n}N_{\alpha_i}(r,\infty)+\left(\frac{1}{q}+\theta(P)\right)\overline{N}_f(r,0)+S(r,f)\\ &= \left(\frac{1}{q}+\theta(P)\right)\overline{N}_f(r,0)+S(r,f).\end{aligned}$$

Combining this with (4.1), we have that

$$d(P)T_f(r) \leq \frac{1}{q}\left(\overline{N}_P(r,\infty)+\sum_{j=1}^{q}\overline{N}_P(r,a_j)\right)+\left(\theta(P)+\frac{1}{q}\right)\overline{N}_f(r,0)+o(T_f(r)).$$

On the other hand, by the definition of the differential polynomial P, we have $\mathrm{Pole}(P) \subset \bigcup_{i=1}^{n}$ Pole $(\alpha_i) \bigcup$ Pole f. Since $\overline{N}_{\alpha_i}(r,\infty) \leq T_{\alpha_i}(r) = o(T_f(r))$ for $i = 1, \cdots, n$, we get that

$$d(P)T_f(r) \leq \frac{1}{q}\left(\overline{N}_P(r,\infty) + \sum_{j=1}^{q} \overline{N}_P\,(r,a_j)\right) + \left(\theta(P) + \frac{1}{q}\right)\overline{N}_f\,(r,0) + o(T_f(r))$$

$$\leq \frac{1}{q}\left(T_f(r) + \sum_{j=1}^{q} \overline{N}_P\,(r,a_j)\right) + \left(\theta(P) + \frac{1}{q}\right)\overline{N}_f\,(r,0) + o(T_f(r)). \qquad (4.2)$$

Therefore,

$$T_f\,(r) \leq \frac{q\theta(P)+1}{qd(P)-1}\overline{N}_f\,(r,0) + \frac{1}{qd(P)-1}\sum_{j=1}^{q}\overline{N}_P\,(r,a_j) + o(T_f(r)).$$

In the case where f is a transcendental entire function, the first inequality in (4.2) becomes

$$d(P)T_f\,(r) \leq \frac{1}{q}\sum_{j=1}^{q}\overline{N}_P\,(r,a_j) + \left(\theta(P) + \frac{1}{q}\right)\overline{N}_f\,(r,0) + o(T_f(r)).$$

This implies that

$$T_f\,(r) \leq \frac{q\theta(P)+1}{qd(P)}\overline{N}_f\,(r,0) + \frac{1}{qd(P)}\sum_{j=1}^{q}\overline{N}_P\,(r,a_j) + o(T_f(r)).$$

We have completed our proof. □

5 Proof of Theorem 1.1

Proof of Theorem 1.1 First, we prove (i). Suppose that there exists a nonconstant meromorphic solution f of

$$Q(z,f) - 1 = R(f).$$

Then $Q(z,f) = R(f) + 1$ is a nonconstant function. Indeed, if $Q(z,f)$ is a constant, then $R(f)$ is a constant, which implies that f is a constant. This is impossible. Applying Theorem 1.9, we have that

$$\|T_f(r) \leq \frac{\theta(Q)+1}{d(Q)-1}\overline{N}_f\,(r,0) + \frac{1}{d(Q)-1}\overline{N}_Q\,(r,1) + o(T_f(r))$$

$$\leq \frac{\theta(Q)+1}{d(Q)-1}\overline{N}_f\,(r,0) + \frac{1}{d(Q)-1}T_{R(f)}(r) + o(T_f(r))$$

$$\leq \frac{\theta(Q)+\deg R+1}{d(Q)-1}T_f(r) + o(T_f(r)).$$

Since $d(Q) > \theta(Q) + \deg R + 2$, then f is a constant. This is a contradiction, and so (i) is proven.

Next, we prove (ii). Suppose that f is a nonconstant entire solution of equation

$$Q(z,f) - 1 = R(f).$$

By the same argument as before, $Q(z,f)$ is a nonconstant function. Applying Theorem 1.9 for the case of the entire function, we get that

$$\|T_f(r) \leq \frac{\theta(Q)+1}{d(Q)}\overline{N}_f\,(r,0) + \frac{1}{d(Q)}\overline{N}_Q\,(r,1) + o(T_f(r))$$

$$\leq \frac{\theta(Q)+1}{d(Q)}\overline{N}_f(r,0) + \frac{1}{d(Q)}\overline{N}_{R(f)}(r,0) + o(T_f(r))$$
$$\leq \frac{\theta(Q)+1}{d(Q)}\overline{N}_f(r,0) + \frac{1}{d(Q)}\sum_{i=1}^{q}\overline{N}_f(r,a_i) + o(T_f(r)).$$

This implies that

$$(\theta(Q)+1)\Theta(0,f) + \sum_{i=1}^{q}\Theta(a_i,f) \leq \theta(Q) - d(Q) + q + 1,$$

provided that $\theta(Q) \geq d(Q) - q - 1$. □

6 The Proof of Theorem 1.11

Proof of Theorem 1.11 With the idea of Hayman [3], we will present our proof as follows:

For each $b \in \mathbb{C}$, set

$$\psi = \frac{D^k f - b}{a f^n}. \tag{6.1}$$

If $D^k f$ is a nonzero constant, then the conclusion is trivial from $n \geq 5$. Now, we consider that $D^k f$ is not a constant, and we obtain that $\psi \not\equiv 0$. Now, we define some divisors:

$$v_0(z) = \begin{cases} Z_b^{D^k f}(z), & \text{if } Z_0^f(z) = 0; \\ 0, & \text{or else.} \end{cases}$$

$$v_1(z) = \begin{cases} Z_b^{D^k f}(z), & \text{if } Z_0^f(z) > 0 \text{ and } Z_\infty^\psi(z) > 0; \\ 0, & \text{or else.} \end{cases}$$

$$v_2(z) = \begin{cases} Z_b^{D^k f}(z), & \text{if } Z_0^f(z) > 0, Z_0^\psi(z) = 0 \text{ and } Z_\infty^\psi(z) = 0; \\ 0, & \text{or else.} \end{cases}$$

$$v_3(z) = \begin{cases} Z_b^{D^k f}(z), & \text{if } Z_0^f(z) > 0 \text{ and } Z_0^\psi(z) > 0; \\ 0, & \text{or else.} \end{cases}$$

Meanwhile, we define the notations $N_0(r), N_1(r), N_2(r)$ and $N_3(r)$ as the corresponding counting functions of the divisors $v_0(z), v_1(z), v_2(z)$ and $v_3(z)$, respectively.

We claim that

(i) $n\overline{Z}_\infty^\psi(z) \leq Z_\infty^\psi(z) + v_1(z)$;

(ii) $(n-k-1)\overline{Z}_0^\psi(z) \leq Z_0^\psi(z) + (n-k-2)v_0(z) + \frac{n-k-1}{n}v_3(z)$.

First, we prove (i).

If $Z_\infty^\psi(z) = 0$, then the proof of claim (i) is trivial. Thus, we assume that $Z_\infty^\psi(z) > 0$. Since $n > k+2$, combining this with (6.1) yields that $Z_0^f(z) > 0$. Without loss of generality, set $Z_0^f(z) = q > 0$. Obviously,

$$n\overline{Z}_\infty^\psi(z) = n. \tag{6.2}$$

Suppose that $Z_b^{D^k f}(z) = p$.

If $p = 0$, it follows from (6.1) that $Z_\infty^\psi(z) \geq n$. Thus, (i) is right.

If $p>0$, by (6.1) and the definition of $v_1(z)$, it is easy to deduce $v_1(z)=p$ and $Z_\infty^\psi(z)=nq-p$. Then $Z_\infty^\psi(z)+v_1(z)=nq\geq n$. Combining this and (6.2), that yields that the claim (i) holds.

Now, we prove (ii).

If $Z_0^\psi(z)=0$, then the proof of claim (ii) is also trivial, so we assume that $Z_0^\psi(z)>0$. Then, with (6.1), we deduce that $Z_b^{D^kf}(z)>0$ or $Z_\infty^f(z)>0$. Of course, $(n-k-1)\overline{Z}_0^\psi(z)=n-k-1$. We will prove the claim by distinguishing three cases.

Case (1) $Z_b^{D^kf}(z)>0$ and $Z_0^f(z)=0$. With the definition of $v_0(z)$, we have that $(n-k-2)v_0(z)\geq n-k-2$. Thus, (ii) is right.

Case (2) $Z_b^{Df}(z)>0$ and $Z_0^f(z)>0$. Since $Z_0^\psi(z)>0$, we deduce that $Z_b^{D^kf}(z)>nZ_0^f(z)\geq n$. Noting the definition of $v_3(z)$, we obtain that $v_3(z)=Z_b^{D^kf}(z)>n$. Furthermore, $\frac{n-k-1}{n}v_3(z)>n-k-1$, which indicates that (ii) is right.

Case (3) $Z_\infty^f(z)>0$. Then $Z_\infty^{f^n}(z)=nZ_\infty^f(z)\geq n$. From Lemma 2.2, we derive that $Z_\infty^{D^kf-b}(z)\leq Z_\infty^f(z)+k$. It follows from (6.1) that

$$Z_0^\psi(z)=Z_\infty^{f^n}(z)-Z_\infty^{D^kf-b}(z)\geq(n-1)Z_\infty^f(z)-k\geq n-(k+1).$$

Thus, the claim (ii) holds.

Therefore, we have completed the proof of the claims.

From the claims, the following two inequalities are immediately derived:

$$n\overline{N}_\psi(r,\infty)\leq N_\psi(r,\infty)+N_1(r);$$

$$(n-k-1)\overline{N}_\psi(r,0)\leq N_\psi(r,0)+(n-k-2)N_0(r)+\frac{n-k-1}{n}N_3(r).$$

By the above two inequalities and Nevanlinna's second fundamental theorem, we have that

$$\begin{aligned}\|T_\psi(r)&\leq\overline{N}_\psi(r,\infty)+\overline{N}_\psi(r,0)+\overline{N}_\psi(r,1)+S(r,f)\\&\leq\frac{1}{n}N_\psi(r,\infty)+\frac{1}{n-k-1}N_\psi(r,0)+\overline{N}_\psi(r,1)\\&\quad+\frac{n-k-2}{n-k-1}N_0(r)+\frac{1}{n}\left[N_1(r)+N_3(r)\right]+S(r,f).\end{aligned}$$

Furthermore, we have that

$$\|\left[1-\frac{1}{n}-\frac{1}{n-k-1}\right]T_\psi(r)\leq\frac{n-k-2}{n-k-1}N_0(r)+\frac{1}{n}\left[N_1(r)+N_3(r)\right]+\overline{N}_\psi(r,1)+S(r,f).\tag{6.3}$$

Next, we will prove the inequality

$$v_0(z)+(n-k-1)Z_\infty^f(z)\leq Z_0^\psi(z).\tag{6.4}$$

If $Z_\infty^f(z)=0$ and $v_0(z)=0$, the inequality holds.

If $Z_\infty^f(z)>0$, then $v_0(z)=0$. By Case (3), it is easy to deduce that $Z_0^\psi(z)\geq(n-1)Z_\infty^f(z)-k\geq(n-k-1)Z_\infty^f(z)$. Thus, the inequality is right.

If $v_0(z)>0$, then $Z_\infty^f(z)=0$. With (6.1), we obtain that $Z_0^\psi(z)=v_0(z)$, which also implies that the inequality holds.

All of the above discussion shows that the inequality (6.4) holds, which leads to

$$N_0(r)+(n-k-1)N_f(r,\infty)\leq N_\psi(r,0).\tag{6.5}$$

On the other hand, we deduce that

$$\begin{aligned} nm_f(r,\infty) = m_{f^n}(r,\infty) &\le m_{\frac{f^n}{D^k f-b}}(r,\infty) + m_{D^k f-b}(r,\infty) \\ &\le m_\psi(r,0) + m_{D^k f}(r,\infty) + O(1) \\ &\le m_\psi(r,0) + m_{\frac{D^k f}{f}}(r,\infty) + m_f(r,\infty) + O(1) \\ &\le m_\psi(r,0) + m_f(r,\infty) + S(r,f). \end{aligned}$$

Thus,

$$(n-k-1)m_f(r,\infty) \le (n-1)m_f(r,\infty) \le m_\psi(r,0) + S(r,f).$$

Combining this and (6.5) yields that

$$\|(n-k-1)T_f(r) \le T_\psi(r) - N_0(r) + S(r,f). \tag{6.6}$$

By (6.3) and (6.6), we deduce that

$$\begin{aligned} \| \left(n-k-3+\frac{k+1}{n}\right) T_f(r) &\le \overline{N}_\psi(r,1) + \frac{1}{n}\left[N_0(r)+N_1(r)+N_3(r)\right] + S(r,f) \\ &\le \overline{N}_\psi(r,1) + \frac{1}{n} N_{D^k f}(r,b) + S(r,f). \end{aligned} \tag{6.7}$$

It follows from Lemma 2.2 that

$$\begin{aligned} \|N_{D^k f}(r,b) &\le T_{D^k f}(r) + O(1) = m_{D^k f}(r,\infty) + N_{D^k f}(r,\infty) + O(1) \\ &\le m_{\frac{D^k f}{f}}(r,\infty) + m_f(r,\infty) + (k+1)N_f(r,\infty) + O(1) \\ &\le (k+1)T_f(r) + S(r,f). \end{aligned} \tag{6.8}$$

Combining (6.7) and (6.8) leads to

$$\|(n-k-3)T_f(r) \le \overline{N}_\psi(r,1) + S(r,f). \tag{6.9}$$

From the condition $n \ge k+4$ and (6.9), we obtain that $\psi - 1$, assuming 0 infinitely often, which implies that $D^k f + a(f)^n - b$ assumes zero infinitely often as well.

Hence, we have completed the proof of the result. □

7 The Proof of Theorem 1.12

Proof of Theorem 1.12 On the contrary, suppose that $f \not\equiv D^k f$. We set

$$\frac{D^k f - a}{f - a} = h. \tag{7.1}$$

Now, we distinguish the following two cases:

Case 1 $h \equiv c$ $(c \ne 1)$ is a constant. From (7.1), we have that

$$\frac{D^k f}{a} - \frac{cf}{a} = 1 - c$$

and hence, by Lemma 2.6 and Lemma 2.10, we get that

$$\begin{aligned} T_{D^k f}(r) &\le T_{\frac{D^k f}{a}}(r) + S(r,f) \\ &\le \overline{N}_{\frac{a}{D^k f}}(r,\infty) + \overline{N}_{\frac{cf}{a}}(r,0) + \overline{N}_{\frac{a}{D^k f}}(r,0) + S(r,f) \\ &\le T_{D^k f}(r) - T_f(r) + 2N_f(r,0) + \overline{N}_f(r,\infty) + S(r,f), \end{aligned}$$

and thus,

$$T_f(r) \le 2N_f(r,0) + \overline{N}_f(r,\infty) + S(r,f),$$

from which we get that

$$2\delta(0,f) + \Theta(\infty,f) \le 2.$$

This contradicts (1.2).

Case 2 h is a non-constant function. From (7.1), we have that

$$\frac{D^k f}{a} - \frac{hf}{a} + h = 1.$$

Set $f_1 = \frac{D^k f}{a}, f_2 = \frac{-hf}{a}, f_3 = h$. Then

$$\sum_{i=1}^{3} f_i \equiv 1. \tag{7.2}$$

Since $f-a$ and $D^k f - a$ share the value 0 CM, $D^k f$ and a do not have some common poles of the same multiplicity, and we know that $h \ne 0$ and h have no zeros. On the other hand, the poles of h must be poles of f or a. Thus,

$$\overline{N}_{\frac{a}{hf}}(r,0) \le \overline{N}_f(r,\infty) + S(r,f), \quad \overline{N}_h(r,\infty) \le \overline{N}_f(r,\infty) + S(r,f). \tag{7.3}$$

From (7.3), Lemma 2.6 and $N_{D^k f}(r,\infty) \le N_f(r,\infty) + k\overline{N}_f(r,\infty)$, we have that

$$\begin{aligned}
&\overline{N}_{\frac{D^k f}{a}}(r,0) + \overline{N}_{\frac{hf}{a}}(r,0) + \overline{N}_h(r,0) + \overline{N}_{\frac{a}{D^k f}}(r,0) + \overline{N}_{\frac{a}{hf}}(r,0) + \overline{N}_h(r,\infty)\\
&\le \overline{N}_{D^k f}(r,0) + \overline{N}_f(r,0) + 2\overline{N}_f(r,\infty) + \overline{N}_h(r,\infty) + S(r,f)\\
&\le T_{D^k f}(r) - T_f(r) + N_f(r,0) + \overline{N}_f(r,0) + 3\overline{N}_f(r,\infty) + S(r,f)\\
&\le T_{D^k f}(r) - T_f(r) + 2N_f(r,0) + 3\overline{N}_f(r,\infty) + S(r,f).
\end{aligned} \tag{7.4}$$

By the definition $\delta(0,f) = 1 - \limsup\limits_{r\to\infty} \frac{N_f(r,0)}{T_f(r)}$ and $\Theta(\infty,f) = 1 - \limsup\limits_{r\to\infty} \frac{\overline{N}_f(r,\infty)}{T_f(r)}$, from (7.4), we can obtain that

$$\begin{aligned}
&\overline{N}_{\frac{D^k f}{a}}(r,0) + \overline{N}_{\frac{hf}{a}}(r,0) + \overline{N}_h(r,0) + \overline{N}_{\frac{a}{D^k f}}(r,0) + \overline{N}_{\frac{a}{hf}}(r,0) + \overline{N}_h(r,\infty)\\
&\le T_{D^k f}(r) - T_f(r) + [2(1-\delta(0,f)) + 3(1-\Theta(\infty,f)) + o(1)]T_f(r) + S(r,f)\\
&= T_{D^k f}(r) - \{2\delta(0,f) + 3\Theta(\infty,f) - 4 + o(1)\}T_f(r)\\
&\le \frac{(k+5-2\delta(0,f)-(3+k)\Theta(\infty,f)+o(1))}{k+1-k\Theta(\infty,f)} T_{\frac{D^k f}{a}}(r).
\end{aligned}$$

Immediately, by Lemma 2.9, and from the above inequality, we can derive that

$$\begin{aligned}
(\lambda + o(1))T(r) \ge\ &\overline{N}_{\frac{D^k f}{a}}(r,0) + \overline{N}_{\frac{hf}{a}}(r,0) + \overline{N}_h(r,0)\\
&+ \overline{N}_{\frac{D^k f}{a}}(r,\infty) + \overline{N}_{\frac{hf}{a}}(r,\infty) + \overline{N}_h(r,\infty),
\end{aligned}$$

where $T(r) = \max\limits_{1\le j\le 3} T_{f_j}(r)$ and

$$\begin{aligned}
\lambda &= \frac{(k+5-2\delta(0,f)-(3+k)\Theta(\infty,f)+o(1))}{k+1-k\Theta(\infty,f)}\\
&\le k+5-2\delta(0,f)-(3+k)\Theta(\infty,f)+o(1) < \frac{1}{2}.
\end{aligned}$$

Hence, from Lemma 2.9, we know that f_1, f_2, f_3 are linearly dependent, so there exist three constants, not all zero, such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0. \tag{7.5}$$

It is obvious that $c_1 \neq 0$. Indeed, if $c_1 = 0$, then $c_2 \neq 0, c_3 \neq 0$, and by (7.5), we can obtain $h\left(c_2(\frac{f}{a}) - c_3\right) = 0$, and since h is non-constant, we get that $f = a\frac{c_3}{c_2}$. Hence, we deduce that $T(r,f) = T\left(r, \frac{ac_3}{c_2}\right) = S(r,f)$, which is impossible. By (7.2) and (7.5), we have that

$$(c_2 - c_1)\,\frac{hf}{a} + (c_1 - c_3)\,h = c_1. \tag{7.6}$$

If $c_2 - c_1 \neq 0, c_1 - c_3 \neq 0$, from (7.6) and by Lemma 2.10, we deduce that

$$\begin{aligned} T(r,f) &\leq T_{\frac{f}{a}}(r) + S(r,f) \\ &\leq \overline{N}_{\frac{a}{f}}(r,\infty) + \overline{N}_h(r,\infty) + \overline{N}_{\frac{f}{a}}(r,\infty) + S(r,f) \\ &\leq N_f(r,0) + 2\overline{N}_f(r,\infty) + S(r,f). \end{aligned}$$

Similarly, this is impossible, because of the condition (1.2).

If $c_2 - c_1 = 0, c_1 - c_3 \neq 0$, we obtain from (7.6) that $h = \frac{c_1}{c_1 - c_3}$, which is a contradiction, because of h is not a constant.

If $c_2 - c_1 \neq 0, c_1 - c_3 = 0$, we obtain from (7.6) that

$$fh = \frac{ac_1}{c_2 - c_1}, \tag{7.7}$$

and by (7.2) and (7.7), we get that

$$\frac{D^k f}{a} + h = \frac{c_2}{c_2 - c_1}.$$

If $c_2 \neq 0$, then, by Lemma 2.10 and Lemma 2.6, we have that

$$\begin{aligned} T_{D^k f}(r) &\leq T_{\frac{D^k f}{a}}(r) + S(r,f) \\ &\leq \overline{N}_{\frac{a}{D^k f}}(r,\infty) + \overline{N}_h(r,0) + \overline{N}_{\frac{D^k f}{a}}(r,\infty) + S(r,f) \\ &\leq T_{D^k f}(r) - T(r,f) + N_f(r,0) + \overline{N}_f(r,\infty) + S(r,f). \end{aligned}$$

Therefore,

$$T_f(r) \leq N_f(r,0) + \overline{N}_f(r,\infty) + S(r,f),$$

and similarly, this is impossible, because of the condition (1.2).

If $c_2 = 0$, then we have that

$$D^k f + ah = 0 \quad \text{and} \quad fh = -a,$$

so we can deduce from the above two equations that

$$D^k f \cdot f = a^2. \tag{7.8}$$

We note that

$$N_f(r,0) \leq N_{fD^k f}(r,0)$$

and

$$2m_f(r,0) = m_{f^2}(r,0) \leq m_{\frac{fD^k f}{f^2}}(r,\infty) + m_{fD^k f}(r,0) = m_{fD^k f}(r,0) + S(r,f),$$

so that

$$T_f\,(r,0) \le T_{fD^k f}(r) + S(r,f). \tag{7.9}$$

Therefore we can get, from (7.8) and (7.9), that

$$T_f(r) \le T_{fD^k f}(r) + S(r,f) = T(r,a^2) + S(r,f) = S(r,f),$$

which is impossible. This proves Theorem 1.12. □

References

[1] Brück R. On entire functions which share one value CM with their derivates. Result in Math, 1996, **30**: 21–24

[2] Gundersen G G, Yang L Z. Entire functions that share one value with one or two of their derivates. J Math Anal Appl, 1998, **223**: 245–260

[3] Hayman W K. Meromorphic Functions. Oxford: Clarendon Press, 1964

[4] Hu P C, Li P, Yang C C. Unicity of Meromorphic Mappings. Berlin: Springer Science and Bussiness Media, 2013

[5] Jin L. Theorems of Picard type for entire functions of several complex variables. Kodai Math J, 2003, **26**: 221–229

[6] Jin L. A unicity theorem for entire functions of several complex variables. Chin Ann Math, 2004, **25B**: 483–492

[7] Li B Q. On Picard's theorem. J Math Anal Appl, 2018, **460**: 561–564

[8] Li B Q, Yang L. On Picard type theorems and entire solutions of differential equations. arXiv: 1809.05553v1

[9] Lü F. Theorems of Picard type for meromorphic function of several complex variables. Complex Variables and Elliptic Equations, 2013, **58**: 1085–1092

[10] Ru M. Nevanlinna Theory and Its Relation to Diophantine Approxmation. Singapore: World Sci Publishing, 2001

[11] Vitter A. The lemma of the logarithmic derivative in several complex variables. Duke Math J, 1977, **44**: 89–104

[12] Yi H X. A question of C.C. Yang on the uniqueness of entire functions. Kodai Math J, 1990, **13**: 39–46

[13] Yi H X. Uniqueness theorems for meromorphic functions whose N-th derivatives share the same 1-points. Complex Variables and Elliptic Equations, 1997, **34**: 421–436

[14] Yang L Z. Further results on entire functions that share one value with their derivates. J Math Anal Appl, 1997, **212**: 529–536

[15] Ye Z. A sharp form of Nevanlinna's second main theorem of several complex variables. Math Z, 1996, **222**: 81–95

[16] Yu K W. On entire and meromorphic functions that share small functions with thire derivatives. J Inequal Pure Appl Math, 2003, **4**(1): Art 21