Acta Mathematica Scientia, 2023, **43B**(2): 640–654
https://doi.org/10.1007/s10473-023-0210-8
©Innovation Academy for Precision Measurement Science and Technology, Chinese Academy of Sciences, 2023

Mathematica Acta Scientia
数学物理学报
http://actams.apm.ac.cn

ANALYSIS AND DISCRETIZATION FOR AN OPTIMAL CONTROL PROBLEM OF A VARIABLE-COEFFICIENT RIESZ-FRACTIONAL DIFFUSION EQUATION WITH POINTWISE CONTROL CONSTRAINTS*

Zhaojie ZHOU (周兆杰) *Fangyuan WANG* (王方圆)
School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China
E-mail:zhouzhaojie@sdnu.edu.cn; wfy_sdnu@163.com

Xiangcheng ZHENG (郑祥成)†
School of Mathematics, Shandong University, Jinan 250100, China
E-mail: zhengxch@outlook.com

Abstract We present a mathematical and numerical study for a pointwise optimal control problem governed by a variable-coefficient Riesz-fractional diffusion equation. Due to the impact of the variable diffusivity coefficient, existing regularity results for their constant-coefficient counterparts do not apply, while the bilinear forms of the state (adjoint) equation may lose the coercivity that is critical in error estimates of the finite element method. We reformulate the state equation as an equivalent constant-coefficient fractional diffusion equation with the addition of a variable-coefficient low-order fractional advection term. First order optimality conditions are accordingly derived and the smoothing properties of the solutions are analyzed by, e.g., interpolation estimates. The weak coercivity of the resulting bilinear forms are proven via the Garding inequality, based on which we prove the optimal-order convergence estimates of the finite element method for the (adjoint) state variable and the control variable. Numerical experiments substantiate the theoretical predictions.

Key words Riesz-fractional diffusion equation; variable coefficient; optimal control; finite element method; Garding inequality; optimal-order error estimate

2010 MR Subject Classification 35R11; 49M25; 65N30

*Received October 29, 2021; revised February 8, 2022. The first author was supported by the National Natural Science Foundation of China (11971276, 12171287), and by the Natural Science Foundation of Shandong Province (ZR2016JL004). The third author was supported by the China Postdoctoral Science Foundation (2021TQ0017, 2021M700244), and by the International Postdoctoral Exchange Fellowship Program (Talent-Introduction Program) (YJ20210019).

†Corresponding author

1 Introduction

In this paper we study the following optimal control problem governed by a Riesz-fractional diffusion equation (RFDE) with variable diffusivity coefficient:

$$\min_{u\in U_{\rm ad}} J(y,u) := \frac{1}{2}\int_\Omega (y(x)-y_d(x))^2 {\rm d}x + \frac{\gamma}{2}\int_\Omega u^2(x){\rm d}x \tag{1.1}$$

subject to

$$\begin{cases} -D\big[K(x)I^{2-\alpha}D\big]y(x) = f(x)+u(x), \quad x\in\Omega := (0,1),\ \alpha\in(1,2), \\ y(0)=y(1)=0. \end{cases} \tag{1.2}$$

Here y refers to the state variable, $Dy(x) = y'(x)$ is the first-order derivative of $y(x)$, y_d represents the target function, and u is the control variable in the admissible set $U_{\rm ad}$ given by

$$U_{\rm ad} = \{u\in L^2(\Omega):\ a\le u(x)\le b \text{ a.e. in } \Omega \text{ with } a,b\in\mathbb{R} \text{ and } a\le b\}. \tag{1.3}$$

$K(x)$ refers to the variable diffusivity coefficient and the operator $I^{2-\alpha} := {}_0I_x^{2-\alpha} + {}_xI_1^{2-\alpha}$, where the left and right fractional integral operators ${}_0I_x^{2-\alpha}$ and ${}_xI_1^{2-\alpha}$ are defined via the Gamma function $\Gamma(\cdot)$:

$$\begin{aligned} {}_0I_x^{2-\alpha}g(x) &= \frac{1}{\Gamma(2-\alpha)}\int_0^x \frac{g(\tau)}{(x-\tau)^{\alpha-1}}{\rm d}\tau, \\ {}_xI_1^{2-\alpha}g(x) &= \frac{1}{\Gamma(2-\alpha)}\int_x^1 \frac{g(\tau)}{(\tau-x)^{\alpha-1}}{\rm d}\tau. \end{aligned}$$

The optimal control problem governed by FDEs plays an important role in many engineering problems, e.g., contaminant transport in heterogeneous media [5, 18, 23] in which the pollution sources are controlled in order that the concentration of pollutants in the water reaches an ideal value with a low cost. From a mathematical point of view, this process can be formulated as an optimal control problem constrained by an RFDE, which models the anomalously diffusive transport of the contaminant in the heterogeneous surrounding.

In recent years optimal control problems governed by FDEs have attracted extensive attention from researchers; see e.g., [3, 4, 10, 11, 13, 19, 20, 25, 28–30, 35]. However, to the best of our knowledge, almost all of the existing works focus on constant-coefficient space-fractional optimal control models, while the corresponding investigations on variable-coefficient problems are relatively meager. The variable-coefficient FDE models are distinguished from their constant-coefficient analogues in the following respects:

(a) The solution structures of the constant-coefficient FDEs are clearly specified in, e.g., [1, 14]; this leads to the regularity estimates of the solutions. However, the variable coefficient complicates the models and invalidates the analysis tools developed in the literature. Though there are some regularity results for variable-coefficient FDEs in weighted Sobolev spaces [32, 34], the corresponding non-weighted estimates, which are required in the error estimates of, e.g., finite element methods, remain untreated.

(b) Commonly-used (weighted) Jacobi spectral methods for constant-coefficient FDEs (cf. [8, 9, 14, 21, 22, 31]) do not apply for their variable-coefficient analogues due to the impact of the variable coefficient. Recently, some indirect spectral methods for variable-coefficient FDEs were developed, in [32, 34], by converting them into constant-coefficient FDEs. However, such transformation methods may not be extended to the study of high-dimensional problems.

Furthermore, the indirect methods are not derived from the variational framework, which makes the error estimates of the discretization of the optimality conditions intricate.

(c) In the finite element method of variable-coefficient FDEs, the corresponding bilinear form may lose the coercivity [26, 27] that is critical in error estimates. In [33], an indirect finite element method for a variable-coefficient FDE was proposed and analyzed by expressing its solution in terms of that of the constant-coefficient FDE. For this reason, such an indirect method could not be extended to high-dimensional cases.

In this paper we address the aforementioned issues to carry out rigorous mathematical and numerical analysis for the optimal control problem (1.1)–(1.2). We first analyze the smoothing properties of the solutions by, e.g., interpolation estimates, and based on this we study a finite element approximation to the optimality conditions of the model. To overcome the loss of coercivity of the resulting bilinear form, we follow [15] to rewrite the state equation (1.2) as a constant-coefficient FDE by adding a variable-coefficient low-order fractional advection term. Instead of assuming that $\left|K'(x)/K(x)\right|$ is small enough as in [15], which ensures the coercivity of the bilinear form and thus facilitates the numerical analysis, we prove the weak coercivity of the bilinear form via the Garding inequality to find the optimal-order convergence rates of the scheme. The proposed numerical analysis techniques could be directly extended and applied in high-dimensional problems; this distinguishes this method from those mentioned in (b)–(c).

The rest of the paper is organized as follows: in Section 2, we introduce the preliminaries to be used subsequently. In Section 3 we derive the first-order optimality conditions and analyze the regularity of the solutions to the optimality system. We then present the finite element approximation of the optimal control problem and prove a prior error estimates for the state variable, the adjoint state variable and the control variable in Section 4. Numerical experiments are carried out in Section 5 to substantiate the theoretical findings.

2 Preliminaries

Let $\mathbb{N}$ be the set of non-negative integers, let $L^p(\Omega)$ with $1 \leq p \leq \infty$ be the space of pth Lebesgue-integrable functions on Ω, let $C^\infty(\Omega)$ denote the space of infinitely differentiable functions on $\Omega := (0,1)$, and let $C_0^\infty(\Omega)$ be the functions in $C^\infty(\Omega)$ that have compact support within Ω. Accordingly, the Sobolev space $W_p^1(\Omega)$ is defined as the collection of L^p functions whose derivatives also belong to $L^p(\Omega)$. In particular, $H^1(\Omega) := W_2^1(\Omega)$. All of these spaces are equipped with standard norms [2]. Let $\omega^\beta(x) := (1-x)^\beta x^\beta$ for some $\beta > -1$ and denote by $L^2_{\omega^\beta}(\Omega)$ the weighted $L^2(\Omega)$ space equipped with the inner product and the norm

$$(q,v)_{\omega^\beta} = \int_\Omega q(x)v(x)\omega^\beta(x)\mathrm{d}x,\ \|q\|_{\omega^\beta} = (q,q)_{\omega^\beta}^{1/2},\ \forall q,v \in L^2_{\omega^\beta}(\Omega).$$

Denote by $H^\theta(\Omega)$ with $0<\theta<1$ the fractional Sobolev space equipped with the inner product, norm and semi-norm

$$\begin{aligned}&(q,q) = \|q\|^2_{L^2(\Omega)},\ |q|_{H^\theta(\Omega)} = \left(\int_\Omega\int_\Omega \frac{(q(x)-q(y))^2}{|x-y|^{1+2\theta}}\mathrm{d}x\mathrm{d}y\right)^{1/2},\\ &\|q\|_{H^\theta(\Omega)} = (\|q\|^2_{L^2(\Omega)} + |q|^2_{H^\theta(\Omega)})^{1/2},\end{aligned}$$

and let $H_0^\theta(\Omega)$ refer to the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{H^\theta(\Omega)}$. Accordingly, the fractional Sobolev space $H^{1+\theta}(\Omega)$ is equipped with the norm [2, 6]

$$\|v\|_{H^{1+\theta}(\Omega)} := \left(\|v\|^2_{H^1(\Omega)} + |v'|^2_{H^\theta(\Omega)}\right)^{1/2}. \tag{2.1}$$

Furthermore, we define the left and right fractional derivative spaces $\dot{H}_l^\theta(\Omega)$ and $\dot{H}_r^\theta(\Omega)$ via the following norms:

$$\begin{aligned} |q|_{\dot{H}_l^\theta(\Omega)} &= \|{}_0I_x^{1-\theta}q'\|_{L^2(\Omega)},\ \ \|q\|_{\dot{H}_l^\theta(\Omega)} = (\|q\|^2_{L^2(\Omega)} + |q|^2_{\dot{H}_l^\theta(\Omega)})^{1/2}, \\ |q|_{\dot{H}_r^\theta(\Omega)} &= \|{}_xI_1^{1-\theta}q'\|_{L^2(\Omega)},\ \ \|q\|_{\dot{H}_r^\theta(\Omega)} = (\|q\|^2_{L^2(\Omega)} + |q|^2_{\dot{H}_r^\theta(\Omega)})^{1/2}. \end{aligned} \tag{2.2}$$

Define the pointwise projection $P_{U_{\text{ad}}}$ onto the admissible set U_{ad} by

$$P_{U_{\text{ad}}}(u) = \max\{a, \min\{b, u\}\},$$

with the property [17]

$$\|P_{U_{\text{ad}}}(u)\|_{H^\theta(\Omega)} \le \|u\|_{H^\theta(\Omega)},\ \forall u \in H^\theta(\Omega),\ 0 \le \theta \le 1. \tag{2.3}$$

Let C represent a generic constant that may assume different values in different cases. The following properties of the aforementioned spaces hold [12]:

Lemma 2.1 For $1/2 < \theta < 1$, the spaces $H_0^\theta(\Omega), \dot{H}_{l,0}^\theta(\Omega)$ and $\dot{H}_{r,0}^\theta(\Omega)$ are equal with equivalent semi-norms and norms.

Lemma 2.2 For $0 < \theta < 1$, the following relation holds for $q \in H_0^\theta(\Omega)$:

$$(D_0I_x^{1-\theta}q, D_xI_1^{1-\theta}q) = -\cos(\theta\pi)\|D_0I_x^{1-\theta}q\|^2_{L^2(\Omega)} = -\cos(\theta\pi)\|D_xI_1^{1-\theta}q\|^2_{L^2(\Omega)}.$$

Lemma 2.3 The left and right fractional integral operators are adjoint in the L^2 sense, i.e., for any $\theta > 0$, it holds that

$$({}_0I_x^\theta q, v) = (q, {}_xI_1^\theta v), \quad \forall q, v \in L^2(\Omega).$$

Lemma 2.4 Suppose that $q \in \dot{H}_{l,0}^\theta(\Omega)$ (or $\dot{H}_{r,0}^\theta(\Omega)$) with $0 < \theta < 1$. Then it holds for $0 < \iota < \theta$ that

$$|q|_{\dot{H}_{l,0}^\iota(\Omega)} \le C|q|_{\dot{H}_{l,0}^\theta(\Omega)} \ \left(\text{or} \ \ |q|_{\dot{H}_{r,0}^\iota(\Omega)} \le C|q|_{\dot{H}_{r,0}^\theta(\Omega)}\right).$$

Lemma 2.5 Suppose that $q \in \dot{H}_{l,0}^\theta(\Omega)$ $\left(\text{or } \dot{H}_{r,0}^\theta(\Omega)\right)$ with $0 < \theta < 1$. Then

$$\|q\|_{L^2(\Omega)} \le C|q|_{\dot{H}_{l,0}^\theta(\Omega)} \ \left(\text{or } \|q\|_{L^2(\Omega)} \le C|q|_{\dot{H}_{r,0}^\theta(\Omega)}\right).$$

Lemma 2.6 ([24]) Suppose that $0 < \theta < 1$ and that $g(x)$ is Lipschitz continuous on Ω. Then there exists a constant $C > 0$, depending on $\|q\|_{L^\infty(\Omega)}$ and the Lipschitz constant of $g(x)$, such that

$$\|gq\|_{H^\theta(\Omega)} \le C\|g\|_{L^\infty(\Omega)}\|q\|_{H^\theta(\Omega)}.$$

In this paper, we make the following assumption on the data:

Assumption A $K \in W^1_\infty(\Omega)$ with a positive lower bound K_*, K' is Lipschitz continuous on Ω and f, $y_d \in L^2(\Omega)$.

3 Analysis of Optimal Control Model

In this section we analyze the optimal control model (1.1)–(1.2). Using the product rule for $K(x)\cdot\big(I^{2-\alpha}Dy(x)\big)$, the variable-coefficient diffusion operator in (1.2) can be rewritten as follows:

$$-D\big[K(x)I^{2-\alpha}D\big]y(x) = -K(x)DI^{2-\alpha}Dy(x) - K'(x)I^{2-\alpha}Dy(x).$$

Consequently, the variable-coefficient RFDE (1.2) can be converted to its constant-coefficient analogue with the addition of a variable-coefficient low order term

$$\mathfrak{L}y(x) := \big(\mathcal{T}+\mathcal{M}_1\big)y(x) := -DI^{2-\alpha}Dy(x) - \frac{K'(x)}{K(x)}I^{2-\alpha}Dy(x) = \frac{f(x)}{K(x)} + \frac{u(x)}{K(x)}.$$

In the rest of the paper, we will frequently use this equivalent form of (1.2) to facilitate the analysis.

3.1 First-Order Optimality Conditions

Theorem 3.1 Suppose that (y,u) is the solution to optimal control problem (1.1)–(1.2). Then the following first-order optimality system holds:

$$\begin{cases} \mathfrak{L}y(x) = \dfrac{f(x)}{K(x)} + \dfrac{u(x)}{K(x)}, \quad x\in\Omega;\;\; y(0)=y(1)=0; \\ \mathfrak{L}^*p(x) = y(x)-y_d(x), \quad x\in\Omega;\;\; p(0)=p(1)=0 \end{cases} \tag{3.1}$$

and

$$\int_\Omega \Big(\gamma u(x) + \frac{p(x)}{K(x)}\Big)(v(x)-u(x))\mathrm{d}x \ge 0, \quad \forall v\in U_{\mathrm{ad}}. \tag{3.2}$$

Here

$$\mathfrak{L}^* := -DI^{2-\alpha}D + DI^{2-\alpha}\frac{K'(x)}{K(x)} =: \mathcal{T}+\mathcal{M}_2.$$

Proof To derive the first order optimality system, let $\hat{J}(u) = J(y(u),u)$, where $y(u)$ is the solution of the state equation associated to u. The optimal control problem (1.1)–(1.2) reduces to the optimization problem $\min\limits_{u\in U_{\mathrm{ad}}}\hat{J}(u)$. Then the first order optimality condition takes the form

$$\hat{J}'(u)(v-u)\ge 0, \forall v\in U_{\mathrm{ad}}.$$

By simple calculations, we have that

$$\begin{aligned}\hat{J}'(u)(v-u) &= \lim_{\lambda\to0^+}\frac{1}{2\lambda}\int_\Omega\Big((y(u+\lambda(v-u))-y_d)^2-(y(u)-y_d)^2\Big)\mathrm{d}x \\ &\quad+\lim_{\lambda\to0^+}\frac{\gamma}{2\lambda}\int_\Omega((u+\lambda(v-u))^2-u^2)\mathrm{d}x \\ &= \int_\Omega(y(u)-y_d)y'(u)(v-u)\mathrm{d}x + \gamma\int_\Omega u(v-u)\mathrm{d}x.\end{aligned}$$

In order to simplify the above inequality, we introduce $z = y'(u)(v-u)$. It follows from the state equation (1.2) that

$$\begin{cases}\mathfrak{L}z = \dfrac{v-u}{K(x)}, \quad x\in\Omega, \\ z(0)=z(1)=0.\end{cases}$$

By adjoint property of fractional integral operators, we obtain from the adjoint state equation in (3.1) that

$$\begin{aligned}&\int_\Omega (y-y_d)(y'(u)(v-u))\mathrm{d}x+\gamma\int_\Omega u(v-u)\mathrm{d}x\\
=&\int_\Omega \mathfrak{L}^*p\cdot z\mathrm{d}x+\gamma\int_\Omega u(v-u)\mathrm{d}x=\int_\Omega \mathfrak{L}z\cdot p\mathrm{d}x+\gamma\int_\Omega u(v-u)\mathrm{d}x\\
=&\int_\Omega (\gamma u+\frac{p}{K})(v-u)\,\mathrm{d}x\geq 0.\end{aligned}$$

Combining the above equations, we obtain that

$$\hat{J}'(u)(v-u)=\int_\Omega(\gamma u+\frac{p}{K})(v-u)\mathrm{d}x\geq 0,\quad \forall v\in U_{\mathrm{ad}},$$

which completes the proof. □

Remark 3.2 We observe that the adjoint state equation in (3.1) can be rewritten in the following equivalent form:

$$\mathfrak{L}^*p(x)=-DI^{2-\alpha}\Big[K(x)D\Big(\frac{p(x)}{K(x)}\Big)\Big]=y(x)-y_d(x). \tag{3.3}$$

Compact with (1.2), this is indeed a different kind of variable-coefficient RFDE if we consider $p(x)/K(x)$ as a new unknown variable. Furthermore, we follow [7] to conclude that the condition (3.2) is equivalent to

$$u=P_{U_{\mathrm{ad}}}(-\frac{p}{\gamma K}). \tag{3.4}$$

3.2 Existing Results for State (Adjoint) Equations

To achieve the regularity of the solutions to the optimal control problem (3.1)–(3.2), we refer the following lemmas for future use:

Lemma 3.3 ([16]) The homogeneous boundary value problem of $\mathcal{T}g=\psi$ with $\psi\in L^2(\Omega)$ admits a unique solution $g\in H^{1/2+\alpha/2-\epsilon}(\Omega)$ for $0<\epsilon\ll 1$ such that

$$\|g\|_{H^{1/2+\alpha/2-\epsilon}(\Omega)}\leq C\|\psi\|_{L^2(\Omega)}.$$

Lemma 3.4 ([34]) Let $f(x)+u(x)\in L^2(\Omega)$. Then the state equation in (3.1) admits a unique solution $y(x)\in L^2_{\omega^{-\alpha/2}}(\Omega)$ satisfying the equivalent problem

$$-I^{2-\alpha}Dy(x):=\frac{1}{K(x)}\int_0^x f(s)+u(s)\mathrm{d}s-\frac{A}{K(x)}, \tag{3.5}$$

where $A:=[K(x)I^{2-\alpha}Dy(x)]|_{x=0}$ satisfies $|A|\leq C\|f+u\|_{L^2(\Omega)}$.

Lemma 3.5 ([33]) Let $y(x)-y_d(x)\in L^2(\Omega)$. Then there exists a unique solution $p(x)\in L^\infty(\Omega)$ to the adjoint state equation in (3.1). In particular, if $\bar{p}(x)$ solves the homogeneous boundary value problem of $\mathcal{T}\bar{p}=y-y_d$, then the $p(x)$ can be represented in terms of $\bar{p}(x)$ as

$$p(x)=K(x)\int_0^x\frac{D\bar{p}(s)}{K(s)}\mathrm{d}s-C^*K(x)\int_0^x\frac{\omega^{\alpha/2-1}(s)}{K(s)}\mathrm{d}s, \tag{3.6}$$

where

$$C^*:=\frac{\int_0^1\frac{D\bar{p}(s)}{K(s)}\mathrm{d}s}{\int_0^1\frac{\omega^{\alpha/2-1}(s)}{K(s)}\mathrm{d}s}$$

satisfies $|C^*|\leq C\|y-y_d\|_{L^2(\Omega)}$.

3.3 Smoothing Properties of the Optimal Control Problem

Lemma 3.6 The function $x^{\alpha/2-1} \in H^{(\alpha-1)/2-\varepsilon}(\Omega)$ for $0<\varepsilon \ll 1$.

Proof Let $\zeta(x) \in C^\infty[0,\infty)$ denote the cutoff function satisfying that $\zeta(x)=1$ on $0 < x \le 1/4$ and $\zeta(x)=0$ on $x \ge 3/4$. Then we have, for some $\sigma>0$, that

$$\zeta_\sigma(x)=\zeta(x/\sigma)=\begin{cases}1, & \text{for } 0<x\le \sigma/4,\\ 0, & \text{for } x\ge 3\sigma/4.\end{cases} \tag{3.7}$$

Let $\nu(x):=x^{\alpha/2-1}=\upsilon(x)+\mu(x)$, where $\upsilon(x)=\zeta_\sigma(x)\nu(x)$ and $\mu(x)=(1-\zeta_\sigma(x))\nu(x)$. It is clear that

$$\|\upsilon\|^2_{L^2(\Omega)} \le C\int_0^{3\sigma/4} x^{\alpha-2}\mathrm{d}x \le C\sigma^{\alpha-1}. \tag{3.8}$$

Next we consider $\mu(x)$ as

$$|\mu'(x)| \le C|1-\zeta_\sigma(x)|x^{\alpha/2-2}+Cx^{\alpha/2-1}|(1-\zeta_\sigma(x))'|. \tag{3.9}$$

Using (3.7), we find that the first right-hand side term of (3.9) vanishes for $x<\sigma/4$ and the second right-hand side term vanishes for $x<\sigma/4$ and $x>3\sigma/4$. Then,

$$\|\mu\|^2_{H^1(\Omega)} \le C\int_\Omega |\mu'|^2\mathrm{d}x \le C\int_{\sigma/4}^1 x^{\alpha-4}\mathrm{d}x + C\int_{\sigma/4}^{3\sigma/4}\sigma^{-2}x^{\alpha-2}\mathrm{d}x \le C(1+\sigma^{\alpha-3}). \tag{3.10}$$

By (3.8) and (3.10), we can get that

$$\begin{aligned}\chi(t,\nu) &:= \inf_{\omega\in H^1(\Omega)}(\|\nu-\omega\|_{L^2(\Omega)}+t\|\omega\|_{H^1(\Omega)})\\ &\le \|\upsilon\|_{L^2(\Omega)}+t\|\mu\|_{H^1(\Omega)} \le C(\sigma^{(\alpha-1)/2}+t(1+\sigma^{(\alpha-3)/2})).\end{aligned}$$

We know that

$$\|u\|^2_{[L^2(\Omega),H^1(\Omega)]_{\rho,2}} = \int_0^\infty t^{-2\rho}\chi^2(t,\nu)\frac{\mathrm{d}t}{t}. \tag{3.11}$$

Note that for $t\ge 1$, we may choose $\omega=0$ such that $\chi(t,\nu)\le \|\nu\|_{L^2(\Omega)}$. For $0<t<1$, we take $\sigma=t$ to obtain that

$$\chi(t,\nu) \le C(t^{(\alpha-1)/2}+t(1+t^{(\alpha-3)/2})) \le Ct^{(\alpha-1)/2}.$$

We incorporate the above estimates to obtain that

$$\|u\|^2_{[L^2(\Omega),H^1(\Omega)]_{\rho,2}} \le C\int_0^1 t^{-2\rho+\alpha-1-1}\mathrm{d}t + C\int_1^\infty t^{-2\rho-1}\mathrm{d}t < \infty$$

for $\rho<(\alpha-1)/2$, which completes the proof. □

Theorem 3.7 Suppose that Assumption A holds. For $0<\epsilon<(\alpha-1)/2$, the solutions (y,p,u) to the optimal control problem (3.1)–(3.2) satisfy

$$y(x)\in H^{\alpha/2+1/2-\epsilon}(\Omega), \quad p(x)\in H^{\alpha/2+1/2-\epsilon}(\Omega), \quad u(x)\in H^1(\Omega),$$

and the following stability estimates:

$$\begin{aligned}&\|y\|_{H^{\alpha/2+1/2-\epsilon}(\Omega)} \le C\|f+u\|_{L^2(\Omega)},\\ &\|p\|_{H^{\alpha/2+1/2-\epsilon}(\Omega)} \le C\|y-y_d\|_{L^2(\Omega)},\\ &\|u\|_{H^1(\Omega)} \le C\|y-y_d\|_{L^2(\Omega)}.\end{aligned} \tag{3.12}$$

Proof We can derive from (3.5) that

$$-DI^{2-\alpha}Dy =: \frac{f+u}{K} + (\frac{1}{K})' \int_0^x f(y)+u(y)\mathrm{d}y - A(\frac{1}{K})'. \tag{3.13}$$

By Assumption A, the right-hand side of (3.13) belongs to $L^2(\Omega)$, which, together with Lemma 3.3, implies that $y \in H^{\alpha/2+1/2-\epsilon}(\Omega)$ with the first stability estimate in (3.12). Then we incorporate $y - y_d \in L^2(\Omega)$ with Lemma 3.3 to conclude that the solution $\bar{p}$ introduced in Lemma 3.5 satisfies

$$\bar{p} \in H^{\alpha/2+1/2-\epsilon}(\Omega). \tag{3.14}$$

Then, in order to determine the regularity of p, we differentiate (3.6) on both sides to obtain that

$$\begin{aligned} p'(x) &= K' \int_0^x \frac{\bar{p}'(s)}{K(s)}\mathrm{d}s + \bar{p}' - C^*\omega^{\alpha/2-1} - K'C^* \int_0^x \frac{\omega^{\alpha/2-1}(s)}{K(s)}\mathrm{d}s \\ &= \frac{K'(x)}{K(x)}\bar{p}(x) + K'(x)\int_0^x \bar{p}(s)\frac{K'(s)}{K^2(s)}\mathrm{d}s + \bar{p}' - C^*\omega^{\alpha/2-1} \\ &\quad - K'(x)C^* \int_0^x \frac{\omega^{\alpha/2-1}(s)}{K(s)}\mathrm{d}s. \end{aligned} \tag{3.15}$$

By (3.14), Lemma 2.6, Lemma 3.6 and the Assumption A, the right-hand side of (3.15) is finite under the semi-norm $|\cdot|_{H^{\alpha/2-1/2-\epsilon}(\Omega)}$. We incorporate this with the definition of $\|\cdot\|_{H^{1/2+\alpha/2-\epsilon}}(\Omega)$ (cf. equation (2.1)) and the observation that the right-hand side of (3.15) belongs to $L^2(\Omega)$ to conclude that $p \in H^{\alpha/2+1/2-\epsilon}(\Omega)$ with the second stability estimate in (3.12). Finally, we apply $p \in H^{\alpha/2+1/2-\epsilon}(\Omega) \subset H^1(\Omega)$, (2.3) and (3.4) to arrive at $u \in H^1(\Omega)$ with the third stability estimate in (3.12), which completes the proof. □

4 Finite Element Approximation to the Optimal Control Model

We develop the Galerkin finite element approximation for optimal control problem (3.1)–(3.2) and prove the corresponding optimal-order error estimates. Define the bilinear forms

$$A(\mu,\nu) := (I^{1-\frac{\alpha}{2}}D\mu, I^{1-\frac{\alpha}{2}}D\nu) - (\frac{K'}{K}I^{2-\alpha}D\mu,\nu),\ \forall\mu,\nu \in H_0^{\frac{\alpha}{2}}(\Omega). \tag{4.1}$$

Then the weak formulation of the control problem reads as $\min\limits_{u\in U_{\mathrm{ad}}} J(y,u)$, subject to

$$A(y,\nu) = (\frac{f}{K} + \frac{u}{K},\nu),\ \forall\nu \in H_0^{\frac{\alpha}{2}}(\Omega). \tag{4.2}$$

To derive the variational problem of first order optimality control condition, we define the following Lagrangian functional:

$$\mathcal{S}(y,p,u) := J(y,u) + (\frac{f}{K} + \frac{u}{K}, p) - A(y,p).$$

Then we have that

$$\begin{cases} A(y,\nu) = (\frac{f}{K} + \frac{u}{K},\nu), & \forall\nu \in H_0^{\frac{\alpha}{2}}(\Omega), \\ A(\nu,p) = (y - y_d,\nu), & \forall\nu \in H_0^{\frac{\alpha}{2}}(\Omega), \\ \int_\Omega (\gamma u + \frac{p}{K})(\nu - u)\mathrm{d}x \geq 0, & \forall\nu \in U_{\mathrm{ad}}. \end{cases} \tag{4.3}$$

To define the finite element scheme, we introduce a uniform partition Ω_e of the interval Ω with the mesh size $h>0$. Let V_h represent the continuous finite element space consisting of piecewise linear polynomials on Ω_e and define the piecewise linear interpolation operator $I_h: H_0^{\frac{\alpha}{2}}(\Omega)\to V_h$ with respect to Ω_e. The following interpolation error estimate holds:

$$\|w-I_hw\|_{H^{\frac{\alpha}{2}}(\Omega)}\le Qh^{\frac{1}{2}-\epsilon}\|w\|_{H^{\frac{\alpha}{2}+\frac{1}{2}-\epsilon}(\Omega)},\quad \forall w\in H^{\frac{\alpha}{2}+\frac{1}{2}-\epsilon}(\Omega). \tag{4.4}$$

Then the finite element approximation for optimal control problem is to find $(y_h,u_h)\in V_h\cap H_0^{\frac{\alpha}{2}}(\Omega)\times U_{\rm ad}$ satisfying $\min\limits_{u_h\in U_{\rm ad}} J(y_h,u_h)$ subject to

$$A(y_h,\nu_h):=(I^{1-\frac{\alpha}{2}}Dy_h,I^{1-\frac{\alpha}{2}}D\nu_h)-(\frac{K'}{K}I^{2-\alpha}Dy_h,\nu_h),\ \forall \nu_h\in V_h.$$

In a manner analogous to the derivations of the continuous optimality conditions, we have the discrete first order optimality conditions

$$\begin{cases} A(y_h,\nu_h)=(\dfrac{f}{K}+\dfrac{u_h}{K},\nu_h), & \forall\nu_h\in V_h,\\ A(\nu_h,p_h)=(y_h-y_d,\nu_h), & \forall\nu_h\in V_h,\\ \displaystyle\int_\Omega(\gamma u_h+\frac{p_h}{K})(\nu_h-u_h){\rm d}x\ge 0, & \forall\nu_h\in U_{\rm ad}. \end{cases} \tag{4.5}$$

Next, to achieve a priori error estimates of the finite element approximation we need to introduce auxiliary problems and results:

$$\begin{cases} A(y_h(u),\nu_h)=(\dfrac{f}{K}+\dfrac{u}{K},\nu_h), & \forall\nu_h\in V_h,\\ A(\nu_h,p_h(u))=(y_h(u)-y_d,\nu_h), & \forall\nu_h\in V_h,\\ A(\nu_h,p_h(y))=(y-y_d,\nu_h), & \forall\nu_h\in V_h. \end{cases} \tag{4.6}$$

Theorem 4.1 Under the Assumption A, the following estimates hold for $v,w\in H_0^{\frac{\alpha}{2}}(\Omega)$:

$$|A(v,w)|\le C\|v\|_{H^{\frac{\alpha}{2}}(\Omega)}\|w\|_{H^{\frac{\alpha}{2}}(\Omega)},\ \|v\|^2_{H^{\frac{\alpha}{2}}(\Omega)}\le C(\|v\|^2_{L^2(\Omega)}+A(v,v)).$$

Proof We apply integration by parts and Lemma 2.3 to obtain, for $w,v\in H_0^{\frac{\alpha}{2}}(\Omega)$, that

$$\begin{aligned}(\mathcal{T}v,w)&=\int_\Omega(I^{2-\alpha}Dv)Dw{\rm d}x\\ &=\int_\Omega({}_0I_x^{1-\alpha/2}Dv)({}_xI_1^{1-\alpha/2}Dw)+({}_xI_1^{1-\alpha/2}Dv)({}_0I_x^{1-\alpha/2}Dw){\rm d}x.\end{aligned}$$

Then, by (2.2), we obtain

$$\begin{aligned}|(\mathcal{T}v,w)|&\le C\|{}_0I_x^{1-\alpha/2}v'\|_{L^2(\Omega)}\|{}_xI_1^{1-\alpha/2}w'\|_{L^2(\Omega)}+C\|{}_xI_1^{1-\alpha/2}v'\|_{L^2(\Omega)}\|{}_0I_x^{1-\alpha/2}w'\|_{L^2(\Omega)}\\ &=C|v|_{\dot H_l^{\frac{\alpha}{2}}(\Omega)}|w|_{\dot H_r^{\frac{\alpha}{2}}(\Omega)}+C|v|_{\dot H_r^{\frac{\alpha}{2}}(\Omega)}|w|_{\dot H_l^{\frac{\alpha}{2}}(\Omega)}.\end{aligned}$$

We apply Lemma 2.1 and $1/2<\alpha/2$ to get that

$$|(\mathcal{T}v,w)|\le C\|v\|_{H^{\frac{\alpha}{2}}(\Omega)}\|w\|_{H^{\frac{\alpha}{2}}(\Omega)}. \tag{4.7}$$

Due to $\alpha-1<\alpha/2$ for $1<\alpha<2$, we apply Lemma 2.1, Lemmas 2.3–2.6 and the zero boundary conditions of v and w to get that

$$|(\mathcal{M}_1v,w)|\le\|\frac{K'}{K}({}_0I_x^{2-\alpha}Dv+{}_xI_1^{2-\alpha}Dv)\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}$$

$$
\begin{aligned}
&\le \|\frac{K'}{K}\|_\infty \|{}_0I_x^{2-\alpha}Dv + {}_xI_1^{2-\alpha}Dv\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} \\
&\le C(|v|_{\dot H_l^{\alpha-1}(\Omega)} + |v|_{\dot H_r^{\alpha-1}(\Omega)})\|w\|_{L^2(\Omega)} \\
&\le C(\|v\|_{\dot H_l^{\frac{\alpha}{2}}(\Omega)} + \|v\|_{\dot H_r^{\frac{\alpha}{2}}(\Omega)})\|w\|_{L^2(\Omega)} \\
&\le C\|v\|_{H^{\frac{\alpha}{2}}(\Omega)}\|w\|_{L^2(\Omega)}. \qquad (4.8)
\end{aligned}
$$

We then set $w = v$ and use Lemmas 2.1–2.2 and the zero boundary conditions of v to obtain

$$
(\mathcal{T}v, v) \ge Q\|v\|^2_{H^{\frac{\alpha}{2}}(\Omega)},\ Q > 0. \qquad (4.9)
$$

By (4.8), (4.9) and the Cauchy-Schwarz inequality, we have that

$$
\begin{aligned}
A(v,v) &\ge Q\|v\|^2_{H^{\frac{\alpha}{2}}(\Omega)} - C\|v\|_{H^{\frac{\alpha}{2}}(\Omega)}\|v\|_{L^2(\Omega)} \\
&\ge Q\|v\|^2_{H^{\frac{\alpha}{2}}(\Omega)} - (\frac{Q}{2}\|v\|^2_{H^{\frac{\alpha}{2}}(\Omega)} + \frac{C^2}{2Q}\|v\|^2_{L^2(\Omega)}) \\
&= \frac{Q}{2}\|v\|^2_{H^{\frac{\alpha}{2}}(\Omega)} - \frac{C^2}{2Q}\|v\|^2_{L^2(\Omega)}. \qquad (4.10)
\end{aligned}
$$

Finally, combining (4.7), (4.8) and (4.10) yields the results for $A(\cdot,\cdot)$.

Due to $(\mathcal{M}_1 v, w) = (\mathcal{M}_2 w, v)$ and

$$
\begin{aligned}
|(\mathcal{M}_2 v, w)| &\le \|\frac{K'}{K}\|_\infty \|v\|_{L^2}\|{}_0I_x^{2-\alpha}Dw + {}_xI_1^{2-\alpha}Dw\|_{L^2} \\
&\le C|v|_{\dot H_l^{\alpha-1}}\|w\|_{H^{\frac{\alpha}{2}}} \le C\|v\|_{H^{\frac{\alpha}{2}}}\|w\|_{H^{\frac{\alpha}{2}}},
\end{aligned}
$$

the estimates for $A(\cdot,\cdot)$ of the adjoint state equation follow from those of $A(\cdot,\cdot)$ in the state equation. □

Theorem 4.2 Under the Assumption A, the following convergence and stability estimates of the auxiliary problems hold for a sufficiently small h and $\epsilon > 0$:

$$
\begin{aligned}
&\|y - y_h(u)\|_{L^2(\Omega)} + h^{1/2-\epsilon}\|y - y_h(u)\|_{H^{\frac{\alpha}{2}}(\Omega)} \le Ch^{1-2\epsilon}\|y\|_{H^{\frac{\alpha}{2}+\frac{1}{2}-\epsilon}(\Omega)}, \\
&\|p - p_h(y)\|_{L^2(\Omega)} + h^{1/2-\epsilon}\|p - p_h(y)\|_{H^{\frac{\alpha}{2}}(\Omega)} \le Ch^{1-2\epsilon}\|p\|_{H^{\frac{\alpha}{2}+\frac{1}{2}-\epsilon}(\Omega)}, \\
&\|y_h(u)\|_{H^{\frac{\alpha}{2}}(\Omega)} \le C\|f + u\|_{L^2(\Omega)}, \quad \|p_h(y)\|_{H^{\frac{\alpha}{2}}(\Omega)} \le C\|y - y_d\|_{L^2(\Omega)}.
\end{aligned}
$$

Proof As the existence and uniqueness of linear problems are equivalent, we prove the uniqueness of solution $y_h(u)$ to show its existence. If there exist two solutions $y_{h1}(u)$ and $y_{h2}(u)$ to the first equation of (4.6), the difference $y_h^*(u) = y_{h1}(u) - y_{h2}(u)$ satisfies that $A(y_h^*(u), \nu_h) = 0$. Next, let y^* be the solution to the homogeneous boundary-value problem of $\mathfrak{L}y^* = 0$. By Lemma 3.4, we have that $y^* = 0$, and thus that $A(y^*, \nu_h) = 0$. Let m be the solution of the homogeneous boundary-value problem of $\mathfrak{L}^* m = y^* - y_h^*(u)$, so we have that

$$
A(\nu, m) = (y^* - y_h^*(u), \nu),\ \forall \nu \in H_0^{\frac{\alpha}{2}}(\Omega).
$$

We take $\nu = y^* - y_h^*(u)$ to obtain

$$
(y^* - y_h^*(u), y^* - y_h^*(u)) = A(y^* - y_h^*(u), m) = A(y^* - y_h^*(u), m - I_h m).
$$

Using Theorem 4.1 and (4.4) we can get that

$$
\|y^* - y_h^*(u)\|^2_{L^2(\Omega)} = A(y^* - y_h^*(u), m - I_h m)
$$

$$\begin{aligned}
&\leq \|y^* - y_h^*(u)\|_{H^{\frac{\alpha}{2}}(\Omega)} \|m - I_h m\|_{H^{\frac{\alpha}{2}}(\Omega)} \\
&\leq Ch^{1/2-\epsilon} \|y^* - y_h^*(u)\|_{H^{\frac{\alpha}{2}}(\Omega)} \|m\|_{H^{\frac{\alpha}{2}+\frac{1}{2}-\epsilon}(\Omega)} \\
&\leq Ch^{1/2-\epsilon} \|y^* - y_h^*(u)\|_{H^{\frac{\alpha}{2}}(\Omega)} \|y^* - y_h^*(u)\|_{L^2(\Omega)}.
\end{aligned}$$

This implies that

$$\|y^* - y_h^*(u)\|_{L^2(\Omega)} \leq Ch^{1/2-\epsilon} \|y^* - y_h^*(u)\|_{H^{\frac{\alpha}{2}}(\Omega)}. \tag{4.11}$$

By Theorem 4.1 and (4.11), we obtain

$$\begin{aligned}
\|y^* - y_h^*(u)\|^2_{H^{\frac{\alpha}{2}}(\Omega)} &\leq C(\|y^* - y_h^*(u)\|^2_{L^2(\Omega)} + A(y^* - y_h^*(u), y^* - y_h^*(u))) \\
&\leq C(h^{1-2\epsilon} \|y^* - y_h^*(u)\|^2_{H^{\frac{\alpha}{2}}(\Omega)} + A(y^* - y_h^*(u), y^* - I_h y^*)) \\
&\leq C(h^{1-2\epsilon} \|y^* - y_h^*(u)\|^2_{H^{\frac{\alpha}{2}}(\Omega)} + \|y^* - y_h^*(u)\|_{H^{\frac{\alpha}{2}}(\Omega)} \|y^* - I_h y^*\|_{H^{\frac{\alpha}{2}}(\Omega)}).
\end{aligned}$$

By (4.4), we can derive, for a sufficiently small h, that

$$\|y^* - y_h^*(u)\|_{H^{\frac{\alpha}{2}}(\Omega)} \leq Ch^{1/2-\epsilon} \|y^*\|_{H^{\frac{\alpha}{2}+\frac{1}{2}-\epsilon}(\Omega)}. \tag{4.12}$$

Thus we have that

$$y^* = y_h^*(u) = 0;$$

that is, $y_h(u)$ exists. Let w be the solution of the dual problem

$$-DI^{2-\alpha} Dw + DI^{2-\alpha} \frac{K'(x)}{K(x)} w = y - y_h(u); \tag{4.13}$$

Then we incorporate $y - y_h(u) \in L^2(\Omega)$ with Lemma 3.3 to conclude that the solution w satisfies $w \in H^{\frac{\alpha}{2}+\frac{1}{2}-\epsilon}(\Omega)$. Next, we have that

$$A(\nu, w) = (y - y_h(u), \nu), \ \forall \nu \in H_0^{\frac{\alpha}{2}}(\Omega).$$

We take $\nu = y - y_h(u)$ to obtain

$$(y - y_h(u), y - y_h(u)) = A(y - y_h(u), w) = A(y - y_h(u), w - I_h w).$$

Using a similar procedure as to that above we can get that

$$\|y - y_h(u)\|_{L^2(\Omega)} \leq Ch^{1/2-\epsilon} \|y - y_h(u)\|_{H^{\frac{\alpha}{2}}(\Omega)} \tag{4.14}$$

and

$$\|y - y_h(u)\|_{H^{\frac{\alpha}{2}}(\Omega)} \leq Ch^{1/2-\epsilon} \|y\|_{H^{\frac{\alpha}{2}+\frac{1}{2}-\epsilon}(\Omega)}. \tag{4.15}$$

The stability estimate of $y_h(u)$ follows from (4.15) and Theorem 3.7. The analysis for the second adjoint state problem in (4.6) can be performed similarly and is thus omitted. □

Theorem 4.3 Under the Assumption A, the following error estimates between the exact and numerical solutions hold for a sufficiently small h and $\epsilon > 0$:

$$\begin{aligned}
&\|y - y_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} + \|u - u_h\|_{L^2(\Omega)} \leq Ch^{1-2\epsilon}, \\
&\|y - y_h\|_{H^{\frac{\alpha}{2}}(\Omega)} + \|p - p_h\|_{H^{\frac{\alpha}{2}}(\Omega)} \leq Ch^{1/2-\epsilon}.
\end{aligned}$$

Proof To derive the error estimate, we first decompose the errors $y-y_h$ and $p-p_h$ into

$$y-y_h=y-y_h(u)+y_h(u)-y_h, \tag{4.16}$$

$$p-p_h=p-p_h(y)+p_h(y)-p_h. \tag{4.17}$$

It is easy to see that $y_h(u)$ and $p_h(y)$ are the finite element approximations of y and p. By Lemma 4.2, we have that

$$\|y-y_h(u)\|_{L^2(\Omega)}+h^{1/2-\epsilon}\|y-y_h(u)\|_{H^{\frac{\alpha}{2}}(\Omega)}\le Ch^{1-2\epsilon}, \tag{4.18}$$

$$\|p-p_h(y)\|_{L^2(\Omega)}+h^{1/2-\epsilon}\|p-p_h(y)\|_{H^{\frac{\alpha}{2}}(\Omega)}\le Ch^{1-2\epsilon}. \tag{4.19}$$

By (4.3) and (4.6) we have that

$$A(y_h(u)-y_h,\nu_h)=(\frac{u}{K}-\frac{u_h}{K},\nu_h). \tag{4.20}$$

If we consider $y_h(u)-y_h$ as the numerical solution of the state equation with the right-hand side term $u-u_h$, which belongs to $L^2(\Omega)$, then a procedure similar to that of Theorem 4.2 yields that

$$\|y_h(u)-y_h\|_{L^2(\Omega)}\le C\|u-u_h\|_{L^2(\Omega)},\ \|y_h(u)-y_h\|_{H^{\frac{\alpha}{2}}(\Omega)}\le C\|u-u_h\|_{L^2(\Omega)}. \tag{4.21}$$

In a similar way, we can derive that

$$\|p_h(y)-p_h\|_{L^2(\Omega)}\le C\|y-y_h\|_{L^2(\Omega)},\ \|p_h(y)-p_h\|_{H^{\frac{\alpha}{2}}(\Omega)}\le C\|y-y_h\|_{L^2(\Omega)}. \tag{4.22}$$

Under (4.16)–(4.17), (4.18)–(4.19), (4.21) and (4.22), this leads to

$$\|y-y_h\|_{L^2(\Omega)}+\|p-p_h\|_{L^2(\Omega)}\le C(h^{1-2\epsilon}+\|u-u_h\|_{L^2(\Omega)}), \tag{4.23}$$

$$\|y-y_h\|_{H^{\frac{\alpha}{2}}(\Omega)}+\|p-p_h\|_{H^{\frac{\alpha}{2}}(\Omega)}\le C(h^{1/2-\epsilon}+\|y-y_h\|_{L^2(\Omega)}). \tag{4.24}$$

Now, it remains to estimate $\|u-u_h\|$. Note that

$$\hat{J}_h'(u)(v-u)=\int_\Omega(\gamma u+\frac{p_h(u)}{K})(v-u)\mathrm{d}x.$$

We can prove that

$$\hat{J}_h'(u)(u-u_h)-\hat{J}_h'(u)(u-u_h)\ge\gamma\|u-u_h\|^2_{L^2(\Omega)}.$$

In fact,

$$\begin{aligned}\hat{J}_h'(u)(u-u_h)-\hat{J}_h'(u_h)(u-u_h)&=\int_\Omega(\gamma u+\frac{p_h(u)}{K}-\gamma u-\frac{p_h}{K})(u-u_h)\mathrm{d}x\\&=\gamma\int_\Omega(u-u_h)^2\mathrm{d}x+\int_\Omega(\frac{p_h(u)}{K}-\frac{p_h}{K})(u-u_h)\mathrm{d}x.\end{aligned}$$

Using the state equation, we deduce that

$$\begin{aligned}&\int_\Omega(\frac{p_h(u)}{K}-\frac{p_h}{K})(u-u_h)\mathrm{d}x=\int_\Omega(p_h(u)-p_h)(\frac{u}{K}-\frac{u_h}{K})\mathrm{d}x\\&=\int_\Omega(p_h(u)-p_h)(-DI^{2-\alpha}D(y_h(u)-y_h))\mathrm{d}x-\int_\Omega(p_h(u)-p_h)(\frac{K'}{K}I^{2-\alpha}D(y_h(u)-y_h))\mathrm{d}x\\&=\int_\Omega(-DI^{2-\alpha}D(p_h(u)-p_h))(y_h(u)-y_h)\mathrm{d}x+\int_\Omega(DI^{2-\alpha}\frac{K'}{K}(p_h(u)-p_h))(y_h(u)-y_h)\mathrm{d}x\\&=\int_\Omega\big(-DI^{2-\alpha}D(p_h(u)-p_h)\mathrm{d}x+DI^{2-\alpha}\frac{K'}{K}(p_h(u)-p_h)\big)(y_h(u)-y_h)\mathrm{d}x\end{aligned}$$

$$= \int_\Omega (y_h(u)-y_h)(y_h(u)-y_h)\mathrm{d}x \geq 0.$$

This implies the result. Then we have that

$$\begin{aligned}\gamma\|u-u_h\|^2_{L^2(\Omega)} &\leq \hat{J}'_h(u)(u-u_h) - \hat{J}'_h(u_h)(u-u_h)\\ &= \int_\Omega (\gamma u + \frac{p_h(u)}{K} - \gamma u_h - \frac{p_h}{K})(u-u_h)\mathrm{d}x\\ &= (\gamma u + \frac{p}{K}, u-u_h) - (\gamma u_h + \frac{p_h}{K}, u-u_h) + (\frac{p_h(u)}{K} - \frac{p}{K}, u-u_h)\\ &\leq 0 + 0 + C\|p-p_h(u)\|_{L^2(\Omega)}\|u-u_h\|_{L^2(\Omega)}.\end{aligned}$$

Thus,

$$\|u-u_h\|_{L^2(\Omega)} \leq C\|p-p_h(u)\|_{L^2(\Omega)}.$$

Letting $p-p_h(u) = p-p_h(y)+p_h(y)-p_h(u)$, by (4.6), we have that

$$A(\nu_h, p_h(y)-p_h(u)) = (y-y_h(u), \nu_h). \tag{4.25}$$

By Theorem 4.2, we can obtain that

$$\|p_h(y)-p_h(u)\|_{L^2(\Omega)} \leq C\|y-y_h(u)\|_{L^2(\Omega)}. \tag{4.26}$$

Then we have that

$$\|u-u_h\|_{L^2(\Omega)} \leq \|p-p_h(y)\|_{L^2(\Omega)} + \|y-y_h(u)\|_{L^2(\Omega)} \leq Ch^{1-2\epsilon}. \tag{4.27}$$

Finally, combining (4.23)–(4.24) and (4.27) yields the results of the theorem. □

5 Numerical Experiments

In this section, we present numerical experiments to compare the experimental rate of convergence of the approximation with the theoretically predicted rate in order to demonstrate our theoretical findings.

Letting $K(x) = e^x$, the solutions to problem (3.1)–(3.2) are given by $y = x^{\frac{\alpha}{2}}(1-x)^{\frac{\alpha}{2}}$, $p = 10x^{\frac{\alpha}{2}}(1-x)^{\frac{\alpha}{2}}$ and $u = \max\{a, \min\{b, -\frac{p}{\gamma K(x)}\}\}$, where $\gamma = 1, a = -1.6$ and $b = -0.2$. The f and y_d are given by

$$\begin{aligned}f(x) &= -2\Gamma(1+\alpha)\cos(\frac{\pi\alpha}{2}) - \cos(\frac{\pi\alpha}{2})\Gamma(1+\alpha)(2x-1) - u,\\ y_d(x) &= y - (-2\Gamma(1+\alpha)\cos(\frac{\pi\alpha}{2}) + \cos(\frac{\pi\alpha}{2})\Gamma(1+\alpha)(2x-1)).\end{aligned}$$

The errors and convergence rates for state, adjoint state and control with different α are shown in Tables 1–5, from which we observe that the convergence rates of the errors in the L^2 norms and the $H^{\frac{\alpha}{2}}$ norms are consistent with the theoretical results presented in Theorem 4.3.

Table 1 Errors and convergence rates of y, p, u with $\alpha = 1.30$

h	$\|y-y_h\|_{L^2}$	rate	$\|p-p_h\|_{L^2}$	rate	$\|u-u_h\|_{L^2}$	rate
1/100	$7.45e-04$		$8.30e-02$		$3.30e-03$	
1/200	$4.01e-04$	0.90	$4.50e-03$	0.90	$1.90e-03$	0.79
1/400	$2.11e-04$	0.93	$2.30e-03$	0.93	$1.10e-03$	0.85
1/800	$1.11e-04$	0.94	$1.20e-03$	0.95	$5.68e-04$	0.91

Table 2 Errors and convergence rates of y, p, u with $\alpha = 1.50$

h	$\|y - y_h\|_{L^2}$	rate	$\|p - p_h\|_{L^2}$	rate	$\|u - u_h\|_{L^2}$	rate
1/100	$4.22e-04$		$4.50e-03$		$1.90e-03$	
1/200	$2.24e-04$	0.91	$2.40e-03$	0.91	$1.10e-03$	0.84
1/400	$1.17e-04$	0.94	$1.20e-03$	0.94	$5.77e-04$	0.91
1/800	$5.99e-05$	0.96	$6.41e-04$	0.96	$2.99e-04$	0.95

Table 3 Errors and convergence rates of y, p, u with $\alpha = 1.80$

h	$\|y - y_h\|_{L^2}$	rate	$\|p - p_h\|_{L^2}$	rate	$\|u - u_h\|_{L^2}$	rate
1/100	$7.53e-05$		$8.00e-04$		$4.01e-04$	
1/200	$4.06e-05$	0.89	$4.31e-04$	0.89	$2.20e-04$	0.87
1/400	$2.12e-05$	0.94	$2.25e-04$	0.94	$1.15e-04$	0.94
1/800	$1.09e-05$	0.96	$1.15e-04$	0.97	$5.87e-05$	0.97

Table 4 Errors and convergence rates of $\|y - y_h\|_{H^{\frac{\alpha}{2}}}$ with different α

$h\diagdown\alpha$	1.3	rate	1.5	rate	1.8	rate
1/100	$2.90e-03$		$1.70e-03$		$4.33e-04$	
1/200	$2.20e-03$	0.41	$1.30e-03$	0.45	$3.10e-04$	0.48
1/400	$1.60e-03$	0.44	$9.13e-04$	0.46	$2.21e-04$	0.49
1/800	$1.20e-03$	0.46	$6.45e-04$	0.48	$1.57e-04$	0.49

Table 5 Errors and convergence rates of $\|p - p_h\|_{H^{\frac{\alpha}{2}}}$ with different α

$h\diagdown\alpha$	1.3	rate	1.5	rate	1.8	rate
1/100	$3.06e-02$		$1.77e-02$		$4.50e-03$	
1/200	$2.23e-02$	0.46	$1.26e-02$	0.49	$3.20e-03$	0.51
1/400	$1.61e-02$	0.46	$9.00e-03$	0.48	$2.20e-03$	0.50
1/800	$1.16e-02$	0.48	$6.50e-03$	0.49	$1.60e-03$	0.50

References

[1] Acosta G, Borthagaray J P. A fractional Laplace equation: regularity of solutions and finite element approximations. SIAM J Numer Anal, 2017, **55**: 472–495

[2] Adams R, Fournier J. Sobolev Spaces. San Diego: Elsevier, 2003

[3] Antil H, Otárola E. A FEM for an optimal control problem of fractional powers of elliptic operators. SIAM J Control Optim, 2015, **53**(6): 3432–3456

[4] Antil H, Otárola E, Salgado A J. A space-time fractional optimal control problem: analysis and discretization. SIAM J Control Optim, 2016, **54**(3): 1295–1328

[5] Benson D A, Wheatcraft S W, Meerschaeert M M. The fractional order governing equations of levy motion. Water Resour Res, 2000, **36**(6): 1413–1423

[6] Brenner S C, Scott L R. The Mathematical Theory of Finite Element Methods. New York: Springer, 2008

[7] Becker R, Kapp H, Rannacher R. Adaptive finite element methods for optimal control of partial differential equations: Basic concept. SIAM J Control Optim, 2000, **39**(1): 113–132

[8] Chen S, Shen J, Wang L L. Generalized Jacobi functions and their applications to fractional differential equations. Math Comp, 2016, **85**: 1603–1638

[9] Chen S, Shen J. An efficient and accurate numerical method for the spectral fractional Laplacian equation. J Sci Comput, 2020, **82**(1): 1–25

[10] D'Elia M, Glusa C, Otárola E. A priori error estimates for the optimal control of the integral fractional Laplacian. SIAM J Control Optim, 2019, **57**: 2775–2798

[11] Du N, Wang H, Liu W B. A fast gradient projection method for a constrained fractional optimal control. J Sci Comput, 2016, **68**: 1–20

[12] Ervin V J, Roop J P. Variational formulation for the stationary fractional advection dispersion equation. Numer Meth for PDE, 2006, **22**: 558–576

[13] Ervin V J. Regularity of the solution to fractional diffusion, advection, reaction equations in weighted Sobolev spaces. J Differ Equ, 2021, **278**: 294–325

[14] Ervin V J, Heuer N, Roop J P. Regularity of the solution to 1-d fractional order diffusion equations. Math Comput, 2018, **87**: 2273–2294

[15] Hao Z P, Park M, Lin G, et al. Finite element method for two-sided fractional differential equations with variable coefficients: Galerkin approach. J Sci Comput, 2019, **79**: 700–717

[16] Hao Z P, Zhang Z Q. Optimal regularity and error estimates of a spectral Galerkin method for fractional advection-diffusion-reaction equations. SIAM J Numer Anal, 2020, **58**(1): 211–233

[17] Kunisch K, Vexler B. Constrained Dirichlet boundary control in L2 for a class of evolution equations. SIAM J Control Optim, 2007, **46**(5): 1726–1753

[18] Klafter J, Sokolov I M. Anomalous diffusion spreads its wings. Phys World, 2005, **18**(8): 29–32

[19] Li S Y, Zhou Z J. Fractional spectral collocation method for optimal control problem governed by space fractional diffusion equation. Appl Math Comput, 2019, **350**: 331–347

[20] Ma L J, Liu B. Dynamic analysis and optimal control of a fractional order singular Leslie-Gower prey-predator model. Acta Math Sci, 2020, **40B**(5): 1525–1552

[21] Mao Z P, Chen S, Shen J. Efficient and accurate spectral method using generalized Jacobi functions for solving Riesz fractional differential equations. Appl Numer Math, 2016, **106**: 165–181

[22] Mao Z P, Karniadakis G E. A spectral method (of exponential convergence) for singular solutions of the diffusion equation with general two-sided fractional derivative. SIAM J Numer Anal, 2018, **56**: 24–49

[23] Meerschaert M M, Sikorskii A. Stochastic Models for Fractional Calculus. Berlin: De Gruyter, 2019

[24] Nezza E D, Palatucci G, Valdinoci E. Hitchhiker's guide to the fractional Sobolev spaces. Bulletin des Sciences Mathématiques, 2012, **136**: 521–573

[25] Wang F Y, Zhang Z Q, Zhou Z J. A spectral Galerkin approximation of optimal control problem governed by fractional advection-diffusion-reaction equations. J Comput Appl Math, 2021, **386**: 113233

[26] Wang H, Yang D P. Wellposedness of variable-coefficient conservative fractional elliptic differential equations. SIAM J Numer Anal, 2013, **51**: 1088–1107

[27] Wang H, Yang D P, Zhu S F. Inhomogeneous Dirichlet boundary-value problems of space-fractional diffusion equations and their finite element approximations. SIAM J Numer Anal, 2014, **52**: 1292–1310

[28] Zhang L, Zhou Z J. Spectral Galerkin approximation of optimal control problem governed by Riesz fractional differential equation. Appl Numer Math, 2019, **143**: 247–262

[29] Zaky M A, Tenreiro Machado J A. On the formulation and numerical simulation of distributed-order fractional optimal control problems. Commun Nonlinear Sci Numer Simul, 2017, **52**: 177–189

[30] Zaky M A. A Legendre collocation method for distributed-order fractional optimal control problems. Nonlinear Dyn, 2018, **91**: 2667–2681

[31] Zhang Z Q. Error estimates of spectral Galerkin methods for a linear fractional reaction-diffusion equation. J Sci Comput, 2019, **78**: 1087–1110

[32] Zheng X C, Ervin V J, Wang H. Numerical approximations for the variable coefficient fractional diffusion equations with non-smooth data. Comput Math Appl Math, 2020, **20**(3): 573–589

[33] Zheng X C, Ervin V J, Wang H. An indirect finite element method for variable-coefficient space-fractional diffusion equations and its optimal-order error estimates. Commun Appl Math Comput, 2020, **2**: 147–162

[34] Zheng X C, Ervin V J, Wang H. Wellposedness of the two-sided variable coefficient Caputo flux fractional diffusion equation and error estimate of its spectral approximation. Appl Numer Math, 2020, **153**: 234–237

[35] Zheng X C, Wang H. An optimal-order numerical approximation to variable-order space-fractional diffusion equations on uniform or graded meshes. SIAM J Numer Anal, 2020, **58**(1): 330–352