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NEW DOOB'S MAXIMAL INEQUALITIES FOR MARTINGALES*

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Abstract Let $1 \le q \le \infty$, b be a slowly varying function and let $\Phi : [0,\infty) \longrightarrow [0,\infty)$ be an increasing convex function with $\Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$. In this paper, we present a new class of Doob's maximal inequality on Orlicz-Lorentz-Karamata spaces $L_{\Phi,q,b}$. The results are new, even for the Lorentz-Karamata spaces with $\Phi(t) = t^p$, the Orlicz-Lorentz spaces with $b \equiv 1$, and weak Orlicz-Karamata spaces with $q = \infty$ in the framework of $L_{\Phi,q,b}$. Moreover, we obtain some even stronger qualitative results that can remove the $\triangle_2$-condition of Liu, Hou and Wang (Sci China Math, 2010, 53(4): 905–916).

Key words martingales; Doob's inequality; Orlicz-Lorentz-Karamata spaces; convex functions

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1 Introduction

Doob's maximal inequality plays a central role in harmonic analysis, probability and ergodic theory. The purpose of this paper is to study Doob's maximal inequalities for martingales, given a martingale $f = (f_n)_{n\ge0}$ that is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the well-known Doob's maximal inequality states that

$$\|Mf\|_{L_p} \le \frac{p}{p-1}\|f\|_{L_p}, \quad p > 1,$$

where $Mf = \sup\limits_{n\ge0} |f_n|$ and $\|f\|_{L_p} = \sup\limits_{n\ge0} \|f_n\|_{L_p}$ (see Doob [4], and also [9, 14, 29]). Doob used this to prove the basic, almost sure convergence properties of martingales.

Over the course of the past few decades, Doob's maximal inequality has attracted considerable attention, and has been rapidly developed to various function spaces, such as weak Orlicz spaces [21], Morrey spaces [13, 25], variable exponent spaces [15, 18] and Musielak-Orlicz-Lorentz spaces [16]. We point out that these results need the $\triangle_2$-condition or analogous behaviors. However, Doob's maximal inequality on Orlicz spaces [22] and variable exponent spaces [30, 31] also holds, in some sense, for no restricting $\triangle_2$-condition. Inspired by this, the

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main aim of this paper is to establish Doob's maximal inequality for a new class of space without the $\triangle_2$-condition. As applications of the main conclusion, we obtain some even stronger qualitative results that can remove the $\triangle_2$-condition in [21], and we also extend Doob's maximal inequality to weak Orlicz-Karamata spaces and Orlicz-Lorentz spaces. Note that our approach is based on the weights of Muckenhoupt [24], which is very different from those of [21].

Throughout this paper, we denote by C the absolute positive constant that is independent of the main parameters involved, but whose value may differ from line to line. The symbol $f \lesssim g$ stands for the inequality $f \le Cg$. When we write $f \approx g$, this stands for $f \lesssim g \lesssim f$.

2 Notations and Main Results

In this section, we state some notations and present the main results of the paper.

2.1 Orlicz Functions

Let Φ be an Orlicz function on $[0,\infty)$, i.e., a continuous increasing and convex function satisfying that $\Phi(0)=0$ and $\lim\limits_{r\to\infty}\Phi(r)=\infty$. Recall that Φ is said to satisfy the $\triangle_2$-condition if there is a constant C such that $\Phi(2t)\le C\Phi(t)$ for all $t>0$. We denote this by $\Phi \in\triangle_2$.

We will work with some standard indices associated with Orlicz functions. Given an Orlicz function Φ, since Φ is monotonic, $\Phi(r)$ is defined for each $r>0$, except for a countable set of points in which we take $\Phi'(r)$ as the derivative from the right. Then, the lower and upper Simonenko indices of Φ (see [28]) are defined, respectively, by

$$p_\Phi=\inf_{t>0}\frac{t\Phi'(t)}{\Phi(t)} \quad \text{and} \quad q_\Phi=\sup_{t>0}\frac{t\Phi'(t)}{\Phi(t)}.$$

The following properties for Simonenko indices of Φ will be used in the sequel:

(i) $1\le p_\Phi\le q_\Phi\le\infty$;

(ii) the following characterizations of p_Φ and q_Φ hold:

$$p_\Phi=\sup\Big\{p>0:\ t^{-p}\Phi(t) \text{ is non-decreasing for all } t>0\Big\},$$
$$q_\Phi=\inf\Big\{q>0:\ t^{-q}\Phi(t) \text{ is non-increasing for all } t>0\Big\};$$

(iii) $\Phi\in\triangle_2$ if and only if $q_\Phi<\infty$. Moreover, Φ is said to be strictly convex if $p_\Phi>1$.
(See [8, 23, 27] for more information on Orlicz functions and the indices.)

2.2 Slowly Varying Functions

A Lebesgue measurable function $b:[1,\infty)\longrightarrow(0,\infty)$ is said to be a slowly varying function if, for any given $\epsilon>0$, the function $t^{\epsilon}b(t)$ is equivalent to a non-decreasing function, and the function $t^{-\epsilon}b(t)$ is equivalent to a non-increasing function on $[1,\infty)$.

A detailed study of Karamata Theory, and the properties and examples of slowly varying functions, can be found in [3, 6, 26]. Given a slowly varying function b on $[1,\infty)$, we denote by γ_b the positive function defined by $\gamma_b(t)=b\big(\max\{t,1/t\}\big)$, for all $t>0$.

Proposition 2.1 ([5]) Let b be a slowly varying function. For any given $\epsilon>0$, the function $t^{\epsilon}\gamma_b(t)$ is equivalent to a non-decreasing function, and $t^{-\epsilon}\gamma_b(t)$ is equivalent to a non-increasing function on $(0,1]$.

Remark 2.2 ([5, 32]) Let $r>0$. Then $\gamma_b(rt)\approx\gamma_b(t)$ for all $t>0$.

2.3 Orlicz-Lorentz-Karamata Spaces

Now we present a class of new spaces based on the Orlicz function and the slowly varying function. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let f be an $\mathcal{F}$-measurable function defined on Ω. The non-increasing rearrangement f^* of f is defined as

$$f^*(t) = \inf\{s \geq 0 : \mathbb{P}\big(\{|f| > s\}\big) \leq t\}, \qquad t \geq 0,$$

with the convention that $\inf \emptyset = \infty$.

Definition 2.3 Let $0 < q \leq \infty$, b be a slowly varying function and let Φ be an Orlicz function on $[0, \infty)$. The Orlicz-Lorentz-Karamata space, denoted by $L_{\Phi,q,b}$, consists of those measurable functions f with $|||f|||_{\Phi,q,b} < \infty$, where

$$|||f|||_{\Phi,q,b} = \begin{cases} \left(q \int_0^1 \Big(\frac{1}{\Phi^{-1}(1/t)} \gamma_b(t) f^*(t) \Big)^q \frac{\mathrm{d}t}{t} \right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup\limits_{t>0} \frac{1}{\Phi^{-1}(1/t)} \gamma_b(t) f^*(t) & \text{if } q = \infty. \end{cases} \tag{2.1}$$

Remark 2.4 The Orlicz-Lorentz-Karamata spaces $L_{\Phi,q,b}$ are the generalizations of various function spaces achieved by taking different Φ, q and b. We now list several examples.

(1) The weak Orlicz space wL_Φ defined by Liu, Hou and Wang [21] is as follows:

$$wL_\Phi = \Big\{ f : \exists\, C > 0 \text{ s.t. } \Phi(t/C)\mathbb{P}(|f| > t) < \infty, \ \forall\, t > 0 \Big\},$$

equipped with the quasi-norm

$$\|f\|_{wL_\Phi} = \inf \big\{ C > 0 : \ \Phi(t/C)\mathbb{P}(|f| > t) \leq 1, \ \forall\, t > 0 \big\}.$$

It follows from Remark 3.1 and Proposition 3.1 in [1] that $\|f\|_{wL_\Phi} = |||f\,|||_{\Phi,\infty,1}$.

Thus $L_{\Phi,q,b}$ can be reduced to the weak Orlicz space wL_Φ when $q = \infty$ and $b \equiv 1$.

(2) The weak Orlicz-Karamata space $wL_{\Phi,b}$ introduced by Zhou, Wu and Jiao [32] is as follows:

$$wL_{\Phi,b} = \Big\{ f : \|f\|_{wL_{\Phi,b}} = \sup_{t>0} t \|\chi_{\{|f|>t\}}\|_{L_\Phi} \gamma_b\big(\mathbb{P}(|f| > t)\big) < \infty \Big\},$$

where

$$\|g\|_{L_\Phi} = \inf \Big\{ C > 0 : \int_\Omega \Phi\Big(\frac{|g|}{C}\Big) \mathrm{d}\mathbb{P} \leq 1 \Big\}.$$

We claim that $L_{\Phi,q,b} = wL_{\Phi,b}$ as $q = \infty$. Indeed, for any $\mathcal{F}$-measurable function f, there exists a sequence of non-negative simple functions $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \uparrow |f|$ a.e.. Moreover, $d_{f_n} \uparrow d_f$ and $f_n^* \uparrow f^*$. This implies that $|||f_n|||_{L_{\Phi,\infty,b}} \to |||f|||_{L_{\Phi,\infty,b}}$ and $\|f_n\|_{wL_{\Phi,b}} \to \|f\|_{wL_{\Phi,b}}$, via Remark 2.2. Therefore, it suffices to verify that $L_{\Phi,\infty,b} = wL_{\Phi,b}$ for non-negative simple functions. Now let

$$f(\omega) = \sum_{i=1}^N \alpha_i \chi_{A_i}(\omega),$$

where $\{A_i\}_{i=1}^N$ is a family of disjoint measurable sets and $\{\alpha_j\}_{j=1}^N \subset \mathbb{R}$ satisfies $0 \leq \alpha_j \leq \alpha_i$ as $1 \leq i \leq j \leq N$. For any $t \geq 0$, we have that

$$\mathbb{P}(|f| > t) = \sum_{j=1}^N \beta_j \chi_{[\alpha_{j+1}, \alpha_j)}(t),$$

where $\alpha_{N+1}=0$ and $\beta_j=\sum\limits_{i=1}^{j}\mathbb{P}(A_i)$ for $1\le j\le N$. Also, one can see that

$$f^*(t)=\sum_{j=1}^{N}a_j\chi_{[\beta_{j-1},\beta_j)}(t),$$

where $\beta_0=0$. Since $\Phi^{-1}(t)$ is increasing on $(0,\infty)$, then we get that

$$\begin{aligned}\|f\|_{wL_{\Phi,b}}&=\sup_{t>0}t\|\chi_{\{|f|>t\}}\|_{L_\Phi}\gamma_b\big(\mathbb{P}(|f|>t)\big)=\sup_{t>0}\frac{t\gamma_b\big(\mathbb{P}(|f|>t)\big)}{\Phi^{-1}\big(1/\mathbb{P}(|f|>t)\big)}\\&=\sup_{t>0}\sum_{j=1}^{N}\frac{t\gamma_b(\beta_j)}{\Phi^{-1}(1/\beta_j)}\chi_{[\alpha_{j+1},\alpha_j)}(t)=\max_{1\le j\le N}\frac{\alpha_j\gamma_b(\beta_j)}{\Phi^{-1}(1/\beta_j)}\end{aligned}$$

and

$$\begin{aligned}|||f|||_{\Phi,\infty,b}&=\sup_{t>0}\frac{\gamma_b(t)}{\Phi^{-1}(1/t)}f^*(t)=\sup_{t>0}\sum_{j=1}^{N}\frac{\alpha_j\gamma_b(t)}{\Phi^{-1}(1/t)}\chi_{[\beta_{j-1},\beta_j)}(t)\\&=\sup_{t>0}\sum_{j=1}^{N}\alpha_j\underbrace{\frac{(1/t)^{p_\Phi-1}}{\Phi^{-1}(1/t)}}_{\uparrow}\underbrace{t^{p_\Phi-1}\gamma_b(t)}_{\uparrow}\chi_{[\beta_{j-1},\beta_j)}(t)=\max_{1\le j\le N}\frac{\alpha_j\gamma_b(\beta_j)}{\Phi^{-1}(1/\beta_j)},\end{aligned}$$

which implies that $|||f|||_{\Phi,\infty,b}=\|f\|_{wL_{\Phi,b}}$.

(3) If $\Phi(t)=t^p$ for $0<p<\infty$, then $L_{\Phi,q,b}$ is the Lorentz-Karamata space $L_{p,q,b}$, which was introduced in [6]. The Lorentz-Karamata spaces offer not only a more general and unified insight into Lebesgue spaces, Lorentz spaces, Lorentz-Zygmund spaces (see [2]) and even generalized Lorentz-Zygmund spaces (see [5]), but also provide a framework in which it is easier to appreciate the central issues pertaining to different results; see [5, 7, 10]. The study of Lorentz-Karamata spaces has recently attracted more and more attention in martingale theory; see [12, 17, 19, 20].

(4) If $b\equiv 1$, then $L_{\Phi,q,b}$ becomes the Orlicz-Lorentz space $L_{\Phi,q}$, which appeared in [11]. In this case, $|||\cdot|||_{\Phi,q,1}$ is denoted by $|||\cdot|||_{\Phi,q}$, and $\|\cdot\|_{wL_{\Phi,1}}=\|\cdot\|_{wL_\Phi}=|||\cdot|||_{\Phi,\infty,1}=|||\cdot|||_{\Phi,\infty}$.

2.4 Main Results

We shall now establish Doob's maximal inequality for the Orlicz-Lorentz-Karamata space, which is the main result of this paper.

Theorem 2.5 Let $1\le q\le\infty$, b be a slowly varying function and let Φ be an Orlicz function with $p_\Phi>1$. Then we have that

$$|||f|||_{\Phi,q,b}\le|||Mf|||_{\Phi,q,b}\lesssim|||f|||_{\Phi,q,b},\qquad\forall\ f=(f_n)_{n\ge0}\in L_{\Phi,q,b}.$$

As an immediate application of the last theorem and Remark 2.4, it is easy to conclude and to improve Doob's maximal inequality on several function spaces for martingales.

Corollary 2.6 Let Φ be an Orlicz function with $p_\Phi>1$. Then we have that

$$\|Mf\|_{wL_\Phi}\lesssim\|f\|_{wL_\Phi},\qquad\forall\ f=(f_n)_{n\ge0}\in wL_\Phi.\tag{2.2}$$

Remark 2.7 We should mention that Liu, Wang and Hou [21] proved (2.2) for when $p_\Phi>1$ and $\Phi\in\triangle_2$. It follows from Corollary 2.6 that, in paper [21], the $\triangle_2$-condition can be removed.

Corollary 2.8 Let b be a slowly varying function and let Φ be an Orlicz function with $p_\Phi > 1$. Then we have that

$$\|Mf\|_{wL_{\Phi,b}} \lesssim \|f\|_{wL_{\Phi,b}}, \qquad \forall\, f = (f_n)_{n\geq 0} \in wL_{\Phi,b}.$$

Corollary 2.9 Let $p > 1$, $1 \leq q < \infty$ and let b be a slowly varying function. Then we have that

$$\|Mf\|_{L_{p,q,b}} \lesssim \|f\|_{L_{p,q,b}}, \qquad \forall\, f = (f_n)_{n\geq 0} \in L_{p,q,b}.$$

Corollary 2.10 Let $1 \leq q < \infty$ and let Φ be an Orlicz function with $p_\Phi > 1$. Then we have that

$$|||f|||_{\Phi,q} \leq |||Mf|||_{\Phi,q} \lesssim |||f|||_{\Phi,q}, \qquad \forall\, f = (f_n)_{n\geq 0} \in L_{\Phi,q}.$$

3 Proof of Main Result

In order to prove our main result, we first present another characterization of the functional $|||\cdot|||_{\Phi,q,b}$ under suitable conditions.

Theorem 3.1 Let $1 \leq q \leq \infty$ and let Φ be an Orlicz function with $p_\Phi > 1$. Then $|||f|||_{\Phi,q,b}$ and

$$\|f\|_{\Phi,q,b,*} = \begin{cases} \left(q\displaystyle\int_0^1 \left(\frac{\gamma_b(t)}{\Phi^{-1}(1/t)} f^{**}(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} & \text{if } 0 < q < \infty, \\ \displaystyle\sup_{t>0} \frac{\gamma_b(t)}{\Phi^{-1}(1/t)} f^{**}(t) & \text{if } q = \infty \end{cases}$$

are equivalent (quasi)-norms. Here f^{**} denotes the maximal non-increasing rearrangement of f, which is defined as

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(x)\mathrm{d}x, \qquad t > 0.$$

Proof Obviously, $f^*(t) \leq f^{**}(t)$ for $t > 0$. Thus it is sufficient to prove that

$$\|f\|_{\Phi,q,b,*} \lesssim |||f\,|||_{\Phi,q,b}\,. \tag{3.1}$$

From the definition of $\|\cdot\|_{\Phi,q,b,*}$ and $|||\cdot|||_{\Phi,q,b}$, we see that

$$\begin{aligned}\|f\|_{\Phi,q,b,*} &= \left(q\int_0^\infty \left[\frac{\gamma_b(t)}{\Phi^{-1}(1/t)}\frac{1}{t}\int_0^t f^*(x)\mathrm{d}x\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} \\ &= q^{1/q}\left(\int_0^\infty \left[U(t)\int_0^t g(x)\mathrm{d}x\right]^q \mathrm{d}t\right)^{1/q}\end{aligned}$$

and

$$|||f|||_{\Phi,q,b} = \left(q\int_0^1 \left(\frac{\gamma_b(t)}{\Phi^{-1}(1/t)} f^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} = q^{1/q}\left(\int_0^\infty \Big(V(t)g(t)\Big)^q \mathrm{d}t\right)^{1/q},$$

where

$$U(t) := \frac{\gamma_b(t)}{\Phi^{-1}\big(1/t\big)} t^{-1-1/q}, \;\; V(t) := \frac{\gamma_b(t)}{\Phi^{-1}\big(1/t\big)} t^{-1/q} \;\text{ and }\; g(t) := f^*(t).$$

Applying Theorem 1.1 in [24], we know that, if the estimation

$$\sup_{r>0}\left[\int_r^\infty \big|U(t)\big|^q \mathrm{d}t\right]^{1/q}\left[\int_0^r \big|V(t)\big|^{-q'}\mathrm{d}t\right]^{1/q'} \tag{3.2}$$

is finite, then one can get that

$$\Big[\int_0^\infty \Big|U(t)\int_0^t g(x)\mathrm{d}x\Big|^q \mathrm{d}t\Big]^{1/q} \le C\Big[\int_0^\infty \big|V(t)g(t)\big|^q \mathrm{d}t\Big]^{1/q},$$

where $q'=q/(q-1)$.

Hence, in order to get Inequality (3.1), we just compute the formula (3.2). Now let us estimate

$$\Big[\int_r^\infty \big|U(t)\big|^q \mathrm{d}t\Big]^{1/q} \quad \text{and} \quad \Big[\int_0^r \big|V(t)\big|^{-q'} \mathrm{d}t\Big]^{1/q'}.$$

The estimations above are divided into the following three cases:

Case 1 We first consider the case of $1<q<\infty$. Since $\frac{\Phi(t)}{t^{p_\Phi}}$ is non-increasing on $(0,\infty)$, then $\frac{(1/t)^{1/p_\Phi}}{\Phi^{-1}(1/t)}$ is non-increasing on $(0,\infty)$. Hence we obtain that

$$\begin{aligned}
\Big[\int_r^\infty \big|U(t)\big|^q \mathrm{d}t\Big]^{1/q} &= \Big[\int_r^\infty \Big[\frac{\gamma_b(t)}{\Phi^{-1}\big(1/t\big)}t^{-1-1/q}\Big]^q \mathrm{d}t\Big]^{1/q} \\
&= \Big[\int_r^\infty \underbrace{\Big(\frac{(1/t)^{1/p_\Phi}}{\Phi^{-1}\big(1/t\big)}\Big)}_{\downarrow}{}^q \underbrace{\big(t^{\frac{1-p_\Phi}{2p_\Phi}}\gamma_b(t)\big)}_{\downarrow}{}^q t^{\frac{q(1-p_\Phi)}{2p_\Phi}-1}\mathrm{d}t\Big]^{1/q} \\
&\le \frac{(1/r)^{1/p_\Phi}}{\Phi^{-1}\big(1/r\big)} r^{\frac{1-p_\Phi}{2p_\Phi}}\gamma_b(r)\Big[\int_r^\infty t^{\frac{q(1-p_\Phi)}{2p_\Phi}-1}\mathrm{d}t\Big]^{1/q} \\
&= \Big(\frac{2p_\Phi}{q(p_\Phi-1)}\Big)^{1/q}\frac{\gamma_b(r)}{\Phi^{-1}\big(1/r\big)}\frac{1}{r}
\end{aligned}$$

and

$$\begin{aligned}
\Big[\int_0^r \big|V(t)\big|^{-q'} \mathrm{d}t\Big]^{1/q'} &= \Big[\int_0^r \Big(\frac{\Phi^{-1}(1/t)}{\gamma_b(t)}\Big)^{\frac{q}{q-1}} t^{\frac{1}{q-1}}\mathrm{d}t\Big]^{(q-1)/q} \\
&= \Big[\int_0^r \underbrace{\Big(\frac{\Phi^{-1}(1/t)}{(1/t)^{1/p_\Phi}}\Big)}_{\uparrow}{}^{\frac{q}{q-1}} \underbrace{\Big(t^{\frac{1-p_\Phi}{2p_\Phi}}\gamma_b(t)\Big)}_{\uparrow}{}^{-\frac{q}{q-1}} t^{\frac{q(p_\Phi-1)}{2p_\Phi(q-1)}-1}\mathrm{d}t\Big]^{(q-1)/q} \\
&\le \frac{\Phi^{-1}(1/r)}{(1/r)^{1/p_\Phi}}\Big(r^{\frac{1-p_\Phi}{2p_\Phi}}\gamma_b(r)\Big)^{-1}\Big[\int_0^r t^{\frac{q(p_\Phi-1)}{2p_\Phi(q-1)}-1}\mathrm{d}t\Big]^{(q-1)/q} \\
&= \Big(\frac{2p_\Phi(q-1)}{q(p_\Phi-1)}\Big)^{\frac{q-1}{q}}\frac{\Phi^{-1}(1/r)}{(1/r)^{1/p_\Phi}}\Big(r^{\frac{1-p_\Phi}{2p_\Phi}}\gamma_b(r)\Big)^{-1} r^{\frac{p_\Phi-1}{2p_\Phi}} \\
&= \Big(\frac{2p_\Phi(q-1)}{q(p_\Phi-1)}\Big)^{\frac{q-1}{q}}\frac{\Phi^{-1}(1/r)}{\gamma_b(r)}r.
\end{aligned}$$

This means that

$$\sup_{r>0}\Big[\int_r^\infty \big|U(t)\big|^q \mathrm{d}t\Big]^{1/q}\Big[\int_0^r \big|V(t)\big|^{-q'} \mathrm{d}t\Big]^{1/q'} \le \frac{2p_\Phi}{q(p_\Phi-1)}(q-1)^{\frac{q-1}{q}}.$$

Thus we see that Inequality (3.1) holds for $q>1$.

Case 2 The case of $q=1$ will now be considered. In this situation,

$$U(t):=\frac{\gamma_b(t)}{\Phi^{-1}\big(1/t\big)}t^{-2},\ \ V(t):=\frac{\gamma_b(t)}{\Phi^{-1}\big(1/t\big)}t^{-1} \ \text{ and }\ g(t):=f^*(t).$$

In a similar manner as to Case 1, we can obtain that

$$\int_r^{\infty} \big|U(t)\big|\mathrm{d}t = \int_r^{\infty} \frac{\gamma_b(t)}{\Phi^{-1}\big(1/t\big)} t^{-2}\mathrm{d}t \le \frac{2p_\Phi}{p_\Phi - 1} \frac{\gamma_b(r)}{\Phi^{-1}\big(1/r\big)} \frac{1}{r}$$

and

$$\sup_{0<t<r} \big|V(t)\big|^{-1} = \sup_{0<t<r} \left[\frac{\Phi^{-1}\big(1/t\big)}{\gamma_b(t)} t\right] = \frac{\Phi^{-1}\big(1/r\big)}{\gamma_b(r)} r.$$

Thus we have that

$$\sup_{r>0} \left\{ \int_r^{\infty} \big|U(t)\big|\mathrm{d}t \cdot \sup_{0<t<r} \big|V(t)\big|^{-1} \right\} \le \frac{2p_\Phi}{p_\Phi - 1},$$

which implies that Inequality (3.1) holds for $q = 1$.

Case 3 Now we consider the situation where $q = \infty$. In this setting, we have that

$$U(t) := \frac{\gamma_b(t)}{\Phi^{-1}\big(1/t\big)} t^{-1}, \ \ V(t) := \frac{\gamma_b(t)}{\Phi^{-1}\big(1/t\big)} \ \text{ and } \ g(t) := f^*(t).$$

In a similar manner as to Case 1, we can get that

$$\sup_{r<t<\infty} \big|U(t)\big| \le \frac{\gamma_b(r)}{\Phi^{-1}\big(1/r\big)} \frac{1}{r}$$

and

$$\begin{aligned}\int_0^r \big|V(t)\big|^{-1}\mathrm{d}t &= \int_0^r \frac{\Phi^{-1}\big(1/t\big)}{\gamma_b(t)}\mathrm{d}t = \int_0^r \frac{\Phi^{-1}\big(1/t\big)}{\big(1/t\big)^{1/p_\Phi}} \frac{1}{t^{-\frac{p-1}{2p}}\gamma_b(t)} t^{-\frac{p+1}{2p}}\mathrm{d}t \\ &\le \frac{\Phi^{-1}\big(1/r\big)}{\big(1/r\big)^{1/p_\Phi}} \frac{1}{r^{-\frac{p-1}{2p}}\gamma_b(r)} \int_0^r t^{-\frac{p+1}{2p}}\mathrm{d}t = \frac{2p_\Phi}{p_\Phi - 1} \frac{\Phi^{-1}\big(1/r\big)}{\gamma_b(r)} r.\end{aligned}$$

Then we obtain that

$$\sup_{r>0} \left\{ \Big(\sup_{r<t<\infty} \big|U(t)\big|\Big) \cdot \int_0^r \big|V(t)\big|^{-1}\mathrm{d}t \right\} \le \frac{2p_\Phi}{p_\Phi - 1}.$$

This completes the proof. □

Lemma 3.2 ([22, Theorem 3.6.3]) Let $f = (f_n)_{n\ge 0} \in L_1$. Then $(Mf)^*(t) \le f^{**}(t),\ t > 0$.

Now we prove Theorem 2.5.

Proof Obviously, $|||f|||_{\Phi,q,b} \le |||Mf|||_{\Phi,q,b}$. We consider the case where $1 \le q < \infty$. It follows from Theorem 3.1 and Lemma 3.2 that

$$\begin{aligned}|||Mf|||_{\Phi,q,b} &= \left(q\int_0^1 \Big(\frac{1}{\Phi^{-1}(1/t)}\gamma_b(t)(Mf)^*(t)\Big)^q \frac{\mathrm{d}t}{t}\right)^{1/q} \\ &\le \left(q\int_0^1 \Big(\frac{1}{\Phi^{-1}(1/t)}\gamma_b(t)f^{**}(t)\Big)^q \frac{\mathrm{d}t}{t}\right)^{1/q} \\ &= \big\|f\big\|_{\Phi,q,b,*} \lesssim |||f\,|||_{\Phi,q,b}\,.\end{aligned}$$

When $q = \infty$, we have that

$$\begin{aligned}|||Mf|||_{\Phi,\infty,b} &= \sup_{t>0} \frac{1}{\Phi^{-1}(1/t)}\gamma_b(t)(Mf)^*(t) \le \sup_{t>0} \frac{1}{\Phi^{-1}(1/t)}\gamma_b(t)f^{**}(t) \\ &= \big\|f\big\|_{\Phi,\infty,b,*} \lesssim |||f\,|||_{\Phi,\infty,b}\,.\end{aligned}$$

This completes the proof of the theorem. □

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