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# OPINION DYNAMICS ON SOCIAL NETWORKS*

Dedicated to Professor Banghe LI on the occasion of his 80th birthday

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**Abstract** Opinion dynamics has recently attracted much attention, and there have been a lot of achievements in this area. This paper first gives an overview of the development of opinion dynamics on social networks. We introduce some classical models of opinion dynamics in detail, including the DeGroot model, the Krause model, $0-1$ models, sign networks and models related to Gossip algorithms. Inspired by some real life cases, we choose the unit circle as the range of the individuals' opinion values. We prove that the individuals' opinions of the randomized gossip algorithm in which the individuals' opinion values are on the unit circle reaches consensus almost surely.

**Key words** opinion dynamics; social networks; graph theory

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## 1 Introduction

As a part of a community, we interact with other people and exchange opinions with others everyday. We have our own opinions about hot topics in the world, and we also influence and are influenced by other people. Some people are easily influenced by others, while some are stubborn. Some people are eager to find someone with the same opinions, while some people are more likely to interact with those who have different opinions [1]. There are people who are submissive to social norms, but there are also people who are rebellious [2]. We can see that all different kinds of opinions spread among people throughout society, and these opinions interact with each other in some way. Opinion dynamics is a field for analyzing the above phenomena. Over the past several decades, there has been growing interest in opinion dynamics as an interdisciplinary study in fields such as control theory, sociology, economics, and biology [3–8].

As networks grow more and more complex, distributive algorithms have become more and more important. Gossip protocol, in which information exchange is only carried out in pairs between two nodes, has been widely used to aggregate and spread information distributively,

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and has been used in distributed computation, optimization, and social networks [9–11]. Gossip algorithms usually consist of two parts: the rules for the selection of pairs of nodes, and updates on the values of the pair of nodes. The name for this algorithm comes from the office gossip. Now, gossip protocol has been used to provide distributive computation in fields like optimization, control theory, signal processing, and artificial intelligence [12–14].

Here, we give a review of the basic models in opinion dynamics. The aim of this paper is to introduce the basic models in opinion dynamics and give some new insights or even extend upon some of these. In Section 2, we introduce some classical models in opinion dynamics, including the DeGroot model, the Krause model, signed networks, and fashion games. Models relevant to gossip algorithms are discussed in Section 3. In the last part of Section 3, we provide a new model in which opinion values are defined on the unit circle instead of by real numbers. We categorize these models in Section 4.

## 2 Models of Opinion Dynamics

In this section, we introduce some classical models in opinion dynamics.

### 2.1 DeGroot Model

The DeGroot model, the basic model of opinion dynamics, was proposed by DeGroot as early as 1974 [15]. The DeGroot model describes the process in which agents in a social network change their own opinions by informing each other of their opinions, collecting all of opinions they received, and finally forming a common opinion. Consider a social network described by a graph $G = (V, E)$ consisting of $n$ agents. The edge of the graph is used to indicate the interactive relationship between the nodes. If agents $i$ and $j$ can interact with each other, or in other words, can exchange information, then $\{i, j\} \in E$. In graph theory, we also say that agent $i$ and agent $j$ are neighbors. At time $t$, each agent has a specific opinion, $x_i(t) \in \mathbb{R}, t = 0, 1, \cdots$. Since each agent has different background, their opinions are also different, and they usually reflect their own information. If an agent $i$ is informed of the opinions $x_j(t), j = 1, \cdots, n,\ j \neq i$ of all their neighbors in the social network, then agent $i$ will naturally change their opinions to suit the opinions and judgments of other members. Let $w_{ij}$ denote the weight that an agent $i$ assigns to an agent $j$ when they changes their opinion; this is chosen by the individual according to the relative importance of the opinion to all members in the network (including themselves). Then at the time $t + 1$, agent $i$'s opinions will be updated to

$$x_i(t+1) = \sum_{j=1}^{n} w_{ij} x_j(t),$$

where $w_{ij} \geq 0$ and $\sum\limits_{j=1}^{n} w_{ij} = 1$. Let $x(t) = (x_1(t), \cdots, x_n(t))^{\mathrm{T}}$ and let $W$ be the matrix where the $(i, j)$-entry is $w_{ij}, i = 1, \cdots, n, j = 1, \cdots, n$. Then the above opinion update rule can be expressed in the following form:

$$x(t+1) = Wx(t). \tag{2.1}$$

Obviously, $W$ is a random matrix, i.e., the sum of each row is 1. The DeGroot model depicts the interaction between individuals in the social network $G = (V, E)$, and the opinion of an agent $i$ at each update moment is the weighted average of its own opinion and the opinions of all its neighbors at the previous moment. Since it is assumed that the network topology does not change with time in the process of opinion updating, the DeGroot model is called a time-invariant social network model. Furthermore, given an initial opinion $x(0)$, the opinion dynamics of the social network can be expressed as

$$x(t) = W^t x(0), \tag{2.2}$$

which is only related to the weight that each agent in this network gives to all members of the entire social network.

The model is convergent if and only if for any initial opinion $x(0)$, there is an opinion vector $x^* \in \mathbb{R}^n$ such that

$$\lim_{t\to\infty} x(t) = x^*.$$

The model can reach consensus if and only if, for any initial opinion $x(0)$, there is a real value $a$ such that

$$\lim_{t\to\infty} x(t) = a\mathbf{1},$$

where $\mathbf{1}$ is the vector with all entries being 1. It can be seen that consensus is a special case of convergence; i.e., consensus can only be reached when the $n$ components of $x^*$ all converge to the same value $a$.

According to (2.1) and (2.2), the opinion evolution process on the social network $G$ is a discrete time Markov chain, and the random matrix $W$ is its state transition matrix. DeGroot [15] gives two sufficient conditions for consensus. First, if there is a positive integer $k$ such that at least one column of the matrix $W^k$ is positive, then each agent's opinion in the social network $G$ can reach consensus. Second, if all the states of the Markov chain are positive recurrent, irreducible, and aperiodic, then agents in the social network $G$ can reach consensus. Furthermore, when consensus is reached, the common opinion of the social network can be precisely calculated.

### 2.2 Krause Model

In the DeGroot model, the topology of the social network is fixed; that is to say, the influence of the social network remains constant. In real life, however, the influence between people is always changing. The Krause model [16, 17] and its extensions [18] are time-variant models in social networks.

Just as in the DeGroot model, we use the real value $x_i(t)$ to represent the opinions of agent $i$. However, each agent in the Krause model only interacts with those agents it regards as necessary in order to communicate. Specifically, each agent $i$ has a confidence bound, denoted by $\varepsilon_i$. The opinions of agent $i$ are only influenced by those agents whose opinions differ from agent $i$'s opinion by no more than $\varepsilon_i$. Therefore, for agent $i$, the agents that will have an influence on its opinion at time $t$ can be represented by the set $I(i,t)$, that is

$$I(i,t) = \{1 \le j \le n : |x_i(t) - x_j(t)| \le \varepsilon_i\},$$

where $|\cdot|$ represents the absolute value of a real number. Having these notations, the opinion formation of agent $i$ can be described as

$$x_i(t+1) = |I(i,t)|^{-1} \sum_{j\in I(i,t)} x_j(t), \tag{2.3}$$

where $|I(i,t)|$ is the number of elements in the finite set $I(i,t)$; that is, agent $i$ adjusts their opinion in period $t+1$ by taking a weighted average with weight $|I(i,t)|^{-1}$ for the opinion of agent $j \in I(i,t)$ at time $t$.

Note that, in the Krause model, the agents interacting with agent $i$ at different times are not exactly the same. If a graph is used to represent the interaction between individuals, the graph is constantly changing. This is to say the network topology of the Krause model is not fixed, but that it changes over time.

To simplify, we can assume that the confidence bound of all agents are equal, i.e., $\varepsilon_i = \varepsilon,\ i = 1, \cdots, n$. An important property of the Krause model is that if the opinions of agent $i$ and agent $j$ satisfy the relation $x_i(t) \le x_j(t)$ at time $t$, then $x_i(t+1) \le x_j(t+1)$. In other words, the order of agent opinion values does not change over time. Taking advantage of this property, it is possible to renumber the agents so that the order of agent opinion values is consistent with the sequence number of the agents, i.e., $x_1(0) \le x_2(0) \le \cdots \le x_n(0)$. Using this order relationship, opinion dynamics in the network system can be studied, even if the network topology formed between agents changes over time.

For a sorted opinion vector $x(t) = (x_1(t), \cdots, x_n(t))$, we say that there is a split (or crack) between agents $i$ and $i+1$ if $|x_{i+1}(t) - x_i(t)| > \varepsilon$. If at time $t$ there is a split between agent $i$ and $i+1$, then, according to the update equation (2.3), $x_{i+1}(t+1)$ will not be less than $x_{i+1}(t)$ and $x_i(t+1)$ will not be greater than $x_i(t)$. Thus, if there exists a split at some time $t$, the split will exist forever after $t$. Then the network of opinion dynamics can be split into two smaller independent networks, one consisting of agent $1, \cdots, i$ and the other consisting of agent $i+1, \cdots, n$.

Since $x(0) = (x_1(0), \cdots, x_n(0))$ is assumed to be sorted, the opinion $x_1(0)$ is nondecreasing and bounded above by $x_n(0)$; as a result, it converges to a value $x_1^*$. Let $k$ be the highest index for which $x_k(t)$ converges to $x_1^*$. According to the fundamental properties in mathematical analysis, we have that

$$\lim_{t\to+\infty} x_j(t) = x_1^*$$

for $j = 1, \cdots, k$. According to the definition of $k$, $x_{k+1}(t)$ will not converge to $x_1^*$. Intuitively, we can imagine that at some time there should be a split between $k$ and $k+1$. In fact, this can be easily proved by contradiction.

Thus, we can divide the whole networks into two smaller networks, i.e., one consisting of agent $1, \cdots, k$, and the other consisting of agent $k+1, \cdots, n$. It is known that the opinions among agents $1, \cdots, k$ reach consensus, which is $x_1^*$. The same analysis method can be used for the remaining networks consisting of agents $k+1, \cdots, n$. Finally, it can be proven that for every agent $i$, $x_i(t)$ converges to a limit $x_i^*$ in finite time. Moreover, for any agents $i$ and $j$, either $x_i^* = x_j^*$ or $|x_i^* - x_j^*| > \varepsilon$.

Blondel et al. [18] also generalize the model (2.3) so that each agent $i$ has an associated weight $w_i$, and updates their opinion in the following way:

$$x_i(t+1) = \frac{\sum\limits_{j:|x_i(t)-x_j(t)|<1} w_j x_j(t)}{\sum\limits_{j:|x_i(t)-x_j(t)|<1} w_j}. \tag{2.4}$$

Here, the confidence bound $\varepsilon$ is equal to 1. It can be verified that the properties and convergence results of model (2.3) are also applicable to model (2.4).

A set of agents whose opinions converge to a common value is called a cluster, and the sum of the weights of all agents in the cluster is called the weight of the cluster. Let $\bar{x}$ be a vector of agent opinions at equilibrium. Add a perturbed agent indexed by 0 with weight $\delta$ and initial opinion $\tilde{x}_0$. Let the system (2.4) evolve again, until it converges to a new, perturbed equilibrium $\bar{x}'$ , and then remove the perturbing agent. Define $\Delta_{\tilde{x}_0,\delta} = \sum_i w_i|\bar{x}_i - \bar{x}'_i|$ as the distance between the original and perturbed equilibria. The original equilibrium vector $\bar{x}$ is said to be stable if $\lim_{\delta\to 0}\sup_{\tilde{x}_0}\Delta_{\tilde{x}_0,\delta} = 0$. Furthermore, $\bar{x}$ is stable if and only if, for any two clusters $A$ and $B$ with weights $W_A$ and $W_B$, respectively, the following holds: either $W_A = W_B$ and the inter-cluster distance is greater than or equal to 2, or $W_A \neq W_B$ and the inter-cluster distance is strictly greater than $1 + \frac{\min(W_A,W_B)}{\max(W_A,W_B)}$.

Furthermore, Blondel et al. [18] introduced the continuous Krause model to study a society with large population. There are so many people that we can use an interval of real numbers to approximately represent them. More specifically, agents are indexed by $I = [0,1]$, and the opinions of all agents are nonnegative and bounded above by a positive constant $L$. Let $x_t(\alpha)$ be the opinion of agent $\alpha \in I$ at time $t$. Let $X$ be the set of measurable functions $x : I \to \mathbb{R}$, and let $X_L \subset X$ be the set of measurable functions $x : I \to [0,L]$. The evolution of the opinions is described by

$$x_{t+1}(\alpha) = \frac{\int_{\beta:(\alpha,\beta)\in C_{x_t}} x_t(\beta)\mathrm{d}\beta}{\int_{\beta:(\alpha,\beta)\in C_{x_t}} \mathrm{d}\beta}, \tag{2.5}$$

where $C_x \subseteq I^2$ is defined for any $x \in X$ by $C_x := \{(\alpha,\beta) \in I^2 : |x(\alpha) - x(\beta)| < 1\}$.

As in the discrete-agent model (2.3), if $x_t(\alpha) \le x_t(\beta)$ holds for agents $\alpha$ and $\beta$ at some time $t$, then the same relation continues to hold at all subsequent times. Furthermore, if the initial opinion vector $x(0) \in X_L$ only takes a finite number of values, the continuous-agent model (2.5) coincides with the weighted discrete-agent model (2.4), with the same range of initial opinions, and where each discrete agent's weight is set as equal to the measure of the set of indices $\alpha$ for which $x_0(\alpha)$ takes the corresponding value. For the continuous Krause model, there are some conjectures which are still unsolved in [18].

### 2.3 Signed Social Networks

The interactions of agents in the above models are all positive. However, in real world, we are often affected by subjective emotions when we communicate with others. When confronted with trustworthy friends, we are usually willing to listen to their opinions and update our views. On the contrary, when confronted with strangers, we are often suspicious, have a negative attitude, and are reluctant to listen to their opinions; sometimes confrontation even occurs. To this end, signed networks are proposed [19–21]. In signed networks, each interaction link has a sign (positive or negative) to reflect whether the agents interacting with each other are friends or not.

Let $G = (V,E)$ be an undirected graph. Each edge in $E$ is associated with a sign, positive or negative, defining $G$ as a signed graph. The positive and negative edges are collected in the sets $E^+$ and $E^-$, respectively. Then $G^+ = (V,E^+)$ and $G^- = (V,E^-)$ are, respectively, termed as positive and negative subgraphs. Suppose that $G$ is a connected graph and that $G^-$ contains at least one edge. Write $N_i^+ = \{j : \{i,j\} \in E^+\}$ as the positive neighbor set of agent $i$, and

$N_i^- = \{j : \{i,j\} \in E^-\}$ as the negative neighbor set of agent $i$. The set $N_i^+ \cup N_i^-$ then contains all nodes that interact with node $i$ in the graph $G$. Correspondingly, the number of neighbors of agent $i$, denoted by $d_i$, is equal to $d_i^+ + d_i^-$, where $d_i^+ = |N_i^+|, d_i^- = |N_i^-|$.

At moments $t = 0, 1, \cdots$, each agent $i$ has an opinion $x_i(t) \in \mathbb{R}$. The opinion updating rule is specified by the sign of the links. Consider a particular link $\{i,j\} \in E$. If the sign of $\{i,j\}$ is positive, each node $s \in \{i,j\}$ updates its opinion by

$$x_s(t+1) = x_s(t) + \alpha(x_{-s}(t) - x_s(t)) = (1-\alpha)x_s(t) + \alpha x_{-s}(t),$$

where $-s \in \{i,j\} \setminus \{s\}$, $\alpha \in (0,1)$. If the sign of $\{i,j\}$ is negative, each node $s \in \{i,j\}$ updates its value by either the opposing rule

$$x_s(t+1) = x_s(t) + \beta(-x_{-s}(t) - x_s(t)) = (1-\beta)x_s(t) - \beta x_{-s}(t), \tag{2.6}$$

or the repelling rule

$$x_s(t+1) = x_s(t) - \beta(x_{-s}(t) - x_s(t)) = (1+\beta)x_s(t) - \beta x_{-s}(t), \tag{2.7}$$

with $\beta \geq 0$.

It is observed that the positive interaction is consistent with the DeGroot model, suggesting that the opinions of trusted social members are attractive to each other. There are two kinds of negative interaction rules: the opposing rule states that the agent will be attracted by the opposite of its neighbor's opinion if they share a negative link; and the repelling rule indicates that the opinions of two agents are mutually exclusive. The two parameters, $\alpha$ and $\beta$, reflect the strength of positive and negative links, respectively.

Structural balance is a fundamental concept in the study of signed graphs. We say that a signed graph $G$ is structurally balanced if there is a partition of the node set into two nonempty and mutually disjoint subsets $V = V_1 \bigcup V_2$, where every edge between the two node subsets $V_1$ and $V_2$ is negative, and every edge within each $V_i$ is positive, $i = 1, 2$.

Then we can introduce the first model with the opposing rule (2.6), along with the negative links, where $x_i(t)$ is updated as follows:

$$\begin{aligned} x_i(t+1) &= x_i(t) + \alpha \sum_{j\in N_i^+} (x_j(t) - x_i(t)) - \beta \sum_{j \in N_i^-} (x_j(t) + x_i(t)) \\ &= (1 - \alpha d_i^+ - \beta d_i^-)x_i(t) + \alpha \sum_{j\in N_i^+} x_j(t) - \beta \sum_{j\in N_i^-} x_j(t). \end{aligned} \tag{2.8}$$

Assume that $0 < \alpha + \beta < 1/\max\limits_{i\in V} d_i$. Then under the opposing rule (2.6), the following statements hold for any initial opinion vector $x(0)$ [21]:

(i) If $G$ is structurally balanced, then

$$\lim_{t\to\infty} x_i(t) = \frac{1}{n}\left(\sum_{j\in V_1} x_j(0) - \sum_{j\in V_2} x_j(0)\right),\ i \in V_1,$$

$$\lim_{t\to\infty} x_i(t) = -\frac{1}{n}\left(\sum_{j\in V_1} x_j(0) - \sum_{j\in V_2} x_j(0)\right),\ i \in V_2.$$

(ii) If $G$ is not structurally balanced, then

$$\lim_{t\to\infty} x_i(t) = 0,\ i \in V.$$

For the second model with the repelling rule (2.7), along with the negative links, $x_i(t)$ is updated as follows:

$$\begin{aligned} x_i(t+1) &= x_i(t) + \alpha \sum_{j\in N_i^+} (x_j(t) - x_i(t)) - \beta \sum_{j\in N_i^-} (x_j(t) - x_i(t)) \\ &= (1 - \alpha d_i^+ + \beta d_i^-) x_i(t) + \alpha \sum_{j\in N_i^+} x_j(t) - \beta \sum_{j\in N_i^-} x_j(t). \end{aligned} \quad (2.9)$$

With a further assumption that $G^+$ is connected, for any $0 < \alpha < 1/\max_{i\in V} d_i^+$, there exists $\beta_* > 0$ such that

(i) if $\beta < \beta_*$, then average consensus is reached for (2.9) in the sense that $\lim_{t\to\infty} x_i(t) = \sum_{j=1}^{n} x_i(0)/n$ for all initial value $x(0)$;

(ii) if $\beta > \beta_*$, then $\lim_{t\to\infty} \|x(t)\| = \infty$ for almost all initial values w.r.t. the Lebesgue measure.

The above model and results can be generalized to directed signed networks. A directed graph $\mathcal{G} = (V, \mathcal{E})$ is called a signed digraph if each of its links $(i,j) \in \mathcal{E}$ has a positive or negative sign. The positive and negative neighbor sets of node $i$ are $N_i^+ = \{j : (j,i) \in E^+\}$ and $N_i^- = \{j : (j,i) \in E^-\}$, respectively. The network dynamics (2.8) and (2.9) are then readily defined for the digraph $\mathcal{G}$. In addition, by replacing undirected edges with directed edges, the structural balance of directed graphs can be obtained.

Assume that $0 < \alpha + \beta < 1/\max_{i\in V} d_i$ and that $\mathcal{G}$ is strongly connected. For the model with network dynamics (2.8) over $\mathcal{G}$, the following statements hold for any initial value $x(0)$ [21].

(i) If $\mathcal{G}$ is structurally balanced, then there are $n$ positive numbers $w_1, \cdots, w_n$ with $\sum_{i=1}^{n} w_i = 1$ such that

$$\lim_{t\to\infty} x_i(t) = \frac{1}{n}\left(\sum_{j\in V_1} w_j x_j(0) - \sum_{j\in V_2} w_j x_j(0)\right), \ i \in V_1,$$

$$\lim_{t\to\infty} x_i(t) = -\frac{1}{n}\left(\sum_{j\in V_1} w_j x_j(0) - \sum_{j\in V_2} w_j x_j(0)\right), \ i \in V_2.$$

(ii) If $\mathcal{G}$ is not structurally balanced, then

$$\lim_{t\to\infty} x_i(t) = 0, \ i \in V.$$

Here, the value of $(w_1, \cdots, w_n)$ depends on $\alpha$ and $\beta$.

Consider the model with network dynamics (2.9) over a digraph $\mathcal{G}$. Suppose that $\mathcal{G}^+$ is strongly connected, and fix that $0 < \alpha < 1/\max_{i\in V} d_i^+$. There exists $\beta_* > 0$ such that, for any $\beta < \beta_*$, there are $q_1(\beta), \cdots, q_n(\beta) \in \mathbb{R}^+$ with $\sum_{i=1}^{n} q_i(\beta) = 1$ satisfying that a consensus is reached at

$$\lim_{t\to\infty} x_i(t) = \sum_{j=1}^{n} q_i(\beta) x_i(0), \ i \in V$$

for all initial value $x(0)$.

We can see that different models have different properties on convergence. They can be used to describe different social phenomena. Other properties, such as rates of convergence, have also been studied. The topologies of the above models are all fixed, and can also be extended to time-varying structures.

### 2.4 Fashion Game

A fashion game [22–25] is essentially a deterministic $0-1$ type opinion dynamics model. In this kind of social network, the value of the opinions of agents are no longer arbitrary real numbers, but binary numbers: either 0 or 1. For example, in the early 1980s, there were two kinds of trousers that were popular in some cities of China. One was straight-leg, and the other was flared. Each person's choice of trousers was influenced by what their neighbors wore. The fashion trend of flared trousers took off at that time. When the fashion trend faded, straight-leg trousers became the dominant choice. However, in recent years, flared trousers have become popular again. Such fashion evolution is very common, and the fashion game can give a good explanation for it.

Regarding fashion, there are two viewpoints that are both extremely popular but almost opposite to each other. Some think that fashion is a distinctive or peculiar manner or way. Others take fashion to be the prevailing custom or style. People falling into the first class are called rebels, and the latter are called conformists. Since fashion comes from comparison with others, and the range of fashion from which people compare is almost always confined to their friends, relatives, colleagues, and neighbors, that fashion works through a social network is very natural [26, 27].

Let $G=(V,E)$ be a social network, where $V=\{1,2,\cdots,n\}$ is the set of agents and $E\subseteq V\times V$ is the set of edges (no self-loops are allowed). Each agent is faced with a binary choice of opinions 0 or 1, and they can only form opinions that affect themselves through the opinions of their neighbors. Time is discrete for $t=0,1,2,\cdots$ and every agent has an opinion at time $t$, which is denoted by $x_i(t)$ for agent $i$. Denote the set of neighbors of player $i$ by $N_i$. For agent $i\in V$, they are either a conformist or a rebel, and its type does not change over time. Conformists tend to take the most common views among their neighbors, while rebels tend to take the opposite. At time $t$ all agents obtain their neighbors' opinions. If an agent's neighbors have equal numbers of agents with opinions 0 and 1, then this agent will not change their opinion at time $t+1$. Otherwise, a conformist agent $i$ will update their opinion at time $t+1$ to the more selected opinion value at time $t$ for agents in $N_i$, while a rebel agent $j$ will update their opinion at time $t+1$ to the less selected opinion value at time $t$ for agents in $N_j$.

More precisely, giving an opinion profile $x(t)=(x_1(t),\cdots,x_n(t))\in\{0,1\}^n$, the utility of agent $i$ is defined by

$$u_i(x(t))=|L_i(x(t))|-|D_i(x(t))|,$$

where

$$L_i(x(t))=\begin{cases}\{j\in N_i:x_j(t)=x_i(t)\}, & i \text{ is a comformist},\\ \{j\in N_i:x_j(t)\neq x_i(t)\}, & i \text{ is a rebel}\end{cases}$$

is the set of neighboring agents that $i$ likes, and $D_i(x(t))=N_i\setminus L_i(x(t))$ is the set of neighboring agents that $i$ dislikes. The $u_i(x(t))$ can also be referred to as the satisfaction degree. Agent $i$ is satisfied at time $t$ if $u_i(x(t))\geq 0$. Otherwise, this agent is dissatisfied. When the agent $i$ is dissatisfied at time $t$, they will change their opinion at time $t+1$, otherwise this agent will maintain their own opinion. Thus, these opinion values are updated synchronously according to

$$x_i(t+1) = \begin{cases} 1 - x_i(t), & u_i(x(t)) < 0, \\ x_i(t), & u_i(x(t)) \geq 0. \end{cases}$$

It is worth mentioning that the iterative rules of this model are inspired by matching pennies from game theory, which is why the researchers call it the fashion game. Since the network topology of the considered system is fixed, the value updating process is deterministic; the opinion vector at a specific time is only dependent on the opinion vector at the last time, and the opinion vector has only a finite number of values, i.e., $2^n$; the system will eventually enter a cycle, that is, at a certain moment $T$, $x(T) = x(s)$ holds for some $0 \leq s < T$, and the evolution of the opinion vector of all agents is the same as before. This is called the fashion cycle.

Zhigang Cao et al. [24] introduced the homophily index to further study the fashion cycle, and concluded that a lower homophily index usually promotes the appearance of fashion cycles.

## 3 Gossip Algorithms

Gossip algorithms can be regarded as a special kind of algorithm in social networks, and they are widely used in modern distributed systems. As the name suggests, the gossip algorithm is inspired by gossip: in an office, when a person gossips about a piece of social news, everyone in the office will know the news for a limited time. The gossip protocol, which characterizes a manner of information dissemination and aggregation on social networks, was proposed in 1987 by Alan Demers [28]. For the gossip algorithm, the key indicators are the convergence and speed of the convergence of the algorithm. This section will introduce some classic gossip algorithms [29–31, 39], and make certain extensions on the basis of existing algorithms.

### 3.1 Deterministic Gossip Algorithms

Consider a network $G = (V, E)$ with the node set $V = \{1, \cdots, n\}$. The value node $i$ holds at time $t$ is denoted as $x_i(t) \in \mathbb{R}$ for discrete time $t = 0, 1, 2, \cdots$. The global network state is then given by $x(t) = (x_1(t), \cdots, x_n(k))^T$. Unlike the DeGroot model, in the deterministic gossip algorithm, only two selected agents update their values or opinions to the average of the values they held prior to the interaction at time $t+1$, and the values of all other agents remain unchanged. Therefore, at time $t+1$, the update rule for agents can be expressed as

$$\begin{aligned} x_i(t+1) &= \frac{x_i(t) + x_j(t)}{2}; \\ x_j(t+1) &= \frac{x_j(t) + x_i(t)}{2}; \\ x_l(t+1) &= x_l(t),\ l \in V \setminus \{i, j\}. \end{aligned}$$

To simplify the update rule, a matrix set

$$M_n := \left\{ I_n - \frac{(e_i - e_j)(e_i - e_j)^T}{2} : \{i, j\} \in E \right\}$$

is introduced, where $I_n$ is the $n \times n$ identity matrix, and $e_m = (0 \cdots 0\ 1\ 0 \cdots 0)^T$ is the $n \times 1$ unit vector whose $m$'th component is 1.

For a deterministic gossip algorithm, we have a given function $W(t)$ from nonnegative integers to $M_n$. Then the update rule can be expressed in the following matrix form:

$$x(t+1) = W(t)x(t), \quad W(t) \in M_n, \ t = 0, 1, \cdots . \tag{3.1}$$

We can see that this has the same matrix form as the DeGroot model. The same method can be used to analyze the speed of convergence.

It is straightforward to ask the question of whether the deterministic gossip algorithm is convergent in finite time. More precisely, we say that algorithm (3.1) achieves finite-time convergence with respect to initial value $x(0) \in \mathbb{R}^n$ if there exists an integer $T(x^0) \geq 0$ such that $x(T) = W_{T-1} \cdots W_0 x(0) \in \text{span}\{\mathbf{1}\}$, and global finite-time convergence if such $T(x^0) \geq 0$ exists for every initial value $x(0) \in \mathbb{R}^n$. Here $\mathbf{1}$ is the vector with all entries being 1.

There exists a deterministic gossip algorithm that converges globally in finite time if and only if the number of network nodes is a power of two. For $n = 2^m$ nodes, the fastest gossip algorithms take a total of $mn = n \log_2 n$ node updates to converge. Moreover, if there exists no integer $m \geq 0$ such that $n = 2^m$, then, for almost all initial values, there exists no deterministic gossip algorithm with finite-time convergence.

For deterministic gossip algorithms, if only one of the two selected interacting agents updates their opinion, we say it an asymmetric deterministic Gossip algorithm. The model description of the asymmetric deterministic gossip algorithm and the conclusion of finite time convergence can be found in [30].

### 3.2 Randomized Gossip Algorithms

We can see that in the above subsection, the value of $W(t)$ is a prefixed matrix. This can be extended to randomized algorithms; that is, $W(t)$ is not a matrix but a distribution on $M_n$.

We consider a special case as an example. At any time, an agent is selected at random and then this agent selects one of their neighbors randomly to communicate. We can use a nonnegative random matrix $P = (p_{ij}) \in \mathbb{R}^{n \times n}$ to characterize this kind of randomized gossip algorithm. Here $p_{ij} > 0$ if and only if $\{i, j\} \in E$, and a random matrix is a matrix in which the sum of any row elements is 1. Furthermore, we assume that the biggest eigenvalue of $P$ is 1, and the absolute value of all other $n-1$ eigenvalues are less than 1. The randomized gossip algorithm relevant to $P$ can be described as follows: at time $t$, agent $i$ is selected randomly with probability $\frac{1}{n}$, and then agent $j$ as one of agent $i$'s neighbor is selected with probability $p_{ij}$. Then, agent $i$ and $j$ both update their opinion values to their average value, i.e.,

$$x_i(t+1) = x_j(t+1) = \frac{x_i(t) + x_j(t)}{2}.$$

Let $x(t)$ be the opinion vector at time $t$ for all agents. The updating rule can be described as

$$x(t+1) = M(t)x(t),$$

where $M(t)$ is a random variable which is

$$I_n - \frac{(e_i - e_j)(e_i - e_j)^T}{2}$$

with probability $\frac{1}{n} p_{ij}$.

It is easy to see that

$$x(t+1) = M(t)M(t-1) \cdots M(0)x(0) = \phi(t)x(0),$$

where $\phi(t)$ is defined by $M(t)M(t-1)\cdots M(0)$. The sum of all entries in $x(t)$ is unchanged over time. We use $x_{\rm ave}$ to denote the average opinion values of all agents. If consensus occurs, the opinion values of all agents should converge to $x_{\rm ave}$. Thus, we must have $\lim\limits_{t\to\infty}\phi(t)=\frac{\mathbf{1}\mathbf{1}^T}{n}$. Let $\overline{M}$ denote the expectation of the variable $M(t)$. Note that $M(t)$, $t=0,1,2,\cdots$ are a series of independent identical distributed random variables. We have $E(\phi(t))=\prod\limits_{i=0}^{t}E(M(i))=\overline{M}^{t+1}$. Thus, we can infer that the expectation of $\phi(t)$ converges to $\frac{\mathbf{1}\mathbf{1}^T}{n}$ if and only if $\lim\limits_{t\to\infty}\overline{M}^t=\frac{\mathbf{1}\mathbf{1}^T}{n}$. A theorem in matrix analysis shows that the convergence of these algorithms is determined by

$$\rho\left(\overline{M}-\frac{\mathbf{1}\mathbf{1}^T}{n}\right)<1,$$

where $\rho(A)$ represent the largest absolute value of eigenvalues of the matrix $A$.

When we get the convergence of the algorithm, it is more valuable to study the speed of convergence. Boyd [29] introduced the concept of $\epsilon$-average time: for any $0<\epsilon<1$, the $\epsilon$-average time of an algorithm relevant to matrix $P$ is denoted by $T_{\rm ave}(\epsilon,P)$, and defined by

$$\sup_{x(0)}\inf\left\{t:\mathbb{P}\left(\frac{\|x(t)-x_{\rm ave}\mathbf{1}\|}{\|x(0)\|}\geq\epsilon\right)\leq\epsilon\right\},$$

where $\|\cdot\|$ is the $l_2$ norm. Intuitively, $\epsilon$-average time describes the shortest time needed for $x(t)$ reach to the $\epsilon$ neighborhood of $x_{\rm ave}\mathbf{1}$ with high probability, regardless of the initial value $x(0)$.

Theorem [29] shows that

$$\frac{0.5\log\epsilon^{-1}}{\log\lambda_2(M)^{-1}}\leq T_{\rm ave}(\epsilon,P)\leq\frac{3\log\epsilon^{-1}}{\log\lambda_2(M)^{-1}},$$

where $\lambda_2(\cdot)$ is the second largest eigenvalue, $M\triangleq I-\frac{1}{2n}D+\frac{P+P^T}{2n}$, and $D$ is a diagonal matrix with $D_i=\sum\limits_{j=1}^{n}[P_{ij}+P_{ji}]$.

### 3.3 Randomized Boolean Gossip Model

In this subsection, we introduce the randomized boolean gossip model [31]. Like fashion games, every agent holds binary values 0 or 1. However randomized boolean gossip models are random algorithms, just as their names indicate, while fashion games are deterministic. Moreover, the evolution of opinions is based on boolean logic.

Also, we consider a social network $G=(V,E)$ with $n$ agents $V=\{1,2,\cdots,n\}$. There is no self-loop in $E$, and $G$ is connected. Time is discrete like $t=0,1,2,\cdots$. At any moment, agent $i$ holds a binary opinion $x_i(t)$. At every time, a pair of agents, $i$ and $j$, are randomly selected such that $\{i,j\}\in E$. They update their opinion according to a boolean function on their opinions. Note that there are 16 boolean functions that map $\{0,1\}^2$ into $\{0,1\}$. We use that $\odot_k$ specifies a binary boolean function and that $a\odot_k b$ represents the value of $\odot_k(a,b)$. The set of all 16 boolean functions is denoted by $H$.

Let $C\neq\emptyset$ be a subset of $H$ which specifies all possible updating rules between two nodes. Assume that $C$ contains $q$ elements. Then $C$ can be represented by

$$\odot_{C_1},\cdots,\odot_{C_q}.$$

Suppose that agents $i$ and $j$ are selected at time $t$. Introduce $p_{kl}>0$ for $1\leq k,l\leq q$ satisfying $\sum\limits_{k,l}p_{kl}=1$. At time $t$, agent $i$ and $j$ jointly choose $(\odot_{C_k},\odot_{C_l})\in C\times C$ with probability $p_{kl}$ to update their opinion, i.e.,

$$\begin{cases} x_i(t+1) = x_i(t) \odot_{C_k} x_j(t), \\ x_j(t+1) = x_j(t) \odot_{C_l} x_i(t), \\ x_m(t+1) = x_m(t), \ m \notin \{i,j\}. \end{cases}$$

As the set $C$ can be arbitrary subsets of the set of all boolean functioins $H$, this randomized boolean gossip model not only can be used to analyze the evolution of social opinions [32], but also can be used to explain gene regulation [33] and virus spreading [34].

Let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t)),\ t = 0, 1, \cdots$ denote the stochastic process determined by the above model. This process defines a Markov chain with $2^n$ states, $\mathcal{M}_G(C) = (S_n, P)$, where

$$S_n = \{[s_1 \cdots s_n] : s_i \in \{0,1\}, i \in V\}$$

is the state space, and $P = [P_{[s_1\cdots s_n][q_1\cdots q_n]}] \in \mathbb{R}^{2^n \times 2^n}$ is the transition matrix such that

$$P_{[s_1\cdots s_n][q_1\cdots q_n]} \triangleq \mathbb{P}(x(t+1) = [q_1 \cdots q_n] | x(t) = [s_1 \cdots s_n]).$$

Next, we consider a special case. The set $C$ does not involve the negation rule $\neg$, and we use the conventional notation $\wedge$ to represent that boolean "AND" operation and $\vee$ to represent "OR" operation. We term such types of boolean operations as positive boolean dynamics, and define that $C_{pst} = \{\vee, \wedge\}$. For this kind of randomized boolean gossip model, the Markov chain is an absorbing chain with two absorbing states, $[0\cdots 0]$ and $[1\cdots 1]$. Thus, from the standard theory of Markov chains, for $x_0 = x(0) \in S_n \setminus \{[0\cdots 0], [1\cdots 1]\}$, we can find a Bernoulli random variable $x_*$, such that

$$P\left(\lim_{t\to\infty} x_i(t) = x_*, \forall i \in V\right) = 1.$$

For other binary valued opinion dynamics, Yildiz [35] studied models with stubborn agents who never change their opinions.

### 3.4 Clique Gossip Algorithms

Cliques are complete subgraphs and are very common in social, computer, and engineering networks. They have been applied to beamforming and clustering in wireless sensor networks [36, 37] and quantum networks [38]. Yang Liu et al. [39] introduced the concept of cliques in gossip algorithms to speed up the gossip algorithms. Intuitively, in clique gossip algorithms, all of the agents in the clique update their opinions simultaneously, while in gossip algorithms, only one pair of agents updates their opinions.

Consider the network $G(V, E)$, where $V = \{1, \cdots, n\}$ is the set of all agents and $E$ represents the relationship between agents as before. Assume that $E$ is without any loop and that $G$ is connected. Agent $i$ holds an opinion $x_i(t) \in \mathbb{R}$ for discrete time $t = 0, 1, \cdots$. Let $H_G^* = \{C_{\mu_1}, \cdots, C_{\mu_d}\}$ be a clique coverage of $G$, which means that each $C_{\mu_l}$ is a clique in $G$ and every agent in $G$ must belong to at least one clique in $H_G^*$. For each clique $C_{\mu_l} \in H_G^*$, we assign a number $A_{ij}(\mu_l)$ for each edge $\{i,j\} \in E$ in the induced subgraph $G[C_{\mu_l}]$ and assign a number $A_{ii}(\mu_l)$ for every agent $i$. Introduce the function $\sigma(\cdot)\colon \mathbb{Z}^{\geq 0} \to \{\mu_1, \mu_1, \cdots, \mu_d\}$. If at time $t$ the clique $C_{\sigma(t)} \in H_G^*$ is selected, then agents update their opinions as follows:

$$x_i(t+1) = \begin{cases} \sum\limits_{j \in C_{\sigma(t)}} A_{ij}(\sigma(t)) x_j(t), & i \in C_{\sigma(t)}, \\ x_i(t), & i \notin C_{\sigma(t)}. \end{cases}$$

We see that the updating rule can be represented by matrix forms

$$x(t+1) = M_{\sigma(t)} x(t).$$

If all agents in the clique update their opinions as their average opinion in the clique, the updating rule is

$$x_i(t+1) = \begin{cases} \sum\limits_{j \in C_{\sigma(t)}} x_j(t)/|C_{\sigma(t)}|, & i \in C_{\sigma(t)}, \\ x_i(t), & i \notin C_{\sigma(t)}. \end{cases}$$

Just as randomized gossip algorithms, the above updating rule can be randomized. If $\sigma(t)$ is prefixed, this algorithm is deterministic. If $\sigma(t)$ is a distribution on $H_G^*$, then this is a randomized clique gossip algorithm.

The finite time convergence was considered. One can find an $m$-regular clique coverage $H_G^*$ which leads to a globally finite-time convergent clique gossip averaging algorithm if and only if $n$ is divisible by $m$ with the same prime factors as $m$.

For randomized clique gossip algorithms, $M(t)$ has the value

$$I - \frac{1}{2m_l} \sum_{(i,j) \in W_{\mu_l}} (e_i - e_j)(e_i - e_j)^T,$$

with probability $p_l$, $l = 1, 2, \cdots, d$. Here

$$W_{\mu_l} \triangleq C_{\mu_l} \times C_{\mu_l} \setminus \{(i,i) : i \in C_{\mu_l}\}.$$

Using the same method as for the randomized gossip algorithms, we have

$$x(t+1) = M(t)M(t-1) \cdots M(0)x(0) = \phi(t)x(0).$$

We also use $\overline{M}$ to represent the expectation of $M(t)$. We also have that

$$E(\phi(t)) = \prod_{i=0}^{t} E(M(i)) = \overline{M}^{t+1},$$

so the convergence of these algorithms is still determined by

$$\rho\left(\overline{M} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right) < 1.$$

We can define $T_{\text{ave}}(\epsilon)$ as $\epsilon$-average time of a convergent randomized clique gossip algorithm as

$$\sup_{x(0)} \inf \left\{ t : P\left( \frac{\|x(t) - x_{\text{ave}}\mathbf{1}\|}{\|x(0)\|} \geq \epsilon \right) \leq \epsilon \right\},$$

and we also know that it satisfies that

$$\frac{0.5 \log \epsilon^{-1}}{\log \lambda_2(\overline{M}))^{-1}} \leq T_{\text{ave}}(\epsilon) \leq \frac{3 \log \epsilon^{-1}}{\log \lambda_2(\overline{M})^{-1}}.$$

This shows that the second largest eigenvalue of $\overline{M}$ determines the speed of the convergence.

Although the aim of introducing cliques gossip algorithms is to boost the speed of the algorithms, an interesting phenomenon shows that the involvement of cliques does not always accelerate the computation. For complete graphs, the introduction of regular cliques does improve the performance of the algorithms. The following example shows that the second largest eigenvalue may be less than the one in the relevant randomized gossip algorithm.

Consider the social network $G(V, E)$ as shows in the figure, where $V = \{1, \cdots, 7\}$. The clique coverage is $H_G^* = \{C_1, C_2, C_3\}$, where three cliques have different sizes.

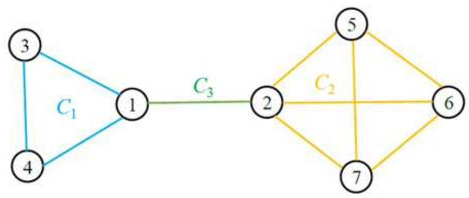


Figure 1 Social networks with 7 agents

For the randomized gossip algorithm, one of the 10 edges is selected with equal probability at time $t$. The two agents of this edge communicate and update their opinions as the average opinion. We can obtain that $\lambda_2(\overline{M}_{\text{Gossip}}) = 0.980084$.

For the randomized clique gossip algorithm, one of the three cliques is selected at time $t$ randomly. The cliques $C_1$, $C_2$, and $C_3$ are going to be selected with probabilities $p$, $q$, and $1-p-q$, respectively. All of the agents of the selected clique communicate and update their opinions as the average opinion of the clique. Then,

(1) when $p = 1/20,\ q = 1/10$, we have $\lambda_2(\overline{M}_{\text{Clique Gossip}}) = 0.9819708$;

(2) when $p = 3/10,\ q = 3/5$, $\lambda_2(\overline{M}_{\text{Clique Gossip}}) = 0.9754117$;

(3) when $p = 9/20,\ q = 1/2$, $\lambda_2(\overline{M}_{\text{Clique Gossip}}) = 0.9864559$.

We see that in this social network, the speed of convergence of clique gossip algorithms is relevant to the probability of the cliques being chosen. Under different $p$ and $q$, we cannot be assured that randomized clique gossip algorithms are faster than randomized gossip algorithms.

### 3.5 Randomized Gossip Algorithm on the Unit Circle

Whether it is the classic social network model or the above-mentioned Gossip algorithm, agent opinion values are one-dimensional real values, so there exists an order relationship. However, some real-life instances, such as the well-known rock-paper-scissors game, do not have an order relationship. They can no longer be described by one-dimensional real values. As another example, some people like to go out for activities at three o'clock in the afternoon every day, while others go at five o'clock. The value range of these time points is a circle on the clock. Inspired by this, we take agent opinion values from the unit circle and study the consensus of agent opinions.

Consider a social network $G = (V, E)$ consisting of $n$ agents. Time is slotted and the value that agent $i$ holds at time $t$ is denoted by $v_i(t) = (x_i(t), y_i(t)) \in \mathbb{R}^2$, where $x_i^2(t) + y_i^2(t) = 1$. It is not difficult to find that for any agent $i$ and time $t$, there is a unique angle $\theta_i(t) \in [0, 2\pi)$ such that $x_i(t) = \cos\theta_i(t),\ y_i(t) = \sin\theta_i(t)$. That is, we can use $\theta_i(t)$ to characterize the opinion vector $v_i(t)$ of agent $i$.

Since, for any $r \in \mathbb{R}$, there is a unique $\theta \in [0, 2\pi)$ such that $\frac{r-\theta}{2\pi}$ is an integer, we introduce the notation mod$(r)$ to represent $\theta$.

The evolution of opinions still proceeds in a random gossip manner, that is, at time $t$, agent $i$ is randomly selected from $V$, and agent $j$ is randomly selected from $i$'s neighbor set $N_i$ to communicate with $i$. We also use averages to update the opinions of $i$ and $j$. How do we take the average on the circle? We know that any two points on the unit circle will divide the circumference into two arcs: when the two arcs are equal in length, they are both semicircles; otherwise, the arc larger than the semicircle is the superior arc, and the arc smaller than the semicircle is the inferior arc. The midpoint of both arcs can be called the average of the two viewpoints. Of course, because the distance between the inferior arcs is closer, we think that individuals are more likely to shift their opinions to the midpoint of the inferior arc after communication, and less likely to shift their opinion to the midpoint of the superior arc. Furthermore, if the opinions of these two agents are the same, their opinions should be the same after the communication. We can borrow a model from quantum networks [40] to define the updates of agent $i$ and agent $j$ as follows: agent $i$ and agent $j$ independently update their opinions to

$$\text{mod}\left(\frac{\theta_i(t)+\theta_j(t)}{2}\right),$$

with probability $\cos^2\frac{\theta_i(t)-\theta_j(t)}{4}$, or to

$$\text{mod}\left(\frac{\theta_i(t)+\theta_j(t)}{2}+\pi\right),$$

with probability $\sin^2\frac{\theta_i(t)-\theta_j(t)}{4}$. We also assume that the selection of both agent $i$ and agent $j$ is uniformly random. Then we get the following theorem:

**Theorem** All the nodes reaches consensus almost surely, i.e.,

$$P\left(\lim_{t\to\infty}\|v_i(t)-v_j(t)\|=0, \forall i,j\in V\right)=1,$$

where $\|\cdot\|$ is the $l_2$ norm in $\mathbb{R}^2$.

**Proof** Let $p_{ij}$ be the probability that agent $i$ and agent $j$ are chosen at any specific time. Denote that

$$\mathcal{F}_t=\sigma(\theta_i(s), i\in \mathrm{V}, 0\le s\le t)$$

is the $\sigma$-algebra flow. Let $\mathbb{E}_{kl}[\cdot|\mathcal{F}_t]$ be the conditional expectation at time $t$ when agent $k$ and agent $l$ have been chosen. Let $g(t)=\sum\limits_{1\le i,j\le N}\cos^2\frac{\theta_i(t)-\theta_j(t)}{2}$. We have that

$$\begin{aligned}
E[g(t+1)|\mathcal{F}_t] &= \sum_{\{k,l\}\in E} p_{kl}E_{kl}[g(t+1)|\mathcal{F}_t] \\
&= \sum_{\{k,l\}\in E} p_{kl}E_{kl}[g(t+1)|\theta_m(t), m\in V] \\
&= \sum_{\{k,l\}\in E} p_{kl}E_{kl}\left[\sum_{1\le i,j\le n}\cos^2\frac{\theta_i(t+1)-\theta_j(t+1)}{2}|\theta_m(t), 1\le m\le n\right] \\
&= E_1+E_2+E_3+E_4+E_5+E_6,
\end{aligned}$$

where

$$E_1 = \sum_{\{k,l\}\in E} p_{kl} E_{kl}\left[\cos^2\frac{\theta_k(t+1)-\theta_l(t+1)}{2}|\theta_m(t), 1\le m\le n\right],$$

$$E_2 = \sum_{\{k,l\}\in E} p_{kl} E_{kl}\left[\sum_{i=k,1\le j\le n, j\ne k,l}\cos^2\frac{\theta_i(t+1)-\theta_j(t+1)}{2}|\theta_m(t), 1\le m\le n\right],$$

$$E_3 = \sum_{\{k,l\}\in E} p_{kl} E_{kl}\left[\sum_{j=l,1\le i\le n, i\ne k,l}\cos^2\frac{\theta_i(t+1)-\theta_j(t+1)}{2}|\theta_m(t), 1\le m\le n\right],$$

$$E_4 = \sum_{\{k,l\}\in E} p_{kl} E_{kl}\left[\sum_{1\le i,j\le n, i\ne k, j\ne l}\cos^2\frac{\theta_i(t+1)-\theta_j(t+1)}{2}|\theta_m(t), 1\le m\le n\right],$$

$$E_5 = \sum_{\{k,l\}\in E} p_{kl} E_{kl}\left[\cos^2\frac{\theta_k(t+1)-\theta_k(t+1)}{2}|\theta_m(t), 1\le m\le n\right],$$

$$E_6 = \sum_{\{k,l\}\in E} p_{kl} E_{kl}\left[\cos^2\frac{\theta_l(t+1)-\theta_l(t+1)}{2}|\theta_m(t), 1\le m\le n\right].$$

It is easy to verify that

$$\begin{aligned}
E_1 &= \sum_{\{k,l\}\in E} p_{kl}\left(\cos^4\frac{\theta_k(t)-\theta_l(t)}{4}+\sin^4\frac{\theta_k(t)-\theta_l(t)}{4}\right)\\
&= \sum_{\{k,l\}\in E} p_{kl}\left(\frac{1}{2}+\frac{1}{2}\cos^2\frac{\theta_k(t)-\theta_l(t)}{2}\right),\\
E_4 &= \sum_{\{k,l\}\in E} p_{kl}\left(\sum_{1\le i,j\le n, i\ne k, j\ne l}\cos^2\frac{\theta_i(t)-\theta_j(t)}{2}\right),\\
E_5 &= 1 = \sum_{\{k,l\}\in E} p_{kl}\cos^2\frac{\theta_k(t)-\theta_k(t)}{2},\\
E_6 &= 1 = \sum_{\{k,l\}\in E} p_{kl}\cos^2\frac{\theta_l(t)-\theta_l(t)}{2}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
E_2 &= \sum_{\{k,l\}\in E} p_{kl}\sum_{1\le j\le n, j\ne k,l} E_{kl}\left[\cos^2\frac{\theta_k(t+1)-\theta_j(t+1)}{2}|\theta_m(t), 1\le m\le n\right]\\
&= \sum_{\{k,l\}\in E} p_{kl}\sum_{1\le j\le n, j\ne k,l}\left(\cos^2\frac{(\theta_k(t)+\theta_l(t))/2-\theta_j(t)}{2}\cos^2\frac{\theta_k(t)-\theta_l(t)}{4}\right.\\
&\quad\left.+\sin^2\frac{(\theta_k(t)+\theta_l(t))/2-\theta_j(t)}{2}\sin^2\frac{\theta_k(t)-\theta_l(t)}{4}\right)\\
&= \sum_{\{k,l\}\in E} p_{kl}\sum_{1\le j\le n, j\ne k,l}\frac{1}{2}\left(\cos^2\frac{\theta_k(t)-\theta_j(t)}{2}+\cos^2\frac{\theta_l(t)-\theta_j(t)}{2}\right),
\end{aligned}$$

$$E_3 = \sum_{\{k,l\}\in E} p_{kl}\sum_{1\le i\le n, i\ne k,l} E_{kl}\left[\cos^2\frac{\theta_i(t+1)-\theta_l(t+1)}{2}|\theta_m(t), 1\le m\le n\right]$$

$$
\begin{aligned}
&= \sum_{\{k,l\}\in E} p_{kl} \sum_{1\le i\le n, i\ne k,l} \left(\cos^2 \frac{(\theta_k(t)+\theta_l(t))/2-\theta_i(t)}{2}\cos^2\frac{\theta_k(t)-\theta_l(t)}{4}\right.\\
&\quad \left. + \sin^2 \frac{(\theta_k(t)+\theta_l(t))/2-\theta_i(t)}{2}\sin^2\frac{\theta_k(t)-\theta_l(t)}{4}\right)\\
&= \sum_{\{k,l\}\in E} p_{kl} \sum_{1\le i\le n, i\ne k,l} \frac{1}{2}\left(\cos^2\frac{\theta_k(t)-\theta_i(t)}{2}+\cos^2\frac{\theta_l(t)-\theta_i(t)}{2}\right).
\end{aligned}
$$

Therefore, we can conclude that

$$
\begin{aligned}
\mathbb{E}[g(t+1)|\mathcal{F}_t] &= E_1+E_2+E_3+E_4+E_5+E_6\\
&= \sum_{\{k,l\}\in \mathrm{E}} p_{kl}\left(\sum_{1\le i,j\le N}\cos^2\frac{\theta_i(t)-\theta_j(t)}{2}+\frac{1}{2}\sin^2\frac{\theta_k(t)-\theta_l(t)}{2}\right)\\
&= g(t)+\frac{1}{2}\sum_{\{k,l\}\in \mathrm{E}} p_{kl}\sin^2\frac{\theta_k(t)-\theta_l(t)}{2}\\
&\ge g(t). \qquad (3.2)
\end{aligned}
$$

The inequality (3.2) implies that $g(t)$ is a submartingale. It is easy to verify that for all $t$, $g(t)\le n^2$. Therefore, according to the Martingale Convergence Theorem, $g(t)$ converges to a finite limit with probability 1. Let $G$ represent the limits of $g(t)$. In other words,

$$
P\left(\lim_{t\to\infty} g(t)=G\right)=1.
$$

By Inequality (3.2) and the Dominated Convergent Theorem, we can prove that

$$
\lim_{t\to\infty} E\left[\sum_{\{k,l\}\in E} p_{kl}\sin^2\frac{\theta_k(t)-\theta_l(t)}{2}\right]=0.
$$

Thus, for any $\{k,l\}\in \mathrm{E}$,

$$
\lim_{t\to\infty} E\left[\sin^2\frac{\theta_k(t)-\theta_l(t)}{2}\right]=0.
$$

Then, we can prove that $\sin^2\frac{\theta_k(t)-\theta_l(t)}{2}$ converges to zero in probability. Therefore, we can conclude that $\cos^2\frac{\theta_k(t)-\theta_l(t)}{2}$ converges to one in probability. That is to say, $g(t)$ converges to $n^2$ in probability. As $g(t)$ converges to $G$ almost surely, we know that $g(t)$ converges to $n^2$ almost surely. According to the definition of $g(t)$, $\theta_i(t)$ is equal to $\theta_j(t)$ almost surely, as $t$ increases to infinity. Thus, everyone reaches consensus. □

## 4 Conclusion

In this paper, we introduced a few types of models of social networks. The key differences among these models can be characterized as follows:

(1) The manner of communication between agents. In the DeGroot model, the signed social network models, and fashion games, agents communicate with all their neighbors. In the Krause model, agents only communicate with neighbors whose opinions do not deviate too much from their own. In gossip algorithms, agents just choose one neighbor with whom to communicate, while in clique gossip algorithms, agents choose all neighbors in one clique with which to communicate.

(2) The set of all possible opinions. In fashion games and randomized boolean gossip algorithms, the possible values of opinions are just 0 and 1. For other models, the possible set is the real numbers. In the last subsection, we introduced the unit circle as the possible set for the first time.

(3) Consists of Deterministic or randomized.

(4) The network topology. The underlying graph can be directed, undirected, or even signed. Moreover, the network topology can be time-variant or time-invariant.

Different combinations of these characteristics form different models, which can then be used to describe different social network phenomena. Opinion dynamics is an interdisciplinary field that has implications for economics, computer science, biology, and mathematics. The technical tools used for studying these models come from graph theory, matrix analysis, combinatorics, probability, and even artificial intelligence. It is a thriving field that deserves a wider attention.

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