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# GLOBAL WELL-POSEDNESS OF A PRANDTL MODEL FROM MHD IN GEVREY FUNCTION SPACES*

Dedicated to Professor Banghe LI on the occasion of his 80th birthday

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**Abstract** We consider a Prandtl model derived from MHD in the Prandtl-Hartmann regime that has a damping term due to the effect of the Hartmann boundary layer. A global-in-time well-posedness is obtained in the Gevrey function space with the optimal index 2. The proof is based on a cancellation mechanism through some auxiliary functions from the study of the Prandtl equation and an observation about the structure of the loss of one order tangential derivatives through twice operations of the Prandtl operator.

**Key words** magnetic Prandtl equation; Gevrey function space; global well-posedness; auxiliary functions; loss of derivative

**2010 MR Subject Classification** 76W05; 35M33; 35Q35

## 1 Introduction and Main Result

We study a Prandtl type system with a damping term induced by a Hartmann magnetic field that was derived by Gérard-Varet and Prestipino in [11]. Suppose that the fluid domain is $\mathbb{R}^n_+ = \{(x,y)\in\mathbb{R}^n;\ x\in\mathbb{R}^{n-1}, y>0\}$. Denote by $\boldsymbol{u}$ and $v$ the tangential and normal components of the velocity fields. Then the dimensionless magnetic Prandtl model in $\mathbb{R}^n_+$ is given by

$$\begin{cases}(\partial_t+\boldsymbol{u}\cdot\partial_x+v\partial_y-\partial_y^2)\boldsymbol{u}+\partial_x p+\boldsymbol{u}-\boldsymbol{U}=0,\\ \partial_x\cdot\boldsymbol{u}+\partial_y v=0,\\ (\boldsymbol{u},v)|_{y=0}=(\boldsymbol{0},0),\qquad \lim\limits_{y\to+\infty}\boldsymbol{u}=\boldsymbol{U},\\ \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0,\end{cases}\tag{1.1}$$

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where $\partial_x p$ and $\boldsymbol{U}$ are the traces of a given Euler flow satisfying the Bernoulli law

$$(\partial_t + \boldsymbol{U} \cdot \partial_x) \boldsymbol{U} + \partial_x p = 0.$$

System (1.1) is derived from the incompressible MHD system in the mixed Prandtl/Hartmann regime, where the leading order equations are

$$\begin{cases} (\partial_t + \boldsymbol{u} \cdot \partial_x + v\partial_y - \partial_y^2)\boldsymbol{u} + \partial_x p = \partial_y \boldsymbol{b}, \\ \partial_y \boldsymbol{u} + \partial_y^2 \boldsymbol{b} = 0, \\ \partial_x \cdot \boldsymbol{u} + \partial_y v = 0, \\ (\boldsymbol{u}, v)|_{y=0} = (\boldsymbol{0}, 0), \quad \lim\limits_{y \to +\infty} (\boldsymbol{u}, \boldsymbol{b}) = (\boldsymbol{U}, \boldsymbol{B}), \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \end{cases} \tag{1.2}$$

where $\boldsymbol{b}$ stands for the tangential magnetic field satisfying that $\partial_y \boldsymbol{b}$ trends to 0 as $y \to +\infty$. By the second equation in (1.2) and the boundary conditions of $(\boldsymbol{u}, \boldsymbol{b})$, the magnetic fields $\boldsymbol{b}$ can be determined in terms of $\boldsymbol{u}$; that is,

$$\partial_y \boldsymbol{b} = -(\boldsymbol{u} - \boldsymbol{U}), \quad \boldsymbol{b} = \boldsymbol{B} + \int_y^{+\infty} (\boldsymbol{u} - \boldsymbol{U}) \mathrm{d}y.$$

Then system (1.2) reduces to (1.1). We refer to [11] for the detailed argument. Compared with the classical Prandtl system, there is an extra damping term in (1.1); this does not lead to any additional difficulty in the local-in-time existence and uniqueness theory. Hence, the local well-posedness theories of the classical Prandtl system in Sobolev or Gevrey function spaces established in [2, 5, 16, 28] hold for (1.1). On the other hand, we can expect a global-in-time solution to (1.1) because of the damping effect. In fact, some results in this direction were obtained in Xie-Yang [37] and Chen-Ren-Wang-Zhang [3] in the analytic and Sobolev spaces, respectively, about the 2D global stability of the Hartmann layer that satisfies Oleinik's monotonicity condition. This paper aims to study the global-in-time property of the system for general data without any structural assumption.

Compared with the local-in-time well-posedness theory (see, for instance, [2, 4–7, 9, 12–14, 17, 20, 24, 38–41] and references therein), the global-in-time property of the Prandtl system is much less known so that it is far from being well understood. Here we refer to an early work on weak solutions by Xin-Zhang [38], and a work on analytic solutions by Paicu-Zhang [29] and the improvement to Gevrey class 2 by Wang-Wang-Zhang [36]. We also mention some other related works [13, 39, 41]. For the Prandtl system with a suitable background magnetic field, the local solutions in Sobolev or Gevrey function spaces were obtained in [19, 21, 23, 25], and the global analytic solution was established recently in [15, 26]. Note that all of these global-in-time existence results are in 2D setting under some suitable structural conditions on the initial data. Hence, the global-in-time properties of these systems in 3D setting remain unknown.

On the other hand, there have been some recent studies on the global well-posedness of hydrostatic Navier-Stokes equations (also called hydrostatic Prandtl equations); these can be used to describe a large scale motion in atmospheric and oceanic sciences, and are derived as a limit of the Navier-Stokes equations in a very thin domain. The hydrostatic Navier-Stokes equations have the same degeneracy structure as in the classical Prandtl system, so that the analytic regularity is sufficient to obtain the local well-posedness theory for general initial data without any structural assumption. Moreover, due to the damping effect, by combining the vertical diffusion and the Poincaré inequality in the vertical interval, the global-in-time property of the hydrostatic Navier-Stokes equations was recently verified by Paicu-Zhang-Zhang [31]. However, differently from the classical Prandtl equations, the analytic regularity is necessary for the well-posedness theory of the hydrostatic Prandtl equations without any structural assumption (cf. Renardy's work [32]). Hence, in order to investigate the well-posedness theory in a larger function space than analytic, some structural conditions are required. We mention that the well-posedness in Sobolev space of the hydrostatic Navier-Stokes equations remains unknown. Under the convex assumption, Masmoudi-Wong [27] established the well-posedness in a Sobolev space of the hydrostatic Euler equations which is the inviscid version of the hydrostatic Navier-Stokes equation. However, under the same convex assumption, the well-posedness has been proved for the hydrostatic Navier-Stokes equation only in the Gevrey function space by Gérard-Varet-Masmoudi-Vicol in [10], with the Gevrey index being up to 9/8, and in [8, 35] up to 3/2 that is believed to be optimal. Furthermore, the global well-posedness theory of the hyperbolic version of the 2D hydrostatic Navier-Stokes equation was obtained recently by Aarach [1] and Paicu-Zhang [30] in analytic and Gevrey function spaces, respectively, with the extension to the 3D case in [22]. A similar well-posedness result on the hyperbolic Prandtl equations in Gevrey class was proven in [18]. These results imply that the hyperbolic feature may lead to some kind of stability effect compared with the parabolic counterparts.

This work aims to combine some kind of intrinsic hyperbolic structure with the extra magnetic damping term to derive the global-in-time well-posedness for general Gevrey classes with sharp index 2 initial data without any structural assumption. More precisely, inspired by [5, 16], we introduce some auxiliary functions to cancel the nonlocal terms involving the loss of tangential derivatives, and then investigate the intrinsic hyperbolic feature for evolution equations of the auxiliary functions.

In the discussion that follows, we assume that $\boldsymbol{U}=\boldsymbol{0}$ and that $\partial_x p=\boldsymbol{0}$. We expect the approach can be applied to the case with a more general outer flow satisfying some suitable conditions; that is, we consider system (1.1) as

$$\begin{cases}(\partial_t+\boldsymbol{u}\cdot\partial_x+v\partial_y-\partial_y^2)\boldsymbol{u}+\boldsymbol{u}=0,\\ \partial_x\cdot\boldsymbol{u}+\partial_y v=0,\\ (\boldsymbol{u},v)|_{y=0}=(\boldsymbol{0},0),\qquad \lim\limits_{y\to+\infty}\boldsymbol{u}=\boldsymbol{0},\\ \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0.\end{cases}\tag{1.3}$$

Note (1.3) is a degenerate parabolic system with a loss of tangential derivatives in the nonlocal normal velocity given by

$$v(t,x,y)=-\int_0^y\partial_x\cdot\boldsymbol{u}(t,x,\tilde{y})\mathrm{d}\tilde{y}$$

as the main degeneracy feature of the Prandtl type equations.

**Notations** In what follows, we will use $\|\cdot\|_{L^2}$ and $(\cdot,\cdot)_{L^2}$ to denote the norm and inner product of $L^2=L^2(\mathbb{R}^n_+)$, and use the notation $\|\cdot\|_{L^2_x}$ and $(\cdot,\cdot)_{L^2_x}$ when the variable $x$ is specified. Similar notation will be used for $L^\infty$. In addition, we use $L^p_xL^q_y=L^p(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))$ for the classical Sobolev space. For a vector-valued function $A=(A_1,A_2,\cdots,A_k)$, we use the convention that $\|A\|_{L^2}^2 = \sum\limits_{1\le j\le k} \|A_j\|_{L^2}^2$. Throughout the paper, $\langle y\rangle := (1+y^2)^{1/2}$.

**Definition 1.1** Let $\ell > 1/2$ be a given number and let $\tau = \tau_N$ be a given weight function defined by

$$\tau(y) = \tau_N(y) = \left(N + y^2\right)^{1/2}, \tag{1.4}$$

with $N \ge 1$ a fixed integer depending only on $\ell$ such that

$$\frac{\ell+3}{N^{1/2}} + \frac{\ell^2+\ell}{N} \le \frac{1}{8}. \tag{1.5}$$

The space $X_r$ of Gevrey functions with the Gevrey index 2 consists of all smooth (scalar or vector-valued) functions $h$ such that the norm $\|h\|_{X_r} < +\infty$, where

$$\|h\|_{X_r}^2 = \sum_{0\le j\le 3} \sum_{\alpha\in\mathbb{Z}_+^{n-1}} L_{r,|\alpha|+j}^2 \|\tau^{\ell+j}\partial_x^\alpha \partial_y^j h\|_{L^2}^2,$$

with

$$L_{r,k} = \frac{r^{k+1}(k+1)^{10}}{(k!)^2}, \quad k\in\mathbb{Z}_+,\ r>0. \tag{1.6}$$

**Theorem 1.2** Let the dimension be $n=2$ or 3, and let the initial datum $\boldsymbol{u}_0$ in (1.3) belong to $X_{2\rho_0}$ for some $\rho_0 > 0$, compatible with the boundary condition in (1.3). Suppose that

$$\|\boldsymbol{u}_0\|_{X_{2\rho_0}} \le \varepsilon_0$$

for some sufficiently small $\varepsilon_0 > 0$. Then the magnetic Prandtl model (1.3) admits a unique global solution $\boldsymbol{u} \in L^\infty\left([0,+\infty[; X_\rho\right)$ with

$$\rho = \rho(t) = \frac{\rho_0}{2} + \frac{\rho_0}{2} e^{-t/12}. \tag{1.7}$$

Moreover,

$$\sup_{t\ge 0} e^{t/4}\|\boldsymbol{u}(t)\|_{X_{\rho(t)}} + \left(\int_0^{+\infty} e^{t/2}\|\partial_y \boldsymbol{u}(t)\|_{X_{\rho(t)}}^2 \mathrm{d}t\right)^{1/2} \le \frac{4(\rho_0+1)}{\rho_0}\varepsilon_0.$$

**Remark 1.3** Note that the same argument shows that the global well-posedness property holds for a Gevrey function space with the Gevrey index in the interval $[1,2]$.

## 2 A Priori Estimate in 2D

To have a clear presentation, we give a detailed proof of Theorem 1.2 in 2D. For $n=2$, we have the scalar tangential velocity $u$, and then the magnetic Prandtl model (1.3) can be written as

$$\begin{cases} (\partial_t + u\partial_x + v\partial_y - \partial_y^2)u + u = 0, \\ \partial_x u + \partial_y v = 0, \\ (u,v)|_{y=0} = (0,0), \quad \lim\limits_{y\to+\infty} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{2.1}$$

The key part is to derive an a priori estimate for (2.1) so that the existence and uniqueness stated in Theorem 1.2 follows from a standard argument. Hence, for brevity, we only present the proof of the following a priori estimates for solutions to (2.1) with Gevrey regularity:

**Theorem 2.1** Let $X_\rho$ be the Gevrey space given in Definition 1.1. Suppose that the initial datum $u_0$ in (2.1) belongs to $X_{2\rho_0}$ for some $\rho_0 > 0$, and let $u \in L^\infty([0,+\infty[;\ X_\rho)$ be any solution to (2.1) satisfying that

$$\int_0^\infty \left(\|u(t)\|_{X_{\rho(t)}}^2 + \|\partial_y u(t)\|_{X_{\rho(t)}}^2\right)\mathrm{d}t < +\infty, \tag{2.2}$$

where $\rho$ is defined by (1.7). If

$$\|u_0\|_{X_{2\rho_0}} \le \varepsilon_0 \tag{2.3}$$

for some sufficiently small $\varepsilon_0 > 0$, then

$$\sup_{t\ge 0} e^{t/4}\|u(t)\|_{X_{\rho(t)}} + \left(\int_0^{+\infty} e^{t/2}\|\partial_y u(t)\|_{X_{\rho(t)}}^2 \mathrm{d}t\right)^{1/2} \le \frac{4(\rho_0+1)}{\rho_0}\varepsilon_0.$$

### 2.1 Methodologies and Auxiliary Functions

The main difficulty for the well-posedness of Prandtl type equations comes from the loss of tangential derivatives. To overcome the tangential degeneracy, the abstract Cauchy-Kowalewski Theorem is an effective method, for example, for the local existence and uniqueness in an analytic setting, cf. [33]. However, it is not trivial to relax the analyticity regularity to a larger function space such as Gevrey space for well-posedness. For this, some intrinsic structure of the system needs to be used, cf. [5, 16]. As was shown in [18, 22, 30], the well-posedness is well expected in the Gevrey class rather than the analytic setting for the hyperbolic Prandtl equations without any structural assumption. This indicates that the hyperbolic feature may act as a stabilizing factor for the Gevrey well-posedness theory. In this paper, with the extra damping term in magnetic Prandtl model, we will prove the global-in-time existence in a Gevrey function space by exploring the intrinsic hyperbolic feature for auxiliary functions.

In order to clarify the stability effect of the hyperbolic feature, let us use the following toy model with a hyperbolic factor $\partial_t^2$ to illustrate the main idea in the proof:

$$\partial_t^2 h + h\partial_x h - \partial_y^2 h = 0, \quad h|_{t=0} = h_0 \ \text{ and } \ \partial_t h|_{t=0} = h_1, \tag{2.4}$$

where one order $x$-derivative is lost in twice-time differentiation. Denoting that $g = \partial_t h$, the above Cauchy problem can be rewritten as

$$\begin{cases} \partial_t h = g, \\ \partial_t g - h\partial_x h - \partial_y^2 h = 0, \\ (h,g)|_{t=0} = (h_0, h_1). \end{cases} \tag{2.5}$$

To overcome the loss of the $x$-derivative, we introduce a decreasing function of radius$\rho = \rho(t)$. In what follows, $\rho$ depends on time $t$, but we only write it as $\rho$ instead of $\rho(t)$ for simplicity of the notations. Moreover, we denote by $\rho'$ and $\rho''$ the first and the second time derivatives of $\rho$, respectively. Now we derive estimates on the Gevrey norm

$$\sum_{m=0}^{+\infty} \frac{\rho^{m+1}}{m!^2}\left(\|\partial_x^m h\|_{L^2} + \|\partial_x^m\partial_y h\|_{L^2} + \|\partial_x^m g\|_{L^2}\right),$$

where $(h,g)$ solves the Cauchy problem (2.5). By direct calculation and using the fact that $\rho' \le 0$, we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho^{2(m+1)}}{m!^4}\left(\|\partial_x^m h\|_{L^2}^2 + \|\partial_x^m\partial_y h\|_{L^2}^2 + \|\partial_x^m g\|_{L^2}^2\right)\right)$$

$$\begin{aligned}
&= \rho' \frac{m+1}{\rho} \frac{\rho^{2(m+1)}}{m!^4} \left( \|\partial_x^m h\|_{L^2}^2 + \|\partial_x^m \partial_y h\|_{L^2}^2 + \|\partial_x^m g\|_{L^2}^2 \right) \\
&\quad + \frac{\rho^{2(m+1)}}{m!^4} \left( h\partial_x^{m+1} h,\ \partial_x^m g \right)_{L^2} + \text{ l.o.t.} \\
&\leq \rho' \frac{m+1}{\rho} \frac{\rho^{2(m+1)}}{m!^4} \left( \|\partial_x^m \partial_y h\|_{L^2}^2 + \|\partial_x^m g\|_{L^2}^2 \right) \\
&\quad + \frac{(m+1)^2}{\rho} \|h\|_{L^\infty} \left( \frac{\rho^{m+2}}{(m+1)!^2} \|\partial_x^{m+1} h\|_{L^2} \right) \left( \frac{\rho^{m+1}}{m!^2} \|\partial_x^m g\|_{L^2} \right) + \text{ l.o.t.},
\end{aligned}$$

where l.o.t. refers to the lower-order terms that can be controlled directly. Moreover, we have ( cf. Subsection 2.2 for details) that

$$\rho' \frac{m+1}{\rho} \frac{\rho^{2(m+1)}}{m!^4} \|\partial_x^m g\|_{L^2}^2 \leq \rho'^3 \frac{(m+1)^3}{\rho^3} \frac{\rho^{2(m+1)}}{m!^4} \|\partial_x^m h\|_{L^2}^2 + \text{ l.o.t..}$$

In summary,

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\rho^{2(m+1)}}{m!^4} \left( \|\partial_x^m h\|_{L^2}^2 + \|\partial_x^m \partial_y h\|_{L^2}^2 + \|\partial_x^m g\|_{L^2}^2 \right) \right) \\
&\leq \frac{1}{2} \rho' \frac{m+1}{\rho} \frac{\rho^{2(m+1)}}{m!^4} \left( \|\partial_x^m \partial_y h\|_{L^2}^2 + \|\partial_x^m g\|_{L^2}^2 \right) + \frac{1}{2} \rho'^3 \frac{(m+1)^3}{\rho^3} \frac{\rho^{2(m+1)}}{m!^4} \|\partial_x^m h\|_{L^2}^2 \\
&\quad + \frac{1}{2} \|h\|_{L^\infty} \frac{m+1}{\rho} \frac{\rho^{2(m+1)}}{m!^4} \|\partial_x^m g\|_{L^2}^2 + \frac{1}{2} \|h\|_{L^\infty} \frac{(m+1)^3}{\rho^3} \frac{\rho^{2(m+2)}}{(m+1)!^4} \|\partial_x^{m+1} h\|_{L^2}^2 + \text{ l.o.t..}
\end{aligned}$$

If we define a norm $|||\cdot|||_{Y_\rho}$ by

$$|||h|||_{Y_\rho}^2 = \sum_{m=0}^{+\infty} \frac{m+1}{\rho} \frac{\rho^{2(m+1)}}{m!^4} \left( \|\partial_x^m \partial_y h\|_{L^2}^2 + \|\partial_x^m \partial_t h\|_{L^2}^2 \right) + \sum_{m=0}^{+\infty} \frac{(m+1)^3}{\rho^3} \frac{\rho^{2(m+1)}}{m!^4} \|\partial_x^m h\|_{L^2}^2, \tag{2.6}$$

then it follows from the above inequality that

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left( \sum_{m=0}^{+\infty} \frac{\rho^{2(m+1)}}{m!^4} \left( \|\partial_x^m h\|_{L^2}^2 + \|\partial_x^m \partial_y h\|_{L^2}^2 + \|\partial_x^m g\|_{L^2}^2 \right) \right) \\
&\leq \frac{1}{2} \left( \max\left\{ \rho', \rho'^3 \right\} + \|h\|_{L^\infty} \right) |||h|||_{Y_\rho}^3 + \text{ l.o.t.} \leq \text{ l.o.t.},
\end{aligned}$$

provided that $\|h\|_{L^\infty}$ is bounded by $|\max\left\{\rho', \rho'^3\right\}|$ because $\rho' < 0$.

Differently from the hyperbolic toy model (2.4), the magnetic Prandtl model is a parabolic initial-boundary problem. If we perform estimates for the 2D magnetic Prandtl model (2.1) directly, the above energy estimate cannot be closed in the Gevrey norm $\|\cdot\|_{X_\rho}$. In order to overcome the loss of derivative difficulty, as in [5, 16], some anxilliary functions are needed. More precisely, for a solution $u \in L^\infty([0, +\infty[;\ X_\rho)$ to (2.1) satisfying the conditions in Theorem 2.1, let $\mathcal{U}$ be a solution to the problem

$$\begin{cases} \left( \partial_t + u\partial_x + v\partial_y - \partial_y^2 \right) \displaystyle\int_0^y \mathcal{U} \mathrm{d}\tilde{y} + \int_0^y \mathcal{U} \mathrm{d}\tilde{y} = -\partial_x^3 v, \\ \mathcal{U}|_{t=0} = 0, \quad \partial_y \mathcal{U}|_{y=0} = \mathcal{U}|_{y\to+\infty} = 0. \end{cases} \tag{2.7}$$

The existence of $\mathcal{U}$ follows from the standard parabolic theory. In fact, one can first apply the existence theory for linear parabolic equations to construct a solution $f$ to the initial-boundary problem

$$\begin{cases} \big(\partial_t + u\partial_x + v\partial_y - \partial_y^2\big) f + f = -\partial_x^3 v, \\ f|_{t=0} = 0, \quad f|_{y=0} = \partial_y f|_{y\to+\infty} = 0, \end{cases} \tag{2.8}$$

and then set $f = \int_0^y \mathcal{U}(t,x,\tilde{y})\mathrm{d}\tilde{y}$. Moreover, under condition (2.2), we use the parabolic regularity theory to conclude that, for $\ell > 1/2$ and for any $m \geq 0$,

$$\begin{cases} \langle y\rangle^{-\ell}\,\partial_x^m \int_0^y \mathcal{U}(t,x,\tilde{y})\mathrm{d}\tilde{y} = \langle y\rangle^{-\ell}\,\partial_x^m f \in L^2\left([0,+\infty[;L^2\right), \\ \partial_x^m \mathcal{U} = \partial_y \partial_x^m f \in L^2\left([0,+\infty[;L_x^2 H_y^1\right). \end{cases} \tag{2.9}$$

The above auxiliary functions are slightly different from those introduced in [16] which were inspired by [5]. As in [16]. by $\mathcal{U}$ and

$$\lambda = \partial_x^3 u - (\partial_y u)\int_0^y \mathcal{U}(t,x,\tilde{y})\mathrm{d}\tilde{y}, \tag{2.10}$$

we can cancel the term involving $v$ with the highest tangential derivative as shown in Subsection 2.2. The two auxiliary functions have the relation

$$\big(\partial_t + u\partial_x + v\partial_y - \partial_y^2\big)\mathcal{U} + \mathcal{U} = \partial_x\lambda + (\partial_x\partial_y u)\int_0^y \mathcal{U}(t,x,\tilde{y})\mathrm{d}\tilde{y} + (\partial_x u)\mathcal{U}. \tag{2.11}$$

In addition,

$$\big(\partial_t + u\partial_x + v\partial_y - \partial_y^2\big)\partial_x\lambda + \partial_x\lambda = -4(\partial_x u)\partial_x^4 u - 3(\partial_x^3 v) + \text{ l.o.t..} \tag{2.12}$$

Recall that the main structure of (2.4) or (2.5) is that the loss of the one order $x$-derivative occurs in the twice-time differentiation. If $\mathcal{U}$ behaves like the 3-order $x$-derivative of $u$, then (2.11)–(2.12) admit a similar structure as shown in toy model (2.5). More precisely, we have the loss of one order $x$-derivative in the twice application of the Prandtl operator instead of a time differentiation so that the pair $(\mathcal{U}, \partial_x\lambda)$ plays a similar role as $(h,g)$ in (2.5). Inspired by the triple norm defined in (2.6), we define the norms on the solutions and the auxiliary functions as follows:

**Definition 2.2** Let $\mathcal{U}, \lambda$ be defined as in (2.7) and (2.10), respectively. By denoting

$$\vec{a} = (u, \mathcal{U}, \lambda),$$

we define $|\vec{a}|_{X_\rho}, |\vec{a}|_{Y_\rho}$ and $|\vec{a}|_{Z_\rho}$ by

$$|\vec{a}|_{X_\rho}^2 = \|u\|_{X_\rho}^2 + \sum_{m=0}^{+\infty} L_{\rho,m+2}^2 \|\partial_x^m \lambda\|_{L^2}^2 + \sum_{m=0}^{+\infty} L_{\rho,m+3}^2 \|\partial_x^m \mathcal{U}\|_{L^2}^2,$$

$$\begin{aligned} |\vec{a}|_{Y_\rho}^2 = {} & \sum_{0\leq j\leq 3}\sum_{m=0}^{+\infty} \frac{m+j+1}{\rho} L_{\rho,m+j}^2 \|\tau^{\ell+j}\partial_x^m \partial_y^j u\|_{L^2}^2 + \sum_{m=0}^{+\infty} \frac{m+3}{\rho} L_{\rho,m+2}^2 \|\partial_x^m \lambda\|_{L^2}^2 \\ & + \sum_{m=0}^{+\infty} \frac{(m+4)^3}{\rho^3} L_{\rho,m+3}^2 \|\partial_x^m \mathcal{U}\|_{L^2}^2, \end{aligned}$$

$$|\vec{a}|_{Z_\rho}^2 = \|\partial_y u\|_{X_\rho}^2 + \sum_{m=0}^{+\infty} L_{\rho,m+2}^2 \|\partial_y \partial_x^m \lambda\|_{L^2}^2 + \sum_{m=0}^{+\infty} L_{\rho,m+3}^2 \|\partial_y \partial_x^m \mathcal{U}\|_{L^2}^2,$$

where $\|\cdot\|_{X_\rho}$ and $L_{\rho,k}$ are given as in Definition 1.1.

**Remark 2.3** The norms defined above satisfy that

$$\|u\|_{X_\rho} \le |\vec{a}|_{X_\rho} \quad \text{and} \quad \|\partial_y u\|_{X_\rho} \le |\vec{a}|_{Z_\rho}.$$

If $\rho \le 1$, then

$$|\vec{a}|_{X_\rho} \le |\vec{a}|_{Y_\rho}.$$

In view of the above remark, the a priori estimate given in Theorem 2.1 holds from the following theorem:

**Theorem 2.4** Under the same assumption as in Theorem 2.1 we have that

$$\sup_{t\ge 0} e^{t/4}|\vec{a}(t)|_{X_{\rho(t)}} + \Big(\int_0^{+\infty} e^{t/2}|\vec{a}(t)|^2_{Z_{\rho(t)}}\mathrm{d}t\Big)^{1/2} \le \frac{4(\rho_0+1)}{\rho_0}\varepsilon_0.$$

The proof of this theorem is given in Subsections 2.2-2.4. To simplify the notations, we will use $C$ to denote a generic constant which may vary from line to line and depend only on the Sobolev embedding constants and the constants $\ell$ and $\rho_0$ in Definition 1.1 and (1.7). Observe that $X_{r_1} \subset X_{r_2}$ for $r_1 \ge r_2$. Then we can assume, without loss of generality, that the initial radius $\rho_0 \le 1$. We now list some facts that follow directly from the definition (1.7) of $\rho$. For $t \ge 0$,

$$\rho_0/2 \le \rho(t) \le \rho_0 \le 1, \quad \rho'(t) \le \rho'^3 < 0, \quad \rho''(t) - \frac{\rho'(t)^2}{\rho(t)} = \frac{\rho_0 e^{-t/12}}{288(1+e^{-t/12})} \ge 0. \tag{2.13}$$

Recall that $L_{\rho,m}$ is defined in (1.6). Then

$$\forall\, m \ge 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}L_{\rho,m} = \rho'\frac{m+1}{\rho}L_{\rho,m}. \tag{2.14}$$

We will use the following Young's inequality for discrete convolution:

$$\Big[\sum_{m=0}^{\infty}\Big(\sum_{j=0}^{m} p_j q_{m-j}\Big)^2\Big]^{1/2} \le \Big(\sum_{m=0}^{\infty} q_m^2\Big)^{1/2}\sum_{j=0}^{\infty} p_j. \tag{2.15}$$

Here $\{p_j\}_{j\ge 0}$ and $\{q_j\}_{j\ge 0}$ are positive sequences. As an immediate consequence of (2.15),

$$\sum_{m=0}^{\infty}\sum_{j=0}^{m} p_j q_{m-j} r_m \le \Big(\sum_{m=0}^{\infty} q_m^2\Big)^{1/2}\Big(\sum_{m=0}^{\infty} r_m^2\Big)^{1/2}\sum_{j=0}^{\infty} p_j \tag{2.16}$$

holds for any positive sequences $\{p_j\}_{j\ge 0}$, $\{q_j\}_{j\ge 0}$ and $\{r_j\}_{j\ge 0}$.

## 2.2 Estimate on the Auxiliary Functions

The following proposition is the main part of the proof for Theorem: 2.4.

**Proposition 2.5** Under the same assumption as that in Theorem 2.1, we have that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{+\infty}\Big(L^2_{\rho,m+2}\|\partial_x^m\lambda\|^2_{L^2} + L^2_{\rho,m+3}\|\partial_x^m\mathcal{U}\|^2_{L^2}\Big)\\ &+\sum_{m=0}^{+\infty}\Big(L^2_{\rho,m+2}\|\partial_x^m\partial_y\lambda\|^2_{L^2} + L^2_{\rho,m+3}\|\partial_x^m\partial_y\mathcal{U}\|^2_{L^2}\Big)\\ &+\frac{1}{2}\sum_{m=0}^{+\infty}\Big(L^2_{\rho,m+2}\|\partial_x^m\lambda\|^2_{L^2} + L^2_{\rho,m+3}\|\partial_x^m\mathcal{U}\|^2_{L^2}\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{+\infty}\rho'^2\frac{(m+4)^2}{\rho^2}L^2_{\rho,m+3}\|\partial_x^m\mathcal{U}\|^2_{L^2}+\frac{1}{2}\sum_{m=0}^{+\infty}\rho'^2\frac{(m+4)^2}{\rho^2}L^2_{\rho,m+3}\|\partial_x^m\mathcal{U}\|^2_{L^2}\\
\leq{}& \frac{1}{4}\rho'^3\sum_{m=0}^{+\infty}\left(\frac{m+3}{\rho}L^2_{\rho,m+2}\|\partial_x^m\lambda\|^2_{L^2}+\frac{(m+4)^3}{\rho^3}L^2_{\rho,m+3}\|\partial_x^m\mathcal{U}\|^2_{L^2}\right)\\
&+C\left(|\vec{a}|_{X_\rho}+|\vec{a}|^2_{X_\rho}\right)\left(|\vec{a}|^2_{Y_\rho}+|\vec{a}|^2_{Z_\rho}\right).
\end{aligned}
$$

First of all, the auxiliary function $\lambda$ satisfies the following equation:

$$
\begin{aligned}
&\big(\partial_t+u\partial_x+v\partial_y-\partial_y^2\big)\lambda+\lambda\\
={}&-4(\partial_x u)\partial_x^3 u-3(\partial_x^2 v)\partial_x\partial_y u-3(\partial_x^2 u)^2-3(\partial_x v)\partial_x^2\partial_y u+2(\partial_y^2 u)\mathcal{U}+(\partial_y u)\int_0^y\mathcal{U}\mathrm{d}\tilde{y}.
\end{aligned}
$$

That is,

$$
\big(\partial_t+u\partial_x+v\partial_y-\partial_y^2\big)\lambda+\lambda=H, \tag{2.17}
$$

with

$$
\begin{aligned}
H={}&-4(\partial_x u)\lambda+\Big(\partial_y u-4(\partial_x u)\partial_y u\Big)\int_0^y\mathcal{U}\mathrm{d}\tilde{y}+3(\partial_x\partial_y u)\int_0^y\lambda(t,x,\tilde{y})\mathrm{d}\tilde{y}\\
&+3(\partial_x\partial_y u)\int_0^y\left((\partial_y u)(t,x,\tilde{y})\int_0^{\tilde{y}}\mathcal{U}(t,x,r)\mathrm{d}r\right)\mathrm{d}\tilde{y}\\
&-3(\partial_x^2 u)^2-3(\partial_x v)\partial_x^2\partial_y u+2(\partial_y^2 u)\mathcal{U}.
\end{aligned}\tag{2.18}
$$

By the definition (2.10) of $\lambda$ and the fact that $\lambda|_{y=0}=\lambda|_{y\to+\infty}=0$, and the assumptions (2.2) and (2.9), we have, for all $m\geq 0$, that

$$
\partial_x^m\lambda\in L^2\left([0,+\infty[;\ L^2\Big(\mathbb{R}_x;H_0^1(\mathbb{R}_+)\Big)\right)\ \text{ and }\ \partial_x^m\partial_y^2\lambda\in L^2\left([0,+\infty[;L^2\Big(\mathbb{R}_x;H^{-1}(\mathbb{R}_+)\Big)\right),
$$

and

$$
\partial_x^m H,\quad \partial_x^m(u\partial_x\lambda+v\partial_y\lambda)\in L^2\left([0,+\infty[;L^2\right).
$$

This implies that

$$
\partial_t\partial_x^m\lambda\in L^2\left([0,+\infty[;L^2\Big(\mathbb{R}_x;H^{-1}(\mathbb{R}_+)\Big)\right).
$$

Thus,

$$
t\mapsto\|\partial_x^m\lambda(t)\|^2_{L^2}
$$

is absolutely continuous on $[0,+\infty[$, and, moreover,

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^m\lambda\|^2_{L^2}=\langle\partial_t\partial_x^m\lambda,\ \partial_x^m\lambda\rangle,
$$

where $\langle\cdot,\ \cdot\rangle$ stands for the pairing between $L^2\Big(\mathbb{R}_x;H^{-1}(\mathbb{R}_+)\Big)$ and $L^2\Big(\mathbb{R}_x;H_0^1(\mathbb{R}_+)\Big)$. Hence, by (2.14),

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}L^2_{\rho,m+2}\|\partial_x^m\lambda\|^2_{L^2}=\rho'\frac{m+3}{\rho}L^2_{\rho,m+2}\|\partial_x^m\lambda\|^2_{L^2}+L^2_{\rho,m+2}\langle\partial_t\partial_x^m\lambda,\ \partial_x^m\lambda\rangle.
$$

Consequently, we apply $\partial_x^m$ to (2.17) and then take the pairing with $\partial_x^m\lambda$ between

$$
L^2\Big(\mathbb{R}_x;H^{-1}(\mathbb{R}_+)\Big)\quad\text{and}\quad L^2\Big(\mathbb{R}_x;H_0^1(\mathbb{R}_+)\Big),
$$

together with the factor $L^2_{\rho,m+2}$, to get that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{+\infty}L^2_{\rho,m+2}\|\partial_x^m\lambda\|^2_{L^2}+\sum_{m=0}^{+\infty}L^2_{\rho,m+2}\|\partial_x^m\partial_y\lambda\|^2_{L^2}+\sum_{m=0}^{+\infty}L^2_{\rho,m+2}\|\partial_x^m\lambda\|^2_{L^2}
$$

$$= \rho' \sum_{m=0}^{+\infty} \frac{m+3}{\rho} L_{\rho,m+2}^2 \|\partial_x^m \lambda\|_{L^2}^2 + \sum_{m=0}^{+\infty} L_{\rho,m+2}^2 \left(\partial_x^m H,\, \partial_x^m \lambda\right)_{L^2}, \tag{2.19}$$

where $H$ is given in (2.18).

The next lemmas are for the estimate on the last term on the right side of (2.19).

**Lemma 2.6** Let $|\vec{a}|_{X_\rho}, |\vec{a}|_{Y_\rho}$ and $|\vec{a}|_{Z_\rho}$ be given as in Definition 2.2. Under the same assumption as in Theorem 2.1, we have that

$$\sum_{m=0}^{+\infty} L_{\rho,m+2}^2 \Big(\partial_x^m \Big[3(\partial_x\partial_y u)\int_0^y \Big((\partial_y u)(t,x,\tilde{y})\int_0^{\tilde{y}} \mathcal{U}(t,x,r)\mathrm{d}r\Big)\mathrm{d}\tilde{y}\Big],\, \partial_x^m\lambda\Big)_{L^2} \le C|\vec{a}|_{X_\rho}^2 |\vec{a}|_{Y_\rho}^2. \tag{2.20}$$

Similarly, it holds that

$$\begin{aligned}&\sum_{m=0}^{+\infty} L_{\rho,m+2}^2 \Big(\partial_x^m \Big[2(\partial_y^2 u)\mathcal{U} + \Big(\partial_y u - 4(\partial_x u)\partial_y u\Big)\int_0^y \mathcal{U}\mathrm{d}\tilde{y}\Big],\, \partial_x^m\lambda\Big)_{L^2}\\ &\le C|\vec{a}|_{X_\rho}|\vec{a}|_{Z_\rho}^2 + C\big(|\vec{a}|_{X_\rho} + |\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{Y_\rho}^2.\end{aligned}$$

**Proof** It suffices to prove the first estimate because the second one can be obtained similarly.

**Step 1** For simplicity of notation, set

$$\mathscr{L}(u,\mathcal{U}) := \int_0^y \Big((\partial_y u)(t,x,\tilde{y})\int_0^{\tilde{y}} \mathcal{U}(t,x,r)\mathrm{d}r\Big)\mathrm{d}\tilde{y}. \tag{2.21}$$

We first prove the following two estimates for $\mathscr{L}(u,\mathcal{U})$:

$$\Big(\sum_{m=0}^{+\infty} \frac{(m+4)^3}{\rho^3} L_{\rho,m+3}^2 \|\partial_x^m \mathscr{L}(u,\mathcal{U})\|_{L_x^2 L_y^\infty}^2\Big)^{1/2} \le C|\vec{a}|_{X_\rho}|\vec{a}|_{Y_\rho} \tag{2.22}$$

and

$$\Big(\sum_{m=0}^{+\infty} L_{\rho,m+5}^2 \|\partial_x^m \mathscr{L}(u,\mathcal{U})\|_{L^\infty}^2\Big)^{1/2} \le C|\vec{a}|_{X_\rho}^2. \tag{2.23}$$

In fact, Leibniz's formula gives that

$$\frac{(m+4)^{3/2}}{\rho^{3/2}} L_{\rho,m+3} \|\partial_x^m \mathscr{L}(u,\mathcal{U})\|_{L_x^2 L_y^\infty} \le a_m + b_m, \tag{2.24}$$

with

$$\begin{aligned} a_m &= C\sum_{j=0}^{[m/2]} \frac{m!}{j!(m-j)!}\frac{(m+4)^{3/2}}{\rho^{3/2}}\frac{L_{\rho,m+3}}{L_{\rho,j+3}L_{\rho,m-j+3}}\\ &\quad\times\big(L_{\rho,j+3}\|\langle y\rangle^{\ell+1}\partial_x^j\partial_y u\|_{L_x^\infty L_y^2}\big)\big(L_{\rho,m-j+3}\|\partial_x^{m-j}\mathcal{U}\|_{L^2}\big)\\ b_m &= C\sum_{j=[m/2]+1}^{m} \frac{m!}{j!(m-j)!}\frac{(m+4)^{3/2}}{\rho^{3/2}}\frac{L_{\rho,m+3}}{L_{\rho,j+1}L_{\rho,m-j+5}}\\ &\quad\times\big(L_{\rho,j+1}\|\langle y\rangle^{\ell+1}\partial_x^j\partial_y u\|_{L^2}\big)\big(L_{\rho,m-j+5}\|\partial_x^{m-j}\mathcal{U}\|_{L_x^\infty L_y^2}\big),\end{aligned}$$

where $[m/2]$ represents the largest integer less than or equal to $m/2$. Direct computation shows that, for any $0 \le j \le [m/2]$,

$$\frac{m!}{j!(m-j)!}\frac{(m+4)^{3/2}}{\rho^{3/2}}\frac{L_{\rho,m+3}}{L_{\rho,j+3}L_{\rho,m-j+3}} \le \frac{C}{\rho^4}\frac{1}{j+1}\frac{(m-j+4)^{3/2}}{\rho^{3/2}} \le \frac{C}{j+1}\frac{(m-j+4)^{3/2}}{\rho^{3/2}},$$

where we have used the fact that $\rho_0/2 \le \rho \le \rho_0$ in the last inequality. Then

$$\begin{aligned}\sum_{m=0}^{+\infty} a_m^2 &\le C\sum_{m=0}^{+\infty}\Big[\sum_{j=0}^{[m/2]}\frac{L_{\rho,j+3}\|\langle y\rangle^{\ell+1}\partial_x^j\partial_y u\|_{L_x^\infty L_y^2}}{j+1}\frac{(m-j+4)^{3/2}}{\rho^{3/2}}L_{\rho,m-j+3}\|\partial_x^{m-j}\mathcal{U}\|_{L^2}\Big]^2\\ &\le C\Big(\sum_{j=0}^{+\infty}\frac{L_{\rho,j+3}\|\langle y\rangle^{\ell+1}\partial_x^j\partial_y u\|_{L_x^\infty L_y^2}}{j+1}\Big)^2\sum_{j=0}^{+\infty}\frac{(j+4)^3}{\rho^3}L_{\rho,j+3}^2\|\partial_x^j\mathcal{U}\|_{L^2}^2\\ &\le C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Y_\rho}^2,\end{aligned}$$

where we have used Young's inequality (2.15) for discrete convolution. Similarly, by using the estimate

$$\forall\ [m/2]\le j\le m,\quad \frac{m!}{j!(m-j)!}\frac{(m+4)^{3/2}}{\rho^{3/2}}\frac{L_{\rho,m+3}}{L_{\rho,j+1}L_{\rho,m-j+5}}\le\frac{C}{m-j+1},$$

we obtain, by observing $|\vec{a}|_{X_\rho}\le|\vec{a}|_{Y_\rho}$, that

$$\begin{aligned}\sum_{m=0}^{+\infty} b_m^2 &\le C\Big(\sum_{j=0}^{+\infty}\frac{L_{\rho,j+5}\|\partial_x^j\mathcal{U}\|_{L_x^\infty L_y^2}}{j+1}\Big)^2\sum_{j=0}^{+\infty}L_{\rho,j+1}^2\|\langle y\rangle^{\ell+1}\partial_x^j\partial_y u\|_{L^2}^2\\ &\le C|\vec{a}|_{X_\rho}^4\le C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Y_\rho}^2.\end{aligned}$$

Combining the above estimates with (2.24) yields (2.22).

**Step 2** To prove (2.20), Leibniz's formula gives that

$$\sum_{m=0}^{+\infty}L_{\rho,m+2}^2\Big(\partial_x^m\big[3(\partial_x\partial_y u)\mathscr{L}(u,\mathcal{U})\big],\ \partial_x^m\lambda\Big)_{L^2}\le I_1+I_2,\tag{2.25}$$

where

$$\begin{aligned}I_1&=3\sum_{m=0}^{+\infty}\sum_{j=0}^{[m/2]}\binom{m}{j}\frac{L_{\rho,m+2}}{L_{\rho,j+4}L_{\rho,m-j+3}}\frac{\rho^2}{(m-j+4)^{\frac32}(m+1)^{\frac12}}L_{\rho,j+4}\|\partial_x^{j+1}\partial_y u\|_{L_x^\infty L_y^2}\\ &\quad\times\Big(\frac{(m-j+4)^{3/2}}{\rho^{3/2}}L_{\rho,m-j+3}\|\partial_x^{m-j}\mathscr{L}(u,\mathcal{U})\|_{L_x^2L_y^\infty}\Big)\Big(\frac{(m+1)^{1/2}}{\rho^{1/2}}L_{\rho,m+2}\|\partial_x^m\lambda\|_{L^2}\Big),\\ I_2&=3\sum_{m=0}^{+\infty}\sum_{j=[m/2]+1}^{m}\binom{m}{j}\frac{L_{\rho,m+2}}{L_{\rho,j+2}L_{\rho,m-j+5}}L_{\rho,j+2}\|\partial_x^{j+1}\partial_y u\|_{L^2}\\ &\quad\times\big(L_{\rho,m-j+5}\|\partial_x^{m-j}\mathscr{L}(u,\mathcal{U})\|_{L^\infty}\big)\big(L_{\rho,m+2}\|\partial_x^m\lambda\|_{L^2}\big).\end{aligned}$$

Straightforward calculation shows that

$$\forall\ 0\le j\le[m/2],\quad\frac{m!}{j!(m-j)!}\frac{L_{\rho,m+2}}{L_{\rho,j+4}L_{\rho,m-j+3}}\frac{\rho^2}{(m-j+4)^{3/2}(m+1)^{1/2}}\le\frac{C}{j+1}.$$

Thus,

$$\begin{aligned}I_1&\le C\sum_{m=0}^{+\infty}\sum_{j=0}^{[m/2]}\frac{L_{\rho,j+4}\|\partial_x^{j+1}\partial_y u\|_{L_x^\infty L_y^2}}{j+1}\\ &\quad\times\Big(\frac{(m-j+4)^{3/2}}{\rho^{3/2}}L_{\rho,m-j+3}\|\partial_x^{m-j}\mathscr{L}(u,\mathcal{U})\|_{L_x^2L_y^\infty}\Big)\Big(\frac{(m+1)^{1/2}}{\rho^{1/2}}L_{\rho,m+2}\|\partial_x^m\lambda\|_{L^2}\Big)\\ &\le C\sum_{j=0}^{+\infty}\frac{L_{\rho,j+4}\|\partial_x^{j+1}\partial_y u\|_{L_x^\infty L_y^2}}{j+1}\Big(\sum_{m=0}^{+\infty}\frac{(m+4)^3}{\rho^3}L_{\rho,m+3}^2\|\partial_x^m\mathscr{L}(u,\mathcal{U})\|_{L_x^2L_y^\infty}^2\Big)^{\frac12}|\vec{a}|_{Y_\rho}\end{aligned}$$

$$\le C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Y_\rho}^2,$$

where we have used (2.16) in the second inequality and (2.22) in the last inequality.

For $I_2$, by (2.23) and the estimate

$$\forall\ [m/2]+1\le j\le m,\quad \frac{m!}{j!(m-j)!}\frac{L_{\rho,m+2}}{L_{\rho,j+2}L_{\rho,m-j+5}}\le\frac{C}{m-j+1},$$

we have that

$$\begin{aligned}I_2&\le C\Big(\sum_{j=0}^{+\infty}L_{\rho,j+2}^2\|\partial_x^{j+1}\partial_yu\|_{L^2}^2\Big)^{1/2}\Big(\sum_{m=0}^{+\infty}\frac{L_{\rho,m+5}\|\partial_x^m\mathscr{L}(u,\mathcal{U})\|_{L^\infty}}{m+1}\Big)|\vec{a}|_{X_\rho}\\&\le C|\vec{a}|_{X_\rho}^4\le C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Y_\rho}^2.\end{aligned}$$

Substituting the above estimates on $I_1$ and $I_2$ into (2.25) yields the first estimate of (2.20) in Lemma 2.6.

**Step 3** It remains to prove the second estimate in Lemma 2.6. For this, write

$$\begin{aligned}&\sum_{m=0}^{+\infty}L_{\rho,m+2}^2\Big(\partial_x^m\big[2(\partial_y^2u)\mathcal{U}\big],\ \partial_x^m\lambda\Big)_{L^2}\\&\le 2\sum_{m=0}^{+\infty}\sum_{j=0}^{[m/2]}\frac{m!}{j!(m-j)!}\frac{L_{\rho,m+2}}{L_{\rho,j+5}L_{\rho,m-j+3}}\frac{\rho^2}{(m-j+4)^{3/2}(m+1)^{1/2}}(L_{\rho,j+5}\|\partial_x^j\partial_y^2u\|_{L^\infty})\\&\quad\times\Big(\frac{(m-j+4)^{3/2}}{\rho^{3/2}}L_{\rho,m-j+3}\|\partial_x^{m-j}\mathcal{U}\|_{L^2}\Big)\Big(\frac{(m+1)^{1/2}}{\rho^{1/2}}L_{\rho,m+2}\|\partial_x^m\lambda\|_{L^2}\Big)\\&\quad+2\sum_{m=0}^{+\infty}\sum_{j=[m/2]+1}^{m}\frac{m!}{j!(m-j)!}\frac{L_{\rho,m+2}}{L_{\rho,j+2}L_{\rho,m-j+5}}(L_{\rho,j+2}\|\left\langle y\right\rangle^{1/2}\partial_x^j\partial_y^2u\|_{L^2})\\&\quad\times\big(L_{\rho,m-j+5}\|\partial_x^{m-j}\mathcal{U}\|_{L^\infty}\big)\big(L_{\rho,m+2}\|\partial_x^m\lambda\|_{L^2}\big).\end{aligned}\tag{2.26}$$

Similarly to the previous step, we conclude that the first term on the right hand side of (2.26) is bounded from above by $C|\vec{a}|_{X_\rho}|\vec{a}|_{Y_\rho}^2$, and the last term is bounded from above by $C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Z_\rho}$. Thus,

$$\begin{aligned}\sum_{m=0}^{+\infty}L_{\rho,m+2}^2\Big(\partial_x^m\big[2(\partial_y^2u)\mathcal{U}\big],\partial_x^m\lambda\Big)_{L^2}&\le C|\vec{a}|_{X_\rho}|\vec{a}|_{Y_\rho}^2+C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Z_\rho}\\&\le C|\vec{a}|_{X_\rho}|\vec{a}|_{Y_\rho}^2+C|\vec{a}|_{X_\rho}|\vec{a}|_{Z_\rho}^2.\end{aligned}$$

Similarly,

$$\sum_{m=0}^{+\infty}L_{\rho,m+2}^2\Big(\partial_x^m\Big[\Big(\partial_yu-4(\partial_xu)\partial_yu\Big)\int_0^y\mathcal{U}\mathrm{d}\tilde{y}\Big],\ \partial_x^m\lambda\Big)_{L^2}\le C\big(|\vec{a}|_{X_\rho}+|\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{Y_\rho}^2.$$

Combining the above two estimates gives the second estimate in Lemma 2.6. This completes the proof of the lemma. □

**Lemma 2.7** Let $|\vec{a}|_{X_\rho},|\vec{a}|_{Y_\rho}$ and $|\vec{a}|_{Z_\rho}$ be given as in Definition 2.2. Under the same assumption as in Theorem 2.1, we have that

$$\sum_{m=0}^{+\infty}L_{\rho,m+2}^2\Big(\partial_x^m\Big[3(\partial_x\partial_yu)\int_0^y\lambda(t,x,\tilde{y})\mathrm{d}\tilde{y}\Big],\ \partial_x^m\lambda\Big)_{L^2}\le C|\vec{a}|_{X_\rho}|\vec{a}|_{Y_\rho}^2,$$

and

$$\sum_{m=0}^{+\infty} L_{\rho,m+2}^2 \Big(\partial_x^m \big[-4(\partial_x u)\lambda - 3(\partial_x^2 u)^2 - 3(\partial_x v)\partial_x^2\partial_y u\big], \partial_x^m \lambda\Big)_{L^2}$$
$$\le C|\vec{a}|_{X_\rho}|\vec{a}|_{Y_\rho}^2 + C|\vec{a}|_{X_\rho}|\vec{a}|_{Z_\rho}^2.$$

**Proof** By Leibniz's formula, we have that

$$\begin{aligned}
&\sum_{m=0}^{+\infty} L_{\rho,m+2}^2 \Big(\partial_x^m \Big[(\partial_x\partial_y u)\int_0^y \lambda(t,x,\tilde{y})\mathrm{d}\tilde{y}\Big], \partial_x^m \lambda\Big)_{L^2} \\
\le &\sum_{m=0}^{+\infty}\sum_{j=0}^{[m/2]} \binom{m}{j} L_{\rho,m+2}^2 \| \langle y\rangle^\ell \partial_x^{j+1}\partial_y u\|_{L_x^\infty L_y^2} \|\partial_x^{m-j}\lambda\|_{L^2}\|\partial_x^m\lambda\|_{L^2} \\
&+\sum_{m=0}^{+\infty}\sum_{j=[m/2]+1}^{m} \binom{m}{j} L_{\rho,m+2}^2 \| \langle y\rangle^\ell \partial_x^{j+1}\partial_y u\|_{L^2} \|\partial_x^{m-j}\lambda\|_{L_x^\infty L_y^2}\|\partial_x^m\lambda\|_{L^2},
\end{aligned} \tag{2.27}$$

where we have used the fact that

$$\| \langle y\rangle^{-\ell} \int_0^y \partial_x^{m-j}\lambda \mathrm{d}\tilde{y}\|_{L_x^2 L_y^\infty} \le C\|\partial_x^{m-j}\lambda\|_{L^2}$$

for $\ell > 1/2$ with $\langle y\rangle = (1+y^2)^{1/2}$. By

$$\forall\ 0 \le j \le [m/2], \quad \frac{m!}{j!(m-j)!}\frac{L_{\rho,m+2}}{L_{\rho,j+4}L_{m-j+2}} \le \frac{C}{j+1},$$

we have that

$$\begin{aligned}
&\sum_{m=0}^{+\infty}\sum_{j=0}^{[m/2]} \binom{m}{j} L_{\rho,m+2}^2 \| \langle y\rangle^\ell \partial_x^{j+1}\partial_y u\|_{L_x^\infty L_y^2} \|\partial_x^{m-j}\lambda\|_{L^2}\|\partial_x^m\lambda\|_{L^2} \\
\le\ & C\sum_{m=0}^{+\infty}\sum_{j=0}^{[m/2]} \frac{L_{\rho,j+4}\| \langle y\rangle^\ell \partial_x^{j+1}\partial_y u\|_{L_x^\infty L_y^2}}{j+1} (L_{\rho,m-j+2}\|\partial_x^{m-j}\lambda\|_{L^2})(L_{\rho,m+2}\|\partial_x^m\lambda\|_{L^2}) \\
\le\ & C\sum_{j=0}^{+\infty} \frac{L_{\rho,j+4}\| \langle y\rangle^\ell \partial_x^{j+1}\partial_y u\|_{L_x^\infty L_y^2}}{j+1} \sum_{m=0}^{+\infty} L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2 \le C|\vec{a}|_{X_\rho}^3,
\end{aligned} \tag{2.28}$$

where (2.16) is used in the second inequality and

$$\sum_{j=0}^{+\infty} \frac{L_{\rho,j+4}\| \langle y\rangle^\ell \partial_x^{j+1}\partial_y u\|_{L_x^\infty L_y^2}}{j+1} \le C\Big(\sum_{j=0}^{+\infty} L_{\rho,j+4}^2 \| \langle y\rangle^\ell \partial_x^{j+1}\partial_y u\|_{L_x^\infty L_y^2}^2\Big)^{1/2} \le C|\vec{a}|_{X_\rho}$$

is used in the last inequality. Similarly, by noting that

$$\forall\ [m/2]+1 \le j \le m, \quad \frac{m!}{j!(m-j)!}\frac{L_{\rho,m+2}}{L_{\rho,j+2}L_{m-j+4}} \le \frac{C}{m-j+1},$$

we have that

$$\begin{aligned}
&\sum_{m=0}^{+\infty}\sum_{j=[m/2]+1}^{m} \binom{m}{j} L_{\rho,m+2}^2 \| \langle y\rangle^\ell \partial_x^{j+1}\partial_y u\|_{L^2} \|\partial_x^{m-j}\lambda\|_{L_x^\infty L_y^2}\|\partial_x^m\lambda\|_{L^2} \\
\le\ & C\Big(\sum_{j=0}^{+\infty} L_{\rho,j+2}^2 \| \langle y\rangle^\ell \partial_x^{j+1}\partial_y u\|_{L^2}^2\Big)^{\frac12} \Big(\sum_{m=0}^{+\infty} L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2\Big)^{\frac12} \sum_{m=0}^{+\infty}\frac{L_{\rho,m+4}\|\partial_x^m\lambda\|_{L_x^\infty L_y^2}}{m+1} \\
\le\ & C|\vec{a}|_{X_\rho}^3.
\end{aligned}$$

By substituting the above inequality and (2.28) into (2.27) and observing that $|\vec{a}|_{X_\rho} \le |\vec{a}|_{Y_\rho}$, we obtain the first estimate in Lemma 2.7.

A similar argument as that above gives that

$$\sum_{m=0}^{+\infty} L_{\rho,m+2}^2 \Big(\partial_x^m \big[-4(\partial_x u)\lambda - 3(\partial_x^2 u)^2\big],\, \partial_x^m \lambda\Big)_{L^2} \le C|\vec{a}|_{X_\rho}^3,$$

and

$$\begin{aligned}
&\sum_{m=0}^{+\infty} L_{\rho,m+2}^2 \Big(\partial_x^m \big[-3(\partial_x v)\partial_x^2\partial_y u\big],\, \partial_x^m \lambda\Big)_{L^2} \\
&\le C\sum_{m=0}^{+\infty}\sum_{j=0}^{[m/2]} \binom{m}{j} L_{\rho,m+2}^2 \| \langle y\rangle^\ell \partial_x^{j+2} u\|_{L_x^\infty L_y^2} \|\partial_x^{m-j+2}\partial_y u\|_{L^2} \|\partial_x^m \lambda\|_{L^2} \\
&\quad + C\sum_{m=0}^{+\infty}\sum_{j=[m/2]+1}^{m} \binom{m}{j} L_{\rho,m+2}^2 \| \langle y\rangle^\ell \partial_x^{j+2} u\|_{L^2} \|\partial_x^{m-j+2}\partial_y u\|_{L_x^\infty L_y^2} \|\partial_x^m \lambda\|_{L^2} \\
&\le C|\vec{a}|_{X_\rho}^2 |\vec{a}|_{Z_\rho} \le C|\vec{a}|_{X_\rho}^3 + C|\vec{a}|_{X_\rho}|\vec{a}|_{Z_\rho}^2,
\end{aligned}$$

where $v = -\int_0^y \partial_x u(t,x,\tilde{y})\mathrm{d}\tilde{y}$ is used. Combining the above estimates and observing that $|\vec{a}|_{X_\rho} \le |\vec{a}|_{Y_\rho}$ gives the second estimate in Lemma 2.7. Hence, the proof of the lemma is complete. □

By the definition of $H$ given in (2.18), the estimates in Lemmas 2.6 and 2.7 yield that

$$\sum_{m=0}^{+\infty} L_{\rho,m+2}^2 \left(\partial_x^m H,\, \partial_x^m \lambda\right)_{L^2} \le C|\vec{a}|_{X_\rho}|\vec{a}|_{Z_\rho}^2 + C\big(|\vec{a}|_{X_\rho} + |\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{Y_\rho}^2.$$

Therefore, we have, by the equality (2.19), the following corollary:

**Corollary 2.8** (Estimate for $\lambda$) Under the same assumption as in Theorem 2.1, we have that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{+\infty} L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2 + \sum_{m=0}^{+\infty} L_{\rho,m+2}^2\|\partial_x^m\partial_y\lambda\|_{L^2}^2 + \sum_{m=0}^{+\infty} L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2 \\
&\le \rho'\sum_{m=0}^{+\infty}\frac{m+3}{\rho} L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2 + C|\vec{a}|_{X_\rho}|\vec{a}|_{Z_\rho}^2 + C\big(|\vec{a}|_{X_\rho} + |\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{Y_\rho}^2.
\end{aligned}$$

We now turn to derive the estimate on $\mathcal{U}$.

**Lemma 2.9** (Estimate on $\mathcal{U}$) Under the same assumption as in Theorem 2.1, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{+\infty} L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2 + \sum_{m=0}^{+\infty} L_{\rho,m+3}^2\|\partial_x^m\partial_y\mathcal{U}\|_{L^2}^2 + \frac{1}{2}\sum_{m=0}^{+\infty} L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2 \\
&\le \frac{1}{2}\sum_{m=0}^{+\infty} L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2 + C|\vec{a}|_{X_\rho}|\vec{a}|_{Y_\rho}^2.
\end{aligned}$$

**Proof** By (2.9) and the boundary condition that $\partial_y\mathcal{U}|_{y=0} = \mathcal{U}|_{y=+\infty} = 0$, the energy estimate on (2.11) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{+\infty} L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2 + \sum_{m=0}^{+\infty} L_{\rho,m+3}^2\|\partial_x^m\partial_y\mathcal{U}\|_{L^2}^2 + \sum_{m=0}^{+\infty} L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2$$

$$
\begin{aligned}
&\leq \rho' \sum_{m=0}^{+\infty} \frac{m+4}{\rho} L_{\rho,m+3}^2 \|\partial_x^m \mathcal{U}\|_{L^2}^2 + \sum_{m=0}^{+\infty} L_{\rho,m+3}^2 \|\partial_x^{m+1}\lambda\|_{L^2} \|\partial_x^m \mathcal{U}\|_{L^2} \\
&\quad + \sum_{m=0}^{+\infty} L_{\rho,m+3}^2 \Big(\partial_x^m \Big[(\partial_x\partial_y u)\int_0^y \mathcal{U}\mathrm{d}\tilde{y} + (\partial_x u)\mathcal{U}\Big],\ \partial_x^m \mathcal{U}\Big)_{L^2}.
\end{aligned}
$$

Note that the first term on the right side is non-positive and the second one is bounded from above by

$$
\frac{1}{2}\sum_{m=0}^{+\infty} L_{\rho,m+2}^2 \|\partial_x^m \lambda\|_{L^2}^2 + \frac{1}{2}\sum_{m=0}^{+\infty} L_{\rho,m+3}^2 \|\partial_x^m \mathcal{U}\|_{L^2}^2.
$$

For the last term in the above inequality, a similar argument as that for Lemmas 2.6 and 2.7 yields that

$$
\sum_{m=0}^{+\infty} L_{\rho,m+3}^2 \Big(\partial_x^m \Big[(\partial_x\partial_y u)\int_0^y \mathcal{U}\mathrm{d}\tilde{y} + (\partial_x u)\mathcal{U}\Big],\ \partial_x^m \mathcal{U}\Big)_{L^2} \leq C|\vec{a}|_{X_\rho}^3 \leq C|\vec{a}|_{X_\rho}|\vec{a}|_{Y_\rho}^2.
$$

Combining the above estimates completes the proof of the lemma. □

**Proof of Proposition 2.5** Combining the estimates in Corollary 2.8 and Lemma 2.9 gives that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{+\infty}\Big(L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2 + L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2\Big) \\
&\quad + \sum_{m=0}^{+\infty}\Big(L_{\rho,m+2}^2\|\partial_x^m\partial_y\lambda\|_{L^2}^2 + L_{\rho,m+3}^2\|\partial_x^m\partial_y\mathcal{U}\|_{L^2}^2\Big) \\
&\quad + \frac{1}{2}\sum_{m=0}^{+\infty}\Big(L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2 + L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2\Big) \\
&\leq \rho'\sum_{m=0}^{+\infty}\frac{m+3}{\rho}L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2 + C|\vec{a}|_{X_\rho}|\vec{a}|_{Z_\rho}^2 + C\big(|\vec{a}|_{X_\rho} + |\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{Y_\rho}^2 \\
&\leq \frac{\rho'}{2}\sum_{m=0}^{+\infty}\frac{m+3}{\rho}L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2 + \frac{\rho'^3}{4}\sum_{m=0}^{+\infty}\frac{m+3}{\rho}L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2 \\
&\quad + C\Big(|\vec{a}|_{X_\rho} + |\vec{a}|_{X_\rho}^2\Big)\Big(|\vec{a}|_{Y_\rho}^2 + |\vec{a}|_{Z_\rho}^2\Big),
\end{aligned} \tag{2.29}
$$

where we have used the fact that $\rho'/2 \leq \rho'^3/2 \leq \rho'^3/4$ in the last inequality.

It remains to estimate the first term on the right hand side of (2.29) as follows:

$$
\begin{aligned}
&\frac{1}{2}\rho'\sum_{m=0}^{+\infty}\frac{m+3}{\rho}L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2 + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{+\infty}\rho'^2\frac{(m+4)^2}{\rho^2}L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2 \\
&\quad + \frac{1}{2}\rho'^2\sum_{m=0}^{+\infty}\frac{(m+4)^2}{\rho^2}L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2 \\
&\leq \frac{1}{4}\rho'^3\sum_{m=0}^{+\infty}\frac{(m+4)^3}{\rho^3}L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2 + C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Z_\rho}^2 + C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Y_\rho}^2.
\end{aligned} \tag{2.30}
$$

Then, Proposition 2.5 follows by combining (2.29) and (2.30).

To prove (2.30), denote that

$$
P = \partial_t + u\partial_x + v\partial_y - \partial_y^2 + 1. \tag{2.31}
$$

By applying $\partial_x^m$ to (2.11), we obtain that

$$\partial_x^{m+1}\lambda = P\partial_x^m\mathcal{U} + K_m, \tag{2.32}$$

where

$$K_m = \sum_{j=1}^{m}\binom{m}{j}\left[(\partial_x^j u)\partial_x^{m-j+1}\mathcal{U} + (\partial_x^j v)\partial_x^{m-j}\partial_y\mathcal{U}\right] - \partial_x^m\Big[(\partial_x\partial_y u)\int_0^y \mathcal{U}\mathrm{d}\tilde{y} + (\partial_x u)\mathcal{U}\Big]. \tag{2.33}$$

Then, by observing that

$$\frac{1}{2}\rho'\sum_{m=0}^{+\infty}\frac{m+3}{\rho}L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2 \le \frac{1}{2}\rho'\sum_{m=0}^{+\infty}\frac{m+4}{\rho}L_{\rho,m+3}^2\|\partial_x^{m+1}\lambda\|_{L^2}^2,$$

and

$$\frac{1}{2}\rho'\|\partial_x^{m+1}\lambda\|_{L^2}^2 + \frac{1}{2}\rho'\|K_m\|_{L^2}^2 \le \frac{1}{4}\rho'\|P\partial_x^m\mathcal{U}\|_{L^2}^2,$$

because of (2.32) and the fact that $\rho' < 0$, we have that

$$\begin{aligned}&\frac{1}{2}\rho'\sum_{m=0}^{+\infty}\frac{m+3}{\rho}L_{\rho,m+2}^2\|\partial_x^m\lambda\|_{L^2}^2\\ \le{}& \frac{1}{4}\rho'\sum_{m=0}^{+\infty}\frac{m+4}{\rho}L_{\rho,m+3}^2\|P\partial_x^m\mathcal{U}\|_{L^2}^2 - \frac{1}{2}\rho'\sum_{m=0}^{+\infty}\frac{m+4}{\rho}L_{\rho,m+3}^2\|K_m\|_{L^2}^2.\end{aligned} \tag{2.34}$$

The two terms on the right hand side of the above inequality will be estimated as follows:

**(i) Estimate on the last term on the right hand side of** (2.34). Noting that $-\rho'/\rho \le 1$, by the defintion of $K_m$ in (2.33), we have that

$$-\frac{1}{2}\rho'\sum_{m=0}^{+\infty}\frac{m+4}{\rho}L_{\rho,m+3}^2\|K_m\|_{L^2}^2 \le \sum_{m=0}^{+\infty}(m+4)L_{\rho,m+3}^2\|K_m\|_{L^2}^2 \le S_1+S_2+S_3, \tag{2.35}$$

where

$$S_1 = 4\sum_{m=0}^{+\infty}\Big((m+4)^{1/2}L_{\rho,m+3}\sum_{j=1}^{m}\frac{m!}{j!(m-j)!}\|(\partial_x^j u)\partial_x^{m-j+1}\mathcal{U}\|_{L^2}\Big)^2,$$

$$S_2 = 4\sum_{m=0}^{+\infty}\Big((m+4)^{1/2}L_{\rho,m+3}\sum_{j=1}^{m}\frac{m!}{j!(m-j)!}\|(\partial_x^j v)\partial_x^{m-j}\partial_y\mathcal{U}\|_{L^2}\Big)^2,$$

and

$$S_3 = 4\sum_{m=0}^{+\infty}\Big((m+4)^{1/2}L_{\rho,m+3}\Big\|\partial_x^m\big[(\partial_x\partial_y u)\int_0^y\mathcal{U}\mathrm{d}\tilde{y} + (\partial_x u)\mathcal{U}\big]\Big\|_{L^2}\Big)^2.$$

We first estimate $S_1$. Note that

$$\forall\ 1\le j\le [m/2],\quad (m+4)^{1/2}\frac{m!}{j!(m-j)!}\frac{L_{\rho,m+3}}{L_{\rho,j+3}L_{\rho,m-j+4}} \le \frac{C}{j+1}\frac{(m-j+5)^{3/2}}{\rho^{3/2}},$$

and

$$\forall\ [m/2]+1\le j\le m,\quad (m+4)^{1/2}\frac{m!}{j!(m-j)!}\frac{L_{\rho,m+3}}{L_{\rho,j+1}L_{\rho,m-j+6}} \le \frac{C}{m-j+1}.$$

By (2.15), we obtain that

$$S_1 \le C\sum_{m=0}^{+\infty}\Big(\sum_{j=1}^{[m/2]}\frac{L_{\rho,j+3}\|\partial_x^j u\|_{L^\infty}}{j+1}\frac{(m-j+5)^{3/2}}{\rho^{3/2}}L_{\rho,m-j+4}\|\partial_x^{m-j+1}\mathcal{U}\|_{L^2}\Big)^2$$

$$+C\sum_{m=0}^{+\infty}\bigg(\sum_{j=[m/2]+1}^{m}L_{\rho,j+1}\|\partial_x^j u\|_{L_x^2L_y^\infty}\frac{L_{\rho,m-j+6}\|\partial_x^{m-j+1}\mathcal{U}\|_{L_x^\infty L_y^2}}{m-j+1}\bigg)^2$$
$$\leq C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Y_\rho}^2+C|\vec{a}|_{X_\rho}^4\leq C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Y_\rho}^2.$$

Similarly, we have that

$$S_2\leq\sum_{m=0}^{+\infty}\bigg(\sum_{j=1}^{[m/2]}\frac{L_{\rho,j+3}\|\left\langle y\right\rangle^\ell\partial_x^{j+1}u\|_{L_x^\infty L_y^2}}{j+1}L_{\rho,m-j+3}\|\partial_x^{m-j}\partial_y\mathcal{U}\|_{L^2}\bigg)^2$$
$$+\sum_{m=0}^{+\infty}\bigg(\sum_{j=[m/2]+1}^{m}L_{\rho,j+1}\|\left\langle y\right\rangle^\ell\partial_x^{j+1}u\|_{L^2}\frac{L_{\rho,m-j+5}\|\partial_x^{m-j}\partial_y\mathcal{U}\|_{L_x^\infty L_y^2}}{m-j+1}\bigg)^2$$
$$\leq C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Z_\rho}^2,$$

and

$$S_3\leq C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Y_\rho}^2.$$

Substituting the above estimates into (2.35) yields that

$$-\frac{1}{2}\rho'\sum_{m=1}^{+\infty}\frac{m+4}{\rho}L_{\rho,m+3}^2\|K_m\|_{L^2}^2\leq C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Z_\rho}^2+C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Y_\rho}^2. \tag{2.36}$$

**(ii) Estimate on the first term on the right hand side of** (2.34). Direct calculation gives that

$$\rho'\frac{m+4}{\rho}L_{\rho,m+3}^2\|P\partial_x^m\mathcal{U}\|_{L^2}^2=\rho'\frac{m+4}{\rho}\|P\big(L_{\rho,m+3}\partial_x^m\mathcal{U}\big)\|_{L^2}^2+\rho'^3\frac{(m+4)^3}{\rho^3}L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2$$
$$-2\rho'^2\frac{(m+4)^2}{\rho^2}\left(P\big(L_{\rho,m+3}\partial_x^m\mathcal{U}\big),\ L_{\rho,m+3}\partial_x^m\mathcal{U}\right)_{L^2}.$$

For the last term in the above inequality, by (2.31), one has that

$$\left(P\big(L_{\rho,m+3}\partial_x^m\mathcal{U}\big),\ L_{\rho,m+3}\partial_x^m\mathcal{U}\right)_{L^2}\geq\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|L_{\rho,m+3}\partial_x^m\mathcal{U}\|_{L^2}^2+L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2.$$

Thus,

$$\rho'\frac{m+4}{\rho}L_{\rho,m+3}^2\|P\partial_x^m\mathcal{U}\|_{L^2}^2$$
$$\leq\rho'^3\frac{(m+4)^3}{\rho^3}L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2-2\rho'^2\frac{(m+4)^2}{\rho^2}L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2$$
$$-\rho'^2\frac{(m+4)^2}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}t}\|L_{\rho,m+3}\partial_x^m\mathcal{U}\|_{L^2}^2.$$

Here, the last term can be written as

$$-\rho'^2\frac{(m+4)^2}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}t}\|L_{\rho,m+3}\partial_x^m\mathcal{U}\|_{L^2}^2$$
$$=-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\rho'^2\frac{(m+4)^2}{\rho^2}L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2\Big)+2\rho'\left(\rho''-\rho'^2/\rho\right)\frac{(m+4)^2}{\rho^2}L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2$$
$$\leq-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\rho'^2\frac{(m+4)^2}{\rho^2}L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2\Big),$$

where we have used (2.13) in the last inequality. Hence,

$$\rho'\frac{m+4}{\rho}L_{\rho,m+3}^2\|P\partial_x^m\mathcal{U}\|_{L^2}^2$$

$$\leq \rho'^3\frac{(m+4)^3}{\rho^3}L^2_{\rho,m+3}\|\partial_x^m\mathcal{U}\|^2_{L^2}-2\rho'^2\frac{(m+4)^2}{\rho^2}L^2_{\rho,m+3}\|\partial_x^m\mathcal{U}\|^2_{L^2}$$
$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\rho'^2\frac{(m+4)^2}{\rho^2}L^2_{\rho,m+3}\|\partial_x^m\mathcal{U}\|^2_{L^2}\Big).$$

Substituting the above estimate and (2.36) into (2.34) yields (2.30). Thus, the proof of proposition is complete. □

## 2.3 Estimate on the Tangential Velocity

We now derive the estimate on $\|u\|_{X_\rho}$.

**Proposition 2.10** Under the same assumption as in Theorem 2.1, the estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{+\infty}L^2_{\rho,m+k}\|\tau^{\ell+k}\partial_x^m\partial_y^k u\|^2_{L^2}+\frac{1}{4}\sum_{m=0}^{+\infty}L^2_{\rho,m+k}\|\tau^{\ell+k}\partial_x^m\partial_y^{k+1}u\|^2_{L^2}$$
$$+\frac{1}{4}\sum_{m=0}^{+\infty}L^2_{\rho,m+k}\|\tau^{\ell+k}\partial_x^m\partial_y^k u\|^2_{L^2}$$
$$\leq\frac{1}{4}\rho'^3\sum_{m=0}^{+\infty}\frac{m+k+1}{\rho}L^2_{\rho,m+k}\|\tau^{\ell+k}\partial_x^m\partial_y^k u\|^2_{L^2}+C\big(|\vec{a}|_{X_\rho}+|\vec{a}|^2_{X_\rho}\big)|\vec{a}|^2_{Y_\rho}$$

holds for any $0\leq k\leq 3$, where $\tau$ is defined as in (1.4).

**Proof** We will basically give the estimates in the cases of when $k=0$ and $k=3$.

**(a) The case of when $k=0$**. We first derive the estimate on tangential derivatives. Applying $\partial_x^m$ to the first equation in (2.1) gives that

$$\big(\partial_t+u\partial_x+v\partial_y-\partial_y^2\big)\partial_x^m u+\partial_x^m u$$
$$=-\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u)\partial_x^{m-j+1}u-\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)\partial_x^{m-j}\partial_y u-(\partial_x^m v)\partial_y u.$$

Multiplying by $\tau^{2\ell}\partial_x^m u$ on both sides of the above equation and then integrating over $\mathbb{R}^2_+$, we have, by observing $|\partial_y\tau^{2\ell}|\leq N^{-1/2}\ell\tau^{2\ell}$ and $|\partial_y^2\tau^{2\ell}|\leq N^{-1}(\ell+\ell^2)\tau^{2\ell}$, that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tau^\ell\partial_x^m u\|^2_{L^2}+\big\|\tau^\ell\partial_y\partial_x^m u\big\|^2_{L^2}+\big\|\tau^\ell\partial_x^m u\big\|^2_{L^2}$$
$$\leq\frac{1}{2}\frac{\ell}{N^{1/2}}\|v\tau^\ell\partial_x^m u\|_{L^2}\|\tau^\ell\partial_x^m u\|_{L^2}+\frac{1}{2}\frac{\ell+\ell^2}{N}\|\tau^\ell\partial_x^m u\|^2_{L^2}$$
$$-\sum_{j=1}^{m}\binom{m}{j}\Big(\tau^\ell(\partial_x^j u)\partial_x^{m-j+1}u,\ \tau^\ell\partial_x^m u\Big)_{L^2}-\sum_{j=1}^{m-1}\binom{m}{j}\Big(\tau^\ell(\partial_x^j v)\partial_x^{m-j}\partial_y u,\ \tau^\ell\partial_x^m u\Big)_{L^2}$$
$$-\Big(\tau^\ell(\partial_x^m v)\partial_y u,\ \tau^\ell\partial_x^m u\Big)_{L^2}.$$

This, with (1.5), yields that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{+\infty}L^2_{\rho,m}\|\tau^\ell\partial_x^m u\|^2_{L^2}+\sum_{m=0}^{+\infty}L^2_{\rho,m}\|\tau^\ell\partial_y\partial_x^m u\|^2_{L^2}+\frac{1}{2}\sum_{m=0}^{+\infty}L^2_{\rho,m}\|\tau^\ell\partial_x^m u\|^2_{L^2}$$
$$\leq\sum_{m=0}^{+\infty}\rho'\frac{m+1}{\rho}L^2_{\rho,m}\|\tau^\ell\partial_x^m u\|^2_{L^2}+C\sum_{m=0}^{+\infty}L^2_{\rho,m}\|v\tau^\ell\partial_x^m u\|_{L^2}\|\tau^\ell\partial_x^m u\|_{L^2}$$
$$+\sum_{m=0}^{+\infty}\sum_{j=1}^{m}\binom{m}{j}L^2_{\rho,m}\|\tau^\ell(\partial_x^j u)\partial_x^{m-j+1}u\|_{L^2}\|\tau^\ell\partial_x^m u\|_{L^2}$$

$$+\sum_{m=0}^{+\infty}\sum_{j=1}^{m-1}\binom{m}{j}L_{\rho,m}^2\|\tau^\ell(\partial_x^j v)\partial_x^{m-j}\partial_y u\|_{L^2}\|\tau^\ell\partial_x^m u\|_{L^2}$$
$$+\sum_{m=0}^{+\infty}L_{\rho,m}^2\|\tau^\ell(\partial_x^m v)\partial_y u\|_{L^2}\|\tau^\ell\partial_x^m u\|_{L^2}. \tag{2.37}$$

Direct calculation gives that

$$\sum_{m=0}^{+\infty}L_{\rho,m}^2\|v\tau^\ell\partial_x^m u\|_{L^2}\|\tau^\ell\partial_x^m u\|_{L^2}\le C|\vec{a}|_{X_\rho}^3. \tag{2.38}$$

Similarly to the proofs of Lemmas 2.6 and 2.7, we have that

$$\sum_{m=0}^{+\infty}\sum_{j=1}^{m}\binom{m}{j}L_{\rho,m}^2\|\tau^\ell(\partial_x^j u)\partial_x^{m-j+1}u\|_{L^2}\|\tau^\ell\partial_x^m u\|_{L^2}$$
$$+\sum_{m=0}^{+\infty}\sum_{j=1}^{m-1}\binom{m}{j}L_{\rho,m}^2\|\tau^\ell(\partial_x^j v)\partial_x^{m-j}\partial_y u\|_{L^2}\|\tau^\ell\partial_x^m u\|_{L^2}\le C|\vec{a}|_{X_\rho}|\vec{a}|_{Y_\rho}^2. \tag{2.39}$$

It remains to estimate the last term in (2.37). Direct calculation gives that

$$\forall\ 0\le m\le 1,\quad L_{\rho,m}^2\|\tau^\ell(\partial_x^m v)\partial_y u\|_{L^2}\|\tau^\ell\partial_x^m u\|_{L^2}\le C|\vec{a}|_{X_\rho}^3.$$

For $m\ge 2$, by recalling $\mathscr{L}(u,\mathcal{U})$ given in (2.21), we use (2.10) to write that

$$\begin{aligned}\partial_x^m v&=-\int_0^y\partial_x^{m+1}u(t,x,\tilde{y})\mathrm{d}\tilde{y}\\&=-\int_0^y\partial_x^{m-2}\lambda(t,x,\tilde{y})\mathrm{d}\tilde{y}-\partial_x^{m-2}\underbrace{\int_0^y\Big(\partial_y u(t,x,\tilde{y})\int_0^{\tilde{y}}\mathcal{U}(t,x,r)\mathrm{d}r\Big)\mathrm{d}\tilde{y}}_{=\mathscr{L}(u,\mathcal{U})}.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{m=0}^{+\infty}L_{\rho,m}^2\|\tau^\ell(\partial_x^m v)\partial_y u\|_{L^2}\|\tau^\ell\partial_x^m u\|_{L^2}\\&\le\frac{C}{\rho^5}|\vec{a}|_{X_\rho}^3+C\sum_{m=2}^{+\infty}L_{\rho,m}^2\|\tau^{\ell+1}\partial_y u\|_{L_x^\infty L_y^2}\|\partial_x^{m-2}\lambda\|_{L^2}\|\tau^\ell\partial_x^m u\|_{L^2}\\&\quad+C\sum_{m=2}^{+\infty}L_{\rho,m}^2\|\tau^{\ell+1}\partial_y u\|_{L_x^\infty L_y^2}\|\partial_x^{m-2}\mathscr{L}(u,\mathcal{U})\|_{L_x^2L_y^\infty}\|\tau^\ell\partial_x^m u\|_{L^2}\\&\le C|\vec{a}|_{X_\rho}^3+C|\vec{a}|_{X_\rho}\sum_{m=2}^{+\infty}\frac{(m+2)^{3/2}}{\rho^{3/2}}L_{\rho,m+1}\|\partial_x^{m-2}\mathscr{L}(u,\mathcal{U})\|_{L_x^2L_y^\infty}\frac{(m+1)^{1/2}}{\rho^{1/2}}L_{\rho,m}\|\tau^\ell\partial_x^m u\|_{L^2}\\&\le C\Big(|\vec{a}|_{X_\rho}+|\vec{a}|_{X_\rho}^2\Big)|\vec{a}|_{Y_\rho}^2,\end{aligned}$$

where we have used (2.22) in the last inequality. By substituting the above estimates and (2.38)–(2.39) into (2.37), and by using the fact that $\rho'\le\rho'^3\le\rho'^3/4\le 0$, we have that

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{+\infty}L_{\rho,m}^2\|\tau^\ell\partial_x^m u\|_{L^2}^2+\sum_{m=0}^{+\infty}L_{\rho,m}^2\|\tau^\ell\partial_y\partial_x^m u\|_{L^2}^2+\frac12\sum_{m=0}^{+\infty}L_{\rho,m}^2\|\tau^\ell\partial_x^m u\|_{L^2}^2\\&\le\frac14\rho'^3\sum_{m=0}^{+\infty}\frac{m+1}{\rho}L_{\rho,m}^2\|\tau^\ell\partial_x^m u\|_{L^2}^2+C\big(|\vec{a}|_{X_\rho}+|\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{Y_\rho}^2.\end{aligned}$$

Thus, Proposition 2.10 holds for $k=0$.

**(b) The case of $k=3$**. We apply $\partial_x^m\partial_y^2$ to the first equation in (2.1) to obtain that

$$\partial_t\partial_x^m\partial_y^2u-\partial_x^m\partial_y^4u+\partial_x^m\partial_y^2u=-\partial_x^m\partial_y^2\big(u\partial_xu+v\partial_yu\big). \tag{2.40}$$

The assumption (2.2) implies that each term in the above equation belongs to

$$L^2\Big([0,+\infty[;L^2(\langle y\rangle^{\ell+3})\Big)$$

with

$$L^2(\langle y\rangle^{\ell+3}):=\Big\{f\in\mathcal{S}';\ \langle y\rangle^{\ell+3}f\in L^2\Big\}.$$

Then we multiply both sides of (2.40) by $-\partial_y\Big(\tau^{2(\ell+3)}\partial_x^m\partial_y^3u\Big)\in L^2\Big([0,+\infty[;L^2(\langle y\rangle^{-(\ell+3)})\Big)$, and then integrate over $\mathbb{R}^2_+$. By using $\partial_y^2u|_{y=0}=0$, we have that

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}\|\tau^{\ell+3}\partial_x^m\partial_y^3u\|_{L^2}^2+\|\tau^{\ell+3}\partial_x^m\partial_y^4u\|_{L^2}^2+\|\tau^{\ell+3}\partial_x^m\partial_y^3u\|_{L^2}^2\\
=&-\int_{\mathbb{R}^2_+}\big(\partial_x^m\partial_y^4u\big)(\partial_y\tau^{2\ell+6})\partial_x^m\partial_y^3u\,\mathrm dx\mathrm dy\\
&+\int_{\mathbb{R}^2_+}\Big[\partial_x^m\partial_y^2\big(u\partial_xu+v\partial_yu\big)\Big]\partial_y\Big(\tau^{2(\ell+3)}\partial_x^m\partial_y^3u\Big)\,\mathrm dx\mathrm dy.\end{aligned}$$

For the terms on the right hand side of the above equality, we have that

$$-\int_{\mathbb{R}^2_+}(\partial_x^m\partial_y^4u)(\partial_y\tau^{2\ell+6})\partial_x^m\partial_y^3u\,\mathrm dx\mathrm dy\le\frac12\left(\|\tau^{\ell+3}\partial_x^m\partial_y^4u\|_{L^2}^2+\|\tau^{\ell+3}\partial_x^m\partial_y^3u\|_{L^2}^2\right),$$

where we have used the fact that $|\partial_y\tau^{2\ell+6}|\le\frac{\ell+3}{N^{1/2}}\tau^{2\ell+6}\le\tau^{2\ell+6}$, and

$$\begin{aligned}&\int_{\mathbb{R}^2_+}\Big[\partial_x^m\partial_y^2\big(u\partial_xu+v\partial_yu\big)\Big]\partial_y\Big(\tau^{2(\ell+3)}\partial_x^m\partial_y^3u\Big)\,\mathrm dx\mathrm dy\\
\le&\frac14\left(\|\tau^{\ell+3}\partial_x^m\partial_y^4u\|_{L^2}^2+\|\tau^{\ell+3}\partial_x^m\partial_y^3u\|_{L^2}^2\right)+4\|\tau^{\ell+3}\partial_x^m\partial_y^2\big(u\partial_xu+v\partial_yu\big)\|_{L^2}^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}\sum_{m=0}^{+\infty}L^2_{\rho,m+3}\|\tau^{\ell+3}\partial_x^m\partial_y^3u\|_{L^2}^2+\frac14\sum_{m=0}^{+\infty}L^2_{\rho,m+3}\|\tau^{\ell+3}\partial_x^m\partial_y^4u\|_{L^2}^2\\
&+\frac14\sum_{m=0}^{+\infty}L^2_{\rho,m+3}\|\tau^{\ell+3}\partial_x^m\partial_y^3u\|_{L^2}^2\\
\le&\rho'\sum_{m=0}^{+\infty}\frac{m+4}{\rho}L^2_{\rho,m+3}\|\tau^{\ell+3}\partial_x^m\partial_y^3u\|_{L^2}^2+4\sum_{m=0}^{+\infty}L^2_{\rho,m+3}\|\tau^{\ell+3}\partial_x^m\partial_y^2\big(u\partial_xu+v\partial_yu\big)\|_{L^2}^2.\end{aligned}$$

Moreover, similarly to the proof of Lemma 2.6, we can show that

$$4\sum_{m=0}^{+\infty}L^2_{\rho,m+3}\|\tau^{\ell+3}\partial_x^m\partial_y^2\big(u\partial_xu+v\partial_yu\big)\|_{L^2}^2\le C|\vec a|^4_{X_\rho}\le C|\vec a|^2_{X_\rho}|\vec a|^2_{Y_\rho}.$$

Therefore, by noting that $\rho'\le\rho'^3\le\rho'^3/4$, it holds that

$$\frac12\frac{\mathrm d}{\mathrm dt}\sum_{m=0}^{+\infty}L^2_{\rho,m+3}\|\tau^{\ell+3}\partial_x^m\partial_y^3u\|_{L^2}^2+\frac14\sum_{m=0}^{+\infty}L^2_{\rho,m+3}\|\tau^{\ell+3}\partial_x^m\partial_y^4u\|_{L^2}^2$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{m=0}^{+\infty} L_{\rho,m+3}^{2}\|\tau^{\ell+3}\partial_x^m\partial_y^3 u\|_{L^2}^2 \\
&\leq \frac{1}{4}\rho'^3\sum_{m=0}^{+\infty}\frac{m+3}{\rho}L_{\rho,m+3}^{2}\|\tau^{\ell+3}\partial_x^m\partial_y^3 u\|_{L^2}^2 + C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Y_\rho}^2.
\end{aligned}
$$

This proves Proposition 2.10 for $k=3$.

The cases of when $k=1,2$ can discussed similarly, so we omit the details, for brevity. The proof of proposition is complete. □

### 2.4 A Priori Estimate in 2D

We will apply the following abstract version of the bootstrap principle given in [34] to prove Theorem 2.4:

**Proposition 2.11** (Proposition 1.21 of [34]) Letting $I$ be a time interval, and for each $t\in I$ we have two statements: a "hypothesis" $\boldsymbol{H}(t)$ and a "conclusion" $\boldsymbol{C}(t)$. Suppose that we can verify the following four statements:

(i) If $\boldsymbol{H}(t)$ is true for some time $t\in I$, then $\boldsymbol{C}(t)$ is also true for the time $t$.

(ii) If $\boldsymbol{C}(t)$ is true for some $t\in I$, then $\boldsymbol{H}(t')$ holds for all $t'$ in a neighborhood of $t$.

(iii) If $t_1,t_2,\cdots$ is a sequence of times in $I$ which converges to another time $t\in I$ and $\boldsymbol{C}(t_n)$ is true for all $t_n$, then $\boldsymbol{C}(t)$ is true.

(iv) $\boldsymbol{H}(t)$ is true for at least one time $t\in I$.

Then $\boldsymbol{C}(t)$ is true for all $t\in I$.

For each $T\in[0,+\infty[$, let $\boldsymbol{H}(T)$ be the statement

$$
\forall\ t\in[0,T],\quad e^{t/2}|\vec{a}(t)|_{X_{\rho(t)}}^2+\frac{1}{4}\int_0^t e^{s/2}\,|\vec{a}(s)|_{Z_{\rho(s)}}^2\,\mathrm{d}s\leq\frac{2\big(1+\rho_0^2\big)}{\rho_0^2}\varepsilon_0^2, \tag{2.41}
$$

and let $\boldsymbol{C}(T)$ be the statement

$$
\forall\ t\in[0,T],\quad e^{t/2}|\vec{a}(t)|_{X_{\rho(t)}}^2+\frac{1}{4}\int_0^t e^{s/2}\,|\vec{a}(s)|_{Z_{\rho(s)}}^2\,\mathrm{d}s\leq\frac{1+\rho_0^2}{\rho_0^2}\varepsilon_0^2, \tag{2.42}
$$

where $\rho_0,\varepsilon_0$ are the constants given in Theorem 2.1. In the discussion that follows, we will verify that the conditions (i)–(iv) in Proposition 2.11 are satisfied by $\boldsymbol{H}(T)$ and $\boldsymbol{C}(T)$ defined as in (2.41) and (2.42). Note that $\vec{a}|_{t=0}=(u_0,0,\partial_x^3u_0)$. Thus, $\boldsymbol{H}(0)$ holds because of (2.3) and the fact that

$$
\begin{aligned}
|\vec{a}(0)|_{X_{\rho_0}}^2 &= \|u_0\|_{X_{\rho_0}}^2+\sum_{m=0}^{+\infty}L_{\rho_0,m+2}^2\|\partial_x^{m+3}u_0\|_{L^2}^2 \\
&\leq \|u_0\|_{X_{\rho_0}}^2+\sum_{m=0}^{+\infty}\frac{(m+3)^4}{4\rho_0^2}4^{-(m+3)}L_{2\rho_0,m+3}^2\|\partial_x^{m+3}u_0\|_{L^2}^2 \\
&\leq \frac{\rho_0^2+1}{\rho_0^2}\|u_0\|_{X_{2\rho_0}}^2.
\end{aligned} \tag{2.43}
$$

Hence, the condition (iv) in Proposition 2.11 holds. Moreover, the conditions (ii)-(iii) follow from the continuity of the function

$$
t\mapsto e^{t/2}|\vec{a}(t)|_{X_{\rho(t)}}^2+\frac{1}{4}\int_0^t e^{s/2}\,|\vec{a}(s)|_{Z_{\rho(s)}}^2\,\mathrm{d}s.
$$

It remains to check the condition (i) in Proposition 2.11; that is that

$$\boldsymbol{H}(T) \text{ is true for some time } T>0 \Longrightarrow \boldsymbol{C}(T) \text{ is also true for the same time } T.$$

By Definition 2.2, we combine the estimates given in Propositions 2.5 and 2.10 to get that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\vec{a}|_{X_\rho}^2+\frac{1}{4}|\vec{a}|_{Z_\rho}^2+\frac{1}{4}|\vec{a}|_{X_\rho}^2\\
&+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{+\infty}\rho'^2\frac{(m+4)^2}{\rho^2}L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2+\frac{1}{8}\sum_{m=0}^{+\infty}\rho'^2\frac{(m+4)^2}{\rho^2}L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2\\
\le{}&\frac{1}{4}\rho'^3|\vec{a}|_{Y_\rho}^2+C\left(|\vec{a}|_{X_\rho}+|\vec{a}|_{X_\rho}^2\right)|\vec{a}|_{Z_\rho}^2+C\left(|\vec{a}|_{X_\rho}+|\vec{a}|_{X_\rho}^2\right)|\vec{a}|_{Y_\rho}^2\\
\le{}&C\big(|\vec{a}|_{X_\rho}+|\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{Z_\rho}^2+\Big(\frac{1}{4}\rho'^3+C|\vec{a}|_{X_\rho}+C|\vec{a}|_{X_\rho}^2\Big)|\vec{a}|_{Y_\rho}^2.
\end{aligned}$$

Note that

$$C\big(|\vec{a}|_{X_\rho}+|\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{Z_\rho}^2\le 4C\frac{1+\rho_0^2}{\rho_0^2}\varepsilon_0|\vec{a}|_{Z_\rho}^2\le\frac{1}{8}|\vec{a}|_{Z_\rho}^2$$

and that

$$\frac{1}{4}\rho'^3+C|\vec{a}|_{X_\rho}+C|\vec{a}|_{X_\rho}^2\le-\frac{1}{4}\left(\frac{\rho_0}{24}\right)^3e^{-t/4}+4C\frac{1+\rho_0^2}{\rho_0^2}e^{-t/4}\varepsilon_0\le 0,$$

provided $\varepsilon_0$ is sufficiently small. Combining the above estimates implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}e^{t/2}|\vec{a}|_{X_\rho}^2+\frac{1}{8}e^{t/2}|\vec{a}|_{Z_\rho}^2+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}e^{t/2}\sum_{m=0}^{+\infty}\rho'^2\frac{(m+4)^2}{\rho^2}L_{\rho,m+3}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2\le 0.$$

By integrating the above estimate over $[0,t]$ for any $t\in[0,T]$ and using $\mathcal{U}|_{t=0}=0$, we obtain that

$$\forall\ t\in[0,T],\quad e^{t/2}|\vec{a}(t)|_{X_{\rho(t)}}^2+\frac{1}{4}\int_0^t e^{s/2}\,|\vec{a}(s)|_{Z_{\rho(s)}}^2\,\mathrm{d}s\le|\vec{a}(0)|_{X_{\rho_0}}^2\le\frac{1+\rho_0^2}{\rho_0^2}\varepsilon_0^2,$$

where we have used (2.43). This yields (2.42) if (2.41) holds, so that the condition (i) holds. Therefore, by Proposition 2.11, the estimate (2.42) holds for any $T\ge 0$, and the proof of Theorem 2.4 is complete.

## 3 A Priori Estimate in 3D

The discussion on the 3D magnetic Prandtl model is similar to that of the 2D case. For this, we will use vector-valued auxiliary functions instead of the scalar ones used in the previous section. More precisely, denote by $\boldsymbol{u}=(u_1,u_2)$ and $v$ the tangential and normal velocities, respectively, and by $(x,y)$ the spatial variables in $\mathbb{R}^2\times\mathbb{R}_+$ with $x=(x_1,x_2)$. As the counterparts of $\mathcal{U}$ and $\lambda$ defined by (2.7) and (2.10), we define $\boldsymbol{\mathcal{U}}=(\mathcal{U}_1,\mathcal{U}_2)$ and $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ as follows: let $\mathcal{U}_j, j=1,2$, solve the initial-boundary problem

$$\begin{cases}\big(\partial_t+\boldsymbol{u}\cdot\partial_x+v\partial_y-\partial_y^2\big)\displaystyle\int_0^y\mathcal{U}_j(t,x,\tilde{y})\mathrm{d}\tilde{y}+\int_0^y\mathcal{U}_j(t,x,\tilde{y})\mathrm{d}\tilde{y}=-\partial_{x_j}^3v,\\ \mathcal{U}_j|_{t=0}=0,\quad \partial_y\mathcal{U}_j|_{y=0}=\mathcal{U}_j|_{y\to+\infty}=0.\end{cases}$$

Accordingly, set

$$\begin{cases}\lambda_1=\partial_{x_1}^3u_1-(\partial_yu_1)\displaystyle\int_0^y\mathcal{U}_1(t,x,\tilde{y})\mathrm{d}\tilde{y},\quad \lambda_2=\partial_{x_2}^3u_1-(\partial_yu_1)\int_0^y\mathcal{U}_2(t,x,\tilde{y})\mathrm{d}\tilde{y},\\ \lambda_3=\partial_{x_1}^3u_2-(\partial_yu_2)\displaystyle\int_0^y\mathcal{U}_1(t,x,\tilde{y})\mathrm{d}\tilde{y},\quad \lambda_4=\partial_{x_2}^3u_2-(\partial_yu_2)\int_0^y\mathcal{U}_2(t,x,\tilde{y})\mathrm{d}\tilde{y}.\end{cases}$$

Denote that $\vec{\boldsymbol{a}} = (\boldsymbol{u}, \boldsymbol{\mathcal{U}}, \boldsymbol{\lambda})$, and define $|\vec{\boldsymbol{a}}|_{X_\rho}, |\vec{\boldsymbol{a}}|_{Y_\rho}$ and $|\vec{\boldsymbol{a}}|_{Z_\rho}$ as in Definition 2.2.

Then the a priori estimate in Theorem 2.4 also holds for the function $\vec{\boldsymbol{a}}$ as stated in the following theorem:

**Theorem 3.1** Suppose that the initial datum $\boldsymbol{u}_0$ in (1.3) belongs to $X_{2\rho_0}$ for some $\rho_0 > 0$. Let $\boldsymbol{u} \in L^\infty([0, +\infty[;\ X_\rho)$ be a solution to (1.3) satisfying that

$$\int_0^\infty \left(\|\boldsymbol{u}(t)\|^2_{X_{\rho(t)}} + \|\partial_y \boldsymbol{u}(t)\|^2_{X_{\rho(t)}}\right)\mathrm{d}t < +\infty,$$

where $\rho$ is defined by (1.7). If $\|\boldsymbol{u}_0\|_{X_{2\rho_0}} \le \varepsilon_0$ for some sufficiently small $\varepsilon_0 > 0$, then we have that

$$\sup_{t\ge 0} e^{t/4} |\vec{\boldsymbol{a}}(t)|_{X_{\rho(t)}} + \left(\int_0^{+\infty} e^{t/2} |\vec{\boldsymbol{a}}(t)|^2_{Z_{\rho(t)}} \mathrm{d}t\right)^{1/2} \le \frac{4(\rho_0+1)}{\rho_0}\varepsilon_0.$$

Thus,

$$\sup_{t\ge 0} e^{t/4} |\boldsymbol{u}(t)|_{X_{\rho(t)}} + \left(\int_0^{+\infty} e^{t/2} |\partial_y \boldsymbol{u}(t)|^2_{X_{\rho(t)}} \mathrm{d}t\right)^{1/2} \le \frac{4(\rho_0+1)}{\rho_0}\varepsilon_0.$$

The proof of Theorem 3.1 is the same as that of the 2D case, so, for brevity, we omit the details.

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