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ITERATIVE METHODS FOR OBTAINING AN INFINITE FAMILY OF STRICT PSEUDO-CONTRACTIONS IN BANACH SPACES*

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Abstract In this paper, we introduce a general hybrid iterative method to find an infinite family of strict pseudo-contractions in a q-uniformly smooth and strictly convex Banach space. Moreover, we show that the sequence defined by the iterative method converges strongly to a common element of the set of fixed points, which is the unique solution of the variational inequality $\langle(\lambda\varphi-\nu\mathcal{F})\tilde{z}, j_q(z-\tilde{z})\rangle\leq 0$, for $z\in\bigcap\limits_{i=1}^{\infty}\Gamma(S_i)$. The results introduced in our work extend to some corresponding theorems.

Key words MKC; iterative algorithm; strict pseudo-contraction; β-Lipschitzian; δ-strongly monotone; Banach spaces

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1 Introduction

Assume that E is a real Banach space. Letting $J: E\to 2^{E^*}$, we define the normalized duality mapping by

$$J(z)=\{g\in E^*:\langle z,g\rangle=\|z\|^2 \text{ and } \|g\|=\|z\|\},\forall z\in E,$$

where $\langle\cdot,\cdot\rangle$ is the generalized duality pairing, and E^* is the dual space of E. In addition, we will use j to denote the single-valued normalized duality mapping. Let $\{z_k\}$ be a sequence in E.

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Thus we use $z_k \to \bar{z}$ (respectively, $z_k \rightharpoonup \bar{z}$, $z_k \overset{*}{\rightharpoonup} \bar{z}$) to denote strong (respectively, weak, weak*) convergence of the sequence $\{z_k\}$ to $\bar{z}$. Assume that J is single valued if, for each $\{z_k\} \subset E$ with $z_k \rightharpoonup \bar{z}$, one has $J(z_k) \overset{*}{\rightharpoonup} J(\bar{z})$; thus J is said to be weakly sequentially continuous.

Letting $q > 1$, we use J_q to denote the generalized duality mapping, which is given by

$$J_q(z) = \{g \in E^* : \langle z, g\rangle = \|z\|^q \text{ and } \|g\| = \|z\|^{q-1}\},$$

and the following relation holds:

$$J_q(z) = \|z\|^{q-2} J(z),\ z \neq 0.$$

Letting $\mu_E : [0, \infty) \to [0, \infty)$, the modulus of the smoothness of E is given by

$$\mu_E(\theta) := \sup\{\frac{1}{2}(\|u+v\| + \|u-v\|) - 1 : \|u\| \leq 1, \|v\| \leq \theta\}.$$

Moreover, if $\lim\limits_{\theta\to 0^+} \frac{\mu_E(\theta)}{\theta} = 0$, then E is said to be uniformly smooth.

Given $q > 1$, if there is a constant $b > 0$ such that $\mu_E(\theta) \leq b\theta^q$, then E is q-uniformly smooth. The example of such spaces are L_p(or l_p), $p > 1$ and Hilbert spaces. More specifically, for each $p > 1$, L_p(or l_p) is $\min\{p, 2\}$ -uniformly smooth.

Noticing that a q-uniformly smooth Banach space is uniformly smooth, this means that its norm is uniformly Fréchet differentiable [1].

When E is uniformly smooth, the normalized duality mapping j is single valued and norm to norm uniformly continuous on every bounded set.

A mapping $S : E \to E$ is called a $(q) - \gamma$-strict pseudo-contraction if there is a constant $\gamma > 0$ for each $y, z \in E$ and for all $j_q(y-z) \in J_q(y-z)$ such that

$$\langle Sy - Sz, j_q(y-z)\rangle \leq \|y-z\|^q - \gamma\|(I-S)y - (I-S)z\|^q. \tag{1.1}$$

The set of fixed points of the mapping S is denoted by $\Gamma(S)$; that is, $\Gamma(S) = \{z \in E : Sz = z\}$. Clearly, (1.1) is equivalent to the following:

$$\langle (I-S)y - (I-S)z, j_q(y-z)\rangle \geq \gamma\|(I-S)y - (I-S)z\|^q. \tag{1.2}$$

The following well-known theorem is the Banach contraction principle:

Theorem 1.1 ([2]) Suppose that (Y, d) is a complete metric space, and that h is a contractive mapping on Y; that is, there is a constant $\rho \in (0, 1)$ such that $d(h(y), h(z)) \leq \rho d(y, z)$, $\forall y, z \in Y$. Then h has a unique fixed point.

Theorem 1.2 ([3]) Suppose that (Y, d) is a complete metric space, and that φ is a Meir-Keeler contraction (MKC for short) on Y; that is, $\forall \epsilon > 0$, and there is a number $c > 0$ such that $d(y, z) < \epsilon + c$ implies that $d(\varphi(y), \varphi(z)) < \epsilon$, $\forall y, z \in Y$. Then φ has a unique fixed point.

Since the contractions are Meir-Keeler contractions, Theorem 1.2 is one of generalizations of Theorem 1.1.

Assume that E is a q-uniformly smooth and strictly convex Banach space which admits a generalized duality mapping $j_q : E \to E^*$. A mapping $\mathcal{F} : E \to E$ is said to be

(1) β-Lipschitzian, if there is a constant $\beta > 0$ such that

$$\|\mathcal{F}y - \mathcal{F}z\| \leq \beta\|y - z\|, \forall y, z \in E; \tag{1.3}$$

(2) δ-strongly monotone, if there is a constant $\delta > 0$ such that

$$\langle \mathcal{F}y - \mathcal{F}z, j_q(y-z)\rangle \geq \delta\|y-z\|^q, \forall y, z \in E. \tag{1.4}$$

Definition 1.3 Suppose that $\mathcal{A}$ is a strongly positive bounded linear operator in a q-uniformly smooth and strictly convex Banach space E; that is, there is a constant $\tilde{\eta} > 0$ such that

$$\langle \mathcal{A}z, j_q(z)\rangle \geq \tilde{\eta}\|z\|^q, \forall z \in E, \|a_1 I - a_2\mathcal{A}\| = \sup_{\|z\|\leq 1}\{|\langle (a_1 I - a_2\mathcal{A})z, j_q(z)\rangle|\}, \ a_1 \in [0,1], a_2 \in [0,1].$$

Here I is the identity mapping and j_q is the generalized duality mapping.

Remark 1.4 By the definition of $\mathcal{A}$, we can know that $\mathcal{A}$ is a $\|\mathcal{A}\|$-Lipschizian and an $\tilde{\eta}$-strongly monotone operator.

Suppose that H is a real Hilbert space, and that $\tilde{D}$ is a non-empty closed convex subset of H.

In 2010, Jung [4] proposed the following method: for a γ-strict pseudo-contraction $S : \tilde{D} \to H$ such that $\Gamma(S) \neq \emptyset$ and $x_1 = x \in \tilde{D}$,

$$\begin{cases} z_k = \tau_k x_k + (1-\tau_k)P_{\tilde{D}}Wx_k, \\ x_{k+1} = \beta_k\lambda h(x_k) + (I - \beta_k\mathcal{A})z_k, \ \forall\, k \geq 1, \end{cases} \tag{1.5}$$

where $W : \tilde{D} \to H$ is a mapping given by $Wx = \gamma x + (1-\gamma)Sx$, and $\{\beta_k\}$ and $\{\tau_k\} \subset (0,1)$ are sequences which hold $\lim\limits_{k\to\infty} \beta_k = 0$, $\sum\limits_{k=1}^{\infty} \beta_k = \infty$, and $0 < \liminf_{k\to\infty} \tau_k \leq \limsup\limits_{k\to\infty} \tau_k \leq b < 1$ for the constant $b \in (0,1)$. He obtained that the sequence $\{x_k\}$ generated by (1.5) converges strongly to a fixed point $\hat{x}$ of S, which uniquely solves the variational inequality $\langle (\lambda h - \mathcal{A})\hat{x}, z - \hat{x}\rangle \leq 0$, $z \in \Gamma(S)$.

Recently, Tian [5] introduced the iterative algorithm

$$x_{k+1} = \beta_k\lambda h(x_k) + (I - \nu\beta_k\mathcal{F})Sx_k, \ \forall\, k \geq 1, \tag{1.6}$$

where S is a non-expansive mapping on H such that $\Gamma(S) \neq \emptyset$, $\mathcal{F}$ is a β-Lipschitzian and a δ-strongly monotone operator, $\{\beta_k\} \subset (0,1)$ is a sequence which satisfies $\lim\limits_{k\to\infty} \beta_k = 0$, $\sum\limits_{k=1}^{\infty} \beta_k = \infty$, and $\lim\limits_{k\to\infty} \beta_{k+1}/\beta_k = 1$. He proved that $\{x_k\}$ given by (1.6) converges strongly to a point $\hat{x}$ in $\Gamma(S)$, which uniquely solves the variational inequality $\langle (\lambda h - \nu\mathcal{F})\hat{x}, z - \hat{x}\rangle \leq 0$, $z \in \Gamma(S)$.

Very recently, Wang [6] proposed the following algorithm: for $x_1 = x \in \tilde{D}$,

$$\begin{cases} z_k = \tau_k x_k + (1-\tau_k)U_k x_k, \\ x_{k+1} = \beta_k\lambda h(x_k) + (I - \nu\beta_k\mathcal{F})z_k, \ \forall\, k \geq 1. \end{cases} \tag{1.7}$$

Here U_k is a mapping given by (2.14), and $\mathcal{F}$ is a β-Lipschitzian and a δ-strongly monotone operator such that $0 < \nu < \frac{2\delta}{\beta^2}$, $\{\beta_k\}$ and $\{\tau_k\} \subset (0,1)$. In Hilbert spaces, she obtained that if the parameters hold to certain conditions, then $\{x_k\}$ generated by (1.7) converges strongly to a common element of the fixed points of an infinite family of γ_i-strict pseudo-contractions, which uniquely solves the variational inequality $\langle (\lambda h - \nu\mathcal{F})\hat{x}, z - \hat{x}\rangle \leq 0$, for $z \in \bigcap\limits_{i=1}^{\infty} \Gamma(S_i)$.

Inspired and motivated by the above works, we introduce the following general iterative scheme: for $x_1 = x \in E$,

$$\begin{cases} z_k = \tau_k x_k + (1-\tau_k)U_k x_k, \\ x_{k+1} = \beta_k\lambda \varphi(x_k) + (I - \nu\beta_k\mathcal{F})z_k, \ \forall\, k \geq 1. \end{cases} \tag{1.8}$$

Here U_k is a mapping given by (2.14), φ is a Meir-Keeler contraction (MKC for short), and $\mathcal{F}$ is a β-Lipschitzian and a δ-strongly monotone operator such that $0 < \nu < \min\{(\frac{q\delta}{D_q\beta^q})^{\frac{1}{q-1}}, 1\}$. We will show that if the parameters hold to certain conditions (see (D_1)–(D_3) in Theorem 3.1), then $\{x_k\}$ given by (1.8) converges strongly to a common element of the fixed points of an infinite family of γ_i-strict pseudo-contractions, which uniquely solves the variational inequality $\langle(\lambda\varphi - \nu\mathcal{F})\tilde{z}, j_q(z-\tilde{z})\rangle \le 0$ for $z \in \bigcap\limits_{i=1}^{\infty}\Gamma(S_i)$. Our results generalize the theories of Wang inself space[6] in the following two respects:

(i) we extend the results of Wang [6] from Hilbert spaces to q-uniformly smooth and strictly convex Banach spaces;

(ii) we extend the results of Wang [6] from a contractive mapping to a Meir-Keeler contraction (MKC for short).

The rests of this work is organized as follows: in the next section, we introduce the notations and preliminary results upon which we rely. In the final section, we study the convergence of the proposed methods.

2 Preliminaries

In this part, we mainly recall some lemmas which are useful for proving our main ideas.

Lemma 2.1 ([7]) Given $q > 1$, where E is a q-uniformly smooth space, there is a constant $D_q > 0$ such that

$$\|y+z\|^q \le \|y\|^q + q\langle z, j_q(y)\rangle + D_q\|z\|^q, \quad \forall y, z \in E. \tag{2.1}$$

Lemma 2.2 ([8]) Suppose that E is a Banach space, and that D is a convex subset of E. If $\varphi : D \to D$ is an MKC, then for every $\varepsilon > 0$, there is a number $c \in (0,1)$ such that

$$\|y-z\| \ge \varepsilon \text{ implies } \|\varphi(y)-\varphi(z)\| \le c\|y-z\|, \quad \forall y, z \in D.$$

If, for any sequence $\{z_k\}$ in a Banach space E, $z_k \rightharpoonup \tilde{z}$ implies

$$\limsup_{k\to\infty}\|z_k - \tilde{z}\| < \limsup_{k\to\infty}\|z_k - z\|,$$

then $\forall z \in E$ with $z \ne \tilde{z}$. Then E satisfies Opial's condition [9]. Banach spaces which satisfy Opial's condition are all spaces $l^p(1 < p < \infty)$ and Hilbert spaces. However, $L^p[0, 2\pi]$ with $1 < p \ne 2$ cannot satisfy Opial's condition. It is well-known that if E admits a weak sequentially continuous duality mapping, it satisfies Opial's condition; see [10].

Lemma 2.3 ([11]) Suppose that E is a reflexive Banach space which satisfies Opial's condition, and that D is a non-empty closed convex subset of E. If S is a non-expansive mapping from D to E, then $I - S$ is demiclosed at zero; that is, $z_k \rightharpoonup z$ and $\|z_k - Sz_k\| \to 0$, and hence $z = Sz$. Moreover, Gu inself space [20] extended this conclusion from a non-expansive mapping to a asymptotic non-expansive mapping.

Lemma 2.4 ([12]) Suppose that $\mathcal{F}$ is a β-Lipschitzian and a δ-strongly monotone operator in a q-uniformly smooth Banach space E such that $\beta > 0$, $\delta > 0$, $0 < a < 1$ and $0 < \nu < \min\{(\frac{q\delta}{D_q\beta^q})^{\frac{1}{q-1}}, 1\}$. Then $T = (I - a\nu\mathcal{F}) : E \to E$ is a contractive mapping with a coefficient $1 - \alpha\theta$ and $\theta = \frac{q\nu\delta - D_q\nu^q\beta^q}{q}$.

Suppose that E is a q-uniformly smooth Banach space, and that $\mathcal{F}: E \to E$ is a β-Lipschitzian and a δ-strongly monotone operator such that $\beta > 0, \delta > 0$, and $S : E \to E$ is a non-expansive mapping. Let $\varphi : E \to E$ be an MKC with $0 < a < 1$, $0 < \nu < \min\{(\frac{q\delta}{D_q\beta^q})^{\frac{1}{q-1}}, 1\}$, $0 < \lambda < \frac{q\nu\delta - D_q\nu^q\beta^q}{q} = \theta$. A mapping T_a on E is then defined as

$$T_a x = a\lambda\varphi(x) + (I - a\nu\mathcal{F})Sx, \quad x \in E.$$

It is easy to get that T_a is a contractive mapping. In fact, by Lemma 2.4, we obtain that

$$\begin{aligned}\|T_a x - T_a z\| &\le a\lambda\|\varphi(x) - \varphi(z)\| + \|(I - a\nu\mathcal{F})Sx - (I - a\nu\mathcal{F})Sz\| \\ &\le a\lambda\|x - z\| + (1 - a\theta)\|x - z\| \\ &= [1 - a(\theta - \lambda)]\|x - z\|\end{aligned}$$

for all $x, z \in E$. Therefore, we have a unique point x_a which is a unique solution of the fixed point equation

$$x_a = a\lambda\varphi(x_a) + (I - a\nu\mathcal{F})Sx_a, \quad x_a \in E. \tag{2.2}$$

Lemma 2.5 Suppose that E is a q-uniformly smooth Banach space which admits a weak sequentially continuous duality mapping $j_q : E \to E^*$. Given that $S : E \to E$ is a non-expansive mapping with $\Gamma(S) \neq \emptyset$ and that $\varphi : E \to E$ is an MKC, $\mathcal{F}$ is a β-lipschitzian and a δ-strongly monotone operator on E. Let $0 < \lambda < \theta$. Then, $\{x_a\}$ given by $x_a = a\lambda\varphi(x_a) + (I - a\nu\mathcal{F})Sx_a$ (as $a \to 0$) converges strongly to a fixed point p of S, which is a unique solution of the following variational inequality:

$$\langle(\nu\mathcal{F} - \lambda\varphi)p, j_q(p - x)\rangle \le 0, \quad x \in \Gamma(S). \tag{2.3}$$

Proof Assume that both $\tilde{z} \in \Gamma(S)$ and $\hat{z} \in \Gamma(S)$ are solutions of (2.3). Without loss of generality, we suppose that there exists a constant σ such that $\|\hat{z} - \tilde{z}\| \ge \sigma$. Therefore, from Lemma 2.2, there exists a constant c such that $\|\varphi(\hat{z}) - \varphi(\tilde{z})\| \le c\|\hat{z} - \tilde{z}\|$. By (2.3), we have that

$$\langle(\nu\mathcal{F} - \lambda\varphi)\tilde{z}, j_q(\tilde{z} - \hat{z})\rangle \le 0, \qquad \langle(\nu\mathcal{F} - \lambda\varphi)\hat{z}, j_q(\hat{z} - \tilde{z})\rangle \le 0. \tag{2.4}$$

Adding up (2.4), we get that

$$\langle(\nu\mathcal{F} - \lambda\varphi)\hat{z} - (\nu\mathcal{F} - \lambda\varphi)\tilde{z}, j_q(\hat{z} - \tilde{z})\rangle \le 0.$$

We observe that

$$\begin{aligned}\langle(\nu\mathcal{F} - \lambda\varphi)\hat{z} - (\nu\mathcal{F} - \lambda\varphi)\tilde{z}, j_q(\hat{z} - \tilde{z})\rangle &= \langle\nu\mathcal{F}\hat{z} - \nu\mathcal{F}\tilde{z}, j_q(\hat{z} - \tilde{z})\rangle - \langle\lambda\varphi(\hat{z}) - \lambda\varphi(\tilde{z}), j_q(\hat{z} - \tilde{z})\rangle \\ &\ge \nu\delta\|\hat{z} - \tilde{z}\|^q - \lambda\|\varphi(\hat{z}) - \varphi(\tilde{z})\|\|\hat{z} - \tilde{z}\|^{q-1} \\ &\ge \nu\delta\|\hat{z} - \tilde{z}\|^q - \lambda c\|\hat{z} - \tilde{z}\|^q \\ &\ge (\nu\delta - \lambda c)\|\hat{z} - \tilde{z}\|^q \\ &\ge (\nu\delta - \lambda c)\sigma^q \\ &> 0.\end{aligned}$$

Hence, $\hat{z} = \tilde{z}$, and we have proved the uniqueness. In the sequel, the unique solution of (2.3) is denoted by $\tilde{z}$.

Notice that $\{x_a\}$ is bounded. In fact, fix $z \in \Gamma(S)$ and $\sigma_1 > 0$ for every $a \in (0, 1)$. When $(\|x_a - z\| \ge \sigma_1)$, from Lemma 2.2, we know that there exists a constant c_1 such that

$$\|\varphi(x_a) - \varphi(z)\| \le c_1\|x_a - z\|,$$

and thus, from Lemma 2.4, we get that

$$\begin{aligned}\|x_a - z\| &= \|a\lambda\varphi(x_a) + (I - a\nu\mathcal{F})Sx_a - z\| \\ &= \|a(\lambda\varphi(x_a) - \nu\mathcal{F}z) + (I - a\nu\mathcal{F})Sx_a - (I - a\nu\mathcal{F})z\| \\ &\le a\|\lambda\varphi(x_a) - \nu\mathcal{F}z\| + (1 - a\theta)\|x_a - z\| \\ &\le a\|\lambda\varphi(x_a) - \lambda\varphi(z)\| + a\|\lambda\varphi(z) - \nu\mathcal{F}z\| + (1 - a\theta)\|x_a - z\| \\ &\le a\lambda c_1\|x_a - z\| + a\|\lambda\varphi(z) - \nu\mathcal{F}z\| + (1 - a\theta)\|x_z - z\|,\end{aligned}$$

and therefore, $\|x_a - z\| \le \|\lambda\varphi(z) - \nu\mathcal{F}z\|/(\theta - \lambda c_1)$. We have, in any case, that

$$\|x_a - z\| \le \max\{\sigma_1, \|\lambda\varphi(z) - \nu\mathcal{F}z\|/(\theta - \lambda c_1)\},$$

so $\{x_a\}$ is bounded.

Next, we show that $x_a \to \tilde{z}(\tilde{z} \in \Gamma(S))$ as $a \to 0$.

Owing to the fact that E is reflexive and that $\{x_a\}$ is bounded, there is a subsequence $\{x_{a_k}\}$ of $\{x_a\}$ such that $x_{a_k} \rightharpoonup z^*$. By $x_a - Sx_a = a(\lambda\varphi(x_a) - \nu\mathcal{F}Sx_a)$, we get that $x_{a_k} - Sx_{a_k} \to 0$ as $a_k \to 0$. In addition, because E satisfies Opial's condition, by Lemma 2.3 we have $z^* \in \Gamma(S)$. We show that

$$\|x_{a_k} - z^*\| \to 0. \tag{2.5}$$

By the method of contradiction, there exists a constant σ_0 and a subsequence $\{x_{a_t}\}$ of $\{x_{a_k}\}$ such that $\|x_{a_t} - z^*\| \ge \sigma_0$. From Lemma 2.2, there exists a constant c_{σ_0} such that $\|\varphi(x_{a_t}) - \varphi(z^*)\| \le c_{\sigma_0}\|x_{a_t} - z^*\|$. We observe that

$$x_{a_t} - z^* = a_t(\lambda\varphi(x_{a_t}) - \nu\mathcal{F}z^*) + (I - a_t\nu\mathcal{F})Sx_{a_t} - (I - a_t\nu\mathcal{F})z^*, \tag{2.6}$$

and from this, we get that

$$\begin{aligned}\|x_{a_t} - z^*\|^q &= a_t\langle\lambda\varphi(x_{a_t}) - \nu\mathcal{F}z^*, j_q(x_{a_t} - z^*)\rangle \\ &\quad + \langle(I - a_t\nu\mathcal{F})Sx_{a_t} - (I - a_t\nu\mathcal{F})z^*, j_q(x_{a_t} - z^*)\rangle \\ &\le a_t\langle\lambda\varphi(x_{a_t}) - \nu\mathcal{F}z^*, j_q(x_{a_t} - z^*)\rangle + (1 - a_t\theta)\|x_{a_t} - z^*\|^q.\end{aligned} \tag{2.7}$$

Thus, it can be seen that

$$\begin{aligned}\|x_{a_t} - z^*\|^q &\le \frac{1}{\theta}\langle\lambda\varphi(x_{a_t}) - \nu\mathcal{F}z^*, j_q(x_{a_t} - z^*)\rangle \\ &\le \frac{1}{\theta}\langle\lambda\varphi(x_{a_t}) - \lambda\varphi(z^*), j_q(x_{a_t} - z^*)\rangle + \frac{1}{\theta}\langle\lambda\varphi(z^*) - \nu\mathcal{F}z^*, j_q(x_{a_t} - z^*)\rangle \\ &\le \frac{1}{\theta}[\lambda c_{\sigma_0}\|x_{a_t} - z^*\|^q + \langle\lambda\varphi(z^*) - \nu\mathcal{F}z^*, j_q(x_{a_t} - z^*)\rangle].\end{aligned} \tag{2.8}$$

Hence,

$$\|x_{a_t} - z^*\|^q \le \frac{\langle\lambda\varphi(z^*) - \nu\mathcal{F}z^*, j_q(x_{a_t} - z^*)\rangle}{\theta - \lambda c_{\sigma_0}}. \tag{2.9}$$

Using the fact that the duality mapping $j_q : E \to E^*$ is single valued and weakly sequentially continuous, from (2.9) we have that $x_{a_t} \to z^*$. This is a contradiction. Therefore, we obtain $x_{a_k} \to z^*$.

Now, we show that z^* is a solution of the variational inequality (2.3). Because

$$x_a = a\lambda\varphi(x_a) + (I - a\nu\mathcal{F})Sx_a, \tag{2.10}$$

we get that

$$(\nu\mathcal{F}-\lambda\varphi)x_a=-\frac{1}{a}[(I-S)x_a-a\nu\mathcal{F}x_a+a\nu\mathcal{F}Sx_a]. \tag{2.11}$$

We notice that

$$\begin{aligned}\langle (I-S)x_a-(I-S)x, j_q(x_a-x)\rangle &= \|x_a-x\|^q-\langle Sx_a-Sx, j_q(x_a-x)\rangle\\ &\geq \|x_a-x\|^q-\|Sx_a-Sx\|\|x_a-x\|^{q-1}\\ &\geq \|x_a-x\|^q-\|x_a-x\|^q\\ &\geq 0.\end{aligned} \tag{2.12}$$

Therefore, for $x\in\Gamma(S)$,

$$\begin{aligned}\langle(\nu\mathcal{F}-\lambda\varphi)x_a, j_q(x_a-x)\rangle &= -\frac{1}{a}\langle (I-S)x_a-a\nu\mathcal{F}x_a+a\nu\mathcal{F}Sx_a, j_q(x_a-x)\rangle\\ &= -\frac{1}{a}\langle (I-S)x_a-(I-S)x, j_q(x_a-x)\rangle\\ &\quad +\langle(\nu\mathcal{F}-\nu\mathcal{F}S)x_a, j_q(x_a-x)\rangle\\ &\leq \langle(\nu\mathcal{F}-\nu\mathcal{F}S)x_a, j_q(x_a-x)\rangle.\end{aligned} \tag{2.13}$$

Now, let us replace a in (2.13) with a_k, and letting $k\to\infty$, we observe that $(\nu\mathcal{F}-\nu\mathcal{F}S)x_{a_k}\to(\nu\mathcal{F}-\nu\mathcal{F}S)z^*=0$ for $z^*\in\Gamma(S)$, so we get

$$\langle(\nu\mathcal{F}-\lambda\varphi)z^*, j_q(z^*-x)\rangle\leq 0.$$

Thus $z^*\in\Gamma(S)$, which is a solution of (2.3). Therefore, $z^*=\tilde{z}$, by uniqueness. In conclusion, we have proved that every cluster point of $\{x_a\}$ (at $a\to 0$) equals $\tilde{z}$, and hence, $x_a\to\tilde{z}$ as $a\to 0$. □

Lemma 2.6 ([13]) Suppose that $\{y_k\}$ and $\{z_k\}$ are bounded sequences in a Banach space E, and that $\{w_k\}$ is a sequence in $[0,1]$ which adheres to the following condition:

$$0<\liminf_{k\to\infty} w_k\leq\limsup_{k\to\infty} w_k<1.$$

Let $y_{k+1}=w_ky_k+(1-w_k)z_k, k\geq 0$, and let $\limsup\limits_{k\to\infty}(\|z_{k+1}-z_k\|-\|y_{k+1}-y_k\|)\leq 0$. Then $\lim\limits_{k\to\infty}\|z_k-y_k\|=0$.

Lemma 2.7 ([14, 15]) Suppose that $\{t_k\}$ is a sequence of non-negative real numbers which satisfies that

$$t_{k+1}\leq(1-\alpha_k)t_k+\alpha_k\beta_k+\xi_k,\quad k\geq 0.$$

Here $\{\alpha_k\}$, $\{\beta_k\}$ and $\{\xi_k\}$ adhere to the following conditions:

(i) $\{\alpha_k\}\subset[0,1]$ and $\sum\limits_{k=0}^{\infty}\alpha_k=\infty$;

(ii) $\limsup\limits_{k\to\infty}\beta_k\leq 0$ or $\sum\limits_{k=0}^{\infty}\alpha_k\beta_k<\infty$;

(iii) $\xi_k\geq 0(k\geq 0)$, $\sum\limits_{k=0}^{\infty}\xi_k<\infty$.

Thus $\lim\limits_{k\to\infty}t_k=0$.

Lemma 2.8 ([16]) Suppose that E is a q-uniformly smooth Banach space, that D is a non-empty closed convex subset of E, and that $S:D\to D$ is a γ-strict pseudo-contraction. Given a mapping S' on D with $S'z=rz+(1-r)Sz$, $\forall z\in D$ and $r\in(0,\nu]$, $\nu=\min\{1,\{\frac{q\gamma}{D_q}\}^{\frac{1}{q-1}}\}$, so S' is called a non-expansive mapping, and $F(S')=F(S)$.

We consider the mapping U_k given by

$$\begin{cases} W_{k,k+1} = I, \\ W_{k,k} = \lambda_k S'_k W_{k,k+1} + (1-\lambda_k)I, \\ W_{k,k-1} = \lambda_{k-1} S'_{k-1} W_{k,k} + (1-\lambda_{k-1})I, \\ \vdots \\ W_{k,j} = \lambda_j S'_j W_{k,j+1} + (1-\lambda_j)I, \\ W_{k,j-1} = \lambda_{j-1} S'_{j-1} W_{k,j} + (1-\lambda_{j-1})I, \\ \vdots \\ W_{k,2} = \lambda_2 S'_2 W_{k,3} + (1-\lambda_2)I, \\ U_k = W_{k,1} = \lambda_1 S'_1 W_{k,2} + (1-\lambda_1)I. \end{cases} \tag{2.14}$$

Here $\{\lambda_k\}$ is a real sequence with $0 \le \lambda_k \le 1$, $S'_i = \alpha_i I + (1-\alpha_i)S_i$, and $S_i : D \to D$ is a γ_i-strict pseudo-contraction with $\alpha_i \in (0, \min\{1, (\frac{\gamma_i q}{D_q})^{\frac{1}{q-1}}\})$. From Lemma 2.8, we can obtain that S'_i is a non-expansive mapping such that $F(S_i) = F(S'_i)$. Therefore, it is easy see that U_k is a non-expansive mapping.

With respect to U_k, we obtain the following important lemmas:

Lemma 2.9 ([17]) Suppose that E is a strictly convex Banach space, and that D is a non-empty closed convex subset of E. Set that $\{S'_i : D \to D\}$ is a family of infinite non-expansive mappings with $\bigcap_{i=1}^{\infty} \Gamma(S'_i) \neq \emptyset$, and that $\{\lambda_i\}$ is a real sequence such that $0 < \lambda_i \le m < 1$ for every $i = 1, 2, \cdots$. Then, for any $z \in D$ and $j \in N$, $\lim_{k\to\infty} U_{k,j} z$ exists.

By Lemma 2.9, a mapping $U : D \to D$ is defined as follows:

$$Uz := \lim_{k\to\infty} U_k z = \lim_{k\to\infty} W_{k,1} z, \quad z \in D.$$

Such a mapping U is said to be the modified U-mapping obtained by $S_1, S_2, \cdots, \lambda_1, \lambda_2, \cdots$ and $\alpha_1, \alpha_2, \cdots$.

Lemma 2.10 ([17]) Suppose that E is a strictly convex Banach space, and that D is a non-empty closed convex subset of E. Set that $\{S'_i : D \to D\}$ is a family of infinite non-expansive mappings with $\bigcap_{i=1}^{\infty} \Gamma(S'_i) \neq \emptyset$, and that $\{\lambda_i\}$ is a real sequence such that $0 < \lambda_i \le m < 1$, $\forall i \ge 1$. Then $\Gamma(U) = \bigcap_{i=1}^{\infty} \Gamma(S'_i)$.

From Lemmas 2.8–2.10, we get that $\Gamma(U) = \bigcap_{i=1}^{\infty} \Gamma(S'_i) = \bigcap_{i=1}^{\infty} \Gamma(S_i)$.

Lemma 2.11 ([18]) Suppose that E is a strictly convex Banach space, and that D is a non-empty closed convex subset of E. Set that $S'_1, S'_2, \cdots$ are non-expansive mappings of D into itself such that $\bigcap_{i=1}^{\infty} \Gamma(S'_i) \neq \emptyset$, and that $\lambda_1, \lambda_2, \cdots$ are real numbers such that $0 < \lambda_i \le m < 1$, $\forall i \ge 1$. Then, if $\mathcal{G}$ is any bounded subset of D, we have that

$$\lim_{k\to\infty} \sup_{z\in\mathcal{G}} \|Uz - U_k z\| = 0.$$

3 Main Result

Now we study the strong convergence results for an infinite family of strict pseudo-contractions in a q-uniformly smooth and strictly convex Banach space.

Theorem 3.1 Suppose that E is a q-uniformly smooth and strictly convex Banach space which admits a weak sequentially continuous duality mapping $j_q : E \to E^*$. Set that $\{S_i : E \to E\}$ is a γ_i-strict pseudo-contraction such that $\bigcap_{i=1}^{\infty} \Gamma(S_i) \neq \emptyset$, and that $\{\lambda_i\}$ is a real sequence with $0 < \lambda_i \leq m < 1$, $\forall i \geq 1$. Take that $\mathcal{F}$ is a β-Lipschitzian and a δ-strongly monotone operator on E such that $0 < \nu < \min\{(\frac{q\delta}{D_q\beta^q})^{\frac{1}{q-1}}, 1\}$, and that φ is an MKC on E with $0 < \lambda < \frac{q\nu\delta - D_q\nu^q\beta^q}{q} = \theta$. Let $\{\beta_k\}$ and $\{\tau_k\} \subset (0,1)$ be sequences which adhere to the following conditions:

(D$_1$) $\lim\limits_{k\to\infty} \beta_k = 0$;

(D$_2$) $\sum\limits_{k=1}^{\infty} \beta_k = \infty$;

(D$_3$) $0 < \liminf\limits_{k\to\infty} \tau_k \leq \limsup\limits_{k\to\infty} \tau_k \leq b < 1$, and $b \in (0,1)$.

Then, $\{x_k\}$ generated by (1.8) converges strongly to $\tilde{z} \in \bigcap_{i=1}^{\infty} \Gamma(S_i)$, which uniquely solves the variational inequality

$$\langle (\lambda\varphi - \nu\mathcal{F})\tilde{z}, j_q(x - \tilde{z})\rangle \leq 0, \quad x \in \bigcap_{i=1}^{\infty} \Gamma(S_i).$$

Proof The rest of our proof consists of the following five steps:

Step 1 We prove that $\{x_k\}$ is bounded. Actually, letting $x \in \bigcap_{i=1}^{\infty} \Gamma(S_i)$, it follows from (1.8) that

$$\begin{aligned} \|z_k - x\| &= \|\tau_k(x_k - x) + (1-\tau_k)(U_k x_k - x)\| \\ &\leq \tau_k\|x_k - x\| + (1-\tau_k)\|U_k x_k - x\| \\ &\leq \|x_k - x\|. \end{aligned} \tag{3.1}$$

Thus, by (1.8), (3.1) and Lemma 2.4, we have that

$$\begin{aligned} \|x_{k+1} - x\| &= \|\beta_k\lambda\varphi(x_k) + (I - \nu\beta_k\mathcal{F})z_k - x\| \\ &= \|\beta_k\lambda\varphi(x_k) - \nu\beta_k\mathcal{F}x + \nu\beta_k\mathcal{F}x + (I - \nu\beta_k\mathcal{F})z_k - x\| \\ &= \|\beta_k(\lambda\varphi(x_k) - \nu\mathcal{F}x) + (I - \nu\beta_k\mathcal{F})z_k - (I - \nu\beta_k\mathcal{F})x\| \\ &\leq (1 - \beta_k\theta)\|z_k - x\| + \beta_k[\|\lambda\varphi(x_k) - \lambda\varphi(x)\| + \|\lambda\varphi(x) - \nu\mathcal{F}x\|] \\ &\leq (1 - \beta_k\theta)\|x_k - x\| + \beta_k\lambda\|x_k - x\| + \beta_k\|\lambda\varphi(x) - \nu\mathcal{F}x\| \\ &\leq [1 - \beta_k(\theta - \lambda)]\|x_k - x\| + \beta_k\|\lambda\varphi(x) - \nu\mathcal{F}x\| \\ &\leq [1 - \beta_k(\theta - \lambda))]\|x_k - x\| + \beta_k(\theta - \lambda)\frac{\|\lambda\varphi(x) - \nu\mathcal{F}x\|}{\theta - \lambda} \\ &\leq \max\{\|x_k - x\|, \frac{\|\lambda\varphi(x) - \nu\mathcal{F}x\|}{\theta - \lambda}\}, \quad k \geq 1. \end{aligned}$$

By induction, we get that

$$\|x_k - x\| \leq \max\{\|x_1 - x\|, \frac{\|\lambda\varphi(x) - \nu\mathcal{F}x\|}{\theta - \lambda}\}, \quad k \geq 1,$$

so we have that $\{x_k\}$ is bounded. We also get that $\{z_k\}$, $\{U_kx_k\}$, $\{\nu\mathcal{F}z_k\}$ and $\varphi(x_k)$ are all bounded. Without loss of generality, we suppose that $\{x_k\},\{z_k\},\{U_kx_k\},\{\nu\mathcal{F}z_k\},\varphi(x_k)\subset\mathcal{G}$, where $\mathcal{G}$ is a bounded set of E.

Step 2 We claim that $\lim\limits_{k\to\infty}\|x_{k+1}-x_k\|=0$. To this end, set $y_k=(x_{k+1}-\tau_kx_k)/(1-\tau_k)$ such that $x_{k+1}=\tau_kx_k+(1-\tau_k)y_k$. We note that

$$\begin{aligned}
y_{k+1}-y_k&=\frac{x_{k+2}-\tau_{k+1}x_{k+1}}{1-\tau_{k+1}}-\frac{x_{k+1}-\tau_kx_k}{1-\tau_k}\\
&=\frac{\beta_{k+1}\lambda\varphi(x_{k+1})+(I-\nu\beta_{k+1}\mathcal{F})z_{k+1}-\tau_{k+1}x_{k+1}}{1-\tau_{k+1}}\\
&\quad-\frac{\beta_k\lambda\varphi(x_k)+(I-\nu\beta_k\mathcal{F})z_k-\tau_kx_k}{1-\tau_k}\\
&=\frac{\beta_{k+1}}{1-\tau_{k+1}}(\lambda\varphi(x_{k+1})-\nu\mathcal{F}z_{k+1})+\frac{z_{k+1}-\tau_{k+1}x_{k+1}}{1-\tau_{k+1}}\\
&\quad-\frac{\beta_k}{1-\tau_k}(\lambda\varphi(x_k)-\nu\mathcal{F}z_k)-\frac{z_k-\tau_kx_k}{1-\tau_k}\\
&=\frac{\beta_{k+1}}{1-\tau_{k+1}}(\lambda\varphi(x_{k+1})-\nu\mathcal{F}z_{k+1})+\frac{[\tau_{k+1}x_{k+1}+(1-\tau_{k+1})U_{k+1}x_{k+1}]-\tau_{k+1}x_{k+1}}{1-\tau_{k+1}}\\
&\quad-\frac{\beta_k}{1-\tau_k}(\lambda\varphi(x_k)-\nu\mathcal{F}z_k)-\frac{[\tau_kx_k+(1-\tau_k)U_kx_k]-\tau_kx_k}{1-\tau_k}\\
&=\frac{\beta_{k+1}}{1-\tau_{k+1}}(\lambda\varphi(x_{k+1})-\nu\mathcal{F}z_{k+1})-\frac{\beta_k}{1-\tau_k}(\lambda\varphi(x_k)-\nu\mathcal{F}z_k)+U_{k+1}x_{k+1}-U_kx_k.
\end{aligned}\tag{3.2}$$

Then, by (3.2), we get that

$$\begin{aligned}
\|y_{k+1}-y_k\|\le{}&\frac{\beta_{k+1}}{1-\tau_{k+1}}(\|\lambda\varphi(x_{k+1})\|+\|\nu\mathcal{F}z_{k+1}\|)+\frac{\beta_k}{1-\tau_k}(\|\lambda\varphi(x_k)\|+\|\nu\mathcal{F}z_k\|)\\
&+\|U_{k+1}x_{k+1}-U_kx_k\|
\end{aligned}\tag{3.3}$$

for all $k\ge1$.

From (2.14), we obtain that

$$\begin{aligned}
\|U_{k+1}x_k-U_kx_k\|&=\|\lambda_1S_1'W_{k+1,2}x_k-\lambda_1S_1'W_{k,2}x_k\|\\
&\le\lambda_1\|W_{k+1,2}x_k-W_{k,2}x_k\|\\
&=\lambda_1\|\lambda_2S_2'W_{k+1,3}x_k-\lambda_2S_2'W_{k,3}x_k\|\\
&\le\lambda_1\lambda_2\|W_{k+1,3}x_k-W_{k,3}x_k\|\\
&\le\cdots\le\lambda_1\lambda_2\cdots\lambda_k\|W_{k+1,k+1}x_k-W_{k,k+1}x_k\|\\
&\le L_1\prod_{i=1}^k\lambda_i.
\end{aligned}$$

Here, $L_1\ge0$ is a constant which satisfies that $\|W_{k+1,k+1}x_k-W_{k,k+1}x_k\|\le L_1$, $\forall k\ge1$.

Therefore, we obtain that

$$\begin{aligned}
\|U_{k+1}x_{k+1}-U_kx_k\|&\le\|U_{k+1}x_{k+1}-U_{k+1}x_k\|+\|U_{k+1}x_k-U_kx_k\|\\
&\le\|x_{k+1}-x_k\|+\|U_{k+1}x_k-U_kx_k\|\\
&\le\|x_{k+1}-x_k\|+L_1\prod_{i=1}^k\lambda_i.
\end{aligned}\tag{3.4}$$

Putting (3.4) into (3.3), we get that

$$\|y_{k+1}-y_k\| \le L_2(\frac{\beta_{k+1}}{1-\tau_{k+1}}+\frac{\beta_k}{1-\tau_k})+\|x_{k+1}-x_k\|+L_1\prod_{i=1}^{k}\lambda_i. \tag{3.5}$$

Here, $L_2=\sup\{\|\lambda\varphi(x_k)\|+\|\nu\mathcal{F}z_k\|, k\ge 1\}$. Then, by (3.5), we obtain that

$$\|y_{k+1}-y_k\|-\|x_{k+1}-x_k\| \le L_2(\frac{\beta_{k+1}}{1-\tau_{k+1}}+\frac{\beta_k}{1-\tau_k})+L_1\prod_{i=1}^{k}\lambda_i. \tag{3.6}$$

Noticing the conditions (D_1), (D_3), (3.6) and $0<\lambda_i\le m<1$, we have that

$$\limsup_{k\to\infty}(\|y_{k+1}-y_k\|-\|x_{k+1}-x_k\|)\le 0.$$

Therefore, from Lemma 2.6, we get that

$$\lim_{k\to\infty}\|y_k-x_k\|=0. \tag{3.7}$$

From (D_3) and (3.7), we obtain that

$$\lim_{k\to\infty}\|x_{k+1}-x_k\|=\lim_{k\to\infty}(1-\tau_k)\|y_k-x_k\|=0.$$

Step 3 We show that $\lim_{k\to\infty}\|x_k-Ux_k\|=0$. Note that

$$\begin{aligned}\|x_k-U_kx_k\| &\le \|x_k-x_{k+1}\|+\|x_{k+1}-z_k\|+\|z_k-U_kx_k\|\\ &=\|x_k-x_{k+1}\|+\|x_{k+1}-z_k\|+\tau_k\|x_k-U_kx_k\|.\end{aligned}$$

By Step 2, (D_1) and (D_3), we get that

$$\begin{aligned}(1-b)\|x_k-U_kx_k\| \le (1-\tau_k)\|x_k-U_kx_k\| &\le \|x_k-x_{k+1}\|+\|x_{k+1}-z_k\|\\ &\le \|x_k-x_{k+1}\|+\beta_k\|\lambda\varphi(x_k)-\nu\mathcal{F}z_k\|\to 0 \text{ as } (k\to\infty),\end{aligned}$$

which means that

$$\|x_k-U_kx_k\|\to 0 \ \ (\text{as } k\to\infty). \tag{3.8}$$

In addition, we know that

$$\begin{aligned}\|x_k-Ux_k\| &\le \|x_k-U_kx_k\|+\|U_kx_k-Ux_k\|\\ &\le \|x_k-U_kx_k\|+\sup_{z\in\mathcal{G}}\|U_kz-Uz\|.\end{aligned} \tag{3.9}$$

From (3.8), (3.9) and Lemma 2.11, we get that

$$\lim_{k\to\infty}\|x_k-Ux_k\|=0.$$

Step 4 We show that $\limsup_{k\to\infty}\langle\lambda\varphi\tilde z-\nu\mathcal{F}\tilde z, j_q(x_k-\tilde z)\rangle\le 0$; here, $\tilde z=\lim_{a\to 0}x_a$ with $x_a=a\lambda\varphi(x_a)+(I-a\nu\mathcal{F})Ux_a$.

Due to the fact that $\{x_k\}$ is bounded, there is a subsequence $\{x_{k_t}\}$ of $\{x_k\}$ which converges weakly to x, and such that $\limsup_{k\to\infty}\langle\lambda\varphi\tilde z-\nu\mathcal{F}\tilde z, j_q(x_k-\tilde z)\rangle=\lim_{t\to\infty}\langle\lambda\varphi\tilde z-\nu\mathcal{F}\tilde z, j_q(x_{k_t}-\tilde z)\rangle$. From $\|x_k-Ux_k\|\to 0$, we obtain that $Ux_{k_t}\rightharpoonup x$. Since E admits a weak sequentially continuous duality mapping, it satisfies Opial's condition; see [10]. By Lemma 2.3, we obtain that $x\in\Gamma(U)$.

Therefore, from Lemma 2.5 and the fact that j_q is a weakly sequentially continuous duality mapping, we have that

$$\begin{aligned}\limsup_{k\to\infty}\langle\lambda\varphi\tilde{z}-\nu\mathcal{F}\tilde{z},j_q(x_k-\tilde{z})\rangle&=\lim_{t\to\infty}\langle\lambda\varphi\tilde{z}-\nu\mathcal{F}\tilde{z},j_q(x_{k_t}-\tilde{z})\rangle\\&=\langle\lambda\varphi\tilde{z}-\nu\mathcal{F}\tilde{z},j_q(x-\tilde{z})\rangle\leq 0.\end{aligned}\tag{3.10}$$

Step 5 We prove that $\{x_k\}$ converges strongly to $\tilde{z}$. By contradiction, there exists a constant $\sigma_0>0$ such that

$$\limsup_{k\to\infty}\|x_k-\tilde{z}\|\geq\sigma_0.$$

Case 1 Fix $\sigma_1(\sigma_1<\sigma_0)$. Fix, for some $k\geq N\in\mathbb{N}$ such that $\|x_k-\tilde{z}\|\geq\sigma_0-\sigma_1$, and for the other $k\geq N\in\mathbb{N}$ such that $\|x_k-\tilde{z}\|<\sigma_0-\sigma_1$, set

$$L_k=\frac{q\langle\lambda\varphi\tilde{z}-\lambda\mathcal{F}\tilde{z},j_q(x_{k+1}-\tilde{z})\rangle}{(\sigma_0-\sigma_1)^q}.$$

By (3.10), we have that $\limsup\limits_{k\to\infty}L_k\leq 0$. Thus, there exists a number N, when $k>N$, such that $L_k\leq\theta-\lambda$. We extract a number $k_0\geq N$ which satisfies $\|x_{k_0}-\tilde{z}\|<\sigma_0-\sigma_1$. Thus we can estimate $\|x_{k_0+1}-\tilde{z}\|$. Note that

$$\begin{aligned}\|x_{k_0+1}-\tilde{z}\|^q&=\|\beta_{k_0}\lambda\varphi(x_{k_0})+(I-\nu\beta_{k_0}\mathcal{F})z_{k_0}-\tilde{z}\|^q\\&=\langle(I-\nu\beta_{k_0}\mathcal{F})z_{k_0}-(I-\nu\beta_{k_0}\mathcal{F})\tilde{z}+\beta_{k_0}[\lambda\varphi(x_{k_0})-\nu\mathcal{F}\tilde{z}],j_q(x_{k_0+1}-\tilde{z})\rangle\\&=\langle(I-\nu\beta_{k_0}\mathcal{F})z_{k_0}-(I-\nu\beta_{k_0}\mathcal{F})\tilde{z},j_q(x_{k_0+1}-\tilde{z})\rangle\\&\quad+\beta_{k_0}\langle\lambda\varphi(x_{k_0})-\nu\mathcal{F}\tilde{z},j_q(x_{k_0+1}-\tilde{z})\rangle\\&=\langle(I-\nu\beta_{k_0}\mathcal{F})z_{k_0}-(I-\nu\beta_{k_0}\mathcal{F})\tilde{z},j_q(x_{k_0+1}-\tilde{z})\rangle\\&\quad+\beta_{k_0}\langle\lambda\varphi(x_{k_0})-\lambda\varphi(\tilde{z}),j_q(x_{k_0+1}-\tilde{z})\rangle+\beta_{k_0}\langle\lambda\varphi(\tilde{z})-\nu\mathcal{F}\tilde{z},j_q(x_{k_0+1}-\tilde{z})\rangle\\&\leq(1-\beta_{k_0}\theta)\|x_{k_0}-\tilde{z}\|\|x_{k_0+1}-\tilde{z}\|^{q-1}+\beta_{k_0}\lambda\|\varphi(x_{k_0})-\varphi(\tilde{z})\|\|x_{k_0+1}-\tilde{z}\|^{q-1}\\&\quad+\beta_{k_0}\langle\lambda\varphi(\tilde{z})-\nu\mathcal{F}\tilde{z},j_q(x_{k_0+1}-\tilde{z})\rangle\\&<[1-\beta_{k_0}(\theta-\lambda)](\sigma_0-\sigma_1)\|x_{k_0+1}-\tilde{z}\|^{q-1}+\beta_{k_0}\langle\lambda\varphi(\tilde{z})-\nu\mathcal{F}\tilde{z},j_q(x_{k_0+1}-\tilde{z})\rangle\\&\leq\frac{1}{q}[1-\beta_{k_0}(\theta-\lambda)]^q(\sigma_0-\sigma_1)^q+\frac{q-1}{q}\|x_{k_0+1}-\tilde{z}\|^q\\&\quad+\beta_{k_0}\langle\lambda\varphi(\tilde{z})-\nu\mathcal{F}\tilde{z},j_q(x_{k_0+1}-\tilde{z})\rangle.\end{aligned}$$

From Young's inequality, we derive that

$$\begin{aligned}\|x_{k_0+1}-\tilde{z}\|^q&<[1-\beta_{k_0}(\theta-\lambda)]^q(\sigma_0-\sigma_1)^q+q\beta_{k_0}\langle\lambda\varphi(\tilde{z})-\nu\mathcal{F}\tilde{z},j_q(x_{k_0+1}-\tilde{z})\rangle\\&<[1-\beta_{k_0}(\theta-\lambda)](\sigma_0-\sigma_1)^q+q\beta_{k_0}\langle\lambda\varphi(\tilde{z})-\nu\mathcal{F}\tilde{z},j_q(x_{k_0+1}-\tilde{z})\rangle\\&=[1-\beta_{k_0}(\theta-\lambda-L_k)](\sigma_0-\sigma_1)^q\\&\leq(\sigma_0-\sigma_1)^q.\end{aligned}$$

Thus, we obtain that

$$\|x_{k_0+1}-\tilde{z}\|<\sigma_0-\sigma_1.$$

By induction, we have that

$$\|x_k-\tilde{z}\|<\sigma_0-\sigma_1,\quad\forall k\geq k_0.$$

This contradicts the fact that the $\limsup\limits_{k\to\infty}\|x_k-\tilde{z}\|\geq\sigma_0$.

Case 2 Fix $\sigma_1\,(\sigma_1 < \sigma_0)$. Set $\|x_k - \tilde{z}\| \geq \sigma_0 - \sigma_1$, $\forall k \geq N \in \mathbb{N}$. By Lemma 2.2, there exists a constant $c \in (0,1)$ such that

$$\|\varphi(x_k) - \varphi(\tilde{z})\| \leq c\|x_k - \tilde{z}\|, \quad k \geq N.$$

Following on from (1.8), we get that

$$\begin{aligned}
\|x_{k+1} - \tilde{z}\|^q &= \|\beta_k \lambda \varphi(x_k) + (I - \nu\beta_k \mathcal{F})z_k - \tilde{z}\|^q \\
&= \langle (I - \nu\beta_k\mathcal{F})z_k - (I - \nu\beta_k\mathcal{F})\tilde{z} + \beta_k[\lambda\varphi(x_k) - \nu\mathcal{F}\tilde{z}], j_q(x_{k+1} - \tilde{z})\rangle \\
&= \langle (I - \nu\beta_k\mathcal{F})z_k - (I - \nu\beta_k\mathcal{F})\tilde{z}, j_q(x_{k+1} - \tilde{z})\rangle \\
&\quad + \beta_k\langle \lambda\varphi(x_k) - \nu\mathcal{F}\tilde{z}, j_q(x_{k+1} - \tilde{z})\rangle \\
&= \langle (I - \nu\beta_k\mathcal{F})z_k - (I - \nu\beta_k\mathcal{F})\tilde{z}, j_q(x_{k+1} - \tilde{z})\rangle \\
&\quad + \beta_k\langle \lambda\varphi(x_k) - \lambda\varphi(\tilde{z}), j_q(x_{k+1} - \tilde{z})\rangle + \beta_k\langle \lambda\varphi(\tilde{z}) - \nu\mathcal{F}\tilde{z}, j_q(x_{k+1} - \tilde{z})\rangle \\
&\leq (1 - \beta_k\theta)\|x_k - \tilde{z}\|\|x_{k+1} - \tilde{z}\|^{q-1} + \beta_k\lambda c\|x_k - \tilde{z}\|\|x_{k+1} - \tilde{z}\|^{q-1} \\
&\quad + \beta_k\langle \lambda\varphi(\tilde{z}) - \nu\mathcal{F}\tilde{z}, j_q(x_{k+1} - \tilde{z})\rangle \\
&= [1 - \beta_k(\theta - \lambda c)]\|x_k - \tilde{z}\|\|x_{k+1} - \tilde{z}\|^{q-1} + \beta_k\langle \lambda\varphi(\tilde{z}) - \nu\mathcal{F}\tilde{z}, j_q(x_{k+1} - \tilde{z})\rangle \\
&\leq [1 - \beta_k(\theta - \lambda c)]\frac{1}{q}\|x_k - \tilde{z}\|^q + \frac{q-1}{q}\|x_{k+1} - \tilde{z}\|^q \\
&\quad + \beta_k\langle \lambda\varphi(\tilde{z}) - \nu\mathcal{F}\tilde{z}, j_q(x_{k+1} - \tilde{z})\rangle.
\end{aligned}$$

From Young's inequality, we derive that

$$\|x_{k+1} - \tilde{z}\|^q \leq [1 - \beta_k(\theta - \lambda c)]\|x_k - \tilde{z}\|^q + q\beta_k\langle \lambda\varphi(\tilde{z}) - \nu\mathcal{F}\tilde{z}, j_q(x_{k+1} - \tilde{z})\rangle. \tag{3.11}$$

Applying Lemma 2.7 to (3.11), we can obtain that $x_k \to \tilde{z}$, as $k \to \infty$. This contradicts the fact that $\|x_k - \tilde{z}\| \geq \sigma_0 - \sigma_1$. Hence, $\{x_k\}$ converges strongly to $\tilde{z} \in \bigcap_{i=1}^{\infty} \Gamma(S_i)$. By $\tilde{z} = \lim_{a\to 0} x_a$ and Lemma 2.5, we obtain that $\tilde{z}$ uniquely solves the variational inequality $\langle \lambda\varphi\tilde{z} - \nu\mathcal{F}\tilde{z}, j_q(x - \tilde{z})\rangle \leq 0$, $x \in \bigcap_{i=1}^{\infty} \Gamma(S_i)$. □

Lemma 3.2 ([19]) Assume that E is a q-uniformly smooth Banach space which admits a weak sequentially continuous duality mapping $j_q : E \to E^*$. Given that $S : E \to E$ is a non-expansive mapping such that $\Gamma(S) \neq \emptyset$ and that $\varphi : E \to E$ is an MKC, $\mathcal{A}$ is a strongly positive bounded linear operator with a coefficient $\theta > 0$. Suppose that $0 < \lambda < \theta$. Then the sequence $\{x_a\}$ given by $x_a = a\lambda\varphi(x_a) + (I - a\mathcal{A})Sx_a$ (as $a \to 0$) converges strongly to a fixed point $\tilde{z}$ of S, which is a unique solution of the variational inequality

$$\langle (\mathcal{A} - \lambda\varphi)\tilde{z}, j_q(\tilde{z} - x)\rangle \leq 0, \quad x \in \Gamma(S). \tag{3.12}$$

When $\mathcal{F}$ reduces to a strongly positive bounded linear operator $\mathcal{A}$ and $\nu = 1$ in (1.8), we can obtain the following results:

Corollary 3.3 Assume that E is a q-uniformly smooth and strictly convex Banach space which admits a weak sequentially continuous duality mapping $j_q : E \to E^*$. Suppose that $\{S_i : E \to E\}$ is a γ_i-strict pseudo-contraction such that $\bigcap_{i=1}^{\infty} \Gamma(S_i) \neq \emptyset$ and that $\{\lambda_i\}$ is a real sequence such that $0 < \lambda_i \leq m < 1$, $\forall i \geq 1$. Set that $\mathcal{A}$ is a strongly positive bounded linear operator on E with a coefficient $0 < \tilde{\eta} < 1$ and that φ is an MKC such that $0 < \lambda < \tilde{\eta}$. Let

$\{\alpha_k\}$ and $\{\beta_k\} \subset (0,1)$ be sequences which adhere to the conditions (D$_1$), (D$_2$) and (D$_3$). Let $\{x_k\}$ be a sequence defined by $x_1 = x \in E$ as follows:

$$\begin{cases} z_k = \tau_k x_k + (1-\tau_k) U_k x_k, \\ x_{k+1} = \beta_k \lambda \varphi(x_k) + (I - \beta_k \mathcal{A}) z_k, \ \forall\, k \geq 1. \end{cases}$$

Then $\{x_k\}$ converges strongly to $\tilde{z} \in \bigcap\limits_{i=1}^{\infty} \Gamma(S_i)$, which uniquely solves the variational inequality $\langle \lambda\varphi\tilde{z} - \mathcal{A}\tilde{z}, j_q(x-\tilde{z})\rangle \leq 0$, $x \in \bigcap\limits_{i=1}^{\infty} \Gamma(S_i)$.

Proof By the same steps as those used to prove Theorem 3.1, and replacing Lemma 2.5 with Lemma 3.2 in Step 4, we easily get the results of Corollary 3.3. □

References

[1] Diestel J. Geometry of Banach Spaces: Selected Topics. Lecture Notes in Math 485. Springer-Verlag, 1975

[2] Banach S. Surles operations dans les ensembles abstraits et leur application aux equations integrales. Fund Math, 1922, **3**: 133–181

[3] Meir A, Keeler E. A theorem on contraction mappings. J Math Anal Appl, 1969, **28**: 326–329

[4] Jung J S. Strong convergence of iterative methods for k-strictly pseudo-contractive mappings in Hilbert spaces. Appl Math Comput, 2010, **215**: 3746–3753

[5] Tian M. A general iterative algorithm for nonexpansive mappings in Hilbert spaces. Nonlinear Anal, 2010, **73**: 689–694

[6] Wang S. A general iterative method for obtaining an infinite family of strictly pseudo-contractive mappings in Hilbert spaces. Appl Math Lett, 2011, **24**: 901–907

[7] Cai G, Hu C S. Strong convergence theorems of a general iterative process for a finite family of λ_i-strict pseudo-contractions in q-uniformly smooth Banach spaces. Comput Math Appl, 2010, **59**(1): 149–160

[8] Suzuki T. Moudafi's viscosity approximations with Meir-Keeler contractions. J Math Anal Appl, 2007, **325**(1): 342–352

[9] Opial Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull Amer Math Soc, 1967, **73**: 591–597

[10] Gossez J P, Dozo E L. Some geometric properties related to the fixed point theory for nonexpansive mappings. Pacific J Math, 1972, **40**: 565–573

[11] Jung J S. Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces. J Math Anal Appl, 2005, **302**(2): 509–520

[12] Wen M, Hu CS, Cui AG, et al. Convergence of an explicit scheme for nonexpansive semigroups in Banach spaces. J Fixed Point Theory Appl, 2019, **21**: 52

[13] Suzuki T. Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigroups without Bochner integral. J Math Anal Appl, 2005, **35**: 227–239

[14] Liu L S. Iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces. J Math Anal Appl, 1995, **194**: 114–125

[15] Xu H K. Iterative algorithms for nonlinear operators. J London Math Sov, 2002, **66**: 240–256

[16] Zhang H, Su Y. Strong convergence theorems for strict pseudo-contractions in q-uniformly smooth Banach spaces. Nonlinear Anal, 2009, **70**(9): 3236–3242

[17] Shimoji K, Takahashi W. Strong convergence to common fixed points of infinite nonexpansive mappings and applications. Taiwanese J Math, 2001, **5**(2): 387–404

[18] Chang S S. A new method for solving equilibrium problem and variational inequality problem with application to optimization. Nonlinear Anal, 2009, **70**: 3307–3319

[19] Song Y L. Strong convergence theorems of a new general iterative process with Meir-Keeler contractions for a countable family of λ_i-strict pseudocontractions in q-uniformly smooth Banach spaces. Fixed Point Theory App, 2010, **19**: 354202

[20] Gu F. Convergence of implicit iterative process with errors for a finite family of asymptotically nonexpansive mappings. Acta Math Sci, 2006, **26B**: 1131–1143