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EXISTENCE RESULTS FOR SINGULAR FRACTIONAL p-KIRCHHOFF PROBLEMS*

Mingqi XIANG (向明启)
College of Science, Civil Aviation University of China, Tianjin 300300, China
E-mail: *xiangmingqi_hit@163.com*

Vicenţiu D. RĂDULESCU
Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland
Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania
Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia
E-mail: *radulescu@inf.ucv.ro*

Binlin ZHANG (张彬林)†
College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China
E-mail: *zhangbinlin2012@163.com*

Abstract This paper is concerned with the existence and multiplicity of solutions for singular Kirchhoff-type problems involving the fractional p-Laplacian operator. More precisely, we study the following nonlocal problem:

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \dfrac{|x|^{\alpha_1 p}|y|^{\alpha_2 p}|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)\mathcal{L}_p^s u = |x|^\beta f(u) & \text{in } \Omega, \\ u=0 & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases}$$

where $\mathcal{L}_p^s$ is the generalized fractional p-Laplacian operator, $N\geq 1$, $s\in(0,1)$, $\alpha_1,\alpha_2,\beta\in\mathbb{R}$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, and $M:\mathbb{R}_0^+\to\mathbb{R}_0^+$, $f:\Omega\to\mathbb{R}$ are continuous functions. Firstly, we introduce a variational framework for the above problem. Then, the existence of least energy solutions is obtained by using variational methods, provided that the nonlinear term f has $(\theta p-1)$-sublinear growth at infinity. Moreover, the existence of infinitely many solutions is obtained by using Krasnoselskii's genus theory. Finally, we obtain the existence and multiplicity of solutions if f has $(\theta p-1)$-superlinear growth

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†Corresponding author

at infinity. The main features of our paper are that the Kirchhoff function may vanish at zero and the nonlinearity may be singular.

Key words Fractional Kirchhoff equation; singular problems; variational and topological methods

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1 Introduction and Main Results

Let $N \geq 1$, $p \geq 1$, $q \geq 1$, $\tau > 0$, $0 \leq a \leq 1$, $\alpha,\ \beta,\ \gamma \in \mathbb{R}$ be such that

$$\frac{1}{\tau}+\frac{\gamma}{N},\ \frac{1}{p}+\frac{\alpha}{N},\ \ \frac{1}{q}+\frac{\beta}{N}>0$$

and

$$\frac{1}{\tau}+\frac{\gamma}{N}=a\left(\frac{1}{p}+\frac{\alpha-1}{N}\right)+(1-a)\left(\frac{1}{q}+\frac{\beta}{N}\right).$$

In the case $a > 0$, we assume in addition that, with $\gamma = a\sigma + (1-a)\beta$, $0 \leq \alpha - \sigma \leq 1$ if $\frac{1}{\tau}+\frac{\gamma}{N}=\frac{1}{p}+\frac{\alpha-1}{N}$.

Caffarelli, Kohn and Nirenberg [5] proved the following well-known Caffarelli-Kohn-Nirenberg inequality:

$$\left\||x|^{\gamma}u\right\|_{L^{\tau}(\mathbb{R}^N)} \leq C\left\||x|^{\alpha}\nabla u\right\|^{a}_{L^{p}(\mathbb{R}^N)}\left\||x|^{\beta}u\right\|^{1-a}_{L^{q}(\mathbb{R}^N)} \quad \text{for } u \in C_0^1(\mathbb{R}^N).$$

In particular, if $a = 1$, this inequality becomes

$$\left\||x|^{\gamma}u\right\|_{L^{\tau}(\mathbb{R}^N)} \leq C\left\||x|^{\alpha}\nabla u\right\|_{L^{p}(\mathbb{R}^N)} \quad \text{for } u \in C_0^1(\mathbb{R}^N).$$

After that, existence and multiplicity of solutions for singular elliptic problems have been investigated by using the Caffarelli-Kohn-Nirenberg inequality. Indeed, due to the Caffarelli-Kohn-Nirenberg inequality, one can study the existence and multiplicity of solutions for some singular elliptic equations like

$$-\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) = b(x)f(u),$$

where $a(x)$ is a nonnegative function satisfying $\inf\limits_{x} a(x) = 0$ and b is a function satisfying $\inf\limits_{x} b(x) = 0$. For instance, Felli and Schneider in [17] considered the equation

$$-\mathrm{div}(|x|^{-2a}\nabla u) - \lambda|x|^{2(1+a)}u = (1+\varepsilon k(x))|x|^{bp}u^{p-1} \ \text{ in } \mathbb{R}^N.$$

The authors obtained the existence of positive solutions and non-radial solutions as ε small enough. Ghergu and Rădulescu [16] studied the singular elliptic equation:

$$-\mathrm{div}(|x|^{-2a}\nabla u) = K(x)|x|^{-bp}|u|^{p-2}u + \lambda g(x) \ \text{ in } \mathbb{R}^N.$$

Under suitable assumptions on K, the authors obtained two distinct solutions as λ small enough by using Ekeland's variational principle and the mountain pass theorem. In [10], Chu et al. studied the existence and the qualitative properties of solutions for the singular p-Laplacian type problem

$$\begin{cases} -\mathrm{div}(|x|^{-\beta}a(x,\nabla u)) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the operator $\operatorname{div}(|x|^{-\beta}a(x,\nabla u))$ is a general form of the singular p-Laplacian $\operatorname{div}(|x|^{\alpha p}|\nabla u|^{p-2}\nabla u)$ and f satisfies $(p-1)$-sublinear growth at infinity. The authors obtained two nontrivial solutions by using variational methods. In [9], Caristi et al. discussed the following nonlocal degenerate problem:

$$\begin{cases}-M\left(\int_\Omega |x|^{-ap}|\nabla u(x)|^p\mathrm{d}x\right)\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)=\lambda|x|^{-p(a+1)+c}f(u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function satisfying that $m_0t^{\alpha-1}\le M(t)\le m_1t^{\alpha-1}$ for all $t\in\mathbb{R}^+$, where $m_1>m_0$ and $1<\alpha<\min\left\{\frac{N}{N-p},\frac{N-p(a+1)+c}{N-p(a+1)}\right\}$, and the nonlinear term f satisfies the following conditions:

(A_1) there exists a constant $\nu>\frac{m_1\alpha p}{m_0}$ such that $0<\nu F(t)\le tf(t)$ for all $t\in\mathbb{R}\setminus\{0\}$;

(H_1) $\lim\limits_{|t|\to\infty}\frac{f(t)}{|t|^{\alpha p-1}}=0$.

Under the above conditions, the authors obtained the existence and multiplicity of solutions. However, it seems that assumptions (A_1) and (H_1) can not hold simultaneously. The paper [21] extended the Caffarelli-Kohn-Nirenberg inequality to the case of variable exponent Sobolev spaces and obtained the existence of solutions for a class of singular $p(x)$-Laplacian equations by using variational methods.

The issue of the Caffarelli-Kohn-Nirenberg inequalities in fractional Sobolev spaces is quite delicate. Very recently, Nguyen and Squasssina in [28] proved that the following fractional Caffarelli-Kohn-Nirenberg inequality (see also [1] for a special case): Let $s\in(0,1)$, $\alpha_1,\alpha_2,\alpha\in\mathbb{R}$ with $\alpha_1+\alpha_2=\alpha$, and $N\ge1$, $p>1$, $q\ge1$, $\tau>0$, $0<a\le1$, $\beta,\ \gamma\in\mathbb{R}$ be such that

$$\frac{1}{\tau}+\frac{\gamma}{N}=a\left(\frac{1}{p}+\frac{\alpha-s}{N}\right)+(1-a)\left(\frac{1}{q}+\frac{\beta}{N}\right).$$

In the case $a>0$, assume in addition that, with $\gamma=a\sigma+(1-a)\beta$, $0\le\alpha-\sigma$ and $\alpha-\sigma\le1$ if $\frac{1}{\tau}+\frac{\gamma}{N}=\frac{1}{p}+\frac{\alpha-s}{N}$. Under the above assumptions, Nguyen and Squassina in [28] proved that if $\frac{1}{\tau}+\frac{\gamma}{N}>0$, then

$$\||x|^\gamma u\|_{L^\tau(\mathbb{R}^N)}\le C[u]^a_{W^{s,p,\alpha}(\mathbb{R}^N)}\||x|^\beta u\|^{1-a}_{L^q(\mathbb{R}^N)}\quad\text{for } u\in C_0^1(\mathbb{R}^N);\tag{1.1}$$

if $\frac{1}{\tau}+\frac{\gamma}{N}<0$, then

$$\||x|^\gamma u\|_{L^\tau(\mathbb{R}^N)}\le C[u]^a_{W^{s,p,\alpha}(\mathbb{R}^N)}\left\||x|^\beta u\right\|^{1-a}_{L^q(\mathbb{R}^N)}\quad\text{for } u\in C_0^1(\mathbb{R}^N\setminus\{0\}),$$

where

$$[u]_{W^{s,p,\alpha}(\mathbb{R}^N)}=\left(\iint_{\mathbb{R}^{2N}}\frac{|x|^{\alpha_1p}|y|^{\alpha_2p}|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{1/p}.$$

Inspired by the above works, in this paper, we study the following singular fractional Kirchhoff type problem:

$$\begin{cases}M\left(\iint_{\mathbb{R}^{2N}}\frac{|x|^{\alpha_1p}|y|^{\alpha_2p}|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)\mathcal{L}_p^su=|x|^\beta f(u) & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}\tag{1.2}$$

where $N\ge1$, $s\in(0,1)$, $\alpha_1,\alpha_2\in\mathbb{R}$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary containing zero, $M:[0,\infty)\to[0,\infty)$ is a continuous function, $f:\Omega\to\mathbb{R}$ is a continuous function, and $\mathcal{L}_p^s$ is the generalized fractional p-Laplacian operator which, up to a normalization constant, is defined as follows:

$$\langle \mathcal{L}_p^s u, v\rangle = \iint_{\mathbb{R}^{2N}} \frac{|x|^{\alpha_1 p}|y|^{\alpha_2 p}|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y,$$

for all $u, v \in C_0^\infty(\mathbb{R}^N)$. Especially, as $\alpha_1 = \alpha_2 = 0$ and $p = 2$, the above operator reduces to the well-known fractional Laplace operator $(-\Delta)^s$. Furthermore, if $s \to 1^-$, then $(-\Delta)^s$ becomes the classic Laplace operator $-\Delta$ (see [14, Proposition 4.4]).

Since the pioneering work of Caffarelli and Silvestre in [7], a lot of attention has been attracted to investigate problems involving fractional Laplace operator. Especially, much effort has been focused on the subcritical and critical growth of the nonlinearities, which lead us to study various variational problems using the critical point theory. Problems like (1.2) appeared in many fields of real world, for example, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and anomalous diffusion. In fact, fractional Laplace operator can be viewed as the typical outcome of stochastically stabilization of Lévy processes; we refer to [2, 6, 14, 20] for more details.

Problem (1.2) also involves the study of Kirchhoff-type problems. In fact, such problems arise in various models of physical and biological systems. In particular, the existence results concerning Kirchhoff-type problems are more and more abundant in recent years. More precisely, Kirchhoff in [19] established a model governed by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

where $u = u(x,t)$ denotes the lateral displacement, E is the Young modulus, ρ is the mass density, h is the cross-section area, L is the length and p_0 is the initial axial tension. In fact, Equation (1.3) extends the classical D'Alembert wave equation based on a physical consideration; that is, enclosing the effects of the changes in the length of the strings during the vibrations. In particular, Fiscella and Valdinoci in [18] proposed a stationary Kirchhoff model involving the fractional Laplacian by investigating the nonlocal aspect of the tension; see [18, Appendix A] for further details.

Throughout the paper, without explicit mention, we assume that $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function and verifies (M_0) or (M_1) and (M_2) as below.

(M_0) There exist $m_0 > 0$ and $\theta > 1$ such that $M(t) \geq m_0 t^{\theta-1}$ for all $t \geq 0$;

(M_1) For any $d > 0$ there exists $\kappa := \kappa(d) > 0$ such that $M(t) \geq \kappa$ for all $t \geq d$;

(M_2) There exists $\theta \in (1, N/N-sp)$ such that

$$\theta\mathscr{M}(t) \geq M(t)t, \quad \forall t \geq 0,$$

where $\mathscr{M}(t) = \int_0^t M(\tau)\mathrm{d}\tau$.

A simple example of M is given by $M(t) = a_0 + b_0\theta\, t^{\theta-1}$ for all $t \geq 0$ and some $\theta > 1$, where $a_0, b_0 \geq 0$ and $a_0 + b_0 > 0$. When M is of this type, problem (1.2) is called to be degenerate if $a = 0$, while it is named non-degenerate if $a > 0$. In recent years, Kirchhoff-type fractional problems have triggered more and more attention. Existence results for non-degenerate Kirchhoff-type fractional Laplacian problems were given, for example, in [30, 32]. While some recent existence results concerning the degenerate case of Kirchhoff-type fractional

Laplacian equations were obtained; see [4, 8, 12, 22–25, 31, 33, 34] and references therein. It is worth pointing out that the degenerate case is rather interesting and is treated in some famous works concerning Kirchhoff theory; see for instance [13]. From a physical point of view, it seems rational to describe a realistic model by $M(0)=0$, which means that the base tension of the string vanishes.

Throughout the paper, we assume that $f:\Omega\to\mathbb{R}$ is a continuous function. In the following, we enumerate the assumptions concerning the nonlinear term f, but keep in mind that they will not be fulfilled simultaneously:

(f_0) f is odd, that is, $f(-t)=-f(t)$ for all $t\in\mathbb{R}$;

(f_1) $\lim\limits_{|t|\to\infty}\frac{f(t)}{|t|^{\theta p-1}}=0$;

(f_2) there exists $q\in(1,p)$ such that $F(t)\geq|t|^q$, where $F(t):=\int_0^t f(\tau)\mathrm{d}\tau$;

(f_3) there exist $q>\theta p$ and $C>0$ such that

$$|f(t)|\leq C|t|^{q-1},\quad \text{for each } t\in\mathbb{R};$$

(f_4) there exist $\mu>\theta p$ and $T>0$ such that f satisfies the Ambrosetti-Rabinowtiz type condition, i.e.,

$$\mu F(t)\leq tf(t),\quad \text{for all } |t|>T.$$

A simple example of function f satisfying $(f_1)-(f_2)$ is given by

$$f(x,t)=\begin{cases}|t|^{q-2}t+|t|^{r-2}t & \text{if } |t|\leq 1,\\ 2|t|^{r-2}t & \text{if } |t|>1,\end{cases}$$

where $1<r<p<q<p_s^*:=Np/(N-sp)$.

Remark 1.1 From (f_1) one can deduce that f is $(\theta p-1)$-sublinear at infinity, while from (f_4) one can deduce that f is $(\theta p-1)$-superlinear at infinity.

Definition 1.2 We say that $u\in W_0^{s,p}(\Omega,|x|^{\alpha p})$ is a (weak) solution of problem (1.2), if it holds that

$$M\left(\iint_{\mathbb{R}^{2N}}\frac{|x|^{\alpha_1 p}|y|^{\alpha_2 p}|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)\langle\mathcal{L}_p^s u,\varphi\rangle=\int_\Omega|x|^\beta f(u)\varphi\mathrm{d}x$$

for all $\varphi\in W_0^{s,p}(\Omega,|x|^{\alpha p})$.

We always assume that $s\in(0,1)$, $\alpha_1,\alpha_2\in\mathbb{R}$, $\alpha=\alpha_1+\alpha_2$, $N\geq1$, and $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary and $0\in\Omega$. Now we are in a position to introduce two existence results involving the case that the nonlinearity f is $(\theta p-1)$-sublinear at infinity.

Theorem 1.3 Assume that M fulfills (M_0) and f satisfies (f_1)–(f_2). If $\beta>(\alpha-s)\theta p+N(\theta-1)$, then problem (1.2) has a least energy solution in $W_0^{s,p}(\Omega,|x|^{\alpha p})$ with negative energy.

Moreover, we get the existence of infinitely many solutions of problem (1.2).

Theorem 1.4 Assume that M fulfills (M_0) and f satisfies (f_0), (f_1) and (f_2). If $\beta>(\alpha-s)\theta p+N(\theta-1)$, then problem (1.2) has infinitely many solutions in $W_0^{s,p}(\Omega,|x|^{\alpha p})$ with negative energy.

We also obtain the existence and multiplicity of solutions for problem (1.2) when the nonlinearity f is $(\theta p-1)$-superlinear at infinity.

Theorem 1.5 Assume that M fulfills (M_1)–(M_2) and f satisfies (f_3)–(f_4). If $\beta > (\alpha - s)q + N(q/p - 1)$, then problem (1.2) admits a nontrivial mountain pass solution in $W_0^{s,p}(\Omega, |x|^{\alpha p})$.

Theorem 1.6 Assume that M fulfills (M_1)–(M_2) and f satisfies (f_0) and (f_3)–(f_4). If $\beta > (\alpha - s)q + N(q/p - 1)$, then problem (1.2) has infinitely many solutions in $W_0^{s,p}(\Omega, |x|^{\alpha p})$.

Remark 1.7 If $\alpha_1 = \alpha_2 = \alpha$, then we can define $\mathcal{L}_p^s$ as follows: for any $x \in \mathbb{R}^N$

$$\mathcal{L}_p^s u(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|x|^{\alpha p}|y|^{\alpha p}|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \mathrm{d}y$$

along any $u \in C_0^\infty(\mathbb{R}^N)$.

To the best of our knowledge, Theorems 1.3–1.6 are the first existence and multiplicity results for singular Kirchhoff-type problems in the fractional setting.

The rest of the paper is organized as follows: in Section 2, we introduce a variational framework of problem (1.2) and give some necessary properties for the functional setting. In Section 3, we obtain the existence of least energy solution for problem (1.2). In Section 4, the existence of infinitely many solutions is obtained by using genus theory. In Section 5, a mountain pass solution and infinitely many solutions for problem (1.2) are obtained by using the mountain pass theorem and the symmetric mountain pass theorem, respectively.

2 Variational Framework and Preliminary Results

We first provide some basic functional setting that will be used in the next sections. Let $1 < p < \infty$ and define $W_0^{s,p}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{1/p}.$$

Let $W_0^{s,p}(\Omega, |x|^{\alpha p})$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \left(\iint_{\mathbb{R}^{2N}} \frac{|x|^{\alpha_1 p}|y|^{\alpha_2 p}|u(x) - u(y)|^p}{|x - y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{1/p}.$$

Using a similar discussion as in [32], the space $W_0^{s,p}(\Omega, |x|^{\alpha p})$ is a reflexive Banach space. Let $1 < q < \infty$ and $\beta \in \mathbb{R}$. Define the weighted Lebesgue space

$$L^q(\Omega, |x|^\beta) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable} \,\middle|\, \int_\Omega |x|^\beta |u(x)|^q \mathrm{d}x < \infty \right\},$$

endowed with the norm

$$\|u\|_{q,\beta} = \left(\int_\Omega |x|^\beta |u(x)|^q \mathrm{d}x \right)^{1/q}.$$

The next fractional Caffarelli-Kohn-Nirenberg inequality will be used later, which was obtained in [28]. In fact, by taking $a = 1$ in (1.1), we have

Theorem 2.1 Let $s \in (0, 1)$, $1 < p < N/s$, $\alpha > -(N - sp)/p$ and $\alpha - s \le \gamma \le \alpha$. Set $p^*_{\alpha,\gamma,s} := Np/(N - p(\gamma - \alpha + s))$. Then there exists $C(N, \alpha, s) > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{\gamma p^*_{\alpha,\gamma,s}} |u|^{p^*_{\alpha,\gamma,s}} \mathrm{d}x \right)^{1/p^*_{\alpha,\gamma,s}} \le C(N, \alpha, s)\|u\| \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N).$$

Using Theorem 2.1, we have the following embedding theorem:

Theorem 2.2 Let $s \in (0,1)$, $1 < p < N/s$ and $\alpha > -(N-sp)/p$. Then $W_0^{s,p}(\Omega, |x|^{\alpha p})$ is continuously embedded in $L^q(\Omega, |x|^\beta)$, if $1 \le q \le p_s^*$ and $\beta \ge (\alpha - s)q + N(q/p-1)$; the embedding is compact if $1 \le q < p_s^*$ and $\beta > (\alpha - s)q + N(q/p-1)$.

Proof If $q = p_s^*$ and $\beta = (\alpha - s)q + N(q/p-1) = \alpha p_s^*$, then by taking $\gamma = \alpha$ in Theorem 2.1, the embedding $W_0^{s,p}(\Omega, |x|^{\alpha p}) \hookrightarrow L^{p_s^*}(\Omega, |x|^\beta)$ is continuous. If $1 \le q < p_s^*$, then we take $\alpha - s < \gamma < \alpha$ such that

$$q < p_{\alpha,\gamma,s}^* = \frac{Np}{N - p(\gamma - \alpha + s)} < \frac{Np}{N - sp} = p_s^*.$$

Let $u \in W_0^{s,p}(\Omega, |x|^{\alpha p})$. Then by the Hölder inequality, we have

$$\int_\Omega |x|^\beta |u|^q \mathrm{d}x \le \left(\int_\Omega |x|^{(\beta - \gamma q)\frac{p^*_{\gamma,\alpha,s}}{p^*_{\gamma,\alpha,s} - q}} \mathrm{d}x \right)^{\frac{p^*_{\gamma,\alpha,s} - q}{p^*_{\gamma,\alpha,s}}} \left(\int_\Omega |x|^{\gamma p^*_{\gamma,\alpha,s}} |u(x)|^{p^*_{\gamma,\alpha,s}} \right)^{\frac{q}{p^*_{\gamma,\alpha,s}}}. \tag{2.1}$$

Since $\beta > (\alpha - s)q + N(q/p-1)$, we get

$$-(\beta - \gamma q)\frac{p^*_{\gamma,\alpha,s}}{p^*_{\gamma,\alpha,s} - q} < N.$$

Thus, it follows that

$$\left(\int_\Omega |x|^{(\beta - \gamma q)\frac{p^*_{\gamma,\alpha,s}}{p^*_{\gamma,\alpha,s} - q}} \mathrm{d}x \right)^{\frac{p^*_{\gamma,\alpha,s} - q}{p^*_{\gamma,\alpha,s}}} < \infty.$$

It follows from (2.1) and Theorem 2.1 that

$$\int_\Omega |x|^\beta |u|^q \mathrm{d}x \le C \left(\int_\Omega |x|^{\gamma p^*_{\gamma,\alpha,s}} |u(x)|^{p^*_{\gamma,\alpha,s}} \right)^{\frac{q}{p^*_{\gamma,\alpha,s}}} \le C\|u\|^q,$$

which yields that the embedding $W_0^{s,p}(\Omega, |x|^{\alpha p}) \hookrightarrow L^q(\Omega, |x|^\beta)$ is continuous.

Next we show that the embedding $W_0^{s,p}(\Omega, |x|^{\alpha p}) \hookrightarrow L^q(\Omega, |x|^\beta)$ is compact. To this aim, let $\{u_n\}$ be a bounded sequence in $W_0^{s,p}(\Omega, |x|^{\alpha p})$. For any $R > 0$ with $B_R(0) \subset \Omega$ is a ball centered at 0 with radius R. Then $\{u_n\}$ is a bounded sequence in $W_0^{s,p}(\Omega \backslash B_R(0))$. By Theorem 7.1 in [14], we obtain that there is a convergent subsequence of $\{u_n\}$ in $L^q(\Omega \setminus B_R(0))$. By choosing a diagonal sequence, without loss of generality, we assume that $\{u_n\}$ converges in $L^q(\Omega \setminus B_R(0))$ for any $R > 0$.

On the other hand, for $1 \le q < p_s^*$, we take $\alpha - s < \gamma < \alpha$ such that

$$q < p^*_{\alpha,\gamma,s} < p_s^*.$$

Since the embedding is continuous, we obtain that $\{u_n\}$ is bounded in $L^{p^*_{\alpha,\gamma,s}}(\Omega, |x|^{\gamma p^*_{\alpha,\gamma,s}})$. Then by the Hölder inequality, for any $R > 0$ small enough and $n, m \in \mathbb{N}$, we deduce that

$$\begin{aligned}
&\int_{B_R(0)} |x|^\beta |u_m - u_n|^q \mathrm{d}x \\
&\le \left(\int_{B_R(0)} |x|^{(\beta - \gamma q)\frac{p^*_{\alpha,\gamma,s}}{p^*_{\alpha,\gamma,s} - q}} \mathrm{d}x \right)^{\frac{p^*_{\alpha,\gamma,s} - q}{p^*_{\alpha,\gamma,s}}} \left(\int_\Omega |x|^{\gamma p^*_{\alpha,\gamma,s}} |u_m - u_n|^{p^*_{\alpha,\gamma,s}} \mathrm{d}x \right)^{\frac{q}{p^*_{\alpha,\gamma,s}}} \\
&\le C \left(\int_{B_R(0)} |x|^{(\beta - \gamma q)\frac{p^*_{\alpha,\gamma,s}}{p^*_{\alpha,\gamma,s} - q}} \mathrm{d}x \right)^{\frac{p^*_{\alpha,\gamma,s} - q}{p^*_{\alpha,\gamma,s}}}
\end{aligned}$$

$$\leq C\left(\int_0^R t^{N-1+(\beta-\gamma q)\frac{p^*_{\alpha,\gamma,s}}{p^*_{\alpha,\gamma,s}-q}}\mathrm{d}t\right)^{\frac{p^*_{\alpha,\gamma,s}-q}{p^*_{\alpha,\gamma,s}}} = C\left(R^{N+(\beta-\gamma q)\frac{p^*_{\alpha,\gamma,s}}{p^*_{\alpha,\gamma,s}-q}}\right)^{\frac{p^*_{\alpha,\gamma,s}-q}{p^*_{\alpha,\gamma,s}}},$$

where $C>0$ denotes various constants independent of n,m. As $\beta > q(\alpha-s)+N(q/p-q)$, it follows that $N+(\beta-\gamma q)\frac{p^*_{\alpha,\gamma,s}}{p^*_{\alpha,\gamma,s}-q}>0$. Thus, for any $\varepsilon>0$ there exists $R>0$ such that

$$\int_{B_R(0)}|x|^\beta|u_m-u_n|^q\mathrm{d}x<\frac{\varepsilon}{2}\quad \forall n,m\in\mathbb{N}.$$

Then we can choose $n_0\in\mathbb{N}$ such that

$$\int_{\Omega\setminus B_R(0)}|x|^\beta|u_n-u_m|^q\mathrm{d}x\leq C_\beta\int_{\Omega\setminus B_R(0)}|u_n-u_m|^q\mathrm{d}x\leq\frac{\varepsilon}{2},$$

where $C_\beta=R^\beta$ if $\beta<0$ and $C_\beta=(\mathrm{diam}(\Omega))^\beta$ if $\beta>0$. Therefore, we conclude

$$\int_\Omega|x|^\beta|u_n-u_m|^q\mathrm{d}x\leq\varepsilon\quad\forall n,m\in\mathbb{N}.$$

This means that $\{u_n\}$ is a Cauchy sequence in $L^q(\Omega,|x|^\beta)$. □

To study solutions of problem (1.2), we define the associated functional $I: W_0^{s,p}(\Omega,|x|^{\alpha p})\to\mathbb{R}$ as follows:

$$I(u)=\Phi(u)-\Psi(u)\quad\text{for all } u\in W_0^{s,N/s}(\Omega,|x|^{\alpha p}),$$

where

$$\Phi(u)=\frac{1}{p}\mathscr{M}(\|u\|^p)\quad\text{and } \Psi(u)=\int_\Omega|x|^\beta F(u)\mathrm{d}x.$$

By assumption (f_2), for any $\varepsilon>0$ there exists $T_\varepsilon>0$ such that

$$|f(t)|\leq\varepsilon|t|^{\theta p-1},\quad\forall|t|>T_\varepsilon.$$

Thus,

$$|f(t)|\leq\varepsilon|t|^{\theta p-1}+\max_{|t|\leq T_\varepsilon}|f(t)|,\quad\forall t\in\mathbb{R}. \tag{2.2}$$

Furthermore,

$$|F(t)|\leq\frac{\varepsilon}{\theta p}|t|^{\theta p}+\max_{|t|\leq T}|f(t)||t|,\quad\forall(x,t)\in\Omega\times\mathbb{R}. \tag{2.3}$$

Using (2.3), $\beta>(\alpha-s)\theta p+N(\theta-1)$ and Theorem 2.2, one can verify that I is well defined, of class $C^1(W_0^{s,p}(\Omega,|x|^{\alpha p}),\mathbb{R})$ and

$$\langle I'(u),v\rangle=M(\|u\|^p)\langle\mathcal{L}_p^s u,v\rangle-\int_\Omega|x|^\beta f(u)v\mathrm{d}x,$$

for all $u,v\in W_0^{s,p}(\Omega,|x|^{\alpha p})$. Clearly, the critical points of I_λ are exactly the weak solutions of problem (1.2).

3 Proof of Theorem 1.3

In this section, we always assume that M satisfies (M_0) and f satisfies (f_1) and (f_2).

Let us now recall that the functional I satisfies the $(PS)_c$ condition in $W_0^{s,p}(\Omega,|x|^{\theta p})$, if any $(PS)_c$ sequence $\{u_n\}\subset W_0^{s,p}(\Omega,|x|^{\theta p})$, namely a sequence such that $I(u_n)\to c$ and $I'(u_n)\to 0$ as $n\to\infty$, admits a strongly convergent subsequence in $W_0^{s,p}(\Omega,|x|^{\theta p})$.

In order to study the existence of least energy solutions for problem (1.2) in the sublinear case, we will use the following direct method in the calculus of variations:

Theorem 3.1 Let X be a reflexive Banach space with norm $\|\cdot\|_X$. Assume that the functional $J : X \to \mathbb{R}$ is

(i) coercive on X, that is, $J(u) \to \infty$ as $\|u\|_X \to \infty$;

(ii) weakly lower semi-continuous on X, that is, for any $u \in X$ and any sequence $\{u_n\} \subset X$ such that $u_n \rightharpoonup u$ weakly in X,

$$J(u) \leq \liminf_{n\to\infty} J(u_n).$$

Then J is bounded from below on X and attains its infimum in X.

Lemma 3.2 The functional I is weakly lower semi-continuous on $W_0^{s,p}(\Omega, |x|^{\alpha p})$.

Proof We first show that Φ is weakly lower semi-continuous on $W_0^{s,p}(\Omega, |x|^{\alpha p})$. To this aim, we define a functional $H : W_0^{s,p}(\Omega, |x|^{\alpha p}) \to \mathbb{R}$ as

$$H(u) = \iint_{\mathbb{R}^{2N}} \frac{|x|^{\alpha_1 p}|y|^{\alpha_2 p}|u(x) - u(y)|^p}{|x - y|^{N+ps}} \mathrm{d}x\mathrm{d}y \quad \forall u \in W_0^{s,p}(\Omega, |x|^{\alpha p}).$$

It is easy to verify that $H \in C^1(W_0^{s,p}(\Omega, |x|^{\alpha p}), \mathbb{R})$ and H is a convex functional. Furthermore, H is sub-differentiable and the sub-differential denoted by ∂H satisfies $\partial H(u) = \{H'(u)\}$ for all $u \in W_0^{s,p}(\Omega, |x|^{\alpha p})$. Assume that $\{u_n\} \subset W_0^{s,p}(\Omega, |x|^{\alpha p}), u \in W_0^{s,p}(\Omega, |x|^{\alpha p})$ and $u_n \rightharpoonup u$ weakly in $W_0^{s,p}(\Omega, |x|^{\alpha p})$ as $n \to \infty$. By the definition of sub-differential, we have

$$H(u_n) - H(u) \geq \langle H'(u), u_n - u\rangle,$$

which yields that $H(u) \leq \liminf_{n\to\infty} H(u_n)$. Without loss of generality, we assume that $t_0 := \liminf_{n\to\infty} H(u_n) > 0$. Since $\mathscr{M}(t) : [0, \infty) \to (0, \infty)$ is continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|t - t_0| < \delta$, it holds that

$$\mathscr{M}(t_0) - \varepsilon < \mathscr{M}(t) < \mathscr{M}(t_0) + \varepsilon.$$

Choose $t_0 - \delta < t_1 < t_0 < t_2 < t_0 + \delta$. By the assumption on M, we know that $\mathscr{M}$ is a increasing function. It follows that

$$\mathscr{M}(t_0) - \varepsilon < \mathscr{M}(t_1) \leq \mathscr{M}(t) \leq \mathscr{M}(t_2) < \mathscr{M}(t_0) + \varepsilon \quad \forall t \in [t_1, t_2].$$

On the other hand, by $t_0 := \liminf_{n\to\infty} H(u_n)$, we obtain that there are at most finite numbers n such that $H(u_n) > t_1$, and so there are at most finite numbers n such that $\mathscr{M}(H(u_n)) > \mathscr{M}(t_1) > \mathscr{M}(t_0) - \varepsilon$. Moreover, there are infinitely many n such that $H(u_n) < t_2$. Thus, there are infinitely many n such that $\mathscr{M}(H(u_n)) < \mathscr{M}(t_2) < \mathscr{M}(t_0) + \varepsilon$. Then we get $\liminf_{n\to\infty} \mathscr{M}(\|u_n\|^p) = \mathscr{M}(\liminf_{n\to\infty} \|u_n\|^p)$. Therefore, we deduce that $\mathscr{M}(\|u\|^p) \leq \liminf_{n\to\infty} \mathscr{M}(\|u_n\|^p)$, which means that $\Phi(u) = \frac{1}{p}\mathscr{M}(\|u\|^p)$ is weakly lower semi-continuous on $W_0^{s,p}(\Omega, |x|^{\alpha p})$.

Next we prove that Ψ is weakly continuous on $W_0^{s,p}(\Omega, |x|^{\alpha p})$. By (f_2), there exists $C > 0$ such that $|f(t)| \leq C(1+|t|^{\theta p-1})$ for all $t \in \mathbb{R}$. It follows from Theorem 2.2 that $W_0^{s,p}(\Omega, |x|^{\alpha p}) \hookrightarrow L^{\theta p}(\Omega, |x|^{\beta})$ is compact for $\beta > \theta p(\alpha - s) + N(\theta - 1)$. Using a standard argument, one can deduce that Ψ is weakly continuous on $W_0^{s,p}(\Omega, |x|^{\alpha p})$.

In conclusion, we obtain that $I(u) = \Phi(u) - \Psi(u)$ is a weakly lower semi-continuous functional on $W_0^{s,p}(\Omega, |x|^{\alpha p})$. $\square$

Lemma 3.3 The functional I is coercive and satisfies the $(PS)_c$ condition.

Proof For any $\varepsilon > 0$, by (M_1) and (2.3), we obtain that for all $u \in W_0^{s,p}(\Omega, |x|^{\alpha p})$ with $\|u\| \geq 1$,

$$I(u) \geq \frac{m_0}{\theta p}\|u\|^{\theta p} - \frac{1}{\theta p}\int_\Omega |x|^\beta |u|^{\theta p}\mathrm{d}x - \Big(\max_{|t|\leq T_\varepsilon} f(t)\Big)\int_\Omega |x|^\beta |u|\mathrm{d}x.$$

By Theorem 2.2 and $\beta > (\alpha - s)\theta p + N(\theta - 1)$, there exists $C > 0$ such that

$$\int_\Omega |x|^\beta |u|^{\theta p}\mathrm{d}x \leq C\|u\|^{\theta p}$$

and

$$\int_\Omega |x|^\beta |u|\mathrm{d}x \leq C\|u\|.$$

It follows that

$$I(u) \geq \Big(\frac{m_0}{\theta p} - \frac{\varepsilon}{\theta p}C\Big)\|u\|^{\theta p} - C\Big(\max_{|t|\leq T_\varepsilon} f(t)\Big)\|u\|$$

for all $u \in W_0^{s,p}(\Omega, |x|^{\alpha p})$ with $\|u\| \geq 1$. Now choose $\varepsilon = m_0/(2C)$, we obtain

$$I(u) \geq \frac{m_0}{2\theta p}\|u\|^{\theta p} - C\|u\|,$$

which together with $\theta p > 1$ implies that $I(u) \to \infty$ as $\|u\| \to \infty$. Thus we have proved that I is coercive.

Next we show that I satisfies the $(PS)_c$ condition. To this aim, we assume that $\{u_n\} \subset W_0^{s,p}(\Omega, |x|^{\theta p})$ is $(PS)_c$ sequence; that is, $I(u_n) \to c$ and $I'(u_n) \to 0$ in $(W_0^{s,p}(\Omega, |x|^{\theta p}))^*$. Since I is coercive, $\{u_n\}$ is bounded in $W_0^{s,p}(\Omega, |x|^{\theta p})$. Thus, up to a subsequence, we have

$$u_n \rightharpoonup u \ \text{ weakly in } W_0^{s,p}(\Omega, |x|^{\theta p}).$$

Moreover, by

$$|\langle I'(u_n), u_n - u\rangle| \leq \|I'(u_n)\|\|u_n - u\| \leq C\|I'(u_n)\| \to 0,$$

we deduce

$$\lim_{n\to\infty}\langle I'(u_n), u_n - u\rangle = 0.$$

It follows that

$$M(\|u_n\|^p)\langle \mathcal{L}_p^s u_n, u_n - u\rangle = \int_\Omega |x|^\beta f(u_n)(u_n - u)\mathrm{d}x. \tag{3.1}$$

By (2.2), we have

$$\left|\int_\Omega |x|^\beta f(u_n)(u_n - u)\mathrm{d}x\right| \leq C\int_\Omega |x|^\beta(|u_n - u| + |u_n - u|^{\theta p})\mathrm{d}x,$$

which converges to zero by Theorem 2.2. It follows from (3.1) that

$$\lim_{n\to\infty} M(\|u_n\|^p)\langle \mathcal{L}_p^s u_n, u_n - u\rangle = 0,$$

which, together with the fact that $\langle \mathcal{L}_p^s(u), u_n - u\rangle = 0$, yields that

$$\lim_{n\to\infty} M(\|u_n\|^p)\left[\langle \mathcal{L}_p^s u_n, u_n - u\rangle - \langle \mathcal{L}_p^s(u), u_n - u\rangle\right] = 0. \tag{3.2}$$

If $\inf_{n\geq 1}\|u_n\| > 0$, by (3.2) and (M_1) we have

$$\lim_{n\to\infty}\left[\langle \mathcal{L}_p^s u_n, u_n - u\rangle - \langle \mathcal{L}_p^s(u), u_n - u\rangle\right] = 0.$$

Then using a similar discussion as in [32, Lemma 3.6], we can obtain that $u_n \to u$ in $W_0^{s,p}(\Omega, |x|^{\alpha p})$. If $\inf_{n\geq 1} \|u_n\| = 0$, then up to a subsequence we obtain that $u_n \to 0$ in $W_0^{s,p}(\Omega, |x|^{\alpha p})$. □

Proof of Theorem 1.3 By Theorem 3.1, Lemmas 3.2 and 3.3, the functional I has a global minimizer $u \in W_0^{s,p}(\Omega, |x|^{\alpha p})$, which is a least energy solution of problem (1.2). Now we prove that u is nontrivial. Choose a nonnegative function $v \in W_0^{s,p}(\Omega, |x|^{\alpha p})$ with $\|v\| = 1$ and $0 < \max_{x\in\Omega} v(x) \leq 1$. Then it follows from the definition of I and (f_2) that

$$\begin{aligned} I(tv) &= \frac{1}{p}\mathscr{M}(t^p\|v\|^p) - \int_\Omega |x|^\beta F(tv)\mathrm{d}x \\ &\leq \frac{1}{p}\Big(\max_{\tau\in[0,1]} M(\tau)\Big)t^p - t^q \int_\Omega |x|^\beta |v|^q \mathrm{d}x \\ &< 0 \quad \text{for } 0 < t < 1 \text{ small enough,} \end{aligned}$$

thanks to $p > q$. Thus, we can choose some $t > 0$ such that $I(tv) < 0$. Then by the minimality of u, we have

$$I(u) \leq I(tv) < 0 = I(0),$$

which yields that u is nontrivial. □

4 Proof of Theorem 1.4

In this section we study the existence of infinitely many solutions of problem (1.2). To this end, we mainly use a classical result due to Clark (see [11]). Before stating our result, we first recall some basic notions on Krasnoselskii's genus and its properties.

Denote by X a real Banach space. Set

$$\Gamma = \{A \subset X \setminus \{0\} : A \text{ is compct and } A = -A\}.$$

Definition 4.1 Let $A \in \Gamma$ and $X = \mathbb{R}^k$. The genus $\gamma(A)$ of A is defined by

$$\gamma(A) = \min\{k \geq 1 : \text{ there exists an odd mapping } \phi \in C(A, \mathbb{R}^k \setminus \{0\})\}.$$

If there does not exist such a mapping for any $k \geq 1$, we set $\gamma(A) = \infty$. Note that if A is a subset which consists of finitely many pairs of points, then $\gamma(A) = 1$. Moreover, $\gamma(\emptyset) = 0$.

Now, we list some necessary results of Krasnoselskii's genus.

Lemma 4.2 (1) Let $X = \mathbb{R}^k$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^k$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = k$. In particular, let $\mathbb{S}^{k-1}$ be a $k-1$-dimensional sphere in $\mathbb{R}^k$, then $\gamma(\mathbb{S}^{k-1}) = k$.

(2) Let $A \subset X$, Ω be a bounded neighborhood of 0 in $\mathbb{R}^k$, and assume that there exists an odd mapping $h \in C(A, \partial\Omega)$ with h a homeomorphism. Then $\gamma(A) = k$.

Theorem 4.3 (Clark's theorem [11]) Let $J \in C^1(X, \mathbb{R})$ be a functional satisfying the $(PS)_c$ condition. Furthermore, let us suppose that

(i) J is even, i.e., $J(-u) = J(u)$ for all $u \in X$, and J is bounded from below;

(ii) there is a compact set $A \subset \Gamma$ such that $\gamma(A) = k$ and $\sup_{u\in A} J(u) < J(0)$.

Then J possesses at least k pairs of distinct critical points, and their corresponding critical values are less than $J(0)$.

Proof of Theorem 1.4 Set

$$\Gamma_k = \{A \subset \Gamma : \gamma(A) \geq k\}, \quad c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} I(u), \ k = 1, 2, \ldots,$$

then it follows from Lemma 3.3 that

$$-\infty < c_1 \leq c_2 \leq \ldots \leq c_k \leq c_{k+1} \leq \ldots.$$

Next we prove that $c_k < 0$ for all $k \in \mathbb{N}$. For each k, we choose k disjoint open sets Ω_i such that $\bigcup_{i=1}^{k} \Omega_i \subset \Omega$. For $i = 1, 2, \ldots, k$, let $e_i \in (W_0^{s,p}(\Omega, |x|^{\alpha p}) \bigcap C_0^\infty(\Omega_i)) \setminus \{0\}$, with $\|e_i\| = 1$, and let

$$A_k = \operatorname{span}\{e_1, e_2, \ldots, e_k\}.$$

Since A_k is finite dimensional, all norms on it are equivalent. Thus there exists a positive constant $C > 0$ such that

$$\left(\int_\Omega |x|^\beta |u|^q \mathrm{d}x\right)^{1/q} \geq C\|u\| \ \text{ for all } u \in A_k.$$

By (f_2), we get

$$\begin{aligned} I(tu) &\leq \frac{1}{p}\Big(\max_{\tau\in[0,1]} M(\tau)\Big)t^p - \sum_{i=1}^{k} \int_{\Omega_i} |x|^\beta F(tu_i)\mathrm{d}x \\ &\leq \frac{1}{p}\Big(\max_{\tau\in[0,1]} M(\tau)\Big)t^p - t^q \sum_{i=1}^{k} \int_{\Omega_i} |x|^\beta |u_i|^q \mathrm{d}x \\ &= \frac{1}{p}\Big(\max_{\tau\in[0,1]} M(\tau)\Big)t^p - t^q \int_{\Omega} |x|^\beta u^q \mathrm{d}x \\ &\leq \frac{1}{p}\Big(\max_{\tau\in[0,1]} M(\tau)\Big)t^p - C^q t^q, \end{aligned}$$

for all $u \in S_k$ and $0 < t \leq 1$ small enough, where $S_k = \{u \in A_k : \|u\| = 1\}$. Thus, we can find $t^* = t(k) \in (0,1)$ and $\varepsilon^* = \varepsilon^*(k) > 0$ such that $I(t^*u) \leq -\varepsilon^* < 0$ for all $u \in S_k$. Set $S_k^{t^*} = \{t^*u : u \in S_k\}$. Clearly, $S_k^{t^*}$ is homeomorphic to $\mathbb{S}^{k-1}$. Then $\gamma(S_k^{t^*}) = k$ and so $c_k \leq \sup_{u \in S_k^{t^*}} I(u) < 0 = I(0)$.

Since f is odd, the functional I is even. In view of Lemma 3.3, we know that all assumptions of Theorem 4.3 are satisfied. Then the functional I admits at least k pairs of distinct critical points. Due to the arbitrary of k, we obtain the existence of infinitely many critical points of I. Thus, the proof is complete. □

5 Proofs of Theorems 1.5–1.6

In this section we consider the superlinear case of problem (1.2). Without special mentioning, we always assume that M satisfies (M_1)–(M_2), and f satisfies (f_3)–(f_4).

In the sequel, we shall make use of the following general mountain pass theorem (see [3]):

Theorem 5.1 Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$ with $J(0) = 0$. Suppose that

(i) there exist $\rho, r > 0$ such that $J(u) \geq \rho$ for all $u \in X$, with $\|u\|_X = r$;

(ii) there exists $e \in X$ satisfying $\|e\|_X > \rho$ such that $J(e) < 0$.

Define $\mathscr{H}=\{h\in C^1([0,1];X):h(0)=1,h(1)=e\}$. Then

$$c=\inf_{h\in\mathscr{H}}\max_{0\le t\le 1}J(h(t))\ge\rho$$

and there exists a $(PS)_c$ sequence $\{u_n\}\subset X$.

Now we check that the functional I satisfies the mountain geometry properties (i) and (ii).

Lemma 5.2 There exist $r,\rho>0$ such that $I(u)\ge\rho$ if $\|u\|=r$.

Proof By (M_2), one can deduce

$$\mathscr{M}(t)\ge\mathscr{M}(1)t^{\theta},\quad\text{for all } 0\le t\le 1. \tag{5.1}$$

By (5.1) and (f_3), we obtain

$$I(u)\ge\frac{1}{p}\mathscr{M}(1)\|u\|^{\theta p}-\frac{C}{q}\int_{\Omega}|x|^{\beta}|u|^q\mathrm{d}x\ge\frac{1}{p}\mathscr{M}(1)\|u\|^{\theta p}-\frac{C}{q}\|u\|^q, \tag{5.2}$$

for all $u\in W_0^{s,p}(\Omega,|x|^{\alpha p})$ with $\|u\|\le 1$. Here we have used the fact that the embedding from $W_0^{s,p}(\Omega,|x|^{\alpha p})$ to $L^q(\Omega,|x|^{\beta})$ is continuous by Theorem 2.2, since $q\in(\theta p,p_s^*)$ and $\beta>(\alpha-s)\theta p+N(\theta-1)$. Since $q>\theta p$, we can choose $r\in(0,1)$ small enough such that $\frac{1}{p}\mathscr{M}(1)r^p-\frac{C}{q}r^q>0$. Then it follows from (5.2) that $I(u)\ge\rho:=\frac{1}{p}\mathscr{M}(1)r^p-\frac{C}{q}r^q>0$ for all $u\in W_0^{s,p}(\Omega,|x|^{\alpha p})$, with $\|u\|=r$. □

Lemma 5.3 There exists $e\in W_0^{s,p}(\Omega,|x|^{\alpha p})$ with $\|e\|>r$ such that $I(e)<0$, where r is given by Lemma 5.2.

Proof By (M_2), we have

$$\mathscr{M}(t)\le\mathscr{M}(1)t^{\theta}\quad\text{for all } t\ge 1. \tag{5.3}$$

Choose a nonnegative function $\varphi\in C_0^{\infty}(\Omega)$ such that $\|\varphi\|=1$. Then by (f_4) and (5.3), for all τ, with $\tau>1$, we have

$$I(\tau\varphi)\le\frac{\mathscr{M}(1)}{p}\tau^{\theta p}-\tau^q\frac{F(T)}{T^{\mu}}\int_{\{x\in\Omega:|\varphi(x)|>T\}}\varphi^q\mathrm{d}x+\Big(\sup_{|t|\le T}|F(t)|\Big)\int_{\Omega}|x|^{\beta}\mathrm{d}x.$$

Since $q>\theta p$, fixing $\tau>0$ even large so that we have that $I(e)<0$, where $e=\tau\varphi$. □

Lemma 5.4 The functional I satisfies the $(PS)_c$ condition.

Proof Let $\{u_n\}\subset W_0^{s,p}(\Omega,|x|^{\alpha p})$ be such that

$$I(u_n)\to c\quad\text{and } I'(u_n)\to 0\quad\text{in } (W_0^{s,p}(\Omega,|x|^{\alpha p}))^*$$

as $n\to\infty$. We first show that $\{u_n\}$ is bounded. Arguing by contradiction, we assume that up to a subsequence,

$$1\le\|u_n\|\to\infty\quad\text{as } n\to\infty.$$

Using (f_4) and (M_2), we deduce

$$\begin{aligned}C+\|u_n\|&\ge I(u_n)-\frac{1}{\mu}\langle I'(u_n),u_n\rangle\\&\ge\left(\frac{1}{\theta p}-\frac{1}{\mu}\right)M(\|u_n\|^p)\|u_n\|^p-\int_{\Omega}|x|^{\beta}(F(u_n)-\frac{1}{\mu}f(u_n)u_n)\mathrm{d}x\\&\ge\left(\frac{1}{\theta p}-\frac{1}{\mu}\right)\kappa(1)\|u_n\|^p-\int_{\{x\in\Omega:|u|>T\}}|x|^{\beta}(F(u_n)-\frac{1}{\mu}f(u_n)u_n)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&-\sup\left\{\left|F(t)-\frac{1}{\mu}f(t)t\right| : |t|\le T\right\}\int_\Omega |x|^\beta \mathrm{d}x\\
&\ge \left(\frac{1}{\theta p}-\frac{1}{\mu}\right)\kappa(1)\|u_n\|^p - \sup\left\{\left|F(t)-\frac{1}{\mu}f(t)t\right| : |t|\le T\right\}\int_\Omega |x|^\beta \mathrm{d}x.
\end{aligned}
$$

Dividing the above inequality by $\|u_n\|^p$ and letting n go to infinity, we obtain

$$
0 \ge \left(\frac{1}{\theta p}-\frac{1}{\mu}\right)\kappa(1),
$$

which together with $\mu > \theta p$ yields a contradiction. Thus, $\{u_n\}$ is bounded in $W_0^{s,p}(\Omega, |x|^{\alpha p})$.

Then there exist a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and u such that

$$
\begin{cases}
u_n \rightharpoonup u & \text{weakly in } W_0^{s,p}(\Omega, |x|^{\alpha p}),\\
u_n \to u & \text{strongly in } L^q(\Omega, |x|^\beta),\\
u_n \to u & \text{a.e. in } \Omega.
\end{cases}
$$

We first show that

$$
\lim_{n\to\infty}\int_\Omega |x|^\beta f(u_n)(u_n-u)\mathrm{d}x = 0. \tag{5.4}
$$

Indeed, by (f_3) and the Hölder inequality, we have

$$
\left|\int_\Omega |x|^\beta f(u_n)(u_n-u)\mathrm{d}x\right| \le C\left(\int_\Omega |x|^\beta |u_n|^q \mathrm{d}x\right)^{\frac{q-1}{q}}\left(\int_\Omega |x|^\beta |u_n-u|^q\mathrm{d}x\right)^{\frac{1}{q}}. \tag{5.5}
$$

Using Theorem 2.2, we obtain $\lim\limits_{n\to\infty 0}\int_\Omega |x|^\beta|u_n-u|^q\mathrm{d}x = 0$. Then it follows from (5.5) that (5.4) holds true.

Due to the fact that $\{u_n\}$ is a $(PS)_c$ sequence, we have

$$
\langle I'(u_n), u_n-u\rangle - M(\|u_n\|^p)\langle \mathcal{L}_p^s u, u_n-u\rangle = o(1).
$$

Then,

$$
o(1) = M(\|u_n\|^p)(\langle \mathcal{L}_p^s u_n, u_n-u\rangle - \langle \mathcal{L}_p^s u, u_n-u\rangle) - \int_\Omega |x|^\beta f(u_n)(u_n-u)\mathrm{d}x.
$$

It follows from (5.4) that

$$
\lim_{n\to\infty} M(\|u_n\|^p)(\langle \mathcal{L}_p^s u_n, u_n-u\rangle - \langle \mathcal{L}_p^s u, u_n-u\rangle) = 0.
$$

Then by using a similar discussion as in Lemma 3.3, we conclude that $\|u_n - u\| \to 0$ as $n\to\infty$. In conclusion, the proof is complete. □

Proof of Theorem 1.5 By Lemmas 5.2–5.3 and Theorem 5.1, there exists a $(PS)_c$ sequence $\{u_n\}$ such that $I(u_n)\to c$, $I'(u_n)\to 0$, where $c = \inf\limits_{h\in\mathscr{H}}\max\limits_{0\le t\le 1} I(h(t)) \ge \rho$ and $\mathscr{H} = \{h \in C^1([0,1]; W_0^{s,p}(\Omega, |x|^{\alpha p})) : h(0) = 1, h(1) = e\}$. Furthermore, by Lemma 5.4, there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u \in W_0^{s,p}(\Omega, |x|^{\alpha p})$ such that $u_n \to u$. Moreover, u is a nonnegative solution of problem (1.2). □

We shall use the following symmetric mountain pass theorem to get the existence of infinitely many solutions of problem (1.2) in the superlinear case:

Theorem 5.5 Let X be a real infinite dimensional Banach space and $J \in C^1(X, \mathbb{R})$ a functional satisfying the $(PS)_c$ condition. Assume that J satisfies the following:

(1) $J(0)=0$ and there exist $\rho, r>0$ such that $J(u)\geq\rho$ for all $\|u\|_X=r$;

(2) J is even;

(3) for all finite dimensional subspace $\widetilde{X}\subset X$, there exists $R=R(\widetilde{X})>0$ such that $J(u)<0$ for all $u\in\widetilde{X}\setminus B_R(\widetilde{X})$.

Then J possesses an unbounded sequence of critical values characterized by a minimax argument.

Proof of Theorem 1.6 By (f_4), we have

$$F(t)\geq\frac{F(T)}{T^\mu}|t|^\mu \quad \text{for all } |t|>T.$$

Set

$$\overline{C}=\sup_{|t|\leq T}\left|F(t)-\frac{F(T)}{T^\mu}|t|^\mu\right|.$$

Then, we obtain

$$F(t)\geq\frac{F(T)}{T^\mu}|t|^\mu-\overline{C} \quad \text{for all } t\in\mathbb{R}. \tag{5.6}$$

Let E be a fixed finite dimensional subspace of $W_0^{s,p}(\Omega,|x|^{\alpha p})$. For any $u\in E$ with $\|u\|=1$, and for all $t\geq1$ we have by (5.3) and (5.6) that

$$I(tu)\leq\frac{1}{p}\mathscr{M}(1)t^{\theta p}-\frac{F(T)}{T^\mu}t^\mu\int_\Omega|x|^\beta|u|^\mu\mathrm{d}x-\overline{C}\int_\Omega|x|^\beta\mathrm{d}x\to-\infty \quad \text{as } t\to\infty.$$

Thus,

$$\sup_{\|u\|=R,u\in E}I(u)=\sup_{\|u\|=1,u\in E}I(Ru)\to-\infty$$

as $R\to\infty$. Hence there exists $R_0>0$ so large such that $I(u)<0$ for all $u\in E$, with $\|u\|=R$ and $R>R_0$. Clearly, $I(0)=0$ and I is even. In view of Lemma 5.2, we know that all assumptions of Theorem 5.5 are satisfied. Thus, problem (1.2) admits an unbounded sequence of solutions. □

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