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THE TIME DECAY RATES OF THE CLASSICAL SOLUTION TO THE POISSON-NERNST-PLANCK-FOURIER EQUATIONS IN $\mathbb{R}^3$*

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Abstract In this work, the Poisson-Nernst-Planck-Fourier system in three dimensions is considered. For when the initial data regards a small perturbation around the constant equilibrium state in a $H^3 \cap \dot{H}^{-s}(0 \le s \le 1/2)$ norm, we obtain the time convergence rate of the global solution by a regularity interpolation trick and an energy method.

Key words Poisson-Nernst-Planck-Fourier system; decay rates; energy method

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1 Introduction

The Poisson-Nernst-Planck (PNP) system is a very important mathematical model describing the movement of charges under a concentration gradient and a electrostatic potential generated by themselves. The system has many applications such as in semiconductor technology, chemical science and biology (see [4, 8, 9, 18, 19, 29, 31, 34, 38–40] and the references therein). In the usual PNP model, the temperature is assumed to be homogeneous and independent of time. However, the heating effect plays a very important role in many applications.

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For example, in biological science (see [5, 35]), the charge transport is found to be sensitive to changes in temperature. The ion-channels which are gated by the temperature will adjust the internal homeostasis and disease-related processes when the temperature changes. This suggests that it is necessary to take the influence of the temperature into consideration to fully understand the behavior of the electrothermal motion. In order to model the dynamic process with temperature, the mechanical equation should be coupled with the thermal equation. By an energetic variation method, the authors in [17, 25] deduced the following Poisson-Nernst-Planck-Fourier (PNPF) equations:

$$\begin{cases} \partial_t\rho_i + \nabla\cdot(\rho_i u_i) = 0,\ i = \pm, \\ \nu_i\rho_i(u_i - u_0) = -k_B\nabla(\rho_i T) - z_i e\rho_i\nabla\phi, \\ -\epsilon\Delta\phi = \sum\limits_{i=\pm}\rho_i e z_i, \\ \sum\limits_{i=0,\pm} k_B C_i\rho_i\partial_t T + \sum\limits_{i=0,\pm} k_B C_i\rho_i u_i\cdot\nabla T + \sum\limits_{i=\pm} k_B\rho_i T\nabla\cdot u_i \\ = k\Delta T + \lambda_0|\nabla u_0|^2 + \sum\limits_{i=\pm}\nu_i\rho_i|u_i - u_0|^2. \end{cases} \tag{1.1}$$

Here u_0 is the velocity of the solvent and satisfies

$$\begin{cases} \mathrm{div} u_0 = 0, \\ \nabla P_0 + \sum\limits_{\pm}\nu_i\rho_i(u_0 - u_i) - \lambda_0\Delta u_0 = 0, \end{cases} \tag{1.2}$$

while, ρ_+ and z_+ are the density and the valence of cations, respectively. ρ_- and z_- are the density and valence of anions. ν_+ and ν_- are the diffusion coefficients. e is the elementary charge. k_B is the Boltzmann constant. T is the temperature. ϵ is related to the dielectric constant and the Debye length. $u_\pm$ denotes the flow map of cations and anions. $C_\pm$ and C_0 are the heat capacities of cations and anions of the solvent. k is the thermal conductivity. u_0 is the velocity of the solvent. λ_0 is the solvent viscosity. This system is inferred by employing the given free energy function and entropy production where the conservative forces are deduced by least action principle and the dissipative forces are inferred by the maximum dissipation principle. More details on the process of deducing this system can be found in [25].

While $T = 1$ and u_0 is equal to some constant, the PNPF system will be reduced to the original PNP equations. One can check [23] and its references for a detailed derivation process of the PNP system. There have been a variety of studies on the well-posedness and the properties of the classical PNP equations and some modified versions. Gagneux and Millet [13] studied the well-posedness of the Nernst-Planck-Poisson system in both one dimension and higher dimensions, and obtained the global existence of the solution by using the Schauder-Tikhonov fixed-point method; they also established the properties (energy and entropy laws, influence of an external electrical field, Boltzmann distribution) of the global solution. By employing Schauder's fixed-point theorem, Hsieh [15] obtained the global existence of the weak solution for a modified Poisson-Nernst-Planck system with steric effects in a bounded domain of $\mathbb{R}^d(d \leq 3)$. Ogawa and Shimizu [32] studied the well-posedness for a normalized Poisson-Nernst-Planck system in a $2D$ critical Hardy space $\mathcal{H}^1(\mathbb{R}^2)$, then Deng and Li [6] considered the well-posedness and ill-posedness of this system in a $2D$ Besov space. One can check [3, 12, 27] for the well-posedness theories for some other modified systems. There are lots of works on the stability and the small parameter (as ϵ tends to zero) problems where the boundary layer may be involved; see [2, 11, 16, 22] and the references therein. The steady problems were investigated in [1, 10, 24, 33] and the singularly perturbed problems are discussed in [20, 26]. The authors in [17] consider the global existence of classic solutions to the PNPF system without a solvent equation.

The well-posedness theory and the problem of the time decay rate are important topics in the theory of partial differential equations. The continuity and continuous dependence on initial data of the unique solution are particularly important for problems arising from physical applications. In addition to this, the asymptotic behavior of the solution also attracts a lot of attention. In these cases, the primary mathematical tasks are to characterize the solution of the partial differential equations and show the explicit rate of convergence. While the initial data is near the constant equilibrium state in three dimensions, based on the energy method and the interpolation estimates, we want to investigate the global existence and the algebraic decay rate of the solution. As mentioned before, the purpose of the PNPF system is that it models the influence of the temperature on the movement of the charges. In particular, the system includes the solvent equation, which is much more physically accurate. From the point view of mathematics, the PNPF system couples the drift-diffusion equations with the nonlinear elliptic equation (1.2); the PNPF system (1.1) cannot be simply seen as the PNP equation and a temperature equation. This elliptic equation (1.2) brings difficulty to the proof. Indeed, by the properties of the elliptic operator, we cannot directly get the L^2 estimates of u_0. During the process of obtaining the a priori estimates, we need to avoid the presence of the L^2 norm of u_0. We will explain these things clearly after stating our main result.

According to [25], the PNPF system is deduced by energy varaiation methods. The free energy and entropy dissipation can form a closed inequality. When we try to do the estimates in L^2 space, their L^2 estimates cannot be closed, but this system can be closed in the H^3 norm. The density ρ_0 of the solvent is equal to some positive constant. We propose the initial condition

$$(\rho_+,\ \rho_-,\ T)(x,0) = (\rho_+(0), \rho_-(0), T(0))(x) \to (1/2,\ 1/2,\ 1) \text{ as } |x| \to \infty.$$

Before stating the main result, we introduce some notations. $\|f\|_{L^2}$ denotes the L^2 norm of the function f. In a similar way, we can define the usual Sobolev norm $\|f\|_{H^3}$. The notation $\|(f,g)\|_{H^3}$ means that $\|f\|_{H^3} + \|g\|_{H^3}$. The norm $\|\cdot\|_{\dot{H}^s}$ with $s \leq 0$ is defined by $\|f\|_{\dot{H}^s} := \|\Lambda^s f\|_{L^2}$, and $\Lambda^s f := \mathscr{F}^{-1}(|\xi|^s \hat{f})$, where $\mathscr{F}^{-1}$ is the inverse Fourier transform. We denote $a \lesssim b$ if there exists some constant $\tilde{C}$, independent of a and b, but dependent on the coefficients in (1.1), such that $a \leq \tilde{C} b$. In addition, in this work, the viscous coefficients $\nu_\pm$ are equal and the valences z_+ and z_- are opposite. The main result of this work is as follows:

Theorem 1.1 For some constant $\tilde{c}_0 > 0$ small enough, which is only dependent on the coefficients in equations (1.1), if we have the initial data $(\rho_+(0), \rho_-(0), T(0)) \in H^N$ with $N \geq 3$, $\nabla\phi(0) \in L^2$ and

$$\|\rho_+(0) + \rho_-(0) - 1\|_{H^3}^2 + \|\rho_+(0) - \rho_-(0)\|_{H^3}^2 + \|T(0) - 1\|_{H^3}^2 + \|\nabla\phi(0)\|_{L^2}^2 \leq \tilde{c}_0, \tag{1.3}$$

then system (1.1) admits a unique global classical solution which satisfies the following estimates for any $t>0$:

$$\begin{aligned}&\|\rho_+(t)+\rho_-(t)-1\|_{H^N}^2+\|\rho_+(t)-\rho_-(t)\|_{H^N}^2+\|T(t)-1\|_{H^N}^2+\|\nabla\phi\|_{L^2}^2\\&+\int_0^t\Big(\|\nabla(\rho_+(s)+\rho_-(s))\|_{H^N}^2+\|(\rho_+(s)-\rho_-(s))\|_{H^{N+1}}^2+\|\nabla T(s)\|_{H^N}^2+\|\nabla\phi\|_{L^2}^2\Big)\,\mathrm{d}s\\\le\ &\|\rho_+(0)+\rho_-(0)-1\|_{H^N}^2+\|\rho_+(0)-\rho_-(0)\|_{H^N}^2+\|T(0)-1\|_{H^N}^2+\|\nabla\phi(0)\|_{L^2}^2\,.\end{aligned}\tag{1.4}$$

Moreover, if we have the initial data $(\rho_+(0),\rho_-(0),T(0))\in\dot H^{-s}$ with $0\le s\le 1/2$, the solution has the following time decay rates for $t\ge 0$:

$$\begin{aligned}&\big\|\nabla^k(\rho_+(t)+\rho_-(t)-1,\ \rho_+(t)-\rho_-(t),\ T(t)-1)\big\|_{L^2}\\\le\ &C_0(1+t)^{-\frac{k+s}{2}},\quad\text{for }k=0,1,2,\dots,N-1,\end{aligned}\tag{1.5}$$

and

$$\big\|\nabla^k(\rho_+(t)-\rho_-(t))\big\|_{L^2}\le C_0(1+t)^{-\frac{k+1+s}{2}},\quad\text{for }k=0,1,2,\dots,N-2.\tag{1.6}$$

In the process of getting a priori estimates of the classical solution, the first step is to deduce the linear equations (A.2). Except for the Poisson equation, the solvent equation (1.2) is a nonlinear elliptic equation. This brings new difficulty in terms of the linearization and deducing the energy estimates. Noticing that the solvent equation is a type of steady Stokes equation, with the help of the Leray projector (see [37]), we can represent u_0 by the density function and the electric field. As in [14, 41, 42], while the initial data is small enough, one can establish a priori estimates like

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_k(t)+\mathcal{D}_k(t)\le 0,\tag{1.7}$$

where $\mathcal{E}_k(t)$ and $\mathcal{D}_k$ are defined in (5.2). Based on (1.7), we can prove that

$$\|(\rho_+(t)+\rho_-(t)-1,\ \rho_+(t)-\rho_-(t),\ T(t)-1)\|_{\dot H^{-s}}\le C_0.\tag{1.8}$$

Then, with the help of the interpolation between positive and negative Sobolev norms (see Lemma A.4), one can infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_k+(\mathcal{E}_k)^{1+\frac{1}{k+s}}\le 0,$$

and obtain the time decay rates of the solution.

The nonlinear terms containing u_0 in the right hand side of (2.1) cannot be absorbed by the dissipation $\mathcal{D}_k$, since the dissipation $\mathcal{D}_k$ does not contain u_0. To obtain (1.7) and (1.8), we must make full use of the structure of equations (2.1). With the help of (1.2) and $(1.1)_2$ and the Helmholtz projection, we can bound the L^2 estimates of ∇u_0 by the electic field, which is the key point for closing the energy estimates. In the whole space, however, by the properties of elliptic operator, we cannot get the L^2 estimates of u_0. This is why s is less than one half.

Remark 1.2 This theorem shows that the constant equilibrium state $(1/2,1/2,1)$ is stable in H^3 space. Our result can be generalized to the N ions case and to general constant equilibrium (see [21] for the existence of the general case).

Remark 1.3 This work mainly studies the well-posedness and the time decay rates of the global solution around the constant equilibrium in $\mathbb{R}^3$. When the equilibrium state is no longer constant and with a strictly positive lower bound, one can obtain similar results, but the computation is very complicated and long. We will study this problem in a forthcoming paper.

The next sections are devoted to proving Theorem (1.1). The uniform estimates are deduced by the energy method in Section 2. We will construct the local classical solution and then extend the local solution to a global one in Section 3. The negative Sobolev estimates will be derived in Section 4. In addition, by a regularity interpolation trick, we get the time decay rates of the classical solution in Section 5.

2 The Energy Estimates of the Nonlinear Equations in L^2 Norm

In this section, we will try to deduce the linearized equations and then infer the uniform estimates of the solution to the nonlinear equations (1.1). Motivated by [16], we can set

$$m=\rho_+-\rho_-,\qquad \tilde p=\rho_++\rho_-=1+p,\qquad T=1+\theta,$$

and denote

$$\begin{aligned}
&a_-=(\frac{k_B}{2\nu_+}-\frac{k_B}{2\nu_-}),a_+=(\frac{k_B}{2\nu_+}+\frac{k_B}{2\nu_-}),d_0=\frac{C_++C_-}{2}+C_0\rho_0,\\
&b_-=(\frac{z_+e}{2\nu_+}+\frac{z_-e}{2\nu_-}),b_+=(\frac{z_+e}{2\nu_+}-\frac{z_-e}{2\nu_-}),a_0=ez_+.
\end{aligned}$$

The linearized equations will become

$$\begin{cases}
\partial_t m-a_+\Delta m+\dfrac{b_+a_0}{\epsilon}m=g_1,\\
\partial_t p-a_+\Delta p-a_+\Delta\theta=g_2,\\
-\epsilon\Delta\phi=a_0m,\\
d_0\partial_t\theta-a_+\Delta p-(a_++\dfrac{k}{k_B})\Delta\theta=\displaystyle\sum_{i=3}^{7}g_i,
\end{cases}\tag{2.1}$$

where

$$\begin{aligned}
g_1&=a_+\Delta(m\theta)+b_+\mathrm{div}(p\nabla\phi)-\mathrm{div}(mu_0),\\
g_2&=a_+\Delta(p\theta)+b_+\mathrm{div}(m\nabla\phi)-\mathrm{div}(pu_0),\\
g_3&=-\frac{d_+p+d_-m}{d_0+d_+p+d_-m}(a_+\Delta p+(a_++\frac{k}{k_B})\Delta\theta),\\
g_4&=\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}\big(a_2^+\nabla(\tilde pT)+a_2^-\nabla(mT)+b_2^-\tilde p\nabla\phi\big)\\
&\quad+\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}\big(b_2^+m\nabla\phi-(d_0+d_+p+d_-m)u_0\big),\\
g_5&=\frac{d_0}{d_0+d_+p+d_-m}\cdot\big(a_+\Delta(p\theta)-a_+\nabla p\cdot\nabla\theta(1-\theta)\big)\\
&\quad+\frac{d_0}{d_0+d_+p+d_-m}\big(a_+(\theta+p\theta)\Delta\theta-\frac{b_+a_0}{\epsilon}m^2(1+\theta)\big),\\
g_6&=\frac{d_0}{d_0+d_+p+d_-m}\big[4(a_+\nabla p\cdot\nabla\theta+b_+\nabla m\cdot\nabla\phi)(1+\theta)+a_+|\nabla\theta|^2\big]\\
&\quad+\frac{d_0}{d_0+d_+p+d_-m}\big[a_+p|\nabla\theta|^2+\frac{zeb_+}{k_B}|\nabla\phi|^2(1+p)+2b_+\nabla\theta\cdot\nabla\phi m\big],\\
g_7&=\frac{d_0\lambda_0}{k_B(d_0+d_+p+d_-m)}|\nabla u_0|^2.
\end{aligned}$$

A detailed calculation is included in Appendix A. For convenience, we introduce some notations, i.e., defining

$$\mathcal{E}_N(t)=\|m\|_{H^N}^2+\|p\|_{H^N}^2+\|\theta\|_{H^N}^2+\|\nabla\phi\|_{L^2}^2$$

and

$$\mathcal{D}_N(t)=\|m\|_{H^{N+1}}^2+\|\nabla p\|_{H^N}^2+\|\nabla\theta\|_{H^N}^2+\|\nabla\phi\|_{H^N}^2.$$

In order to derive the uniform estimates for the nonlinear system (2.1), we should assume the following a priori estimates for some sufficiently small $\tilde{c_0}$:

$$\mathcal{E}_3(t)=\|(m,p,\theta)\|_{H^3}^2+\|\nabla\phi\|_{L^2}^2\le C\tilde{c}_0. \tag{2.2}$$

Since $\mathcal{E}_3(t)$ is sufficiently small, it is enough to make sure that $1+p$ and $1+\theta$ have the positive lower and upper bounds.

The main task in this section is to establish the following lemma:

Lemma 2.1 While $\sup\limits_{0\le s\le T}\mathcal{E}_3(s)\le C\tilde{c}_0$ for some $T>0$, there exists some constant $c_d>0$ independent of T such that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^l(m,p,\sqrt{d_0}\theta,\nabla\phi)(t)\right\|_{L^2}^2+c_d\left(\left\|\nabla^{l+1}p\right\|_{L^2}^2+\left\|\nabla^{l+1}\theta\right\|_{L^2}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{H^1}^2\right)\\ \lesssim\ &\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}p\right\|_{L^2}^2+\left\|\nabla^{l+1}\theta\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{H^1}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right).\end{aligned} \tag{2.3}$$

Proof We apply the operator ∇^l with $l\ge 0$ to equations (2.1) and multiply Equations $(2.1)_1$, $(2.1)_2$ and $(2.1)_4$ by $\nabla^l m$, $\nabla^l p$ and $\nabla^l\theta$, respectively, then integrate over $\mathbb{R}^3$, to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^l m\right\|_{L^2}^2+\left\|\nabla^l p\right\|_{L^2}^2+d_0\left\|\nabla^l\theta\right\|_{L^2}^2\right)+a_+\left\|\nabla^{l+1}m\right\|_{L^2}^2+\frac{b_+a_0}{\epsilon}\left\|\nabla^l m\right\|_{L^2}^2\\ &+a_+\left\|\nabla^{l+1}p\right\|_{L^2}^2+\left(a_++\frac{k}{k_B}\right)\left\|\nabla^{l+1}\theta\right\|_{L^2}^2+2a_+\int_{\mathbb{R}^3}\nabla^{l+1}p\cdot\nabla^{l+1}\theta\mathrm{d}x\\ =\ &\int_{\mathbb{R}^3}\nabla^l g_1\cdot\nabla^l m\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^l g_2\cdot\nabla^l p\mathrm{d}x+\sum_{i=3}^{7}\int_{\mathbb{R}^3}\nabla^l g_i\cdot\nabla^l\theta\mathrm{d}x.\end{aligned}$$

Noticing that

$$2\left|\int_{\mathbb{R}^3}\nabla^{l+1}p\nabla^{l+1}\theta\mathrm{d}x\right|\le\frac{a_+k_B}{a_+k_B+\frac{k}{2}}\left\|\nabla^{l+1}p\right\|_{L^2}^2+\frac{a_+k_B+\frac{k}{2}}{a_+k_B}\left\|\nabla^{l+1}\theta\right\|_{L^2}^2,$$

we get that

$$\begin{aligned}&\frac{a_+k}{2a_+k_B+k}\left\|\nabla^{l+1}p\right\|_{L^2}^2+\frac{k}{2k_B}\left\|\nabla^{l+1}\theta\right\|_{L^2}^2\\ \le\ &2a_+\int_{\mathbb{R}^3}\nabla^{l+1}p\nabla^{l+1}\theta\mathrm{d}x+a_+\left\|\nabla^{l+1}p\right\|_{L^2}^2+\left(a_++\frac{k}{k_B}\right)\left\|\nabla^{l+1}\theta\right\|_{L^2}^2.\end{aligned}$$

Let c_d be the most minor among $\frac{a_+k}{2a_+k_B+k}$, $\frac{k}{2k_B}$, a_+ and $\frac{b_+a_0}{\epsilon}$. It follows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^l m\right\|_{L^2}^2+\left\|\nabla^l p\right\|_{L^2}^2+d_0\left\|\nabla^l\theta\right\|_{L^2}^2\right)+c_d\left(\left\|\nabla^l m\right\|_{H^1}^2+\left\|\nabla^{l+1}p\right\|_{L^2}^2+\left\|\nabla^{l+1}\theta\right\|_{L^2}^2\right)\\ \le\ &\int_{\mathbb{R}^3}\nabla^l g_1\cdot\nabla^l m\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^l g_2\cdot\nabla^l p\mathrm{d}x+\sum_{i=3}^{7}\int_{\mathbb{R}^3}\nabla^l g_i\cdot\nabla^l\theta\mathrm{d}x.\end{aligned} \tag{2.4}$$

Before estimating the nonlinear terms on the right hand side of (2.4), we get the estimates of u_0. By virtue of (1.2) and $(1.1)_2$, we have that

$$-\lambda_0\Delta u_0 = -\nabla[P_0 + k_B(p+\theta+p\theta)] - a_0 m\nabla\phi. \tag{2.5}$$

We can conclude that

$$\|\nabla u_0\|_{L^2} \le C\|m\|_{L^3}\|\nabla\phi\|_{L^2}. \tag{2.6}$$

In fact, we can multiply (2.5) by u_0 and employ integration by parts and Young's inequality to get

$$\begin{aligned}\lambda_0\|\nabla u_0\|_{L^2}^2 &= a_0\left|\int_{\mathbb{R}^3} m\nabla\phi\cdot u_0\mathrm{d}x\right|\\ &\lesssim \|m\|_{L^3}\|\nabla\phi\|_{L^2}\|u_0\|_{L^6}\\ &\lesssim \|m\|_{L^3}\|\nabla\phi\|_{L^2}\|\nabla u_0\|_{L^2}\\ &\le C_\eta\|m\|_{L^3}^2\|\nabla\phi\|_{L^2}^2 + \eta\|\nabla u_0\|_{L^2}^2;\end{aligned} \tag{2.7}$$

the term $\eta\|\nabla u_0\|_{L^2}^2$ can be absorbed by the left side of (2.7), since η is sufficiently small.

By [37], let $\mathbb{P}_h$ be the Helmholtz projection from $L^2(\mathbb{R})$ to $L^2_\sigma(\mathbb{R})$, where $L^2_\sigma(\mathbb{R})$ is a subspace of $L^2(\mathbb{R})$ with a divergence free vector in the distribution sense. Noticing that u_0 is a divergence free vector, we then get

$$u_0 = \mathbb{P}_h u_0 = -\frac{a_0}{\lambda_0}(-\Delta)^{-1}\mathbb{P}_h(m\nabla\phi). \tag{2.8}$$

In virtue of (2.8), and with the help of the electric field equation, we can get that

$$\begin{aligned}\left\|\nabla^{l+1}u_0\right\|_{L^2} &\lesssim \left\|\nabla^{l-1}(m\cdot\nabla\phi)\right\|_{L^2}\\ &\lesssim \|m\|_{L^3}\left\|\nabla^{l-1}\nabla\phi\right\|_{L^6} + \|\nabla\phi\|_{L^3}\left\|\nabla^{l-1}m\right\|_{L^6}\\ &\lesssim (\|m\|_{L^3} + \|\nabla\phi\|_{L^3})\left(\left\|\nabla^l m\right\|_{L^2} + \left\|\nabla^l\nabla\phi\right\|_{L^2}\right)\\ &\lesssim \left(\|m\|_{L^2}^{1/2}\|\nabla m\|_{L^2}^{1/2} + \|\nabla\phi\|_{L^2}^{1/2}\left\|\nabla^2\phi\right\|_{L^2}^{1/2}\right)\left(\left\|\nabla^l m\right\|_{L^2} + \left\|\nabla^l\nabla\phi\right\|_{L^2}\right)\\ &\lesssim \sqrt{\mathcal{E}_3}(t)\left(\left\|\nabla^l m\right\|_{L^2} + \left\|\nabla^l\nabla\phi\right\|_{L^2}\right).\end{aligned} \tag{2.9}$$

Now, we turn to estimate the nonlinear terms on the right hand side of (2.4). By employing Hölder's inequality, the product estimates (A.4) of Lemma A.2, and Gagliardo-Nirenberg's inequality (A.3), we can deduce that

$$\begin{aligned}a_+\int_{\mathbb{R}^3}\nabla^l\Delta(m\theta)\cdot\nabla^l m\mathrm{d}x &= -a_+\int_{\mathbb{R}^3}\nabla^l\nabla(m\theta)\cdot\nabla^l\nabla m\mathrm{d}x\\ &\le a_+\left\|\nabla^{l+1}(m\theta)\right\|_{L^2}\left\|\nabla^{l+1}m\right\|_{L^2}\\ &\le a_+\left(\|\theta\|_{L^\infty}\left\|\nabla^{l+1}m\right\|_{L^2} + \|m\|_{L^\infty}\left\|\nabla^{l+1}\theta\right\|_{L^2}\right)\left\|\nabla^{l+1}m\right\|_{L^2}\\ &\le a_+\|(m,\theta)\|_{L^\infty}\left\|\nabla^{l+1}(m,\theta)\right\|_{L^2}^2\\ &\le a_+\|\nabla(m,\theta)\|_{L^2}^{\frac12}\left\|\nabla^2(m,\theta)\right\|_{L^2}^{\frac12}\left\|\nabla^{l+1}(m,\theta)\right\|_{L^2}^2\\ &\le a_+\left(\|\nabla(m,\theta)\|_{L^2} + \left\|\nabla^2(m,\theta)\right\|_{L^2}\right)\left\|\nabla^{l+1}(m,\theta)\right\|_{L^2}^2\\ &\le a_+\sqrt{\mathcal{E}_3(t)}\left\|\nabla^{l+1}(m,\theta)\right\|_{L^2}^2.\end{aligned} \tag{2.10}$$

We derive, by using integration by parts, Hölder's inequality and the product estimates (A.4) of Lemma A.2, that

$$\begin{aligned} b_+\int_{\mathbb{R}^3}\nabla^l\mathrm{div}(p\nabla\phi)\cdot\nabla^l m\mathrm{d}x &\le b_+\left\|\nabla^l(p\nabla\phi)\right\|_{L^2}\left\|\nabla^{l+1}m\right\|_{L^2}\\ &\le b_+\left(\|p\|_{L^\infty}\left\|\nabla^l\nabla\phi\right\|_{L^2}+\|\nabla\phi\|_{L^3}\left\|\nabla^l p\right\|_{L^6}\right)\left\|\nabla^{l+1}m\right\|_{L^2}\\ &\le b_+\left(\|p\|_{L^\infty}+\|\nabla\phi\|_{L^3}\right)\left(\left\|\nabla^{l+1}(p,m)\right\|_{L^2}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right)\\ &\le b_+\sqrt{\mathcal{E}_3(t)}\left(\left\|\nabla^{l+1}(p,m)\right\|_{L^2}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right). \end{aligned}\tag{2.11}$$

With the help of (2.6) and (2.9), we can deduce that

$$\begin{aligned} &\int_{\mathbb{R}^3}\nabla^l\mathrm{div}(mu_0)\cdot\nabla^l m\mathrm{d}x\\ \le&\left\|\nabla^l(mu_0)\right\|_{L^2}\left\|\nabla^{l+1}m\right\|_{L^2}\le\left(\|m\|_{L^3}\left\|\nabla^l u_0\right\|_{L^6}+\|u_0\|_{L^\infty}\left\|\nabla^l m\right\|_{L^2}\right)\left\|\nabla^{l+1}m\right\|_{L^2}\\ \le&\left(\|u_0\|_{L^\infty}+\|m\|_{L^3}\right)\left(\left\|\nabla^{l+1}(u_0,m)\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{L^2}^2\right)\\ \le&\left(\|\nabla u_0\|_{L^2}^{\frac12}\left\|\nabla^2u_0\right\|_{L^2}^{\frac12}+\|m\|_{L^2}^{\frac12}\|\nabla m\|_{L^2}^{\frac12}\right)\left(\left\|\nabla^{l+1}(u_0,m)\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{L^2}^2\right)\\ \le&\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}(u_0,m)\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{L^2}^2\right)\\ \lesssim&\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}m\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{L^2}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right). \end{aligned}\tag{2.12}$$

It follows from (2.10)-(2.12) that

$$\int_{\mathbb{R}^3}\nabla^l g_1\cdot\nabla^l m\mathrm{d}x\lesssim\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{L^2}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right).\tag{2.13}$$

Since the dissipation $\mathcal{D}_N(t)$ does not contain $\nabla^l p$, the term $\mathrm{div}(pu_0)$ in g_2 cannot be estimated the same in (2.12). We must deal with this term carefully. When $l=0$, by the fact that $\mathrm{div}u_0=0$ and (2.6), we can infer that

$$\begin{aligned} \int_{\mathbb{R}^3}\mathrm{div}(pu_0)\cdot p\mathrm{d}x &\lesssim\|p\|_{L^3}\|u_0\|_{L^6}\|\nabla p\|_{L^2}\lesssim\|p\|_{L^3}\|\nabla u_0\|_{L^2}\|\nabla p\|_{L^2}\\ &\lesssim\mathcal{E}_3(t)\left(\|\nabla p\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right). \end{aligned}\tag{2.14}$$

When $l\ge1$, by integration by parts, Hölder's inequality, the product estimates (A.4) of Lemma A.2 and Gagliardo-Nirenberg's inequality (A.3), we get

$$\begin{aligned} &\int_{\mathbb{R}^3}\nabla^l\mathrm{div}(pu_0)\cdot\nabla^l p\mathrm{d}x\\ \lesssim&\left\|\nabla^l(pu_0)\right\|_{L^2}\left\|\nabla^{l+1}p\right\|_{L^2}\lesssim\left(\|p\|_{L^3}\left\|\nabla^l u_0\right\|_{L^6}+\|u_0\|_{L^\infty}\left\|\nabla^l p\right\|_{L^2}\right)\left\|\nabla^{l+1}p\right\|_{L^2}\\ \lesssim&\|p\|_{L^2}^{\frac12}\|\nabla p\|_{L^2}^{\frac12}\left\|\nabla^{l+1}u_0\right\|_{L^2}\left\|\nabla^{l+1}p\right\|_{L^2}+\|\nabla u_0\|_{L^2}^{\frac12}\left\|\nabla^2u_0\right\|_{L^2}^{\frac12}\left\|\nabla^l p\right\|_{L^2}\left\|\nabla^{l+1}p\right\|_{L^2}. \end{aligned}\tag{2.15}$$

By virtue of (2.6) and (2.9) with $l=1$, we can deduce that

$$\begin{aligned} &\|\nabla u_0\|_{L^2}^{\frac12}\left\|\nabla^2u_0\right\|_{L^2}^{\frac12}\left\|\nabla^l p\right\|_{L^2}\\ \lesssim&\|m\|_{L^3}^{\frac12}\|\nabla\phi\|_{L^2}^{\frac12}\|m\nabla\phi\|_{L^2}^{\frac12}\left\|\nabla^l p\right\|_{L^2}\lesssim\|\nabla\phi\|_{L^2}\|m\|_{L^3}^{\frac12}\|m\|_{L^\infty}^{\frac12}\left\|\nabla^l p\right\|_{L^2}\\ \lesssim&\|\nabla\phi\|_{L^2}\left\|\nabla^l m\right\|_{L^2}^{\frac{1}{4l+4}}\left\|\nabla^{\frac{1}{2l+1}}m\right\|_{L^2}^{\frac{2l+1}{4l+4}}\left\|\nabla^l m\right\|_{L^2}^{\frac{3}{4l+4}}\left\|\nabla^{\frac{3}{2l-1}}m\right\|_{L^2}^{\frac{2l-1}{4l+4}}\left\|\nabla^{l+1}p\right\|_{L^2}^{\frac{l}{l+1}}\|p\|_{L^2}^{\frac{1}{l+1}}\\ \lesssim&\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}p\right\|_{L^2}+\left\|\nabla^l m\right\|_{L^2}\right). \end{aligned}\tag{2.16}$$

Plugging (2.16) into (2.15), together with (2.14), we get for $l \geq 0$ that

$$\int_{\mathbb{R}^3} \nabla^l \mathrm{div}(pu_0) \cdot \nabla^l p \mathrm{d}x \lesssim \sqrt{\mathcal{E}_3}(t) \left(1 + \sqrt{\mathcal{E}_3}(t)\right) \left(\left\|\nabla^{l+1} p\right\|_{L^2}^2 + \left\|\nabla^l m\right\|_{L^2}^2 + \left\|\nabla^l \nabla \phi\right\|_{L^2}^2\right). \tag{2.17}$$

The other two terms in g_2 can be estimated by an argument similar to that of (2.10) and (2.11). Thus, we have

$$\left|\int_{\mathbb{R}^3} \nabla^l g_2 \cdot \nabla^l m \mathrm{d}x\right| \lesssim \sqrt{\mathcal{E}_3}(t) \left(1 + \sqrt{\mathcal{E}_3}(t)\right) \left(\left\|\nabla^{l+1} (p,\theta)\right\|_{L^2}^2 + \left\|\nabla^l m\right\|_{H^1}^2 + \left\|\nabla^l \nabla \phi\right\|_{L^2}^2\right). \tag{2.18}$$

For g_i, $i \geq 3$, since the denominator contains $d_0 + d_+ p + d_- m$, the strict positive lower L^∞ bound of $d_0 + d_+ p + d_- m$ is needed. Under the assumption of the a priori estimates (2.2), there exists $\tilde{d}_0$ such that

$$\frac{\tilde{d}_0}{2} \leq d_0 + d_+ p + d_- m \leq \tilde{d}_0. \tag{2.19}$$

In what follows, we deal with g_i ($i \geq 3$). When $l = 0$, it holds that

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} g_3 \theta \mathrm{d}x\right| &= \left|\int_{\mathbb{R}^3} \frac{d_+ p + d_- m}{d_0 + d_+ p + d_- m} (a_+ \Delta p + (a_+ + \frac{k}{k_B}) \Delta \theta) \theta \mathrm{d}x\right| \\
&= \left|\int_{\mathbb{R}^3} \frac{d_+ p + d_- m}{d_0 + d_+ p + d_- m} [a_+ \nabla p + (a_+ + \frac{k}{k_B}) \nabla \theta] \nabla \theta \mathrm{d}x\right| \\
&\quad + \left|\int_{\mathbb{R}^3} \nabla (\frac{d_+ p + d_- m}{d_0 + d_+ p + d_- m}) [a_+ \nabla p + (a_+ + \frac{k}{k_B}) \nabla \theta] \theta \mathrm{d}x\right| \\
&\lesssim \|p \nabla p \nabla \theta\|_{L^1} + \left\||\nabla \theta|^2 p\right\|_{L^1} + \|m \nabla p \nabla \theta\|_{L^1} + \left\|m |\nabla \theta|^2\right\|_{L^1} \\
&\quad + \left\||\nabla p|^2 \theta\right\|_{L^1} + \|\theta \nabla p \nabla \theta\|_{L^1} + \|\nabla m \nabla p \theta\|_{L^1} + \|\theta \nabla m \nabla \theta\|_{L^1} \\
&\lesssim \sqrt{\mathcal{E}(t)} \left(\|\nabla p\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|m\|_{H^1}^2\right).
\end{aligned} \tag{2.20}$$

When $l \geq 1$, we can use integration by parts, Hölder's inequality, the product estimates (A.4) of Lemma A.2 and Gagliardo-Nirenberg's inequality (A.3) to give that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla^l \left[\frac{d_+ p + d_- m}{d_0 + d_+ p + d_- m} (a_+ \Delta p)\right] \cdot \nabla^l \theta \mathrm{d}x \\
= &-\int_{\mathbb{R}^3} \nabla^l \left[\frac{d_+ p + d_- m}{d_0 + d_+ p + d_- m} (a_+ \nabla p)\right] \cdot \nabla^{l+1} \theta \mathrm{d}x \\
&-\int_{\mathbb{R}^3} \nabla^l \left[\nabla \left(\frac{d_+ p + d_- m}{d_0 + d_+ p + d_- m}\right) (a_+ \nabla p)\right] \cdot \nabla^l \theta \mathrm{d}x \\
\lesssim &\left\|\nabla^l \left[\frac{d_+ p + d_- m}{d_0 + d_+ p + d_- m} \nabla p\right]\right\|_{L^2} \left\|\nabla^{l+1} \theta\right\|_{L^2} \\
&+ \left\|\nabla^l \left[\nabla \left(\frac{d_+ p + d_- m}{d_0 + d_+ p + d_- m}\right) \nabla p\right]\right\|_{L^{\frac{6}{5}}} \left\|\nabla^l \theta\right\|_{L^6} \\
\lesssim &\left\|\frac{d_+ p + d_- m}{d_0 + d_+ p + d_- m}\right\|_{L^\infty} \left\|\nabla^l \nabla p\right\|_{L^2} \left\|\nabla^{l+1} \theta\right\|_{L^2} \\
&+ \left\|\nabla^l \left(\frac{d_+ p + d_- m}{d_0 + d_+ p + d_- m}\right)\right\|_{L^6} \|\nabla p\|_{L^3} \left\|\nabla^{l+1} \theta\right\|_{L^2} \\
&+ \left\|\nabla^{l+1} \left(\frac{d_+ p + d_- m}{d_0 + d_+ p + d_- m}\right)\right\|_{L^2} \|\nabla p\|_{L^3} \left\|\nabla^{l+1} \theta\right\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\nabla\left(\frac{d_+p+d_-m}{d_0+d_+p+d_-m}\right)\right\|_{L^3}\left\|\nabla^{l+1}p\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
\lesssim\ &\|(m,p)\|_{L^\infty}\left\|\nabla^{l+1}p\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}+\|\nabla p\|_{L^3}\left\|\nabla^{l+1}(m,p)\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
&+\|\nabla(m,p)\|_{L^3}\left\|\nabla^{l+1}p\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
\lesssim\ &\sqrt{\mathcal{E}_3}(t)\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2. \qquad (2.21)
\end{aligned}
$$

The second term in g_3 can be estimated by the same argument as that in (2.21). Thus, for $l\geq 0$, we finally have

$$
\left|\int_{\mathbb{R}^3}\nabla^l g_3\cdot\nabla^l\theta\mathrm{d}x\right|\lesssim\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{L^2}^2\right). \qquad (2.22)
$$

For the term g_4, when $l=0$, by the facts (2.19) and (2.6), we have that

$$
\begin{aligned}
\int_{\mathbb{R}^3}g_4\theta dx\lesssim\ &\|\nabla\theta\|_{L^2}\|\nabla(\tilde{p}T)+\nabla(mT)+\tilde{p}\nabla\phi\|_{L^2}\|\theta\|_{L^\infty}+\|\nabla\theta\|_{L^2}\|m\nabla\phi\|_{L^2}\|\theta\|_{L^\infty}\\
&+\|\nabla\theta\|_{L^2}\|u_0\|_{L^6}\|\theta\|_{L^3}\\
\lesssim\ &\|\nabla\theta\|_{L^2}\|(m,\tilde{p},T)\|_{L^\infty}\|\nabla(m,p,\theta)\|_{L^2}\|\theta\|_{L^\infty}+\|\nabla\theta\|_{L^2}\|\tilde{p}\|_{L^\infty}\|\nabla\phi\|_{L^2}\|\theta\|_{L^\infty}\\
&+\|\nabla\theta\|_{L^2}\|m\|_{L^\infty}\|\nabla\phi\|_{L^2}\|\theta\|_{L^\infty}+\|\nabla\theta\|_{L^2}\|\nabla u_0\|_{L^2}\|\theta\|_{L^3}\\
\lesssim\ &\|\nabla\theta\|_{L^2}\|\nabla(m,p,\theta)\|_{L^2}\|\theta\|_{L^\infty}+\|\nabla\theta\|_{L^2}\|\nabla\phi\|_{L^2}\|\theta\|_{L^\infty}\\
&+\|\nabla\theta\|_{L^2}\|m\|_{L^\infty}\|\nabla\phi\|_{L^2}\|\theta\|_{L^\infty}+\|\nabla\theta\|_{L^2}\|\nabla u_0\|_{L^2}\|\theta\|_{L^3}\\
\lesssim\ &\mathcal{E}_3(t)\left(\|\nabla(m,p,\theta)\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right). \qquad (2.23)
\end{aligned}
$$

For when $l\geq 1$, we will estimate each term of g_4. By Hölder's inequality, we get that

$$
\int_{\mathbb{R}^3}\nabla^l g_4\cdot\nabla^l\theta\mathrm{d}x\lesssim\left\|\nabla^l g_4\right\|_{L^{\frac{6}{5}}}\left\|\nabla^l\theta\right\|_{L^6}\lesssim\left\|\nabla^l g_4\right\|_{L^{\frac{6}{5}}}\left\|\nabla^{l+1}\theta\right\|_{L^2}. \qquad (2.24)
$$

Since $\tilde{p}=1+p$ and $T=1+\theta$, we can easily get that

$$
\begin{aligned}
&\left\|\nabla^l\left[\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}a_2^+\nabla(\tilde{p}T)\right]\right\|_{L^{\frac{6}{5}}}\\
=\ &\left\|\nabla^l\left[\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}a_2^+\nabla(p+\theta+p\theta)\right]\right\|_{L^{\frac{6}{5}}}\\
\lesssim\ &\left\|\nabla^l\left[\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}\nabla p\right]\right\|_{L^{\frac{6}{5}}}+\left\|\nabla^l\left[\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}\nabla\theta\right]\right\|_{L^{\frac{6}{5}}}\\
&+\left\|\nabla^l\left[\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}\nabla(p\theta)\right]\right\|_{L^{\frac{6}{5}}}. \qquad (2.25)
\end{aligned}
$$

By the product estimates (A.4) of Lemma A.2 and Gagliardo-Nirenberg's inequality (A.3), we have that

$$
\begin{aligned}
&\left\|\nabla^l\left[\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}\nabla p\right]\right\|_{L^{\frac{6}{5}}}\\
\lesssim\ &\left\|\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}\right\|_{L^3}\left\|\nabla^{l+1}p\right\|_{L^2}+\|\nabla p\|_{L^3}\left\|\nabla^l\left(\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}\right)\right\|_{L^2}\\
\lesssim\ &\|\nabla\theta\|_{L^3}\left\|\nabla^{l+1}p\right\|_{L^2}+\|\nabla p\|_{L^3}\left\|\left(\frac{d_0}{d_0+d_+p+d_-m}\right)\right\|_{L^\infty}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
&+\|\nabla p\|_{L^3}\|\nabla\theta\|_{L^3}\left\|\nabla^l\left(\frac{d_0}{d_0+d_+p+d_-m}\right)\right\|_{L^6}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|\nabla\theta\|_{L^3}\left\|\nabla^{l+1}p\right\|_{L^2}+\|\nabla p\|_{L^3}\left\|\nabla^{l+1}\theta\right\|_{L^2}+\|\nabla p\|_{L^3}\|\nabla\theta\|_{L^3}\left\|\nabla^{l+1}(m,p)\right\|_{L^2}\\
&\lesssim \sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left\|\nabla^{l+1}(p,\theta,m)\right\|_{L^2}
\end{aligned}\tag{2.26}
$$

and

$$
\begin{aligned}
&\left\|\nabla^l\left[\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}\nabla(p\theta)\right]\right\|_{L^{\frac{6}{5}}}\\
\lesssim&\left\|\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}\right\|_{L^3}\left\|\nabla^{l+1}(p\theta)\right\|_{L^2}+\|\nabla(p\theta)\|_{L^3}\left\|\nabla^l\left(\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}\right)\right\|_{L^2}\\
\lesssim&\|\nabla\theta\|_{L^3}\left\|\nabla^{l+1}(p\theta)\right\|_{L^2}+\|\nabla(p\theta)\|_{L^3}\left\|\left(\frac{d_0}{d_0+d_+p+d_-m}\right)\right\|_{L^\infty}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
&+\|\nabla(p\theta)\|_{L^3}\|\nabla\theta\|_{L^3}\left\|\nabla^l\left(\frac{d_0}{d_0+d_+p+d_-m}\right)\right\|_{L^6}\\
\lesssim&\|\nabla\theta\|_{L^3}\|(p,\theta)\|_{L^\infty}\left\|\nabla^{l+1}(p,\theta)\right\|_{L^2}+\|(p,\theta)\|_{L^\infty}\|\nabla(p,\theta)\|_{L^3}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
&+\|(p,\theta)\|_{L^\infty}\|\nabla(p,\theta)\|_{L^3}\|\nabla\theta\|_{L^3}\left\|\nabla^{l+1}(m,p)\right\|_{L^2}\\
\lesssim&\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left\|\nabla^{l+1}(p,\theta,m)\right\|_{L^2}.
\end{aligned}\tag{2.27}
$$

We can estimate the rest of the terms of g_4 by arguments similar to those in (2.26)–(2.27) and (2.15)–(2.17). Thus, we have

$$
\int_{\mathbb{R}^3}\nabla^l g_4\cdot\nabla^l\theta\mathrm{d}x\lesssim\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}(p,\theta,m)\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{L^2}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right).\tag{2.28}
$$

In what follows, we will estimate the term g_5. The term g_5 will be divided into four terms. First of all, we want to get the estimates of the first term of g_5. By employing integration by parts and Hölder's inequality, it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\nabla^l\left[\frac{d_0}{d_0+d_+p+d_-m}\Delta(p\theta)\right]\cdot\nabla^l\theta\mathrm{d}x\\
=&-\int_{\mathbb{R}^3}\nabla^l\left[\frac{d_0}{d_0+d_+p+d_-m}\nabla(p\theta)\right]\cdot\nabla^{l+1}\theta\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\nabla^l\left[\nabla\left(\frac{d_0}{d_0+d_+p+d_-m}\right)\nabla(p\theta)\right]\cdot\nabla^l\theta\mathrm{d}x\\
\lesssim&\left\|\nabla^l\left[\frac{d_0}{d_0+d_+p+d_-m}\nabla(p\theta)\right]\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
&+\left\|\nabla^l\left[\nabla\left(\frac{d_0}{d_0+d_+p+d_-m}\right)\nabla(p\theta)\right]\right\|_{L^{\frac{6}{5}}}\left\|\nabla^l\theta\right\|_{L^6}.
\end{aligned}\tag{2.29}
$$

From (2.29), for $l=0$, it is clear that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\left[\frac{d_0}{d_0+d_+p+d_-m}(a_+\Delta(p\theta))\right]\theta\mathrm{d}x\\
\lesssim&\|\nabla(p\theta)\|_{L^2}\|\nabla\theta\|_{L^2}+\left\|\left[\nabla\left(\frac{d_0}{d_0+d_+p+d_-m}\right)\nabla(p\theta)\right]\right\|_{L^{\frac{6}{5}}}\|\theta\|_{L^6}\\
\lesssim&\|(p,\theta)\|_{L^\infty}\|\nabla(p,\theta)\|_{L^2}\|\nabla\theta\|_{L^2}+\|\nabla(p,m)\|_{L^2}\|\nabla(p\theta)\|_{L^3}\|\nabla\theta\|_{L^2}\\
\lesssim&\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\|\nabla(p,\theta,m)\|_{L^2}^2.
\end{aligned}\tag{2.30}
$$

For $l\geq 1$, by employing the product estimates (A.4) of Lemma A.2 and Gagliardo-Nirenberg's inequality (A.3), we can deduce that

$$\begin{aligned}
&\left\|\nabla^l\left[\frac{d_0}{d_0+d_+p+d_-m}\nabla(p\theta)\right]\right\|_{L^2}\\
\lesssim&\left\|\frac{d_0}{d_0+d_+p+d_-m}\right\|_{L^\infty}\left\|\nabla^{l+1}(p\theta)\right\|_{L^2}+\left\|\nabla^l\left(\frac{d_0}{d_0+d_+p+d_-m}\right)\right\|_{L^6}\left\|\nabla(p\theta)\right\|_{L^3}\\
\lesssim&\left\|(p,\theta)\right\|_{L^\infty}\left\|\nabla^{l+1}(p,\theta)\right\|_{L^2}+\left\|\nabla(p,\theta)\right\|_{L^3}\left\|(p,\theta)\right\|_{L^\infty}\left\|\nabla^{l+1}(m,p)\right\|_{L^2}\\
\lesssim&\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2},
\end{aligned}\tag{2.31}$$

and

$$\begin{aligned}
&\left\|\nabla^l\left[\nabla\left(\frac{d_0}{d_0+d_+p+d_-m}\right)\cdot\nabla(p\theta)\right]\right\|_{L^{\frac{6}{5}}}\\
\lesssim&\left\|\nabla(p\theta)\right\|_{L^3}\left\|\nabla^{l+1}\left(\frac{d_0}{d_0+d_+p+d_-m}\right)\right\|_{L^2}+\left\|\nabla\left(\frac{d_0}{d_0+d_+p+d_-m}\right)\right\|_{L^3}\left\|\nabla^{l+1}(p\theta)\right\|_{L^2}\\
\lesssim&\left\|(p,\theta)\right\|_{L^\infty}\left\|\nabla(p,\theta)\right\|_{L^3}\left\|\nabla^{l+1}(p,m)\right\|_{L^2}+\left\|\nabla(p,m)\right\|_{L^3}\left\|(p,\theta)\right\|_{L^\infty}\left\|\nabla^{l+1}(p,\theta)\right\|_{L^2}\\
\lesssim&\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}.
\end{aligned}\tag{2.32}$$

The estimates (2.31)–(2.32), together with (2.30), finally give that

$$\int_{\mathbb{R}^3}\nabla^l\left[\frac{d_0}{d_0+d_+p+d_-m}(a_+\Delta(p\theta))\right]\cdot\nabla^l\theta\mathrm{d}x\lesssim\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2.\tag{2.33}$$

Second, for the second term of g_5, with the help of (2.19), it is clear that for $l=0$,

$$\begin{aligned}
\int_{\mathbb{R}^3}\left[\frac{d_0}{d_0+d_+p+d_-m}(a_+\nabla p\cdot\nabla\theta(1-\theta))\right]\theta\mathrm{d}x&\lesssim\left\|\nabla p\right\|_{L^2}\left\|\nabla\theta\right\|_{L^2}\left\|\theta\right\|_{L^\infty}\\
&\lesssim\sqrt{\mathcal{E}_3}(t)\left\|\nabla(p,\theta)\right\|_{L^2}^2.
\end{aligned}\tag{2.34}$$

While $l\geq 1$, we can use Hölder's inequality, the product estimates (A.4) of Lemma A.2 and Gagliardo-Nirenberg's inequality (A.3) to obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\nabla^l\left[\frac{d_0}{d_0+d_+p+d_-m}(a_+\nabla p\cdot\nabla\theta(1-\theta))\right]\cdot\nabla^l\theta\mathrm{d}x\\
\lesssim&\left\|\nabla^l\left[\frac{d_0}{d_0+d_+p+d_-m}(a_+\nabla p\cdot\nabla\theta(1-\theta))\right]\right\|_{L^{\frac{6}{5}}}\left\|\nabla^l\theta\right\|_{L^6}\\
\lesssim&\left\|\frac{1-\theta}{d_0+d_+p+d_-m}\right\|_{L^\infty}\left\|\nabla^l(\nabla p\cdot\nabla\theta)\right\|_{L^{\frac{6}{5}}}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
&+\left\|\nabla^l\left(\frac{1-\theta}{d_0+d_+p+d_-m}\right)\right\|_{L^2}\left\|\nabla p\cdot\nabla\theta\right\|_{L^3}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
\lesssim&\left\|\nabla(p,\theta)\right\|_{L^3}\left\|\nabla^{l+1}(p,\theta)\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}+\left\|\nabla p\right\|_{L^\infty}\left\|\nabla\theta\right\|_{L^3}\left\|\nabla^l(m,p,\theta)\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
\lesssim&\left\|\nabla(p,\theta)\right\|_{L^3}\left\|\nabla^{l+1}(p,\theta)\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
&+\left\|\nabla p\right\|_{L^\infty}\left\|\nabla^{\frac{l+1}{2l}}\theta\right\|_{L^2}^{\frac{l}{l+1}}\left\|\nabla^{l+1}\theta\right\|_{L^2}^{\frac{1}{l+1}}\left\|(m,p,\theta)\right\|_{L^2}^{\frac{1}{l+1}}\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^{\frac{l}{l+1}}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
\lesssim&\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2.
\end{aligned}\tag{2.35}$$

From (2.34) and (2.35), it holds for $l\geq 0$ that

$$\int_{\mathbb{R}^3}\nabla^l\left[\frac{d_0}{d_0+d_+p+d_-m}(a_+\nabla p\cdot\nabla\theta(1-\theta))\right]\cdot\nabla^l\theta\mathrm{d}x$$

$$\lesssim \sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2. \tag{2.36}$$

Next, we estimate the third term of g_5. It holds, by employing integration by parts, that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\nabla^l\left[\frac{d_0}{d_0+d_+p+d_-m}(a_+(1+p)\theta\Delta\theta)\right]\cdot\nabla^l\theta\mathrm{d}x\\
=&\int_{\mathbb{R}^3}\nabla^l\left[\frac{d_0a_+(1+p)}{d_0+d_+p+d_-m}(\theta\Delta\theta)\right]\cdot\nabla^l\theta\mathrm{d}x\\
=&-\int_{\mathbb{R}^3}\nabla^l\left[\frac{d_0a_+(1+p)}{d_0+d_+p+d_-m}(\theta\nabla\theta)\right]\cdot\nabla^{l+1}\theta\mathrm{d}x-\int_{\mathbb{R}^3}\nabla^l\left[\frac{d_0a_+(1+p)}{d_0+d_+p+d_-m}(\nabla\theta\nabla\theta)\right]\cdot\nabla^l\theta\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\nabla^l\left[\nabla\left(\frac{d_0a_+(1+p)}{d_0+d_+p+d_-m}\right)(\nabla\theta\nabla\theta)\right]\cdot\nabla^l\theta\mathrm{d}x.
\end{aligned}\tag{2.37}$$

For when $l=0$, it is clear that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left[\frac{d_0}{d_0+d_+p+d_-m}(a_+(1+p)\theta\Delta\theta)\right]\theta\mathrm{d}x\\
\lesssim&\|\theta\|_{L^\infty}\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^2}+\|\theta\|_{L^\infty}\|\nabla(m,p)\|_{L^\infty}\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^2}\\
\lesssim&\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\|\nabla\theta\|_{L^2}^2.
\end{aligned}\tag{2.38}$$

For $l\geq 1$, by using Hölder's inequality, the product estimates (A.4) of Lemma A.2 and Gagliardo-Nirenberg's inequality (A.3), we get that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\nabla^l\left[\frac{d_0a_+(1+p)}{d_0+d_+p+d_-m}(\theta\nabla\theta)\right]\cdot\nabla^{l+1}\theta\mathrm{d}x\\
\lesssim&\left\|\nabla^l\left[\frac{d_0a_+(1+p)}{d_0+d_+p+d_-m}(\theta\nabla\theta)\right]\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
\lesssim&\left(\left\|\frac{d_0a_+(1+p)}{d_0+d_+p+d_-m}\right\|_{L^\infty}\left\|\nabla^l(\theta\nabla\theta)\right\|_{L^2}+\left\|\nabla^l\left(\frac{d_0a_+(1+p)}{d_0+d_+p+d_-m}\right)\right\|_{L^6}\|\theta\nabla\theta\|_{L^3}\right)\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
\lesssim&\left(\left\|\nabla^l(\theta\nabla\theta)\right\|_{L^2}+\left\|\nabla^{l+1}(m,p)\right\|_{L^2}\|\theta\|_{L^\infty}\|\nabla\theta\|_{L^3}\right)\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
\lesssim&\left(\|\theta\|_{L^\infty}\left\|\nabla^{l+1}\theta\right\|_{L^2}+\|\nabla\theta\|_{L^3}\left\|\nabla^l\theta\right\|_{L^6}+\left\|\nabla^{l+1}(m,p)\right\|_{L^2}\|\theta\|_{L^\infty}\|\nabla\theta\|_{L^3}\right)\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
\lesssim&\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2.
\end{aligned}\tag{2.39}$$

The other two terms in (2.37) can be dealt with by the routine of using Hölder's inequality and Gagliardo-Nirenberg's inequality (A.3). We can employ an argument similar to that of (2.39) to deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\nabla^l\left[\frac{d_0a_+(1+p)}{d_0+d_+p+d_-m}(\nabla\theta\nabla\theta)\right]\cdot\nabla^l\theta\mathrm{d}x\\
\lesssim&\left\|\nabla^l\left[\frac{d_0a_+(1+p)}{d_0+d_+p+d_-m}(\nabla\theta\nabla\theta)\right]\right\|_{L^{\frac{6}{5}}}\left\|\nabla^l\theta\right\|_{L^6}\\
\lesssim&\left\|\frac{d_0a_+(1+p)}{d_0+d_+p+d_-m}\right\|_{L^\infty}\left\|\nabla^l(\nabla\theta\nabla\theta)\right\|_{L^{\frac{6}{5}}}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
&+\|\nabla\theta\nabla\theta\|_{L^3}\left\|\nabla^l\left(\frac{d_0a_+(1+p)}{d_0+d_+p+d_-m}\right)\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
\lesssim&\|\nabla\theta\|_{L^3}\left\|\nabla^{l+1}\theta\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}+\|\nabla\theta\|_{L^\infty}\|\nabla\theta\|_{L^3}\left\|\nabla^l(p,m)\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}\\
\lesssim&\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2
\end{aligned}\tag{2.40}$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \nabla^l \left[\nabla \left(\frac{d_0 a_+(1+p)}{d_0+d_+p+d_-m}\right)(\nabla\theta\nabla\theta)\right]\cdot\nabla^l\theta \mathrm{d}x \\
\lesssim& \left\|\nabla^l \left[\nabla \left(\frac{d_0 a_+(1+p)}{d_0+d_+p+d_-m}\right)(\nabla\theta\nabla\theta)\right]\right\|_{L^{\frac{6}{5}}} \left\|\nabla^l\theta\right\|_{L^6} \\
\lesssim& \left\|\nabla \left(\frac{d_0 a_+(1+p)}{d_0+d_+p+d_-m}\right)\right\|_{L^3} \left\|\nabla^l(\nabla\theta\nabla\theta)\right\|_{L^2} \left\|\nabla^{l+1}\theta\right\|_{L^2} \\
&+\|\nabla\theta\nabla\theta\|_{L^3} \quad \left\|\nabla^{l+1}\left(\frac{d_0 a_+(1+p)}{d_0+d_+p+d_-m}\right)\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2} \\
\lesssim& \|\nabla(p,m)\|_{L^3}\|\nabla\theta\|_{L^\infty}\left\|\nabla^{l+1}\theta\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2} \\
&+\|\nabla\theta\|_{L^\infty}\|\nabla\theta\|_{L^3} \quad \left\|\nabla^{l+1}(m,p)\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2} \\
\lesssim& \sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2.
\end{aligned} \tag{2.41}
$$

Plugging (2.39)-(2.41) into (2.37), together with (2.38), it holds for $l \geq 0$ that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \nabla^l \left[\frac{d_0}{d_0+d_+p+d_-m}(a_+(1+p)\theta\Delta\theta)\right]\cdot\nabla^l\theta \mathrm{d}x \\
\lesssim& \sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2.
\end{aligned} \tag{2.42}
$$

Finally, we deal with the last term of g_5. For $l=0$, we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \left[\frac{d_0}{d_0+d_+p+d_-m}\frac{b_+a_0}{\epsilon}(1+\theta)m^2\right]\theta \mathrm{d}x \\
\lesssim& \|m\|_{L^2}\|m\|_{L^3}\|\theta\|_{L^6} \lesssim \sqrt{\mathcal{E}_3}(t)\left(\|\nabla\theta\|_{L^2}^2+\|m\|_{L^2}^2\right).
\end{aligned}
$$

When $l \geq 1$, by employing Hölder's inequality, the product estimates (A.4) of Lemma A.2 and Gagliardo-Nirenberg's inequality (A.3), we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \nabla^l \left[\frac{d_0}{d_0+d_+p+d_-m}\frac{b_+a_0}{\epsilon}(1+\theta)m^2\right]\cdot\nabla^l\theta \mathrm{d}x \\
\lesssim& \left\|\nabla^l\left[\frac{d_0}{d_0+d_+p+d_-m}\frac{b_+a_0}{\epsilon}(1+\theta)m^2\right]\right\|_{L^{\frac{6}{5}}}\left\|\nabla^l\theta\right\|_{L^6} \\
\lesssim& \left\|\frac{d_0b_+a_0(1+\theta)}{\epsilon(d_0+d_+p+d_-m)}\right\|_{L^\infty}\left\|\nabla^l(m^2)\right\|_{L^{\frac{6}{5}}}\left\|\nabla^{l+1}\theta\right\|_{L^2} \\
&+\left\|m^2\right\|_{L^3}\left\|\nabla^l\left(\frac{d_0b_+a_0(1+\theta)}{\epsilon(d_0+d_+p+d_-m)}\right)\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2} \\
\lesssim& \|m\|_{L^3}\left\|\nabla^l m\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2}+\|m\|_{L^3}\|m\|_{L^\infty}\left\|\nabla^l(m,p,\theta)\right\|_{L^2}\left\|\nabla^{l+1}\theta\right\|_{L^2} \\
\lesssim& \sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{L^2}^2\right).
\end{aligned}
$$

Thus, for $l \geq 0$, we finally get that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \nabla^l \left[\frac{d_0}{d_0+d_+p+d_-m}\frac{b_+a_0}{\epsilon}(1+\theta)m^2\right]\cdot\nabla^l\theta \mathrm{d}x \\
\lesssim& \sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{L^2}^2\right).
\end{aligned} \tag{2.43}
$$

In terms of (2.33), (2.36), (2.42) and (2.43), we get that

$$
\int_{\mathbb{R}^3} \nabla^l g_5\cdot\nabla^l\theta \mathrm{d}x \lesssim \sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}(m,p,\theta)\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{L^2}^2\right). \tag{2.44}
$$

By an argument similar to that for estimates (2.40), we can get that

$$\begin{aligned}&\int_{\mathbb{R}^3}\nabla^l g_6\cdot\nabla^l\theta\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^l g_7\cdot\nabla^l\theta\mathrm{d}x\\ &\lesssim\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}(p,\theta)\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{H^1}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right).\end{aligned}\tag{2.45}$$

In light of (2.13), (2.18), (2.22), (2.28), (2.44) and (2.45), we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^l m\right\|_{L^2}^2+\left\|\nabla^l p\right\|_{L^2}^2+d_0\left\|\nabla^l\theta\right\|_{L^2}^2\right)+c_d\left(\left\|\nabla^l m\right\|_{H^1}^2+\left\|\nabla^{l+1}p\right\|_{L^2}^2+\left\|\nabla^{l+1}\theta\right\|_{L^2}^2\right)\\ &\lesssim\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}(p,\theta)\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{H^1}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right).\end{aligned}\tag{2.46}$$

Unfortunately, we note that the term $\nabla^l\nabla\phi$ appearing on the right hand side of (2.46) can not be absorbed by the left hand side; that is, the energy estimates cannot be closed. Therefore, we must obtain the estimates of $\nabla\phi$, since this is necessary to close the energy estimates. Plugging $(2.1)_3$ into equation $(2.1)_1$, we get that

$$\partial_t\Delta\phi-a_+\Delta(\Delta\phi)+\frac{a_0b_+}{\epsilon}\Delta\phi=-\frac{a_0}{\epsilon}g_1.\tag{2.47}$$

We can apply the operator ∇^l with $l\geq 0$ to equations (2.47) and multiply by $\nabla^l\phi$ to deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^l\nabla\phi\right\|_{L^2}^2+a_+\left\|\nabla^l m\right\|_{L^2}^2+\frac{a_0b_+}{\epsilon}\left\|\nabla^l\nabla\phi\right\|_{L^2}^2=\frac{a_0}{\epsilon}\int_{\mathbb{R}^3}\nabla^l g_1\cdot\nabla^l\phi\mathrm{d}x.\tag{2.48}$$

Integration by parts, together with Hölder's inequality, implies that

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla^l\Delta(m\theta)\cdot\nabla^l\phi\mathrm{d}x&\lesssim\left\|\nabla^{l+1}(m\theta)\right\|_{L^2}\left\|\nabla^l\nabla\phi\right\|_{L^2}\\ &\lesssim\|(m,\theta)\|_{L^\infty}\left\|\nabla^{l+1}(m,\theta)\right\|_{L^2}\left\|\nabla^l\nabla\phi\right\|_{L^2}\\ &\lesssim\sqrt{\mathcal{E}_3}(t)\left(\left\|\nabla^{l+1}(m,\theta)\right\|_{L^2}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right).\end{aligned}\tag{2.49}$$

A similar argument to that of (2.49) gives rise to

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla^l\mathrm{div}(p\nabla\phi)\cdot\nabla^l\phi\mathrm{d}x&\lesssim\left\|\nabla^l(p\nabla\phi)\right\|_{L^2}\left\|\nabla^l\nabla\phi\right\|_{L^2}\\ &\lesssim\|p\|_{L^\infty}\left\|\nabla^l\nabla\phi\right\|_{L^2}^2+\|\nabla\phi\|_{L^3}\left\|\nabla^l p\right\|_{L^6}\left\|\nabla^l\nabla\phi\right\|_{L^2}\\ &\lesssim\sqrt{\mathcal{E}_3}(t)\left(\left\|\nabla^{l+1}p\right\|_{L^2}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right)\end{aligned}\tag{2.50}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla^l\mathrm{div}(mu_0)\cdot\nabla^l\phi\mathrm{d}x&\lesssim\left\|\nabla^l(mu_0)\right\|_{L^2}\left\|\nabla^l\nabla\phi\right\|_{L^2}\\ &\lesssim\|u_0\|_{L^\infty}\left\|\nabla^l m\right\|_{L^2}\left\|\nabla^l\nabla\phi\right\|_{L^2}+\|m\|_{L^3}\left\|\nabla^l u_0\right\|_{L^6}\left\|\nabla^l\nabla\phi\right\|_{L^2}\\ &\lesssim\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^l m\right\|_{L^2}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right).\end{aligned}\tag{2.51}$$

It follows from (2.48)–(2.51) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^l\nabla\phi\right\|_{L^2}^2+a_+\left\|\nabla^l m\right\|_{L^2}^2+\frac{a_0b_+}{\epsilon}\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\\ &\lesssim\sqrt{\mathcal{E}_3}(t)\left(1+\sqrt{\mathcal{E}_3}(t)\right)\left(\left\|\nabla^{l+1}(p,m,\theta)\right\|_{L^2}^2+\left\|\nabla^l m\right\|_{L^2}^2+\left\|\nabla^l\nabla\phi\right\|_{L^2}^2\right).\end{aligned}\tag{2.52}$$

Summing up (2.52) and (2.46), we can obtain (2.3). □

From Lemma 2.1, we directly obtain the uniform estimates as follows:

Proposition 2.2 If there is some sufficiently small $\tilde{c} > 0$ and some $T > 0$ such that

$$\sup_{0\le t\le T} \mathcal{E}_3(t) \le \tilde{c}, \tag{2.53}$$

then it holds for $N \ge 3$ that

$$\mathcal{E}_N(t) + \int_0^t \mathcal{D}_N(s)\mathrm{d}s \le C\mathcal{E}_N(0), \tag{2.54}$$

which proves (1.4) of Theorem 1.1. By a standard continuity argument, the a priori estimates (2.2) can be closed when we take $N = 3$.

3 Approximate Solutions and Global in Time Solution

3.1 Local in time solution

In this section, we will try to construct the local in time solution to (1.1) under the assumptions of Theorem 1.1. We shall sketch the idea of constructing the approximate solutions by an iteration method. These processes are routine. The approximate system is as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}m_n - a_+\Delta m_n + \dfrac{b_+a_0}{\epsilon}m_n = g_{n-1,1} \\ \dfrac{\partial}{\partial t}p_n - a_+\Delta p_n - a_+\Delta\theta_n = g_{n-1,2}, \\ d_0\dfrac{\partial}{\partial t}\theta_n - a_+\Delta p_n - (a_+ + \dfrac{k}{k_B})\Delta\theta_n = \displaystyle\sum_{i=3}^{7} g_{n-1,i}. \end{cases} \tag{3.1}$$

Here $g_{n-1,i}$ $(1 \le i \le 7)$ are like g_i, in which the unknowns p, m, θ, ϕ, u_0 are replaced by p_{n-1}, m_{n-1}, θ_{n-1}, ϕ_{n-1}, $u_{0,n-1}$. In particular, $u_{0,n-1} = \mathbb{P}_h u_{0,n-1} = -\frac{ze}{\lambda_0}(-\Delta)^{-1}\mathbb{P}_h(m_{n-1}\nabla\phi_{n-1})$.

Let $p_0 = 0, m_0 = 0, \theta_0 = 0, u_{0,0} = 0$, and

$$p_n(0) = \rho_+(0) + \rho_-(0) - 1, m_n(0) = \rho_+(0) - \rho_-(0), \theta_n(0) = T(0) - 1, \quad \forall n \in \mathbb{N}^+.$$

For each $n \in \mathbb{N}^+$, system (3.1) is reduced to the linear parabolic one with source terms. By the parabolic theorem, the existence of a H^3 solution to system (3.1) can be established while the source terms belong to the H^2 space. In what follows, we will show how to deduce the H^2 bound of the source terms for each $n \in \mathbb{N}^+$.

Let

$$\dot{\mathcal{E}}_n(t) := \left(\left\|\nabla^3 m_n\right\|_{L^2}^2 + \left\|\nabla^3 p_n\right\|_{L^2}^2 + d_0\left\|\nabla^3\theta_n\right\|_{L^2}^2\right)(t),$$

and

$$\dot{\mathcal{D}}_n(t) = \left(\left\|\nabla^4 m_n\right\|_{L^2}^2 + \left\|\nabla^4 p_n\right\|_{L^2}^2 + \left\|\nabla^3\theta_n\right\|_{H^1}^2\right)(t).$$

When $n = 1$, the source terms vanish. In a manner similar to that used to deduce (2.4), we obtain that

$$\sup_{0\le s\le t} \dot{\mathcal{E}}_1(s) + 2\int_0^t \dot{\mathcal{D}}_1(s)\mathrm{d}s \le \dot{\mathcal{E}}_1(0).$$

By arguments similar to (2.13), (2.18), (2.22), (2.28), (2.44) and (2.45), and according to (2.54), we can deduce that

$$\int_0^t \sum_{i=1}^{7} \left\|\nabla^2 g_{1,i}(s)\right\|_{L^2}^2 \mathrm{d}s < \infty.$$

By induction, we can infer that, for any $n \in \mathbb{N}^+$ and $t > 0$,

$$\int_0^t \sum_{i=1}^{7} \left\|\nabla^2 g_{n,i}(s)\right\|_{L^2}^2 \mathrm{d}s < \infty.$$

Recalling that there exists the denominator $d_0 + d_+p + d_-m$ in each $g_{n,i}(3 \le i \le 7$, we have that the denominator must be positive with a strict lower bound. This also requires that the initial data should be small in the H^3 sense. With the help of (2.13), (2.18), (2.22), (2.28), (2.44), (2.45) and (2.54), by employing the Picard iteration method and the induction method, we can deduce the uniform upper bound of the H^3 bound of approximate solutions with respect to $n \in \mathbb{N}^+$ for small enough $T > 0$; by the Sobolev embedding inequality, their H^3 norm can be obtained. Then for small enough time, we can prove that (p_n, m_n, θ_n) is a Cauchy sequence in the H^3 norm and the uniqueness of the local in time solution can be obtained.

3.2 Global solution

By combining the local in time existence and the uniform estimates (2.54), we can get the global classical solution in the framework of [28].

4 The Estimates in $\dot{H}^{-s}$ Norms

In this section, we are trying to deduce some useful estimates of the solution in the negative Sobolev norms. These estimates will be very helpful in the process of deducing the time decay estimates of the solution.

Lemma 4.1 It holds for $s \in [0, 1/2]$ that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda^{-s}(m,p,\sqrt{d_0}\theta,\nabla\phi)(t)\right\|_{L^2}^2 + c_d\left(\left\|\Lambda^{-s}(\nabla p,\nabla\theta,\nabla\phi)\right\|_{L^2}^2 + \left\|\Lambda^{-s}m\right\|_{H^1}^2\right)\\ &\lesssim \left(\|m\|_{H^3}^2 + \|\nabla p,\theta\|_{H^2}^2 + \|\nabla\phi\|_{L^2}^2\right)\left\|\Lambda^{-s}(m,p,\theta,\nabla\phi)(t)\right\|_{L^2}^2, \end{aligned} \tag{4.1}$$

and

$$\left\|\Lambda^{-s}(m,p,\sqrt{d_0}\theta,\nabla\phi)\right\|_{L^2} \le C_0. \tag{4.2}$$

Proof We apply the operator Λ^{-s} to equations $(2.1)_1$, $(2.1)_2$ and $(2.1)_4$, and multiply by $\Lambda^{-s}m$, $\Lambda^{-s}p$ and $\Lambda^{-s}\theta$, respectively, then integrate over $\mathbb{R}^3$, to deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\Lambda^{-s}m\right\|_{L^2}^2 + \left\|\Lambda^{-s}p\right\|_{L^2}^2 + d_0\left\|\Lambda^{-s}\theta\right\|_{L^2}^2\right) + a_+\left\|\Lambda^{-s}\nabla m\right\|_{L^2}^2 + \frac{b_+a_0}{\epsilon}\left\|\Lambda^{-s}m\right\|_{L^2}^2\\ &+ a_+\left\|\Lambda^{-s}\nabla p\right\|_{L^2}^2 + \left(a_+ + \frac{k}{k_B}\right)\left\|\Lambda^{-s}\nabla\theta\right\|_{L^2}^2 + 2a_+\int_{\mathbb{R}^3}\Lambda^{-s}\nabla p\cdot\Lambda^{-s}\nabla\theta\mathrm{d}x\\ =& \int_{\mathbb{R}^3}\Lambda^{-s}g_1\cdot\Lambda^{-s}m\mathrm{d}x + \int_{\mathbb{R}^3}\Lambda^{-s}g_2\cdot\Lambda^{-s}p\mathrm{d}x + \sum_{i=3}^{7}\int_{\mathbb{R}^3}\Lambda^{-s}g_i\cdot\Lambda^{-s}\theta\mathrm{d}x.\end{aligned}$$

Note that

$$\begin{aligned}&2\int_{\mathbb{R}^3}\Lambda^{-s}\nabla p\Lambda^{-s}\nabla\theta\mathrm{d}x + a_+\left\|\Lambda^{-s}\nabla p\right\|_{L^2}^2 + \left(a_+ + \frac{k}{k_B}\right)\left\|\Lambda^{-s}\nabla\theta\right\|_{L^2}^2\\ \ge& \frac{a_+k}{2a_+k_B + k}\left\|\Lambda^{-s}\nabla p\right\|_{L^2}^2 + \frac{k}{2k_B}\left\|\Lambda^{-s}\nabla\theta\right\|_{L^2}^2.\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\Lambda^{-s}m\right\|_{L^2}^2+\left\|\Lambda^{-s}p\right\|_{L^2}^2+d_0\left\|\Lambda^{-s}\theta\right\|_{L^2}^2\right)\\
&+c_d\left(\left\|\Lambda^{-s}m\right\|_{H^1}^2+\left\|\Lambda^{-s}\nabla p\right\|_{L^2}^2+\left\|\Lambda^{-s}\nabla\theta\right\|_{L^2}^2\right)\\
\le&\int_{\mathbb{R}^3}\Lambda^{-s}g_1\cdot\Lambda^{-s}m\mathrm{d}x+\int_{\mathbb{R}^3}\Lambda^{-s}g_2\cdot\Lambda^{-s}p\mathrm{d}x+\sum_{i=3}^{7}\int_{\mathbb{R}^3}\Lambda^{-s}g_i\cdot\Lambda^{-s}\theta\mathrm{d}x\\
\lesssim&\left\|\Lambda^{-s}g_1\right\|_{L^2}\left\|\Lambda^{-s}m\right\|_{L^2}+\left\|\Lambda^{-s}g_2\right\|_{L^2}\left\|\Lambda^{-s}p\right\|_{L^2}+\sum_{i=3}^{7}\left\|\Lambda^{-s}g_i\right\|_{L^2}\left\|\Lambda^{-s}\theta\right\|_{L^2}.
\end{aligned}\tag{4.3}$$

Now, we estimate the nonlinear terms on the right-hand side of (4.3). Employing estimates (A.5) in Lemma A.3 and Gagliardo-Nirenberg's inequality (A.3), we can get that

$$\begin{aligned}
\left\|\Lambda^{-s}\Delta(m\theta)\right\|_{L^2}&\lesssim\left\|m\nabla^2\theta+\nabla m\nabla\theta+\theta\nabla^2m\right\|_{L^{\frac{1}{1/2+s/3}}}\\
&\lesssim\|m\|_{L^{3/s}}\left\|\nabla^2\theta\right\|_{L^2}+\|\nabla m\|_{L^{3/s}}\|\nabla\theta\|_{L^2}+\|\theta\|_{L^{3/s}}\left\|\nabla^2m\right\|_{L^2}\\
&\lesssim\|\nabla(m,\theta)\|_{L^2}^{1/2+s}\left\|\nabla^2(m,\theta)\right\|_{L^2}^{1/2-s}\left\|\nabla^2(m,\theta)\right\|_{L^2}\\
&\quad+\left\|\nabla^2m\right\|_{L^2}^{1/2+s}\left\|\nabla^3m\right\|_{L^2}^{1/2-s}\|\nabla\theta\|_{L^2}\\
&\lesssim\|\nabla(m,\theta)\|_{H^2}^2.
\end{aligned}\tag{4.4}$$

An argument similar to that of (4.4) gives that

$$\begin{aligned}
\left\|\Lambda^{-s}\mathrm{div}(p\nabla\phi)\right\|_{L^2}&\lesssim\left\|p\nabla^2\phi+\nabla p\nabla\phi\right\|_{L^{\frac{1}{1/2+s/3}}}\lesssim\|p\|_{L^{3/s}}\left\|\nabla^2\phi\right\|_{L^2}+\|\nabla\phi\|_{L^{3/s}}\|\nabla p\|_{L^2}\\
&\lesssim\|\nabla p\|_{L^2}^{1/2+s}\left\|\nabla^2p\right\|_{L^2}^{1/2-s}\|m\|_{L^2}+\left\|\nabla^2\phi\right\|_{L^2}^{1/2+s}\left\|\nabla^3\phi\right\|_{L^2}^{1/2-s}\|\nabla p\|_{L^2}\\
&\lesssim\|\nabla p\|_{H^1}^2+\|m\|_{H^1}^2
\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}
&\left\|\Lambda^{-s}\left[\frac{d_0\nabla\theta}{d_0+d_+p+d_-m}a_2^+\nabla(\tilde{p}T)\right]\right\|_{L^2}\lesssim\left\|\left[\frac{\nabla\theta}{d_0+d_+p+d_-m}a_2^+\nabla(\tilde{p}T)\right]\right\|_{L^{\frac{1}{1/2+s/3}}}\\
\lesssim&\left\|\frac{\nabla\theta}{d_0+d_+p+d_-m}\right\|_{L^{3/s}}\|\nabla(\tilde{p}T)\|_{L^2}\lesssim\|\nabla\theta\|_{L^{3/s}}\|\nabla(p+\theta+p\theta)\|_{L^2}\\
\lesssim&\|\nabla\theta\|_{L^{3/s}}\|\nabla(p+\theta)\|_{L^2}+\|\nabla\theta\|_{L^{3/s}}\|\nabla(p\theta)\|_{L^2}\\
\lesssim&\left\|\nabla^2\theta\right\|_{L^2}^{1/2+s}\left\|\nabla^3\theta\right\|_{L^2}^{1/2-s}\|\nabla(p,\theta)\|_{L^2}+\|p\|_{L^\infty}\left\|\nabla^2\theta\right\|_{L^2}^{1/2+s}\left\|\nabla^3\theta\right\|_{L^2}^{1/2-s}\|\nabla(p,\theta)\|_{L^2}\\
\lesssim&\|\nabla(p,\theta)\|_{H^2}^2.
\end{aligned}\tag{4.6}$$

Because $\mathrm{div}u_0=0$, we can obtain that

$$\begin{aligned}
\left\|\Lambda^{-s}\mathrm{div}(mu_0)\right\|_{L^2}&\lesssim\|u_0\nabla m\|_{L^{\frac{1}{1/2+s/3}}}\lesssim\|u_0\|_{L^{3/s}}\|\nabla m\|_{L^2}\\
&\lesssim\|\nabla u_0\|_{L^2}^{1/2+s}\left\|\nabla^2u_0\right\|_{L^2}^{1/2-s}\|\nabla m\|_{L^2}\\
&\lesssim\|m\|_{H^1}^2+\|\nabla\phi\|_{L^2}^2.
\end{aligned}\tag{4.7}$$

The rest of the terms in g_i can be estimated in a manner similar to (4.4)–(4.7). Then, we get that

$$\left\|\Lambda^{-s}g_i\right\|_{L^2}\lesssim\|m\|_{H^3}^2+\|\nabla p,\theta\|_{H^2}^2+\|\nabla\phi\|_{L^2}^2.\tag{4.8}$$

By (4.8) and (4.3), we can deduce (4.1).

We denote $\mathcal{E}_{-s}(t)$ to be equivalent to $\|\Lambda^{-s}(m,p,\theta,\nabla\Phi)(t)\|_{L^2}^2$. From (2.54), we can take $N=3$ to get that

$$\int_0^t \Big(\|\nabla p\|_{H^3}^2+\|\nabla\theta\|_{H^3}^2+\|\nabla\phi\|_{L^2}^2+\|m\|_{H^4}^2\Big)\,\mathrm{d}s\le C_0. \tag{4.9}$$

With this fact, we can integrate (4.1) in time to obtain that

$$\begin{aligned}\mathcal{E}_{-s}(t)&\le \mathcal{E}_{-s}(0)+C\int_0^t\Big(\|m\|_{H^3}^2+\|\nabla p,\theta\|_{H^2}^2+\|\nabla\phi\|_{L^2}^2\Big)\sqrt{\mathcal{E}_{-s}(\tau)}\mathrm{d}\tau\\&\le C_0\Big(1+\sup_{0\le\tau\le t}\sqrt{\mathcal{E}_{-s}(\tau)}\Big).\end{aligned}$$

This implies (4.2) for $s\in[0,1/2]$. □

5 Time Decay Rates of the Global Solution

In this section, we will prove the decay rates. First, we use (A.6) of Lemma A.4 and (4.2) to obtain, for $s\in[0,1/2]$ and $k+s>0$, that

$$\begin{aligned}\left\|\nabla^k(m,p,\theta,\nabla\phi)\right\|_{L^2}&\lesssim\|(m,p,\theta,\nabla\phi)\|_{\dot H^{-s}}^{\frac{1}{k+1+s}}\left\|\nabla^{k+1}(m,p,\theta,\nabla\phi)\right\|_{L^2}^{\frac{k+s}{k+1+s}}\\&\le C_0\left\|\nabla^{k+1}(m,p,\theta,\nabla\phi)\right\|_{L^2}^{\frac{k+s}{k+1+s}}.\end{aligned}\tag{5.1}$$

We take

$$\begin{aligned}\mathcal{E}_k&=\left\|\nabla^k m\right\|_{L^2}^2+\left\|\nabla^k p\right\|_{L^2}^2+\left\|\nabla^k\theta\right\|_{L^2}^2+\left\|\nabla^k\nabla\phi\right\|_{L^2}^2,\\\mathcal{D}_k&=\left\|\nabla^{k+1} m\right\|_{L^2}^2+\left\|\nabla^{k+1} p\right\|_{L^2}^2+\left\|\nabla^{k+1}\theta\right\|_{L^2}^2+\left\|\nabla^k\nabla\phi\right\|_{L^2}^2.\end{aligned}\tag{5.2}$$

By (5.1), we obtain that

$$\mathcal{D}_k\ge C_0\left(\mathcal{E}_k\right))^{1+\frac{1}{k+s}}.\tag{5.3}$$

By (2.3) and (5.3), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_k+(\mathcal{E}_k)^{1+\frac{1}{k+s}}\le 0.$$

By solving the inequality above, we obtain that

$$\mathcal{E}_k\le C_0(1+t)^{-(k+s)}.\tag{5.4}$$

Then we prove (1.5). With the help of the electric field equation, we can easily get (1.6) from (1.5). The proof of Theorem 1.1 is completed.

Appendix

In this appendix, we give the main steps for deducing the linearized equations (2.1).

Let

$$d_+=\frac{C_++C_-}{2},\qquad d_-=\frac{C_+-C_-}{2}.$$

Then

$$C_+\rho_++C_-\rho_-+C_0\rho_0=\frac{C_++C_-}{2}\tilde p+\frac{C_+-C_-}{2}m+C_0\rho_0=d_+\tilde p+d_-m+C_0\rho_0.\tag{A.1}$$

Let

$$a_- = \left(\frac{k_B}{2\nu_+} - \frac{k_B}{2\nu_-}\right), a_+ = \left(\frac{k_B}{2\nu_+} + \frac{k_B}{2\nu_-}\right), b_- = \frac{zea_-}{k_B}, b_+ = \frac{zea_+}{k_B}, a_0 = 4ze,$$

$$a_2^- = \left(\frac{C_+k_B}{2\nu_+} - \frac{C_-k_B}{2\nu_-}\right), a_2^+ = \left(\frac{C_+k_B}{2\nu_+} + \frac{C_-k_B}{2\nu_-}\right), b_2^- = \frac{zea_2^-}{k_B}, b_2^+ = \frac{zea_2^+}{k_B},$$

and then, by (A.1) and $(1.1)_2$, we have that

$$\sum_{i=0,\pm} C_i\rho_i u_i = -a_2^+\nabla(\tilde{p}T) - a_2^-\nabla(mT) - b_2^-\tilde{p}\nabla\phi - b_2^+ m\nabla\phi + (d_0 + d_+p + d_-m)u_0.$$

Let

$$G = \frac{k_B|\nabla\rho_+|^2}{\nu_+\rho_+} + \frac{k_B|\nabla\rho_-|^2}{\nu_-\rho_-},$$

and by $(1.1)_2$, we have that

$$\begin{aligned}\sum_{i=\pm}\rho_i T\mathrm{div}u_i = &-(a_+\Delta\tilde{p} + a_-\Delta m)T + GT^2 - (a_+\nabla\tilde{p} + a_-\nabla m)T\nabla T\\ &- T(a_+\tilde{p} + a_-m)\Delta T - (b_-\tilde{p} + b_+m)T\Delta\phi.\end{aligned}$$

It is likely that we can get that

$$\nu_+\rho_+|u_+ - u_0|^2 = \frac{k_B^2}{\nu_+}\frac{|\nabla\rho_+T|^2}{\rho_+} + \frac{2k_B}{\nu_+}\nabla\rho_+T\nabla(k_BT + ze\phi) + \rho_+\frac{|\nabla(k_BT + ze\phi)|^2}{\nu_+},$$

and

$$\nu_-\rho_-|u_- - u_0|^2 = \frac{k_B^2}{\nu_-}\frac{|\nabla\rho_-T|^2}{\rho_-} + \frac{2k_B}{\nu_-}\nabla\rho_-T\nabla(k_BT - ze\phi) + \rho_-\frac{|\nabla(k_BT - ze\phi)|^2}{\nu_-},$$

which imply that

$$\begin{aligned}&\nu_+\rho_+|u_+ - u_0|^2 + \nu_-\rho_-|u_- - u_0|^2\\ =& Gk_BT^2 + 4k_B[(a_+\nabla\tilde{p} + a_-\nabla m)T\nabla T + (b_-\nabla\tilde{p} + b_+\nabla m)T\nabla\phi]\\ &+ k_B|\nabla T|^2(a_+\tilde{p} + a_-\tilde{m}) + ze|\nabla\phi|^2(b_+\tilde{p} + b_-m) + 2k_B\nabla T\nabla\phi(b_-\tilde{p} + b_+m).\end{aligned}$$

Letting $d_0 = d_+ + C_0\rho_0$, $T = 1 + \theta$, and $\tilde{p} = p + 1$, the linearized system becomes

$$\begin{cases}\dfrac{\partial}{\partial t}m - a_+\Delta m - a_-\Delta(p + \theta) + \dfrac{b_+a_0}{\epsilon}m = 0,\\ \dfrac{\partial}{\partial t}p - a_+\Delta(p + \theta) - a_-\Delta m + \dfrac{b_-a_0}{\epsilon}m = 0,\\ d_0\dfrac{\partial}{\partial t}\theta - a_+\Delta(p + \theta) - \dfrac{k}{k_B}\Delta\theta - a_-\Delta m + \dfrac{b_-a_0}{\epsilon}m = 0.\end{cases} \tag{A.2}$$

From the solvent equation (1.2), we have that $-\lambda_0\Delta u_0 = -[\nabla(P_0 + k_B\tilde{p}T) + a_0m\nabla\phi]$. By virtue of the Leray project operator, no linear term can be split from u_0. When $b_- = 0$, we can get the closed L^2 estimates for any ϵ.

In what follows, we list some useful inequalities which are frequently used for obtaining the energy estimates of the solution.

Lemma A.1 (Gagliardo-Nirenberg's inequality) The parameters satisfy that $0 \le k, m \le l$ and $2 \le p \le \infty$, so we have that

$$\|\nabla^m f\|_{L^p} \lesssim \left\|\nabla^k f\right\|_{L^2}^{1-\vartheta} \left\|\nabla^l f\right\|_{L^2}^{\vartheta}, \tag{A.3}$$

where $0 \leq \vartheta \leq 1$ and m satisfies

$$m + 3\Big(\frac{1}{2} - \frac{1}{p}\Big) = k(1-\vartheta) + l\vartheta.$$

Notice that if $p = \infty$, we require that $0 < \vartheta < 1$.

Proof The detailed proof can be seen in [30], p.125. □

Lemma A.2 (Product estimates) For the integer $l \geq 0$, we have that

$$\left\|\nabla^l(fg)\right\|_{L^p} \lesssim \|g\|_{L^{p_0}} \left\|\nabla^l f\right\|_{L^{p_1}} + \left\|\nabla^l g\right\|_{L^{p_2}} \|f\|_{L^{p_3}}. \tag{A.4}$$

The parameters p, p_1, p_2 and $p_3 \in [1, +\infty]$ satisfy $1/p = 1/p_0 + 1/p_1 = 1/p_2 + 1/p_3$.

Proof We can prove this lemma in the same way as Lemma A.1 in [7]. □

Lemma A.3 (Sobolev embeding inequality) If $0 \leq s < 3/2$, it holds that

$$\|f\|_{\dot{H}^{-s}} \lesssim \|f\|_{L^p}, \tag{A.5}$$

where $1 < p \leq 2$ and $1/2 + s/3 = 1/p$.

Proof See [36] p.119, Theorem 1. □

Lemma A.4 Suppose that $s, \ell \geq 0$. Then we have

$$\left\|\nabla^\ell f\right\|_{L^2} \lesssim \left\|\nabla^{\ell+1} f\right\|_{L^2}^{1-\vartheta} \|f\|_{\dot{H}^{-s}}^{\vartheta}, \text{ where } \vartheta = \frac{1}{\ell + 1 + s}. \tag{A.6}$$

Proof One can see Lemma A.4 in [14] for the detailed proof. □

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