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THE GLOBAL EXISTENCE AND A DECAY ESTIMATE OF SOLUTIONS TO THE PHAN-THEIN-TANNER MODEL*

Ruiying WEI (位瑞英)
School of Mathematics and Statistics, Shaoguan University, Shaoguan 512005, China
E-mail: *weiruiying521@163.com*

Yin LI (李银)†
Faculty of Education, Shaoguan University, Shaoguan 512005, China
E-mail: *liyin2009521@163.com*

Zheng-an YAO (姚正安)
Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, China
E-mail: *mcsyao@mail.sysu.edu.cn*

Abstract In this paper, we study the global existence and decay rates of strong solutions to the three dimensional compressible Phan-Thein-Tanner model. By a refined energy method, we prove the global existence under the assumption that the H^3 norm of the initial data is small, but that the higher order derivatives can be large. If the initial data belong to homogeneous Sobolev spaces or homogeneous Besov spaces, we obtain the time decay rates of the solution and its higher order spatial derivatives. Moreover, we also obtain the usual $L^p-L^2(1\le p\le 2)$ type of the decay rate without requiring that the L^p norm of initial data is small.

Key words Phan-Thein-Tanner model; global existence; time decay rates

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1 Introduction

The theory of the Phan-Thein-Tanner model has recently gained quite some attention, and is derived from network theory for polymeric fluid. This type of fluid is described by the

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†Corresponding author

following set of equations:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho(u_t + u\cdot\nabla u) - \mu(\triangle u + \nabla \operatorname{div} u) + \nabla p = \mu_1 \operatorname{div}\tau, \\ \tau_t + u\cdot\nabla\tau + Q(\tau,\nabla u) + (a + b\operatorname{tr}\tau)\tau = \mu_2 D(u), \\ (\rho,u,\tau)|_{t=0} = (\rho_0,u_0,\tau_0), (t,x)\in\mathbb{R}^+\times\mathbb{R}^3. \end{cases} \tag{1.1}$$

The unknowns ρ, u, τ, p are the density, velocity, stress tensor and scalar pressure of the fluid, respectively. $D(u)$ is the symmetric part of ∇u; that is,

$$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).$$

$Q(\tau,\nabla u)$ is a given bilinear form

$$Q(\tau,\nabla u) = \tau\Omega(u) - \Omega(u)\tau + \lambda(D(u)\tau + \tau D(u)),$$

where $\Omega(u)$ is the skew-symmetric part of ∇u, namely,

$$\Omega(u) = \frac{1}{2}(\nabla u - (\nabla u)^T).$$

$\mu > 0$ is the viscosity coefficient and μ_1 is the elastic coefficient. a and μ_2 are associated with the Debroah number $De = \frac{\mu_2}{a}$ (which indicates the relation between the characteristic flow time and elastic time [2]). $\lambda\in[-1,1]$ is a physical parameter; we call the system a co-rotational case when $\lambda = 0$. $b \ge 0$ is a constant related to the rate of creation or destruction for the polymeric network junctions.

To complete system (1.1), the initial data are given by

$$(\rho,u,\tau)|_{t=0} = (\rho_0,u_0,\tau_0), \quad x\in\mathbb{R}^3, \tag{1.2}$$

with the far field behavior

$$(\rho,u,\tau)(t,x) = (\bar\rho,0,0) \text{ as } |x|\to\infty, t\ge 0.$$

Let us review some previous works about model (1.1) and related models. If we ignore the stress tensor, (1.1) reduces to the compressible Navier-Stokes (NS) equations. The convergence rates of solutions for the compressible Navier-Stokes equations to the steady state have been investigated extensively since the first global existence of small solutions in H^3 was improved upon by Matsumura and Nishida [21, 22]. When the initial perturbation is $(\rho_0 - 1, u_0)\in L^p\cap H^N (N\ge 3)$ with $p\in[1,2]$, the L^2 optimal decay rate of the solution to the NS system is

$$\|(\rho-1,u)(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}.$$

For the small initial perturbation belonging to H^3 only, Matsumura [20] employed the weighted energy method to show the L^2 decay rates. Ponce [27] obtained the optimal L^p convergence rate. In [29], Schonbek and Wiegner studied the large time behavior of solutions to the Navier-Stokes equation in $H^m(\mathbb{R}^n)$ for all $n\le 5$. In order to establish optimal decay rates for the higher order spatial derivatives of solutions, for when the initial perturbation is bounded in the $H^{-s}(s\in[0,\frac{3}{2}))$ norm instead of the L^1-norm, Guo and Wang [12] used a general energy method to develop the time convergence rates

$$\|\nabla^l(\rho-1,u)(t)\|_{H^{N-l}} \le C(1+t)^{-\frac{l+s}{2}}$$

for $0 \leq l \leq N-1$. In addition, the decay rate of solutions to the NS system was investigated in [5, 33] (see also the references therein).

If $b=0$, the system (PTT) reduces to the famous Oldroyd-B model (see [25]), which has been studied widely. Most of the results on Oldroyd-B fluids are about the incompressible model. C. Guillopé and J.C. Saut [10, 11] proved the existence of local strong solutions and the global existence of one dimensional shear flows. Later, the smallness restriction on the coupling constant in [10] was removed by Molinet and Talhouk [23]. In [19], F. Lin, C. Liu and P. Zhang proved local existence and global existence (with small initial data) of classical solutions for an Oldroyd system without an artificially postulated damping mechanism. Similar results were obtained in several papers by virtue of different methods; see Z. Lei, C. Liu and Y. Zhou [17], T. Zhang and D. Fang [38], Y. Zhu [41]. D. Fang and R. Zi [6] proved the global existence of strong solutions with a class of large data.

On the other hand, there are relatively few results for the compressible model. Lei [16] proved the local and global existence of classical solutions for a compressible Oldroyd-B system in a torus with small initial data. He also studied the incompressible limit problem and showed that the compressible flows with well-prepared initial data converge to incompressible ones when the Mach number converges to zero. The case of ill-prepared initial data was considered by Fang and Zi [8] in the whole space $\mathbb{R}^d$, $d \geq 2$. Recently, the smallness restriction on a coupling constant was removed by Zi in [39]. On the other hand, for suitable Sobolev spaces, Fang and Zi [7] obtained the unique local strong solution with the initial density vanishing from below and a blow-up criterion for this solution. Zhou, Zhu and Zi [40] proved the existence of a global strong solution provided that the initial data are close to the constant equilibrium state in the H^2-framework and obtained the convergence rates of the solutions. For the compressible Oldroyd type model based on the deformation tensor, see the results [14, 18, 28, 37] and references therein.

In this paper, we focus on the PTT model ($b \neq 0$). To our knowledge, there are a lot of numerical results about the PTT model (see, [1, 9, 26]). Recently, [4] proved that the strong solution in critical Besov spaces exists globally when the initial data are a small perturbation over and around the equilibrium. [3] proved that the strong solution will blow up in finite time and proved the global existence of a strong solution with small initial data. However, to our knowledge, there are few results on the compressible PTT model, especially regarding the large-time behavior. Compared with the incompressible models, the compressible equations of the PTT model are more difficult to deal with because of the strong nonlinearities and interactions among the physical quantities. The main purpose of this paper is to study the global existence and decay rates of smooth solutions for the compressible PTT model. We first establish the global solution of the solutions to (1.1)–(1.2) in the whole space $\mathbb{R}^3$ near the constant equilibrium state under the assumption that the H^3 norm of the initial data is small, but the higher order derivatives can be arbitrarily large. Then we establish the large time behavior appealing to the work of Strain et al. [32], Guo et al. [12], Sohinger et al. [30], Wang [36] and Tan et al. [34, 35]. Moreover, we also obtain the usual $L^p-L^2 (1 \leq p \leq 2)$ type of decay rate without requiring that the L^p norm of the initial data is small.

Throughout the paper, without loss of generality, we set $\mu=\mu_1=\mu_2=a=b=\bar{\rho}=1$. Before stating our main results, we explain the notations and conventions used throughout.

∇^l with an integer $l \geq 0$ stands for the any spatial derivatives of order l. When $l < 0$ or l is not a positive integer, ∇^l stands for Λ^l defined by $\Lambda^s u := \mathscr{F}^{-1}(|\xi|^s \hat{u}(\xi))$, where $\hat{u}$ is the Fourier transform of u and $\mathscr{F}^{-1}$ its inverse. We use $\dot{H}^s(\mathbb{R}^3)$ $(s \in \mathbb{R})$ to denote the homogeneous Sobolev spaces on $\mathbb{R}^3$ with the norm $\|\cdot\|_{\dot{H}^s}$ defined by $\|f\|_{\dot{H}^s} = \|\Lambda^s f\|_{L^2} = \||\xi|^s \hat{f}\|_{L^2}$, $H^s(\mathbb{R}^3)$ to denote the usual Sobolev spaces with the norm $\|\cdot\|_{H^s}$, and $L^p(1 \leq p \leq \infty)$ to denote the usual $L^p(\mathbb{R}^3)$ spaces with the norm $\|\cdot\|_{L^p}$. Finally, we introduce the homogeneous Besov space, letting $\varphi \in C_0^\infty(\mathbb{R}^3_\xi)$ be a cut-off function such that $\varphi(\xi) = 1$ with $|\xi| \leq 1$, and letting $\varphi(\xi) \leq 2$ with $|\xi| \leq 2$. Let $\psi(\xi) = \varphi(\xi) - \varphi(2\xi)$ and $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j \in \mathbb{Z}$. Then, by the construction $\sum\limits_{j\in\mathbb{Z}} \psi_j(\xi) = 1$, if $\xi \neq 0$, we set $\dot{\triangle}_j f = \mathscr{F}^{-1} * f$, so that for $s \in \mathbb{R}$, we define the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^N)$ with the norm $\|\cdot\|_{\dot{B}^s_{p,q}}$ by

$$\|f\|_{\dot{B}^s_{p,q}}(\mathbb{R}^N) = \begin{cases} \left(\sum\limits_{j\in\mathbb{Z}} 2^{jsq} \|\dot{\triangle}_j f\|^q_{L^p(\mathbb{R}^N)}\right)^{\frac{1}{q}}, & 1 \leq p \leq \infty, 1 \leq q < \infty, \\ \operatorname*{ess\,sup}\limits_{j\in\mathbb{Z}} 2^{js} \|\dot{\triangle}_j f\|_{L^p(\mathbb{R}^N)}, & 1 \leq p \leq \infty, q = \infty. \end{cases}$$

We will employ the notation $A \lesssim B$ to mean that $A \leq CB$ for a universal constant $C > 0$ that only depends on the parameters coming from the problem. For the sake of concision, we write $\|(A,B)\|_X := \|A\|_X + \|B\|_X$.

For $N \geq 3$, we define the energy functional by

$$\mathcal{E}_N(t) := \sum_{l=0}^N \|\nabla^l(\varrho, u, \tau)\|^2_{L^2},$$

and the corresponding dissipation rate by

$$\mathcal{D}_N(t) := \sum_{l=1}^N \|\nabla^l \varrho\|^2_{L^2} + \sum_{l=0}^N (\|\nabla^{l+1} u\|^2_{L^2} + \|\nabla^l \tau\|^2_{L^2}).$$

Now, we state our main result about the global existence and decay properties of a solution to the system (1.1)–(1.2) in the following theorems:

Theorem 1.1 Letting $N \geq 3$, and assuming that $(\rho_0 - 1, u_0, \tau_0) \in H^N$, there exists a sufficiently small $\delta_0 > 0$ such that if $\mathcal{E}_3(0) \leq \delta_0$, then the problem (1.1)–(1.2) has a unique global solution $(\rho, u, \tau)(t)$ satisfying

$$\sup_{0\leq t\leq\infty} \mathcal{E}_3(t) + \int_0^\infty \mathcal{D}_3(s)\mathrm{d}s \leq C\mathcal{E}_3(0). \tag{1.3}$$

Furthermore, if $\mathcal{E}_N(0) < \infty$ for any $N \geq 3$, then (1.1)–(1.2) admits a unique solution $(\rho, u, \tau)(t)$ satisfying

$$\sup_{0\leq t\leq\infty} \mathcal{E}_N(t) + \int_0^\infty \mathcal{D}_N(s)\mathrm{d}s \leq C\mathcal{E}_N(0). \tag{1.4}$$

In addition, if the initial data belong to Negative Sobolev or Besov spaces, based on the regularity interpolation method and the results in Theorem 1.1, we can derive some further decay rates of the solution and its higher order spatial derivatives to systems (1.1)–(1.2).

Theorem 1.2 Under all the assumptions in Theorem 1.1, let $(\rho, u, \tau)(t)$ be the solution to the system (1.1)–(1.2) constructed in Theorem 1.1. Suppose that $(\rho_0 - 1, u_0, \tau_0) \in \dot{H}^{-s}$ for some $s\in[0,\frac{3}{2})$ or $(\rho_0-1,u_0,\tau_0)\in\dot{B}^{-s}_{2,\infty}$ for some $s\in(0,\frac{3}{2}]$. Then we have

$$\|(\rho-1,u,\tau)(t)\|_{\dot{H}^{-s}}\leq C_0, \tag{1.5}$$

or

$$\|(\rho-1,u,\tau)(t)\|_{\dot{B}^{-s}_{2,\infty}}\leq C_0. \tag{1.6}$$

Moreover, for $k\geq 0$, if $N\geq k+2$, it holds that

$$\|\nabla^k(\rho-1,u)(t)\|_{L^2}\leq C_0(1+t)^{-\frac{k+s}{2}}. \tag{1.7}$$

$$\|\nabla^k\tau(t)\|_{L^2}\leq C_0(1+t)^{-\frac{k+1+s}{2}}. \tag{1.8}$$

Note that the Hardy-Littlewood-Sobolev theorem (cf. Lemma 4.4) implies that for $p\in(1,2]$, $L^p\subset\dot{H}^{-s}$ with $s=3(\frac{1}{p}-\frac{1}{2})\in[0,\frac{3}{2})$. This, together with Theorem 1.2, means that the L^p-L^2 type of decay result follows as a corollary. However, the imbedding theorem cannot cover the case $p=1$; to amend this, Sohinger-Strain [30] instead introduced the homogeneous Besov space $\dot{B}^{-s}_{2,\infty}$, due to the fact that the endpoint imbedding $L^1\subset\dot{B}^{-\frac{3}{2}}_{2,\infty}$ holds (Lemma 4.5). At this stage, by Theorem 1.2, we have the following corollary of the usual L^p-L^2 type of decay result:

Corollary 1.3 Under the assumptions of Theorem 1.2, if we replace the $\dot{H}^{-s}$ or $\dot{B}^{-s}_{2,\infty}$ assumption by $(\varrho_0,u_0,\tau_0)\in L^p$ for some $p\in[1,2]$, then, for any integer $k\geq 0$, if $N\geq k+2$, the following decay result holds:

$$\|\nabla^k(\varrho,u)(t)\|_{L^2}\leq C_0(1+t)^{-\frac{k+\sigma_p}{2}}, \tag{1.9}$$

$$\|\nabla^k\tau(t)\|_{L^2}\leq C_0(1+t)^{-\frac{k+1+\sigma_p}{2}}. \tag{1.10}$$

Here $\sigma_p:=3(\frac{1}{p}-\frac{1}{2})$.

The rest of our paper is organized as follows: in Section 2, we establish the refined energy estimates for the solution and derive the negative Sobolev and Besov estimates. Furthermore, we use this section to prove Theorem 1.1. Finally, we prove Theorem 1.2 in Section 3.

2 The Global Existence of Solution

In this section, we are going to prove our main result. The proof of local well-posedness for PTT is similar to the Oldroyd-B model (see [7, 13]), so we omit the details here. Theorem 1.1 will be proved by combining the local existence of (ϱ,u,τ) to (1.1)–(1.2) and some a priori estimates as well as the communication argument. We first reformulate system (1.1). We set $\varrho=\rho-1$. Then the initial value problem (1.1)–(1.2) can be rewritten as

$$\begin{cases}\varrho_t+\mathrm{div}u=S_1,\\ u_t+\gamma\nabla\varrho-(\triangle u+\nabla\mathrm{div}u)-\mathrm{div}\tau=S_2,\\ \tau_t+\tau-D(u)=S_3,\end{cases} \tag{2.1}$$

where the nonlinear terms $S_i(i=1,2,3)$ are defined as

$$\begin{aligned}&S_1=-\mathrm{div}(\varrho u),\\ &S_2=-u\cdot\nabla u-f(\varrho)(\triangle u+\nabla\mathrm{div}u)-g(\varrho)\nabla\varrho-f(\varrho)\mathrm{div}\tau,\\ &S_3=-u\cdot\nabla\tau-Q(\tau,\nabla u)-\mathrm{tr}\tau\tau,\end{aligned}$$

with

$$(\varrho, u, \tau)(x, 0) = (\varrho_0, u_0, \tau_0) \to 0 \text{ as } |x| \to \infty, \tag{2.2}$$

and here,

$$\gamma = \frac{P'(1)}{1}, \quad f(\varrho) = \frac{\varrho}{\varrho + 1}, \quad g(\varrho) = \frac{P'(\varrho + 1)}{\varrho + 1} - \frac{P'(1)}{1}. \tag{2.3}$$

For simplicity, in what follows, we set $p'(1) = 1$; that is $\gamma = 1$.

Then, we will derive the nonlinear energy estimates for system (2.1). Hence, we assume that, for sufficiently small $\delta > 0$,

$$\|(\varrho, u, \tau)(t)\|_{H^3} \leq \delta. \tag{2.4}$$

First of all, by (2.4) and Sobolev's inequality, we obtain that

$$\frac{1}{2} \leq \varrho + 1 \leq 2.$$

Hence, we immediately have that

$$|f(\varrho)|, |g(\varrho)| \leq C|\varrho|, \quad |f^{(k)}(\varrho)|, |g^{(k)}(\varrho)| \leq C \quad \text{for any} \quad k \geq 1. \tag{2.5}$$

2.1 Energy estimates

Before establishing the global existence of the solution under the assumption of (2.4), we derive the basic energy estimates for the solution to systems (2.1)–(2.2). We begin with the standard energy estimates.

Lemma 2.1 If

$$\sup_{0 \leq t \leq T} \|(\varrho, u, \tau)(t)\|_{H^3} \leq \delta, \tag{2.6}$$

then, for any integers $k \geq 0$ and $t \geq 0$, we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varrho, u, \tau)\|_{L^2}^2 + (\|\nabla u\|_{L^2}^2 + \|\mathrm{div} u\|_{L^2}^2 + \|\tau\|_{L^2}^2) \lesssim \delta(\|\nabla(\varrho, u)\|_{L^2}^2 + \|\tau\|_{L^2}^2). \tag{2.7}$$

Proof For (2.1) on ϱ, u, and τ, respectively, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|\varrho|^2 + |u|^2 + |\tau|^2)\mathrm{d}x + \int_{\mathbb{R}^3}(|\nabla u|^2 + |\mathrm{div} u|^2 + |\tau|^2)\mathrm{d}x \\ = &\int_{\mathbb{R}^3}\Big\{ - (\varrho \mathrm{div} u + u \cdot \nabla\varrho) \cdot \varrho \\ &- [u \cdot \nabla u + f(\varrho)(\triangle u + \nabla \mathrm{div} u) + g(\varrho)\nabla\varrho + f(\varrho)\mathrm{div}\tau] \cdot u \\ &- (u \cdot \nabla\tau + Q(\tau, \nabla u) + \mathrm{tr}\tau\tau) \cdot \tau\Big\}\mathrm{d}x \\ := &\sum_{i=1}^{9} M_i. \end{aligned} \tag{2.8}$$

We shall estimate each term on the right hand side of (2.8). First, for the term M_1, it is obvious that

$$M_1 = -\int_{\mathbb{R}^3} \varrho \mathrm{div} u \cdot \varrho \mathrm{d}x \lesssim \|\varrho\|_{L^6}\|\nabla u\|_{L^2}\|\varrho\|_{L^3} \lesssim \delta(\|\nabla\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2). \tag{2.9}$$

$$M_2 = -\int_{\mathbb{R}^3} u \cdot \nabla\varrho \cdot \varrho \mathrm{d}x \lesssim \|u\|_{L^3}\|\nabla\varrho\|_{L^2}\|\varrho\|_{L^6} \lesssim \delta\|\nabla\varrho\|_{L^2}^2. \tag{2.10}$$

$$M_3=-\int_{\mathbb{R}^3} u\cdot\nabla u\cdot u\mathrm{d}x\lesssim\|u\|_{L^3}\|\nabla u\|_{L^2}\|u\|_{L^6}\lesssim\delta\|\nabla u\|_{L^2}^2. \tag{2.11}$$

Appealing to Hölder's inequality, Lemma 4.1, Lemma 4.3 and Cauchy's inequality, we obtain that

$$\begin{aligned}M_4&\approx-\int_{\mathbb{R}^3} f(\varrho)\nabla^2u\cdot u\mathrm{d}x\lesssim\|f(\varrho)\|_{L^6}\|\nabla^2u\|_{L^{\frac{3}{2}}}\|u\|_{L^6}\\&\lesssim\|f(\varrho)\|_{L^6}\|\nabla u\|_{L^2}^{\frac{3}{4}}\|\nabla^3u\|_{L^2}^{\frac{1}{4}}\|u\|_{L^6}\lesssim\delta(\|\nabla\varrho\|_{L^2}^2+\|\nabla u\|_{L^2}^2).\end{aligned} \tag{2.12}$$

By the fact of (2.6) and Hölder's and Cauchy's inequalities, we obtain that

$$M_5=-\int_{\mathbb{R}^3} g(\varrho)\nabla\varrho\cdot u\mathrm{d}x\lesssim\|g(\varrho)\|_{L^3}\|\nabla\varrho\|_{L^2}\|u\|_{L^6}\lesssim\delta(\|\nabla\varrho\|_{L^2}^2+\|\nabla u\|_{L^2}^2). \tag{2.13}$$

We integrate by parts and by Lemma 4.3 and Hölder's inequality to get that

$$\begin{aligned}M_6&=-\int_{\mathbb{R}^3} f(\varrho)\mathrm{div}\tau\cdot u\mathrm{d}x\lesssim\|f(\varrho)\|_{L^6}\|\mathrm{div}\tau\|_{L^{\frac{3}{2}}}\|u\|_{L^6}\\&\lesssim\|f(\varrho)\|_{L^6}\|\tau\|_{L^2}^{\frac{3}{4}}\|\nabla^2\tau\|_{L^2}^{\frac{1}{4}}\|u\|_{L^6}\lesssim\delta(\|\nabla\varrho\|_{L^2}^2+\|\nabla u\|_{L^2}^2).\end{aligned} \tag{2.14}$$

$$M_7=-\int_{\mathbb{R}^3} u\cdot\nabla\tau\cdot\tau\mathrm{d}x\lesssim\|u\|_{L^6}\|\nabla\tau\|_{L^3}\|\tau\|_{L^2}\lesssim\delta(\|\nabla u\|_{L^2}^2+\|\tau\|_{L^2}^2). \tag{2.15}$$

$$M_8=-\int_{\mathbb{R}^3} Q(\tau,\nabla u)\cdot\tau\mathrm{d}x\lesssim\|\tau\|_{L^\infty}\|\nabla u\|_{L^2}\|\tau\|_{L^2}\lesssim\delta(\|\nabla u\|_{L^2}^2+\|\tau\|_{L^2}^2). \tag{2.16}$$

$$M_9=-\int_{\mathbb{R}^3} \mathrm{tr}\tau\tau\cdot\tau\mathrm{d}x\lesssim\|\mathrm{tr}\tau\|_{\mathrm{L}^\infty}\|\tau\|_{\mathrm{L}^2}^2\lesssim\delta\|\tau\|_{\mathrm{L}^2}^2. \tag{2.17}$$

Summing up the estimates for M_1–M_9, we deduce (2.7), which yields the desired result. □

Lemma 2.2 Letting all of the assumptions in Lemma 2.1 be in force, for any $k\geq 0$, it holds that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=k+1}^{k+2}\|\nabla^l(\varrho,u,\tau)\|_{L^2}^2+\sum_{l=k+1}^{k+2}(\|\nabla^{l+1}u\|_{L^2}^2+\|\nabla^l\mathrm{div}u\|_{L^2}^2+\|\nabla^l\tau\|_{L^2}^2)\\&\lesssim\delta\sum_{l=k+1}^{k+2}(\|\nabla^l\varrho\|_{L^2}^2+\|\nabla^{l+1}u\|_{L^2}^2+\|\nabla^l\tau\|_{L^2}^2).\end{aligned} \tag{2.18}$$

Proof For any integer $k\geq 0$, by the $\nabla^l(l=k+1,k+2)$ energy estimate, for (2.1) on ϱ,u, and τ, respectively, we get that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|\nabla^l\varrho|^2+|\nabla^lu|^2+|\nabla^l\tau|^2)\mathrm{d}x+\int_{\mathbb{R}^3}(|\nabla^{l+1}u|^2+|\nabla^l\mathrm{div}u|^2+|\nabla^l\tau|^2)\mathrm{d}x\\&=\int_{\mathbb{R}^3}\Big\{-\nabla^l(\varrho\mathrm{div}u+u\cdot\nabla\varrho)\cdot\nabla^l\varrho\\&\quad-\nabla^l[u\cdot\nabla u+f(\varrho)(\triangle u+\nabla\mathrm{div}u)+g(\varrho)\nabla\varrho+f(\varrho)\mathrm{div}\tau]\cdot\nabla^lu\\&\quad-\nabla^l(u\cdot\nabla\tau+Q(\tau,\nabla u)+\mathrm{tr}\tau\tau)\cdot\nabla^l\tau\Big\}\mathrm{d}x\\&:=\sum_{i=1}^9 I_i.\end{aligned} \tag{2.19}$$

We shall estimate each term on the right hand side of (2.19). First, for the term I_1, if $l=1$, we further obtain that

$$\begin{aligned} I_1 &= -\int_{\mathbb{R}^3} (\nabla \varrho \mathrm{div} u) \cdot \nabla \varrho \mathrm{d}x - \int_{\mathbb{R}^3} (\varrho \nabla \mathrm{div} u) \cdot \nabla \varrho \mathrm{d}x \\ &\lesssim \|\nabla \varrho\|_{L^3} \|\nabla \varrho\|_{L^2} \|\mathrm{div} u\|_{L^6} + \|\varrho\|_{L^\infty} \|\nabla \mathrm{div} u\|_{L^2} \|\nabla \varrho\|_{L^2} \\ &\lesssim \delta(\|\nabla \varrho\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2). \end{aligned} \tag{2.20}$$

If $l \geq 2$, then employing the Leibniz formula and by Hölder's inequality we obtain that

$$\begin{aligned} I_1 &= -\int_{\mathbb{R}^3} \nabla^l (\varrho \mathrm{div} u) \cdot \nabla^l \varrho \mathrm{d}x \\ &= -\int_{\mathbb{R}^3} \sum_{s=0}^{l} C_l^s \nabla^s \varrho \nabla^{l-s} \mathrm{div} u \cdot \nabla^l \varrho \mathrm{d}x \\ &\lesssim \sum_{s=0}^{l} C_l^s \|\nabla^s \varrho \nabla^{l-s} \mathrm{div} u\|_{L^2} \|\nabla^l \varrho\|_{L^2}. \end{aligned} \tag{2.21}$$

If $0 \leq s \leq [\frac{l}{2}]$, by using Lemma 4.1, we estimate the first factor in the above to get that

$$\begin{aligned} \|\nabla^s \varrho \nabla^{l-s} \mathrm{div} u\|_{L^2} &\lesssim \|\nabla^s \varrho\|_{L^\infty} \|\nabla^{l-s+1} u\|_{L^2} \\ &\lesssim \|\nabla^\alpha \varrho\|_{L^2}^{1-\frac{s}{l}} \|\nabla^l \varrho\|_{L^2}^{\frac{s}{l}} \|\nabla u\|_{L^2}^{\frac{s}{l}} \|\nabla^{l+1} u\|_{L^2}^{1-\frac{s}{l}} \\ &\lesssim \delta(\|\nabla^l \varrho\|_{L^2} + \|\nabla^{l+1} u\|_{L^2}), \end{aligned} \tag{2.22}$$

where α is defined by

$$\frac{s}{3} = (\frac{\alpha}{3} - \frac{1}{2}) \times (1 - \frac{s}{l}) + (\frac{l}{3} - \frac{1}{2}) \times \frac{s}{l}.$$

Since $0 \leq s \leq [\frac{l}{2}]$, we have that $\alpha = \frac{3l}{2(l-s)} \in [\frac{3}{2}, 3]$.

If $[\frac{l}{2}] + 1 \leq s \leq l$, by using Lemma 4.1 again, we estimate the first factor in the inequalities (2.21) as follows:

$$\begin{aligned} \|\nabla^s \varrho \nabla^{l-s} \mathrm{div} u\|_{L^2} &\lesssim \|\nabla^s \varrho\|_{L^2} \|\nabla^{l-s+1} u\|_{L^\infty} \\ &\lesssim \|\nabla \varrho\|_{L^2}^{\frac{l-s}{l-1}} \|\nabla^l \varrho\|_{L^2}^{1-\frac{l-s}{l-1}} \|\nabla^\alpha u\|_{L^2}^{1-\frac{l-s}{l-1}} \|\nabla^{l+1} u\|_{L^2}^{\frac{l-s}{l-1}} \\ &\lesssim \delta(\|\nabla^l \varrho\|_{L^2} + \|\nabla^{l+1} u\|_{L^2}). \end{aligned} \tag{2.23}$$

Here α is defined by

$$\frac{l-s+1}{3} = (\frac{\alpha}{3} - \frac{1}{2}) \times (1 - \frac{l-s}{l-1}) + (\frac{l+1}{3} - \frac{1}{2}) \times \frac{l-s}{l-1}.$$

Since $[\frac{l}{2}] + 1 \leq s \leq l$, we have that $\alpha = \frac{l-1}{2(s-1)} + 2 \in [\frac{5}{2}, 3]$.

Combining (2.20)–(2.23), and by Cauchy's inequality, we deduce that

$$I_1 \lesssim \delta(\|\nabla^l \varrho\|_{L^2}^2 + \|\nabla^{l+1} u\|_{L^2}^2). \tag{2.24}$$

For the term I_2, employing Lemma 4.2, we infer that

$$\begin{aligned} I_2 &= -\int_{\mathbb{R}^3} (u \cdot \nabla \nabla^l \varrho + [\nabla^l, u] \nabla \varrho) \cdot \nabla^l \varrho \mathrm{d}x \\ &\lesssim \|\nabla u\|_{L^\infty} \|\nabla^l \varrho\|_{L^2}^2 + \|\nabla u\|_{L^\infty} \|\nabla^l \varrho\|_{L^2}^2 + \|\nabla^l u\|_{L^6} \|\nabla \varrho\|_{L^3} \|\nabla^l \varrho\|_{L^2} \\ &\lesssim \delta(\|\nabla^l \varrho\|_{L^2}^2 + \|\nabla^{l+1} u\|_{L^2}^2). \end{aligned} \tag{2.25}$$

For the term I_3,

$$\begin{aligned} I_3 &= -\int_{\mathbb{R}^3} \nabla^l (u\cdot\nabla u)\cdot\nabla^l u \mathrm{d}x \\ &= -\int_{\mathbb{R}^3} \sum_{s=0}^{l} C_l^s \nabla^s u\cdot\nabla^{l-s}\nabla u\cdot\nabla^l u\mathrm{d}x \\ &\lesssim \sum_{s=0}^{l} \|\nabla^s u \nabla^{l-s+1} u\|_{L^{\frac{6}{5}}} \|\nabla^l u\|_{L^6}. \end{aligned} \tag{2.26}$$

If $0\le s\le [\frac{l}{2}]$, by using Lemma 4.1, we estimate the first factor in the above to get that

$$\begin{aligned} \|\nabla^s u\nabla^{l-s+1}u\|_{L^{\frac{6}{5}}} &\lesssim \|\nabla^s u\|_{L^3}\|\nabla^{l+1-s}u\|_{L^2} \\ &\lesssim \|\nabla^\alpha u\|_{L^2}^{1-\frac{s}{l+1}} \|\nabla^{l+1}u\|_{L^2}^{\frac{s}{l+1}} \|u\|_{L^2}^{\frac{s}{l+1}} \|\nabla^{l+1}u\|_{L^2}^{1-\frac{s}{l+1}} \\ &\lesssim \delta \|\nabla^{l+1}u\|_{L^2}, \end{aligned} \tag{2.27}$$

where α is defined by

$$\frac{s}{3}-\frac{1}{3} = (\frac{\alpha}{3}-\frac{1}{2})\times(1-\frac{s}{l+1}) + (\frac{l+1}{3}-\frac{1}{2})\times\frac{s}{l+1}.$$

Since $0\le s\le [\frac{l}{2}]$, we have that $\alpha = \frac{l+1}{2(l+1-s)} \in [\frac{1}{2},1)$.

If $[\frac{l}{2}]+1\le s\le l$, by using Lemma 4.1 again, we estimate the first factor in the inequalities (2.26) to get that

$$\begin{aligned} \|\nabla^s u\nabla^{l-s+1}u\|_{L^{\frac{6}{5}}} &\lesssim \|\nabla^s u\|_{L^2}\|\nabla^{l-s+1}u\|_{L^3} \\ &\lesssim \|u\|_{L^2}^{1-\frac{s}{l+1}} \|\nabla^{l+1}u\|_{L^2}^{\frac{s}{l+1}} \|\nabla^\alpha u\|_{L^2}^{\frac{s}{l+1}} \|\nabla^{l+1}u\|_{L^2}^{1-\frac{s}{l+1}} \\ &\lesssim \delta \|\nabla^{l+1}u\|_{L^2}, \end{aligned} \tag{2.28}$$

where α is defined by

$$\frac{l-s+1}{3}-\frac{1}{3} = (\frac{\alpha}{3}-\frac{1}{2})\times\frac{s}{l+1} + (\frac{l+1}{3}-\frac{1}{2})\times(1-\frac{s}{l+1}).$$

Since $[\frac{l}{2}]+1\le s\le l$, we have that $\alpha=\frac{l+1}{2s}\in(\frac{1}{2},1]$.

Combining (2.27)–(2.28) and by using Cauchy's inequality, we deduce that

$$I_3 \lesssim \delta\|\nabla^{l+1}u\|_{L^2}^2. \tag{2.29}$$

Now, we estimate the term I_4. If $l=1$, we integrate by parts and use Lemma 4.2 and Hölder's inequality to get that

$$I_4 \approx \int_{\mathbb{R}^3} f(\varrho)\nabla^2 u\cdot\nabla^2 u\mathrm{d}x \lesssim \|f(\varrho)\|_{L^6}\|\nabla^2 u\|_{L^3}\|\nabla^2 u\|_{L^2} \lesssim \delta(\|\nabla\varrho\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2). \tag{2.30}$$

If $l\ge 2$, we integrate by parts and employ the Leibniz formula and Hölder's inequality to obtain

$$\begin{aligned} I_4 &\approx \int_{\mathbb{R}^3} \nabla^{l-1}[f(\varrho)\nabla^2 u]\cdot\nabla^{l+1}u\mathrm{d}x \\ &= \int_{\mathbb{R}^3} \sum_{s=0}^{l-1} C_{l-1}^s \nabla^s f(\varrho)\nabla^{l+1-s}u\cdot\nabla^{l+1}u\mathrm{d}x \\ &\lesssim \sum_{s=0}^{l-1} \|\nabla^s f(\varrho)\nabla^{l+1-s}u\|_{L^2}\|\nabla^{l+1}u\|_{L^2}. \end{aligned} \tag{2.31}$$

If $0\le s\le[\frac{l}{2}]$, by using Lemma 4.1 and Lemma 4.3 we estimate the first factor in the above to be

$$\begin{aligned}\|\nabla^s f(\varrho)\nabla^{l+1-s}u\|_{L^2}&\lesssim\|\nabla^s f(\varrho)\|_{L^\infty}\|\nabla^{l+1-s}u\|_{L^2}\\&\lesssim\|\nabla^\alpha\varrho\|_{L^2}^{\frac{l-s}{l-1}}\|\nabla^l\varrho\|_{L^2}^{1-\frac{l-s}{l-1}}\|\nabla u\|_{L^2}^{1-\frac{l-s}{l-1}}\|\nabla^l\varrho\|_{L^2}^{\frac{l-s}{l-1}}\\&\lesssim\delta(\|\nabla^l\varrho\|_{L^2}+\|\nabla^{l+1}u\|_{L^2}),\end{aligned}\tag{2.32}$$

where α is defined by

$$\frac{s}{3}=(\frac{\alpha}{3}-\frac{1}{2})\times\frac{l-s}{l-1}+(\frac{l}{3}-\frac{1}{2})\times(1-\frac{l-s}{l-1}).$$

Since $0\le s\le[\frac{l}{2}]$, we have that $\alpha=1+\frac{3(l-1)}{2(l-s)}\in(\frac{3}{4},3]$.

If $[\frac{l}{2}]+1\le s\le l-1$, by using Lemma 4.1 and Lemma 4.3 again, we obtain that

$$\begin{aligned}\|\nabla^s f(\varrho)\nabla^{l+1-s}u\|_{L^2}&\lesssim\|\nabla^s f(\varrho)\|_{L^2}\|\nabla^{l+1-s}u\|_{L^\infty}\\&\lesssim\|\nabla\varrho\|_{L^2}^{\frac{l-s}{l-1}}\|\nabla^l\varrho\|_{L^2}^{1-\frac{l-s}{l-1}}\|\nabla^\alpha u\|_{L^2}^{1-\frac{l-s}{l-1}}\|\nabla^{l+1}u\|_{L^2}^{\frac{l-s}{l-1}}\\&\lesssim\delta(\|\nabla^l\varrho\|_{L^2}+\|\nabla^{l+1}u\|_{L^2}),\end{aligned}\tag{2.33}$$

where α is defined by

$$\frac{l-s+1}{3}=(\frac{\alpha}{3}-\frac{1}{2})\times(1-\frac{l-s}{l-1})+(\frac{l+1}{3}-\frac{1}{2})\times\frac{l-s}{l-1}.$$

Since $[\frac{l}{2}]+1\le s\le l$, we have that $\alpha=\frac{l-1}{2(s-1)}+2\in(\frac{5}{2},3]$.

Combining (2.30)–(2.33), we deduce that

$$I_4\lesssim\delta(\|\nabla^l\varrho\|_{L^2}^2+\|\nabla^{l+1}u\|_{L^2}^2).\tag{2.34}$$

Next, we estimate the term I_5. If $l=1$, we integrate by parts and use Lemma 4.3 and Hölder's inequality to get that

$$I_5=\int_{\mathbb{R}^3}(g(\varrho)\nabla\varrho)\cdot\nabla^2u\mathrm{d}x\lesssim\|g(\varrho)\|_{L^6}\|\nabla\varrho\|_{L^3}\|\nabla^2u\|_{L^2}\lesssim\delta(\|\nabla\varrho\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2).\tag{2.35}$$

If $l\ge2$, we integrate by parts to find that

$$\begin{aligned}I_5&=\int_{\mathbb{R}^3}\nabla^{l-1}[g(\varrho)\nabla\varrho]\cdot\nabla^{l+1}u\mathrm{d}x\\&=\int_{\mathbb{R}^3}\sum_{s=0}^{l-1}C_{l-1}^s\nabla^s g(\varrho)\nabla^{l-1-s}\nabla\varrho\cdot\nabla^{l+1}u\mathrm{d}x\\&\lesssim\sum_{s=0}^{l-1}\|\nabla^s g(\varrho)\nabla^{l-s}\varrho\|_{L^2}\|\nabla^{l+1}u\|_{L^2}.\end{aligned}\tag{2.36}$$

If $0\le s\le[\frac{l}{2}]$, by using Lemma 4.1 and Lemma 4.3, we estimate the first factor in the above to get that

$$\begin{aligned}\|\nabla^s g(\varrho)\nabla^{l-s}\varrho\|_{L^2}&\lesssim\|\nabla^s g(\varrho)\|_{L^\infty}\|\nabla^{l-s}\varrho\|_{L^2}\\&\lesssim\|\nabla^\alpha\varrho\|_{L^2}^{1-\frac{s}{l}}\|\nabla^l\varrho\|_{L^2}^{\frac{s}{l}}\|\varrho\|_{L^2}^{\frac{s}{l}}\|\nabla^l\varrho\|_{L^2}^{1-\frac{s}{l}}\\&\lesssim\delta\|\nabla^l\varrho\|_{L^2},\end{aligned}\tag{2.37}$$

where α is defined by

$$\frac{s}{3}=(\frac{\alpha}{3}-\frac{1}{2})\times(1-\frac{s}{l})+(\frac{l}{3}-\frac{1}{2})\times\frac{s}{l}.$$

Since $0\le s\le [\frac{l}{2}]$, we have that $\alpha=\frac{3l}{2(l-s)}\in[\frac{3}{2},3]$.

If $[\frac{l}{2}]+1\le s\le l-1$, by using Lemma 4.1 and Lemma 4.3, we get that

$$\begin{aligned}\|\nabla^s g(\varrho)\nabla^{l-s}\varrho\|_{L^2}\|_{L^2}&\lesssim \|\nabla^{l+1}\varrho\|_{L^6}\|\nabla^{k-l-1}g(\varrho)\|_{L^3}\\&\lesssim \|\nabla\varrho\|_{L^2}^{\frac{l-s}{l-1}}\|\nabla^l\varrho\|_{L^2}^{1-\frac{l-s}{l-1}}\|\nabla^\alpha\varrho\|_{L^2}^{1-\frac{l-s}{l-1}}\|\nabla^l\varrho\|_{L^2}^{\frac{l-s}{l-1}}\\&\lesssim \delta\|\nabla^l\varrho\|_{L^2},\end{aligned}\tag{2.38}$$

where α is defined by

$$\frac{l-s}{3}=(\frac{\alpha}{3}-\frac{1}{2})\times(1-\frac{l-s}{l-1})+(\frac{l}{3}-\frac{1}{2})\times\frac{l-s}{l-1}.$$

Since $[\frac{l}{2}]+1\le s\le l$, we have that $\alpha=\frac{l-1}{2(s-1)}+2\in[\frac{5}{2},3]$.

Combining (2.35)–(2.38), we deduce that

$$I_5\lesssim\delta(\|\nabla^l\varrho\|_{L^2}^2+\|\nabla^{l+1}u\|_{L^2}^2).\tag{2.39}$$

We now estimate the term I_6. If $l=1$, by Hölder's inequality we get that

$$I_6=\int_{\mathbb{R}^3}f(\varrho)\text{div}\tau\cdot u\text{d}x\le\|f(\varrho)\|_{L^\infty}\|\text{div}\tau\|_{L^2}\|u\|_{L^2}\lesssim\delta(\|\nabla\tau\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2).\tag{2.40}$$

If $l\ge 2$, we integrate by parts and employ the Leibniz formula and Hölder's inequality to obtain that

$$\begin{aligned}I_6&=\int_{\mathbb{R}^3}\nabla^{l-1}[f(\varrho)\text{div}\tau]\cdot\nabla^{l+1}u\text{d}x\\&=\int_{\mathbb{R}^3}\sum_{s=0}^{l-1}C_{l-1}^s\nabla^s f(\varrho)\cdot\nabla^{l-1-s}\text{div}\tau\cdot\nabla^{l+1}u\text{d}x\\&\lesssim\sum_{s=0}^{l-1}\|\nabla^s f(\varrho)\nabla^{l-1-s}\text{div}\tau\|_{L^2}\|\nabla^{l+1}u\|_{L^2}.\end{aligned}\tag{2.41}$$

If $0\le l\le[\frac{l}{2}]$, by using Lemma 4.1 and Lemma 4.3, we estimate the first factor in the above to establish that

$$\begin{aligned}\|\nabla^s f(\varrho)\nabla^{l-1-s}\text{div}\tau\|_{L^2}&\lesssim\|\nabla^s f(\varrho)\|_{L^\infty}\|\nabla^{l-s}\tau\|_{L^2}\\&\lesssim\|\nabla^\alpha\varrho\|_{L^2}^{1-\frac{s}{l}}\|\nabla^{l+1}\varrho\|_{L^2}^{\frac{s}{l}}\|\tau\|_{L^2}^{\frac{s}{l}}\|\nabla^l\tau\|_{L^2}^{1-\frac{s}{l}}\\&\lesssim\delta(\|\nabla^l\varrho\|_{L^2}+\|\nabla^l\tau\|_{L^2}),\end{aligned}\tag{2.42}$$

where α is defined by

$$\frac{s}{3}=(\frac{\alpha}{3}-\frac{1}{2})\times(1-\frac{s}{l})+(\frac{l}{3}-\frac{1}{2})\times\frac{s}{l}.$$

Since $0\le s\le[\frac{l}{2}]$, we have that $\alpha=\frac{3l}{2(l-s)}\in[\frac{3}{2},1]$.

If $[\frac{l}{2}]+1\le s\le l-1$, by using Lemma 4.1 and Lemma 4.3 again, we obtain that

$$\begin{aligned}\|\nabla^s f(\varrho)\nabla^{l-1-s}\text{div}\tau\|_{L^2}&\lesssim\|\nabla^s f(\varrho)\|_{L^2}\|\nabla^{l-s}\tau\|_{L^\infty}\\&\lesssim\|\nabla\varrho\|_{L^2}^{\frac{l-s}{l-1}}\|\nabla^l\varrho\|_{L^2}^{1-\frac{l-s}{l-1}}\|\nabla^\alpha\tau\|_{L^2}^{1-\frac{l-s}{l-1}}\|\nabla^l\tau\|_{L^2}^{\frac{l-s}{l-1}}\\&\lesssim\delta(\|\nabla^l\varrho\|_{L^2}+\|\nabla^l\tau\|_{L^2}),\end{aligned}\tag{2.43}$$

where α is defined by

$$\frac{l-s}{3}=(\frac{\alpha}{3}-\frac{1}{2})\times(1-\frac{l-s}{l-1})+(\frac{l}{3}-\frac{1}{2})\times\frac{l-s}{l-1}.$$

Since $[\frac{l}{2}]+1\le s\le l-1$, we have that $\alpha=1+\frac{l-1}{2(s-1)}\in(\frac{3}{2},2]$.

Combining (2.41)–(2.43), we deduce that

$$I_6\lesssim\delta(\|\nabla^l\varrho\|_{L^2}^2+\|\nabla^l\tau\|_{L^2}^2+\|\nabla^{l+1}u\|_{L^2}^2). \tag{2.44}$$

We now estimate the term I_7. Using Lemma 4.2 and Hölder's inequality, we get that

$$\begin{aligned} I_7&=-\int_{\mathbb{R}^3}\nabla^l(u\cdot\nabla\tau)\cdot\nabla^l\tau\mathrm{d}x\\ &=-\int_{\mathbb{R}^3}\nabla^l(u\cdot\nabla\nabla^l\tau+[\nabla^l,u]\nabla\tau)\cdot\nabla^l\tau\mathrm{d}x\\ &\lesssim\|\nabla u\|_{L^\infty}\|\nabla^l\tau\|_{L^2}^2+\|\nabla^l u\|_{L^6}\|\nabla\tau\|_{L^3}\|\nabla^l\tau\|_{L^2}\\ &\lesssim\delta(\|\nabla^{l+1}u\|_{L^2}^2+\|\nabla^l\tau\|_{L^2}^2). \end{aligned} \tag{2.45}$$

Similarly to I_1, we can bound

$$I_8\lesssim\delta(\|\nabla^{l+1}u\|_{L^2}^2+\|\nabla^l\tau\|_{L^2}^2). \tag{2.46}$$

We now estimate the term I_9. If $l=1$, we further obtain that

$$I_9=-\int_{\mathbb{R}^3}\nabla(\mathrm{tr}\tau\tau)\cdot\nabla\tau\mathrm{d}x\lesssim\|\nabla\tau\|_{L^\infty}\|\tau\|_{L^2}^2\lesssim\delta\|\nabla\tau\|_{L^2}^2. \tag{2.47}$$

If $l\ge2$, by Hölder's inequality, we get that

$$\begin{aligned} I_9&=-\int_{\mathbb{R}^3}\nabla^l(\mathrm{tr}\tau\tau)\cdot\nabla^l\tau\mathrm{d}x\\ &=\int_{\mathbb{R}^3}\sum_{s=0}^{l}C_l^s\nabla^s\mathrm{tr}\tau\nabla^{l-s}\tau\cdot\nabla^l\tau\mathrm{d}x\\ &\lesssim\sum_{s=0}^{l}\|\nabla^s\mathrm{tr}\tau\nabla^{l-s}\tau\|_{L^2}\|\nabla^l\tau\|_{L^2}. \end{aligned} \tag{2.48}$$

If $0\le s\le[\frac{l}{2}]$, by using Lemma 4.1, we estimate the first factor in the above to get that

$$\begin{aligned} \|\nabla^s\mathrm{tr}\tau\nabla^{l-s}\tau\|_{L^2}&\lesssim\|\nabla^s\mathrm{tr}\tau\|_{L^\infty}\|\nabla^{l-s}\tau\|_{L^2}\\ &\lesssim\|\nabla^\alpha\tau\|_{L^2}^{1-\frac{s}{l}}\|\nabla^l\tau\|_{L^2}^{\frac{s}{l}}\|\tau\|_{L^2}^{\frac{s}{l}}\|\nabla^l\tau\|_{L^2}^{1-\frac{s}{l}}\\ &\lesssim\delta\|\nabla^l\tau\|_{L^2}, \end{aligned} \tag{2.49}$$

where α is defined by

$$\frac{s}{3}=(\frac{\alpha}{3}-\frac{1}{2})\times(1-\frac{s}{l})+(\frac{l}{3}-\frac{1}{2})\times\frac{s}{l}.$$

Since $0\le s\le[\frac{l}{2}]$, we have that $\alpha=\frac{3l}{2(l-s)}\in[\frac{3}{2},3]$.

If $[\frac{l}{2}]+1\le s\le l$, by using Lemma 4.1, we get that

$$\begin{aligned} \|\nabla^s\mathrm{tr}\tau\nabla^{l-s}\tau\|_{L^2}&\lesssim\|\nabla^s\tau\|_{L^2}\|\nabla^{l-s}\tau\|_{L^\infty}\\ &\lesssim\|\nabla\tau\|_{L^2}^{\frac{l-s}{l-1}}\|\nabla^l\tau\|_{L^2}^{1-\frac{l-s}{l-1}}\|\nabla^\alpha\tau\|_{L^2}^{1-\frac{l-s}{l-1}}\|\nabla^l\tau\|_{L^2}^{\frac{l-s}{l-1}}\\ &\lesssim\delta\|\nabla^l\tau\|_{L^2}, \end{aligned} \tag{2.50}$$

where α is defined by

$$\frac{l-s}{3}=(\frac{\alpha}{3}-\frac{1}{2})\times(1-\frac{l-s}{l-1})+(\frac{l}{3}-\frac{1}{2})\times\frac{l-s}{l-1}.$$

Since $[\frac{l}{2}]+1\le s\le l$, we have that $\alpha=\frac{l-1}{2(s-1)}+1\in[\frac{3}{2},2)$.

Combining (2.47)–(2.50), we deduce that

$$I_9\lesssim\delta\|\nabla^l\tau\|_{L^2}^2. \tag{2.51}$$

Summing up the estimates for I_1–I_9, i.e., (2.24), (2.25), (2.29), (2.34), (2.39), (2.44), (2.45), (2.46) and (2.51), we deduce (2.18), which yields the desired result. □

We now recover the dissipative estimates of ϱ by constructing some interactive energy functionals in the following lemma:

Lemma 2.3 Let all of the assumptions in Lemma 2.1 be in force. Then, for any $k\ge 0$, it holds that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=k}^{k+1}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho\mathrm{d}x+\frac{1}{2}\sum_{l=k}^{k+1}\|\nabla^{l+1}\varrho\|_{L^2}^2\\ \le{}& C_l\delta\sum_{l=k}^{k+1}(\|\nabla^{l+1}\varrho\|_{L^2}^2+\|\nabla^{l+2}u\|_{L^2}^2+\|\nabla^{l+1}\tau\|_{L^2}^2)\\ &+\sum_{l=k}^{k+1}(\|\nabla^{l+1}u\|_{L^2}^2+4\|\nabla^{l+2}u\|_{L^2}^2+2\|\nabla^{l+1}\tau\|_{L^2}^2).\end{aligned} \tag{2.52}$$

Proof Applying $\nabla^l(l=k,k+1)$ to $(2.1)_2$, multiplying $\nabla^{l+1}\varrho$, and integrating by parts, we get that

$$\begin{aligned}&\int_{\mathbb{R}^3}\nabla^l u_t\cdot\nabla^{l+1}\varrho\mathrm{d}x+\|\nabla^{l+1}\varrho\|_{L^2}^2\\ ={}&\int_{\mathbb{R}^3}\nabla^l\triangle u\cdot\nabla^{l+1}\varrho\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^l\nabla\mathrm{div}u\cdot\nabla^{l+1}\varrho\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^l\mathrm{div}\tau\cdot\nabla^{l+1}\varrho\mathrm{d}x\\ &-\int_{\mathbb{R}^3}\nabla^l(u\cdot\nabla u)\cdot\nabla^{l+1}\varrho\mathrm{d}x-\int_{\mathbb{R}^3}\nabla^l(f(\varrho)\triangle u)\cdot\nabla^{l+1}\varrho\mathrm{d}x-\int_{\mathbb{R}^3}\nabla^l(f(\varrho)\nabla\mathrm{div}u)\cdot\nabla^{l+1}\varrho\mathrm{d}x\\ &-\int_{\mathbb{R}^3}\nabla^l(g(\varrho)\nabla\varrho)\cdot\nabla^{l+1}\varrho\mathrm{d}x-\int_{\mathbb{R}^3}\nabla^l(f(\varrho)\nabla\mathrm{div}\tau)\cdot\nabla^{l+1}\varrho\mathrm{d}x\\ :={}&\sum_{i=1}^{8}J_i.\end{aligned} \tag{2.53}$$

For the first term on the left-hand side of (2.53), by $(2.1)_1$ and integrating by parts for both the t-and x-variables, we may estimate

$$\begin{aligned}&-\int_{\mathbb{R}^3}\nabla^l u_t\cdot\nabla^{l+1}\varrho\mathrm{d}x\\ ={}&-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho\mathrm{d}x-\int_{\mathbb{R}^3}\nabla^l\mathrm{div}u\cdot\nabla^l\varrho_t\mathrm{d}x\\ ={}&-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho\mathrm{d}x+\|\nabla^l\mathrm{div}u\|_{L^2}^2+\int_{\mathbb{R}^3}\nabla^l\mathrm{div}u\cdot\nabla^l\mathrm{div}(\varrho u)\mathrm{d}x.\end{aligned} \tag{2.54}$$

Thus,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho\mathrm{d}x+\|\nabla^{l+1}\varrho\|_{L^2}^2\\ ={}&\sum_{i=1}^{8}J_i+\|\nabla^l\mathrm{div}u\|_{L^2}^2+\int_{\mathbb{R}^3}\nabla^l\mathrm{div}u\cdot\nabla^l\mathrm{div}(\varrho u)\mathrm{d}x\end{aligned}$$

$$:= \sum_{i=1}^{8} J_i + \|\nabla^l \mathrm{div} u\|_{L^2}^2 + J_9. \tag{2.55}$$

Now, we concentrate our attention on estimating the terms J_1–J_9. First, employing Cauchy's inequality, it holds that

$$J_1 + J_2 \le \frac{1}{4}\|\nabla^{l+1}\varrho\|_{L^2}^2 + 4\|\nabla^{l+2}u\|_{L^2}^2. \tag{2.56}$$

$$J_3 \le \frac{1}{4}\|\nabla^{l+1}\varrho\|_{L^2}^2 + 2\|\nabla^{l+1}\tau\|_{L^2}^2. \tag{2.57}$$

Moreover, taking into account (2.27)–(2.28), we are in a position to obtain that

$$\begin{aligned} J_4 &= \int_{\mathbb{R}^3} \mathrm{div}\nabla^l(u\cdot\nabla u)\cdot\nabla^l\varrho \mathrm{d}x \lesssim \|\mathrm{div}\nabla^l(u\cdot\nabla u)\|_{L^{\frac{6}{5}}}\|\nabla^l\varrho\|_{L^6} \\ &\lesssim \delta(\|\nabla^{l+1}\varrho\|_{L^2}^2 + \|\nabla^{l+2}u\|_{L^2}^2). \end{aligned} \tag{2.58}$$

Similarly to (2.30)–(2.33), using Hölder's inequality, Lemma 4.1 and Lemma 4.3, the terms J_5, J_6 can be estimated as follows:

$$J_5 \lesssim \|\sum_{s=0}^{l} C_s^l \nabla^s f(\varrho)\nabla^{l-s}\triangle u\|_{L^2}\|\nabla^{l+1}\varrho\|_{L^2} \lesssim \delta(\|\nabla^{l+1}\varrho\|_{L^2}^2 + \|\nabla^{l+2}u\|_{L^2}^2). \tag{2.59}$$

$$J_6 \lesssim \|\sum_{s=0}^{l} C_s^l \nabla^s f(\varrho)\nabla^{l-s}\nabla \mathrm{div} u\|_{L^2}\|\nabla^{l+1}\varrho\|_{L^2} \lesssim \delta(\|\nabla^{l+1}\varrho\|_{L^2}^2 + \|\nabla^{l+2}u\|_{L^2}^2). \tag{2.60}$$

Furthermore, taking into account (2.35)–(2.38), applying Hölder's inequality, Lemma 4.1 and Lemma 4.3, we obtain that

$$J_7 \lesssim \|\sum_{s=0}^{l} C_s^l \nabla^s g(\varrho)\nabla^{l-s}\nabla\varrho\|_{L^2}\|\nabla^{l+1}\varrho\|_{L^2} \lesssim \delta\|\nabla^{l+1}\varrho\|_{L^2}^2. \tag{2.61}$$

Similarly to the estimates of (2.42)–(2.43), we further obtain that

$$J_8 \lesssim \|\sum_{s=0}^{l} C_s^l \nabla^s f(\varrho)\nabla^{l-s}\mathrm{div}\tau\|_{L^2}\|\nabla^{l+1}\varrho\|_{L^2} \lesssim \delta(\|\nabla^{l+1}\varrho\|_{L^2}^2 + \|\nabla^{l+1}\tau\|_{L^2}^2). \tag{2.62}$$

Finally, similarly to I_9, by integration by parts and Lemma 4.1, we get that

$$J_9 = \int_{\mathbb{R}^3} \nabla^l \mathrm{div}(\varrho u)\cdot\nabla^l \mathrm{div} u \mathrm{d}x \lesssim \delta(\|\nabla^{l+1}\varrho\|_{L^2}^2 + \|\nabla^{l+1}u\|_{L^2}^2). \tag{2.63}$$

Putting these estimates into (2.55), and summing up with $l = k, k+1$, we finally obtain that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=k}^{k+1}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho \mathrm{d}x + \frac{1}{2}\sum_{l=k}^{k+1}\|\nabla^{l+1}\varrho\|_{L^2}^2 \\ &\le C_l\delta\sum_{l=k}^{k+1}(\|\nabla^{l+1}\varrho\|_{L^2}^2 + \|\nabla^{l+2}u\|_{L^2}^2 + \|\nabla^{l+1}\tau\|_{L^2}^2) \\ &\quad + \sum_{l=k}^{k+1}(\|\nabla^{l+1}u\|_{L^2}^2 + 4\|\nabla^{l+2}u\|_{L^2}^2 + 2\|\nabla^{l+1}\tau\|_{L^2}^2). \end{aligned}$$

Thus, we have completed the proof of Lemma 2.2. □

2.2 Negative Sobolev estimates

In this subsection, we will derive the evolution of the negative Sobolev norms and Besov norms of the solution. In order to estimate the nonlinear terms, we need to restrict ourselves to the fact that $s \in (0, \frac{3}{2}]$. First, for the homogeneous Sobolev space, we will establish the following lemma:

Lemma 2.4 For $s \in (0, \frac{1}{2}]$, we have that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varrho, u, \tau)\|_{\dot{H}^{-s}}^2 + 2\|\nabla u\|_{\dot{H}^{-s}}^2 + \|\tau\|_{\dot{H}^{-s}}^2 \\ &\lesssim \|\nabla(\varrho, u)\|_{H^1}^2\|\varrho\|_{\dot{H}^{-s}} + (\|\nabla(\varrho, u)\|_{H^1}^2 + \|\nabla\tau\|_{L^2}^2)\|u\|_{\dot{H}^{-s}} + (\|\nabla u\|_{H^1}^2 + \|\tau\|_{H^2}^2)\|\tau\|_{\dot{H}^{-s}},\end{aligned} \tag{2.64}$$

and for $s \in (\frac{1}{2}, \frac{3}{2})$, we have that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varrho, u, \tau)\|_{\dot{H}^{-s}}^2 + 2\|\nabla u\|_{\dot{H}^{-s}}^2 + \|\tau\|_{\dot{H}^{-s}}^2 \\ &\lesssim \|(\varrho, u, \tau)\|_{L^2}^{s-\frac{1}{2}}\|\nabla(\varrho, u, \tau)\|_{L^2}^{\frac{3}{2}-s}\|(\nabla\varrho, \nabla u, \nabla^2 u, \tau)\|_{L^2}\|(\varrho, u, \tau)\|_{\dot{H}^{-s}}.\end{aligned} \tag{2.65}$$

Proof Applying Λ^{-s} to (2.1), and multiplying the resulting identities by $\Lambda^{-s}\varrho$, $\Lambda^{-s}u$ and $\Lambda^{-s}\tau$, respectively, summing up them and then integrating over $\mathbb{R}^3$ by parts, we get that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|\Lambda^{-s}\varrho|^2 + |\Lambda^{-s}u|^2 + |\Lambda^{-s}\tau|^2)\mathrm{d}x + \int_{\mathbb{R}^3}(|\nabla\Lambda^{-s}u|^2 + |\mathrm{div}\Lambda^{-s}u|^2 + |\Lambda^{-s}\tau|^2)\mathrm{d}x \\ &= \int_{\mathbb{R}^3}\Lambda^{-s}(-\varrho\mathrm{div}u - u\cdot\nabla\varrho)\cdot\Lambda^{-s}\varrho \\ &\quad - \Lambda^{-s}[u\cdot\nabla u + f(\varrho)(\triangle u + \nabla\mathrm{div}u) + g(\varrho)\nabla\varrho + f(\varrho)\mathrm{div}\tau]\cdot\Lambda^{-s}u \\ &\quad - \Lambda^{-s}[u\cdot\nabla\tau + Q(\tau, \nabla u) + \mathrm{tr}\tau\tau]\cdot\Lambda^{-s}\tau\mathrm{d}x, \\ &:= \sum_{i=1}^{9}K_i.\end{aligned} \tag{2.66}$$

If $s \in (0, \frac{1}{2}]$, then $\frac{1}{2} + \frac{s}{3} < 1$ and $\frac{3}{s} \geq 6$. Then, using Lemma 4.4, together with Hölder's and Young's inequalities, we obtain that

$$\begin{aligned}K_1 &= -\int_{\mathbb{R}^3}\Lambda^{-s}(\varrho\mathrm{div}u)\cdot\Lambda^{-s}\varrho\mathrm{d}x \lesssim \|\varrho\mathrm{div}u\|_{\dot{H}^{-s}}\|\varrho\|_{\dot{H}^{-s}} \\ &\lesssim \|\varrho\mathrm{div}u\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\|\varrho\|_{\dot{H}^{-s}} \lesssim \|\varrho\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2}\|\varrho\|_{\dot{H}^{-s}} \\ &\lesssim \|\nabla\varrho\|_{L^2}^{s+\frac{1}{2}}\|\nabla^2\varrho\|_{L^2}^{\frac{1}{2}-s}\|\nabla u\|_{L^2}\|\varrho\|_{\dot{H}^{-s}} \\ &\lesssim (\|\nabla\varrho\|_{L^2}^2 + \|\nabla^2\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)\|\varrho\|_{\dot{H}^{-s}}.\end{aligned} \tag{2.67}$$

Similarly, we can bound the terms K_2–K_5 by

$$K_2 = -\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla\varrho)\cdot\Lambda^{-s}\varrho\mathrm{d}x \lesssim (\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla\varrho\|_{L^2}^2)\|\varrho\|_{\dot{H}^{-s}}. \tag{2.68}$$

$$K_3 = -\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla u)\cdot\Lambda^{-s}u\mathrm{d}x \lesssim (\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2)\|u\|_{\dot{H}^{-s}}. \tag{2.69}$$

$$\begin{aligned}K_4 &= -\int_{\mathbb{R}^3}\Lambda^{-s}(f(\varrho)(\triangle u + \bigtriangledown\mathrm{div}u))\cdot\Lambda^{-s}\varrho\mathrm{d}x \\ &\lesssim (\|\nabla\varrho\|_{L^2}^2 + \|\nabla^2\varrho\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2)\|u\|_{\dot{H}^{-s}}.\end{aligned} \tag{2.70}$$

$$K_5=-\int_{\mathbb{R}^3}\Lambda^{-s}(g(\varrho)\nabla\varrho)\cdot\Lambda^{-s}u\mathrm{d}x\lesssim(\|\nabla\varrho\|_{L^2}^2+\|\nabla^2\varrho\|_{L^2}^2)\|u\|_{\dot{H}^{-s}}. \tag{2.71}$$

$$K_6=-\int_{\mathbb{R}^3}\Lambda^{-s}(f(\varrho)\mathrm{div}\tau)\cdot\Lambda^{-s}u\mathrm{d}x\lesssim(\|\nabla\varrho\|_{L^2}^2+\|\nabla^2\varrho\|_{L^2}^2+\|\nabla\tau\|_{L^2}^2)\|u\|_{\dot{H}^{-s}}. \tag{2.72}$$

$$K_7=-\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla\tau)\cdot\Lambda^{-s}\tau\mathrm{d}x\lesssim(\|\nabla u\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2+\|\nabla\tau\|_{L^2}^2)\|\tau\|_{\dot{H}^{-s}}. \tag{2.73}$$

$$K_8=-\int_{\mathbb{R}^3}\Lambda^{-s}(Q(\tau,\nabla u))\cdot\Lambda^{-s}\tau\mathrm{d}x\lesssim(\|\nabla\tau\|_{L^2}^2+\|\nabla^2\tau\|_{L^2}^2+\|\nabla u\|_{L^2}^2)\|\tau\|_{\dot{H}^{-s}}. \tag{2.74}$$

$$K_9=-\int_{\mathbb{R}^3}\Lambda^{-s}(\mathrm{tr}\tau\tau)\cdot\Lambda^{-s}\tau\mathrm{d}x\lesssim(\|\nabla\tau\|_{L^2}^2+\|\nabla^2\tau\|_{L^2}^2+\|\tau\|_{L^2}^2)\|\tau\|_{\dot{H}^{-s}}. \tag{2.75}$$

Hence, plugging the estimates (2.67)–(2.75) into (2.66), we deduce (2.64).

Now if $s\in(\frac{1}{2},\frac{3}{2})$, we shall estimate the right-hand side of (2.66) in a different way. Since $s\in(\frac{1}{2},\frac{3}{2})$, we have that $\frac{1}{2}+\frac{s}{3}<1$ and $2<\frac{3}{s}<6$. Then, by Lemmas 4.5 and 4.1, we obtain that

$$\begin{aligned}K_1&=-\int_{\mathbb{R}^3}\Lambda^{-s}(\varrho\mathrm{div}u)\cdot\Lambda^{-s}\varrho\mathrm{d}x\lesssim\|\Lambda^{-s}(\varrho\mathrm{div}u)\|_{L^2}\|\varrho\|_{\dot{H}^{-s}}\\&\lesssim\|\varrho\mathrm{div}u\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\|\Lambda^{-s}\varrho\|_{L^2}\lesssim\|\varrho\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2}\|\varrho\|_{\dot{H}^{-s}}\\&\lesssim\|\varrho\|_{L^2}^{s-\frac{1}{2}}\|\nabla\varrho\|_{L^2}^{\frac{3}{2}-s}\|\nabla u\|_{L^2}\|\varrho\|_{\dot{H}^{-s}}.\end{aligned} \tag{2.76}$$

Similarly, we can bound the remaining terms by

$$K_2=-\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla\varrho)\cdot\Lambda^{-s}\varrho\mathrm{d}x\lesssim\|u\|_{L^2}^{s-\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}-s}\|\nabla\varrho\|_{L^2}\|\varrho\|_{\dot{H}^{-s}}. \tag{2.77}$$

$$K_3=-\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla u)\cdot\Lambda^{-s}u\mathrm{d}x\lesssim\|u\|_{L^2}^{s-\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}-s}\|\nabla u\|_{L^2}\|u\|_{\dot{H}^{-s}}. \tag{2.78}$$

$$K_4=-\int_{\mathbb{R}^3}\Lambda^{-s}(f(\varrho)(\triangle u+\nabla\mathrm{div}u))\cdot\Lambda^{-s}u\mathrm{d}x\lesssim\|\varrho\|_{L^2}^{s-\frac{1}{2}}\|\nabla\varrho\|_{L^2}^{\frac{3}{2}-s}\|\nabla^2u\|_{L^2}\|u\|_{\dot{H}^{-s}}. \tag{2.79}$$

$$K_5=-\int_{\mathbb{R}^3}\Lambda^{-s}(g(\varrho)\nabla\varrho)\cdot\Lambda^{-s}u\mathrm{d}x\lesssim\|\varrho\|_{L^2}^{s-\frac{1}{2}}\|\nabla\varrho\|_{L^2}^{\frac{3}{2}-s}\|\nabla\varrho\|_{L^2}\|u\|_{\dot{H}^{-s}}. \tag{2.80}$$

$$K_6=-\int_{\mathbb{R}^3}\Lambda^{-s}(f(\varrho)\mathrm{div}\tau)\cdot\Lambda^{-s}u\mathrm{d}x\lesssim\|\varrho\|_{L^2}^{s-\frac{1}{2}}\|\nabla\varrho\|_{L^2}^{\frac{3}{2}-s}\|\nabla\tau\|_{L^2}\|u\|_{\dot{H}^{-s}}. \tag{2.81}$$

$$K_7=-\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla\tau)\cdot\Lambda^{-s}\tau\mathrm{d}x\lesssim\|u\|_{L^2}^{s-\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}-s}\|\nabla\tau\|_{L^2}\|\tau\|_{\dot{H}^{-s}}. \tag{2.82}$$

$$K_8=-\int_{\mathbb{R}^3}\Lambda^{-s}(Q(\tau,\nabla u))\cdot\Lambda^{-s}\tau\mathrm{d}x\lesssim\|\tau\|_{L^2}^{s-\frac{1}{2}}\|\nabla\tau\|_{L^2}^{\frac{3}{2}-s}\|\nabla u\|_{L^2}\|\tau\|_{\dot{H}^{-s}}. \tag{2.83}$$

$$K_9=-\int_{\mathbb{R}^3}\Lambda^{-s}(\mathrm{tr}\tau\tau)\cdot\Lambda^{-s}\tau\mathrm{d}x\lesssim\|\tau\|_{L^2}^{s-\frac{1}{2}}\|\nabla\tau\|_{L^2}^{\frac{3}{2}-s}\|\tau\|_{L^2}\|\tau\|_{\dot{H}^{-s}}. \tag{2.84}$$

Hence, plugging estimates (2.76)–(2.84) into (2.66), we deduce (2.65). □

2.3 Negative Besov estimates

We replace the homogeneous Sobolev space by the homogeneous Besov space. Now, we will derive the evolution of the negative Besov norms of the solution (ϱ,u,τ) to (2.1)–(2.2). More precisely, we have

Lemma 2.5 Let all of the assumptions in Lemma 2.1 hold. Then, for $s\in(0,\frac{1}{2}]$, we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varrho,u,\tau)\|_{\dot{B}_{2,\infty}^{-s}}^2+2\|\nabla u\|_{\dot{B}_{2,\infty}^{-s}}^2+\|\tau\|_{\dot{B}_{2,\infty}^{-s}}^2)\lesssim\|\nabla(\varrho,u,\tau)\|_{H^1}^2\|(\varrho,u,\tau)\|_{\dot{B}_{2,\infty}^{-s}}, \tag{2.85}$$

and for $s \in (\frac{1}{2}, \frac{3}{2}]$, we have that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varrho,u,\tau)\|^2_{\dot{B}^{-s}_{2,\infty}}+2\|\nabla u\|^2_{\dot{B}^{-s}_{2,\infty}}+\|\tau\|^2_{\dot{B}^{-s}_{2,\infty}})\\&\lesssim\|(\varrho,u,\tau)\|^{s-\frac{1}{2}}_{L^2}\|\nabla(\varrho,u,\tau)\|^{\frac{3}{2}-s}_{L^2}\|(\nabla\varrho,\nabla u,\nabla^2u,\tau)\|_{L^2}\|(\varrho,u,\tau)\|_{\dot{B}^{-s}_{2,\infty}}.\end{aligned}\tag{2.86}$$

Proof Applying the $\dot{\triangle}_j$ energy estimate of (2.1) with a multiplication of 2^{-2sj} and then taking the supremum over $j\in\mathbb{Z}$, we infer that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varrho,u,\tau)\|^2_{\dot{B}^{-s}_{2,\infty}}+2\|\nabla u\|^2_{\dot{B}^{-s}_{2,\infty}}+\|\tau\|^2_{\dot{B}^{-s}_{2,\infty}})\\&\lesssim-\sup_{j\in\mathbb{Z}}2^{-2sj}\int_{\mathbb{R}^3}\dot{\triangle}_j[\mathrm{div}u+\mathrm{div}(\varrho u)]\cdot\dot{\triangle}_j\varrho\mathrm{d}x\\&\quad-\sup_{j\in\mathbb{Z}}2^{-2sj}\int_{\mathbb{R}^3}\dot{\triangle}_j[\nabla\varrho+u\cdot\nabla u+f(\varrho)(\triangle u+\nabla\mathrm{div}u)+g(\varrho)\nabla\varrho+f(\varrho)\mathrm{div}\tau]\cdot\dot{\triangle}_ju\mathrm{d}x\\&\quad-\sup_{j\in\mathbb{Z}}2^{-2sj}\int_{\mathbb{R}^3}\dot{\triangle}_j[u\cdot\nabla\tau+Q(\tau,\nabla u)+\mathrm{tr}\tau\tau]\cdot\dot{\triangle}_ju\mathrm{d}x\\&\lesssim\|\mathrm{div}(\varrho u)\|_{\dot{B}^{-s}_{2,\infty}}\|\varrho\|_{\dot{B}^{-s}_{2,\infty}}+\|u\cdot\nabla u+f(\varrho)(\triangle u+\nabla\mathrm{div}u)+g(\varrho)\nabla\varrho\\&\quad+f(\varrho)\mathrm{div}\tau\|_{\dot{B}^{-s}_{2,\infty}}\|u\|_{\dot{B}^{-s}_{2,\infty}}+\|u\cdot\nabla\tau+Q(\tau,\nabla u)+\mathrm{tr}\tau\tau\|_{\dot{B}^{-s}_{2,\infty}}\|u\|_{\dot{B}^{-s}_{2,\infty}}.\end{aligned}\tag{2.87}$$

According to Lemma 4.5 and (2.87), the remaining proof of Lemma 2.5 is exactly the same with the proof of Lemma 2.4, except that we allow that $s=\frac{3}{2}$ and replace Lemma 4.4 with Lemma 4.5, and the $\dot{H}^{-s}$ norm by the $\dot{B}^{-s}_{2,\infty}$ norm. □

Next, we will combine all the energy estimates that we have derived in order to prove Theorem 1.1; the key point here is that we only assume that the H^3 norm of initial data is small.

Proof We first close the energy estimates at the H^3-level by assuming that $\sqrt{\mathcal{E}_3(t)}\le\delta$ is sufficiently small. From Lemma 2.2, taking $k=0,1$ in (2.18) and summing up, we deduce that, for any $t\in[0,T]$,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=1}^3\|\nabla^l(\varrho,u,\tau)\|^2_{L^2}+\sum_{l=1}^3(\|\nabla^{l+1}u\|^2_{L^2}+\|\nabla^l\mathrm{div}u\|^2_{L^2}+\|\nabla^l\tau\|^2_{L^2})\\&\lesssim\delta\sum_{l=1}^3(\|\nabla^l\varrho\|^2_{L^2}+\|\nabla^{l+1}u\|^2_{L^2}+\|\nabla^l\tau\|^2_{L^2}).\end{aligned}\tag{2.88}$$

From this, together with Lemma 2.1, we deduce that, for any $t\in[0,T]$,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=0}^3\|\nabla^l(\varrho,u,\tau)\|^2_{L^2}+\sum_{l=0}^3(\|\nabla^{l+1}u\|^2_{L^2}+\|\nabla^l\mathrm{div}u\|^2_{L^2}+\|\nabla^l\tau\|^2_{L^2})\\&\lesssim\delta\sum_{l=1}^3\|\nabla^l\varrho\|^2_{L^2}+\delta\sum_{l=0}^3(\|\nabla^{l+1}u\|^2_{L^2}+\|\nabla^l\tau\|^2_{L^2}).\end{aligned}\tag{2.89}$$

In addition, taking the $k=0,1$ in (2.52) of Lemma 2.3 and summing up, we obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=0}^2\int_{\mathbb{R}^3}\nabla^lu\cdot\nabla^{l+1}\varrho\mathrm{d}x+\frac{1}{2}\sum_{l=0}^2\|\nabla^{l+1}\varrho\|^2_{L^2}\\&\le C_l\delta\sum_{l=0}^2(\|\nabla^{l+1}\varrho\|^2_{L^2}+\|\nabla^{l+2}u\|^2_{L^2}+\|\nabla^{l+1}\tau\|^2_{L^2})\end{aligned}$$

$$+\sum_{l=0}^{2}(\|\nabla^{l+1}u\|_{L^2}^2+4\|\nabla^{l+2}u\|_{L^2}^2+2\|\nabla^{l+1}\tau\|_{L^2}^2). \tag{2.90}$$

Taking into account the smallness of δ, by a linear combination of (2.89) and (2.90), we deduce that there exists an instant energy functional $\tilde{\mathcal{E}}_3(t)$ equivalent to $\mathcal{E}_3(t)$ such that

$$\tilde{\mathcal{E}}_3(t)+\int_0^t\mathcal{D}_3(s)\mathrm{d}s\le C\tilde{\mathcal{E}}_3(0),\qquad \forall t\in[0,T]. \tag{2.91}$$

By a standard continuity argument, we then close the a priori estimates if we assume, at the initial time, that $\tilde{\mathcal{E}}_3(0)\le\delta_0$ is sufficiently small. This concludes the unique global small $\tilde{\mathcal{E}}_3$ solution.

From the global existence of the $\tilde{\mathcal{E}}_3$ solution, we shall deduce the global existence of the $\tilde{\mathcal{E}}_N$ solution. For $N\ge 3, t\in[0,\infty]$, applying Lemma 2.2 and taking $k=0,1,\dots,N-2$, we infer that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=1}^{N}\|\nabla^l(\varrho,u,\tau)\|_{L^2}^2+\sum_{l=1}^{N}(\|\nabla^{l+1}u\|_{L^2}^2+\|\nabla^l\mathrm{div}u\|_{L^2}^2+\|\nabla^l\tau\|_{L^2}^2)\\&\lesssim\delta\sum_{l=1}^{N}(\|\nabla^l\varrho\|_{L^2}^2+\|\nabla^{l+1}u\|_{L^2}^2+\|\nabla^l\tau\|_{L^2}^2).\end{aligned} \tag{2.92}$$

From this, together with Lemma 2.1, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=0}^{N}\|\nabla^l(\varrho,u,\tau)\|_{L^2}^2+\sum_{l=0}^{N}(\|\nabla^{l+1}u\|_{L^2}^2+\|\nabla^l\mathrm{div}u\|_{L^2}^2+\|\nabla^l\tau\|_{L^2}^2)\\&\lesssim\delta\sum_{l=1}^{N}\|\nabla^l\varrho\|_{L^2}^2+\delta\sum_{l=0}^{N}(\|\nabla^{l+1}u\|_{L^2}^2+\|\nabla^l\tau\|_{L^2}^2).\end{aligned} \tag{2.93}$$

Furthermore, by Lemma 2.3, and taking $k=0,1,\dots,N-2$, we have that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=0}^{N-1}\int_{\mathbb{R}^3}\nabla^l u\cdot\nabla^{l+1}\varrho\mathrm{d}x+\frac{1}{2}\sum_{l=0}^{N-1}\|\nabla^{l+1}\varrho\|_{L^2}^2\\&\le C_l\delta\sum_{l=0}^{N-1}(\|\nabla^{l+1}\varrho\|_{L^2}^2+\|\nabla^{l+2}u\|_{L^2}^2+\|\nabla^{l+1}\tau\|_{L^2}^2)\\&\quad+\sum_{l=0}^{N-1}(\|\nabla^{l+1}u\|_{L^2}^2+4\|\nabla^{l+2}u\|_{L^2}^2+2\|\nabla^{l+1}\tau\|_{L^2}^2).\end{aligned} \tag{2.94}$$

By a linear combination of (2.93) and (2.94), we infer that there exists an instant energy functional $\tilde{\mathcal{E}}_N(t)$ taht is equivalent to $\mathcal{E}_N(t)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_N(t)+\bar{\lambda}\mathcal{D}_N(t)\le 0 \tag{2.95}$$

for some $\bar{\lambda}\in(0,1)$. This implies (1.4). Thus, we have completed the proof of Theorem 1.1. □

3 Convergence Rate of the Solution

Having in hand the conclusion of Theorem 1.1, Lemma 2.4 and Lemma 2.5, we now proceed to prove the various time decay rates of the unique global solution to (2.1)–(2.2).

Proof of Theorem 1.2 In what follows, for convenience of presentation, we define a family of energy functionals and the corresponding dissipation rates as

$$\mathcal{E}_k^{k+2} := \sum_{l=k}^{k+2} \|\nabla^l(\varrho, u, \tau)\|_{L^2}^2, \tag{3.1}$$

and

$$\mathcal{D}_k^{k+2}(t) := \sum_{l=k+1}^{k+2} \|\nabla^l \varrho\|_{L^2}^2 + \sum_{l=k}^{k+2}(\|\nabla^{l+1} u\|_{L^2}^2 + \|\nabla^l \tau\|_{L^2}^2). \tag{3.2}$$

Taking into accounts Lemmas 2.1–2.3, we have that, for $k - 0, 1, \ldots, N-2$,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=k}^{k+2} \|\nabla^l(\varrho, u, \tau)\|_{L^2}^2 + \sum_{l=k}^{k+2}(\|\nabla^{l+1} u\|_{L^2}^2 + \|\nabla^l \mathrm{div} u\|_{L^2}^2 + \|\nabla^l \tau\|_{L^2}^2) \\
\lesssim{}& \delta \sum_{l=k+1}^{k+2} \|\nabla^l \varrho\|_{L^2}^2 + \delta \sum_{l=k}^{k+2}(\|\nabla^{l+1} u\|_{L^2}^2 + \|\nabla^l \tau\|_{L^2}^2),
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l=k}^{k+1} \int_{\mathbb{R}^3} \nabla^l u \cdot \nabla^{l+1} \varrho \mathrm{d}x + \frac{1}{2}\sum_{l=k}^{k+1} \|\nabla^{l+1} \varrho\|_{L^2}^2 \\
\le{}& C\delta \sum_{l=k}^{k+1}(\|\nabla^{l+1} \varrho\|_{L^2}^2 + \|\nabla^{l+2} u\|_{L^2}^2 + \|\nabla^{l+1} \tau\|_{L^2}^2) \\
&+ \sum_{l=k}^{k+1}(\|\nabla^{l+1} u\|_{L^2}^2 + 4\|\nabla^{l+2} u\|_{L^2}^2 + 2\|\nabla^{l+1} \tau\|_{L^2}^2).
\end{aligned} \tag{3.4}$$

By a linear combination of (3.3) and (3.4), since δ is small, we deduce that there exists an instant energy functional $\tilde{\mathcal{E}}_k^{k+2}$ that is equivalent to $\mathcal{E}_k^{k+2}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_k^{k+2}(t) + \mathcal{D}_k^{k+2}(t) \le 0. \tag{3.5}$$

We note that $\mathcal{D}_k^{k+2}$ is weaker than $\tilde{\mathcal{E}}_k^{k+2}$, which prevents the exponential decay of the solution. We need to bound the missing terms in the energy; that is, bound $\|\nabla^l(\varrho, u)\|_{L^2}^2$ in terms of $\mathcal{D}_k^{k+2}$. From this, we can then derive the time decay rate from (3.5). To this end, we need the Sobolev interpolation between the negative and positive Sobolev norms. We assume for the moment that we have proved (1.5) and (1.6). Using Lemma 4.6 for $s > 0$ and $k + s \ge 0$, we have that

$$\|\nabla^k(\varrho, u)\|_{L^2} \le C\|(\varrho, u)\|_{\dot{H}^{-s}}^{\frac{1}{k+s+1}} \|\nabla^{k+1}(\varrho, u)\|_{L^2}^{\frac{k+s}{k+s+1}} \le C\|\nabla^{k+1}(\varrho, u)\|_{L^2}^{\frac{k+s}{k+s+1}}. \tag{3.6}$$

Similarly, applying Lemma 4.7, for $s > 0$ and $k + s \ge 0$, we have that

$$\|\nabla^k(\varrho, u)\|_{L^2} \le C\|(\varrho, u)\|_{\dot{B}_{2,\infty}^{-s}}^{\frac{1}{k+s+1}} \|\nabla^{k+1}(\varrho, u)\|_{L^2}^{\frac{k+s}{k+s+1}} \le C\|\nabla^{k+1}(\varrho, u)\|_{L^2}^{\frac{k+s}{k+s+1}}. \tag{3.7}$$

As a consequence, from (3.6)–(3.7), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_k^{k+2}(t) + (\mathcal{E}_k^{k+2})^{1+\alpha}(t) \le 0, \tag{3.8}$$

where $\alpha = \frac{1}{k+s}, k = 0, 1, \ldots, N-2$. Solving this inequality directly, we are in a position to obtain that

$$\mathcal{E}_k^{k+2}(t) \le ((\mathcal{E}_k^{k+2}(0))^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} = C_0(1+t)^{-(k+s)}.$$

This proves the decay (1.7). Regarding (1.8), applying ∇^k to $(2.1)_3$ and multiplying the resulting identity by $\nabla^k\tau$, then integrating over $\mathbb{R}^3$, we have that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla^k\tau|^2\mathrm{d}x+\int_{\mathbb{R}^3}|\nabla^k\tau|^2\mathrm{d}x&=-\int_{\mathbb{R}^3}\nabla^k(u\cdot\nabla\tau+Q(\tau,\nabla u)+\mathrm{tr}\tau\tau)\cdot\nabla^k\tau\mathrm{d}x\\&:=\sum_{i=1}^{3}P_i.\end{aligned}\tag{3.9}$$

For the term P_1, similarly as to I_7, using (1.7), we deduce that

$$P_1\lesssim\delta(\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^k\tau\|_{L^2}^2)\lesssim\delta\|\nabla^k\tau\|_{L^2}^2+(1+t)^{-(k+1+s)}.\tag{3.10}$$

Similarly as for P_1, the term P_2 can be estimated as

$$P_2\lesssim\delta(\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^k\tau\|_{L^2}^2)\lesssim\delta\|\nabla^k\tau\|_{L^2}^2+(1+t)^{-(k+1+s)}.\tag{3.11}$$

For the term P_3, similarly as to I_9, we deduce that

$$P_3\lesssim\delta\|\nabla^k\tau\|_{L^2}^2.\tag{3.12}$$

Combining (3.10)–(3.12), we deduce from (3.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k\tau\|_{L^2}^2+\|\nabla^k\tau\|_{L^2}^2\le C(1+t)^{-(k+1+s)}.$$

This, together with Gronwall's inequality, implies (1.8).

Finally, we turn back to the proof of (1.5) and (1.6). First, we propose to prove (1.5) by Lemma 2.4. However, we are not able to prove it for all $s\in[0,\frac{3}{2}]$ at this moment, so we must distinguish the argument by the value of s. First, this is trivial for the case $s=0$, then, for $s\in(0,\frac{1}{2}]$, integrating (2.64) in time, and by (1.3), we obtain that

$$\begin{aligned}\|(\varrho,u,\tau)\|_{\dot H^{-s}}^2&\lesssim\|(\varrho_0,u_0,\tau_0)\|_{\dot H^{-s}}^2+\int_0^t\mathcal{D}_3(\upsilon)\|(\varrho,u,\tau)\|_{\dot H^{-s}}\mathrm{d}\upsilon\\&\le C(1+\sup_{0\le\upsilon\le t}\|(\varrho,u,\tau)\|_{\dot H^{-s}}).\end{aligned}\tag{3.13}$$

This, together with Cauchy's inequality, implies (1.5) for $s\in(0,\frac{1}{2}]$, and thus verifies (1.7) for $s\in(0,\frac{1}{2}]$. Next, let $s\in(\frac{1}{2},1)$, though note that the arguments for the case $s\in(0,\frac{1}{2}]$ cannot be applied to this case. However, observe that we have $(\varrho_0,u_0,\tau_0)\in\dot H^{-\frac{1}{2}}$, due to the fact that $\dot H^{-s}\cap L^2\in\dot H^{-q}$ for any $q\in[0,s]$. At this stage, from (1.7), it holds that for $k\ge0$ and $N\ge k+2$,

$$\|\nabla^k(\rho-1,u,\tau)(t)\|_{L^2}\le C_0(1+t)^{-\frac{k+\frac{1}{2}}{2}}.\tag{3.14}$$

Thus, integrating (2.65) in time for $s\in(\frac{1}{2},1)$ and applying (3.14) yields that

$$\begin{aligned}\|(\varrho,u,\tau)\|_{\dot H^{-s}}^2&\lesssim\|(\varrho_0,u_0,\tau_0)\|_{\dot H^{-s}}^2+\int_0^t\sqrt{\mathcal{D}_3(\upsilon)}\|(\varrho,u,\tau)(\upsilon)\|_{L^2}^{s-\frac{1}{2}}\\&\quad\times\|\nabla(\varrho,u,\tau)(\upsilon)\|_{L^2}^{\frac{3}{2}-s}\|(\varrho,u,\tau)(\upsilon)\|_{\dot H^{-s}}\mathrm{d}\upsilon\\&\le C(1+\sup_{0\le\upsilon\le t}\|(\varrho,u,\tau)\|_{\dot H^{-s}}\int_0^t(1+\upsilon)^{-2(1-\frac{s}{2})}\mathrm{d}\upsilon)\\&\le C(1+\sup_{0\le\upsilon\le t}\|(\varrho,u,\tau)\|_{\dot H^{-s}}).\end{aligned}\tag{3.15}$$

In the last inequality, we used the fact that $s \in (\frac{1}{2}, 1)$, so the time integral is finite. By Cauchy's inequality, this implies (1.5) for $s \in (\frac{1}{2}, 1)$. From this, we also verify (1.7) for $s \in (\frac{1}{2}, 1)$. Finally, letting $s \in [1, \frac{3}{2})$, we choose s_0 such that $s - \frac{1}{2} < s_0 < 1$. Then $(\varrho_0, u_0, \tau_0) \in \dot{H}^{-s_0}$, and from (1.7), it holds that

$$\|\nabla^k(\rho - 1, u, \tau)(t)\|_{L^2} \leq C_0(1+t)^{-\frac{k+s_0}{2}} \tag{3.16}$$

for $k \geq 0$ and $N \geq k+2$. Therefore, similarly to (3.15), using (3.16) and (2.65) for $s \in (1, \frac{3}{2})$, we conclude that

$$\begin{aligned} \|(\varrho, u, \tau)\|^2_{\dot{H}^{-s}} &\leq C(1 + \sup_{0 \leq \upsilon \leq t} \|(\varrho, u, \tau)\|_{\dot{H}^{-s}} \int_0^t (1+\upsilon)^{-(s_0 + \frac{3}{2} - s)} \mathrm{d}\upsilon) \\ &\leq C(1 + \sup_{0 \leq \upsilon \leq t} \|(\varrho, u, \tau)\|_{\dot{H}^{-s}}). \end{aligned} \tag{3.17}$$

Here, we have taken into account the fact that $s - s_0 < \frac{1}{2}$, so the time integral in (3.17) is finite. This implies (3.14) for $s \in (1, \frac{3}{2})$, and thus we have proved (1.7) for $s \in (1, \frac{3}{2})$. The rest of the proof is exactly same as above; we only need to replace Lemma 4.6 and Lemma 2.4 by Lemma 4.7 and Lemma 2.5, respectively. Then we can deduce (1.6) for $s \in (0, \frac{3}{2}]$. For the sake of brevity, we omit the details here. Thus, we have completed the proof of Theorem 1.2. □

4 Appendix: Analysis Tools

In this subsection we collect some auxiliary results. First, we will extensively use the Sobolev interpolation of the Gagliardo-Nirenberg inequality.

Lemma 4.1 ([24]) Letting $0 \leq m, \alpha \leq l$, we have that

$$\|\nabla^\alpha f\|_{L^p} \lesssim \|\nabla^m f\|^{1-\theta}_{L^q} \|\nabla^l f\|^\theta_{L^r}, \tag{4.1}$$

where $0 \leq \theta \leq 1$ and α satisfies that

$$\frac{\alpha}{3} - \frac{1}{p} = (\frac{m}{3} - \frac{1}{q})(1-\theta) + (\frac{l}{3} - \frac{1}{r})\theta.$$

Here, when $p = \infty$, we require that $0 < \theta < 1$.

We recall the following commutator estimate:

Lemma 4.2 ([15]) Letting $m \geq 1$ be an integer and defining the commutator

$$[\nabla^m, f]g = \nabla^m(fg) - f\nabla^m g,$$

we have that

$$\|[\nabla^m, f]g\|_{L^p} \lesssim \|\nabla f\|_{L^{p_1}} \|\nabla^{m-1} g\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}} \|g\|_{L^{p_4}}, \tag{4.2}$$

and for $m \geq 0$, that

$$\|\nabla^m(fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|\nabla^m g\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}} \|g\|_{L^{p_4}}, \tag{4.3}$$

where $p, p_2, p_3 \in (1, \infty)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

We now recall the following elementary but useful inequality:

Lemma 4.3 ([36]) Assume that $\|\varrho\|_{L^\infty} \leq 1$ and $p > 1$. Let $g(\varrho)$ be a smooth function of ϱ with bounded derivatives of any order. Then, for any integer $m \geq 1$, we have that

$$\|\nabla^m g(\varrho)\|_{L^p} \lesssim \|\nabla^m \varrho\|_{L^p}. \tag{4.4}$$

If $s \in [0, \frac{3}{2})$, the Hardy-Littlewood-Sobolev theorem implies the following L^p type inequality:

Lemma 4.4 ([31]) Let $0 \le s < \frac{3}{2}, 1 < p \le 2, \frac{1}{2} + \frac{s}{3} = \frac{1}{p}$. Then

$$\|f\|_{\dot{H}^{-s}} \lesssim \|f\|_{L^p}. \tag{4.5}$$

In addition, for $s \in (0, \frac{3}{2}]$, we will use the following result:

Lemma 4.5 ([12]) Let $0 < s \le \frac{3}{2}, 1 \le p < 2, \frac{1}{2} + \frac{s}{3} = \frac{1}{p}$. Then,

$$\|f\|_{\dot{B}^{-s}_{2,\infty}} \lesssim \|f\|_{L^p}. \tag{4.6}$$

We will employ the following special Sobolev interpolation:

Lemma 4.6 ([37]) Letting $s \ge 0$ and $l \ge 0$, we have that

$$\|\nabla^l f\|_{L^2} \le \|\nabla^{l+1} f\|_{L^2}^{1-\theta} \|f\|_{\dot{H}^{-s}}^{\theta}, \quad \text{where} \quad \theta = \frac{1}{l+1+s}. \tag{4.7}$$

Lemma 4.7 ([30]) Letting $s \ge 0$ and $l \ge 0$, we have that

$$\|\nabla^l f\|_{L^2} \le \|\nabla^{l+1} f\|_{L^2}^{1-\theta} \|f\|_{\dot{B}^{-s}_{2,\infty}}^{\theta}, \quad \text{where} \quad \theta = \frac{1}{l+1+s}. \tag{4.8}$$

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