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AN AVERAGING PRINCIPLE FOR STOCHASTIC DIFFERENTIAL DELAY EQUATIONS DRIVEN BY TIME-CHANGED LÉVY NOISE*

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Abstract In this paper, we aim to derive an averaging principle for stochastic differential equations driven by time-changed Lévy noise with variable delays. Under certain assumptions, we show that the solutions of stochastic differential equations with time-changed Lévy noise can be approximated by solutions of the associated averaged stochastic differential equations in mean square convergence and in convergence in probability, respectively. The convergence order is also estimated in terms of noise intensity. Finally, an example with numerical simulation is given to illustrate the theoretical result.

Key words Averaging principle; stochastic differential equation; time-changed Lévy noise; variable delays

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1 Introduction

Non-Gaussian type Lévy processes not only allow their trajectories to change continuously most of the time, but also allow jump discontinuities occurring at random times. Hence, stochastic differential equations (SDEs) driven by Lévy noise have been utilised to formulate and to analyse many practical systems arising in many branches of science and engineering (see, e.g., Applebaum [1]). At the same time, time-changed semimartingales have attracted considerable attention, and their various generalizations have been widely used to model anomalous diffusions arising in physics, finance, hydrology, and cell biology (see the recent monograph by Umarov, Hahn and Kobayashi [18]). Kobayashi [8] investigated stochastic integrals with respect to a time-changed semimartingale and derived the time-changed Itô formula for SDEs driven by a time-changed semimartingale. When the original semimartingale is a standard Brownian

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motion, then it is well known that the transition probability density of the time-changed Brownian motion satisfies a time-fractional partial differential equation (Nane and Ni [13]). This is a very interesting feature and it is very useful in modelling and describing phenomena in applied areas (Mijena and Nane [12]). SDEs driven by time-changed Lévy noise capture more flexibility in modelling, and thus have become a hot and also very important topic (see, e.g., [3, 8, 9, 14, 15]).

Meanwhile, the averaging principle provides a powerful tool in order to strike a balance between realistically complex models and comparably simpler models which are more amenable to analysis and simulation. The fundamental idea of the stochastic averaging principle is to approximate the original stochastic system by a simpler stochastic system; this was initiated by Khasminskii in the seminal work [7]. To date, the stochastic averaging principle has been developed for many more general types of stochastic differential equations (see, e.g., [4, 10, 11, 16, 17, 19, 21], just to mention a few).

Although there are many papers in the literature devoted to study of the stochastic averaging principle for stochastic differential equations with or without delays and driven by Brownian motion, fractional Brownian motion, and Lévy processes, as well as more general stochastic measures inducing semimartingales and so on (see, e.g., [16] and references therein), as we know, there has not been any consideration of an averaging principle for stochastic differential equations driven by time-changed Lévy noise with variable delays. Significantly though, due to their stochasticity, the stochastic differential equations with delays driven by time-changed Lévy processes are potentially useful and important for modelling complex systems in diverse areas of applications. A typical example is stochastic modelling for ecological systems, wherein time-changed Lévy processes as well as delay properties capture certain random but non-Markovian features and phenomena exploited in the real world (see, e.g., [2]). Compared to the classical stochastic differential equations driven by Brownian motion, fractional Brownian motion, and Lévy processes, the stochastic differential equations with delays driven by time-changed Lévy processes are much more complex, therefore, a stochastic averaging principle for such stochastic equations is naturally interesting and would also be very useful. This is what motivates the present paper, which aims to establish a stochastic averaging principle for the stochastic differential equations with delays driven by time-changed Lévy processes. The main difficulty here is that the scaling properties of the time-changed Lévy processes are intrinsically complicated, so it is difficult to construct the approximating averaging equations for the general equations. One remedy for this is to select the involved noises in a proper scaling pattern, and then to establish the averaging principle by deriving the relevant convergence for the averaging principle. In this paper, based on our delicate choice of noises, we succeed in showing that the stochastic differential equations with delays driven by time-changed Lévy processes can be approximated by the associated averaging stochastic differential equations both in mean square convergence and in convergence in probability.

Take a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\geq 0})$ satisfying the usual hypotheses of completeness and right continuity. Fix $m, n \in \mathbb{N}$. Let $B(t) = (B_1(t), B_2(t), \cdots, B_m(t))^T$ be an m-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion. Let $\{D(t), t \geq 0\}$ be a right continuous left limit increasing $\{\mathcal{F}_t\}_{t\geq 0}$-Lévy process with a Lévy symbol $1 < \alpha < 2$, called a subordinator starting from 0 with the Laplace transform $\mathbb{E}(e^{-\lambda D(t)}) = e^{-t\phi(\lambda)}, \quad \lambda > 0$, where the Laplace exponent

$\phi(\lambda)=\int_0^\infty(1-e^{-\lambda x})\mu(\mathrm{d}x)$ with a σ-finite measure μ on $(0,\infty)$ is such that $\int_0^\infty(1\wedge x)\mu(\mathrm{d}x)<\infty$. Define its generalized inverse as $E_t:=\inf\{\tau>0:D(\tau)>t\}$, which is known as the first hitting time process. The time change E_t is continuous and nondecreasing, however, it is not Markovian. The composition $B\circ E=(B_{E_t})_{t\geq 0}$, called a time-changed Brownian motion, is a square integrable martingale with respect to the natural filtration $\{\mathcal{F}_{E_t}\}_{t\geq 0}$ for the process $\{E_t\}$.

Next, recall that a Lévy measure ν on $\mathbb{R}^n\setminus\{0\}$ is a σ-finite measure satisfying $\int_{\mathbb{R}^n\setminus\{0\}}(|y|^2\wedge 1)\nu(\mathrm{d}y)<\infty$. In this paper, we specify the Lévy measure on $\mathbb{R}^n\setminus\{0\}$ by $\nu(\mathrm{d}y):=\frac{\mathrm{d}y}{|y|^{n+1}}$, let N be the $\{\mathcal{F}_t\}_{t\geq 0}$-Poisson random measure associated with ν (see, e.g., [1]), and let $\widetilde{N}(\mathrm{d}t,\mathrm{d}y):=N(\mathrm{d}t,\mathrm{d}y)-\frac{\mathrm{d}t\mathrm{d}y}{|y|^{n+1}}$ be the compensated $\{\mathcal{F}_t\}_{t\geq 0}$-martingale measure; both N and $\widetilde{N}$ are independent of the Brownian motion B. In fact, $\widetilde{N}$ is nothing but the 1-stable Lévy motion or a Cauchy process. Here we would like to point out that the selection of $\nu(\mathrm{d}y)=\frac{\mathrm{d}y}{|y|^{n+1}}$ is rather restrictive in terms of the general structure of Lévy processes (see, e.g., [1]), but it turns out that this is the only proper choice for constructing the right associated averaging stochastic differential equations in our paper.

Let $\tau>0$ and $C([-\tau,0];\mathbb{R}^n)$ be the family of continuous $\mathbb{R}^n$-valued functions φ defined on $[-\tau,0]$ with norm $\|\varphi\|=\sup\limits_{-\tau\leq\theta\leq 0}|\varphi(\theta)|$.

Motivated by the above discussion, in this short paper we want to establish an averaging principle for SDEs driven by time-changed Lévy noise with variable delays

$$\begin{aligned}\mathrm{d}x(t)&=f(t,E_t,x(t-),x(t-\delta(t)))\mathrm{d}E_t+g(t,E_t,x(t-),x(t-\delta(t)))\mathrm{d}B_{E_t}\\&\quad+\int_{|z|<c}h(t,E_t,x(t-),x(t-\delta(t)),z)\widetilde{N}(\mathrm{d}E_t,\mathrm{d}z),\ t\in[0,T],\end{aligned}\tag{1.1}$$

with the initial value $x(0)=\xi=\{\xi(\theta):-\tau\leq\theta\leq 0\}\in C([-\tau,0];\mathbb{R}^n)$ fulfilling $\xi(0)\in\mathbb{R}^n$ and $\mathbb{E}\|\xi\|^2<\infty$, where the functions $f:[0,T]\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, $g:[0,T]\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$, $h:[0,T]\times\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})\to\mathbb{R}^n$ are measurable continuous functions, $\delta:[0,T]\to[0,\tau]$, and the constant $c>0$ is the maximum allowable jump size.

The rest of the paper is organised as follows: in the next section, we will present appropriate conditions to the relevant SDEs (1.1) and briefly formulate a time-changed Gronwall's inequality in our setting for later use. Section 3 is devoted to our main results and their proofs. In Section 4, the last section, an example is given to illustrate the theoretical results in Section 3.

2 Preliminaries

In order to derive the main results of this paper, we require that the functions $f(t_1,t_2,x,y)$, $g(t_1,t_2,x,y)$ and $h(t_1,t_2,x,y,z)$ satisfy the following assumptions:

Assumption 2.1 For any $x_1,x_2,y_1,y_2\in\mathbb{R}^n$, there exists a positive bounded function $\varphi(t)$ such that

$$\begin{aligned}&|f(t_1,t_2,x_1,y_1)-f(t_1,t_2,x_2,y_2)|\vee|g(t_1,t_2,x_1,y_1)-g(t_1,t_2,x_2,y_2)|\\&\leq\varphi(t_1)(|x_1-x_2|+|y_1-y_2|),\end{aligned}\tag{2.1}$$

and

$$\int_{|z|<c}|h(t_1,t_2,x_1,y_1,z)-h(t_1,t_2,x_2,y_2,z)|^2\nu(\mathrm{d}z)\leq\varphi(t)(|x_1-x_2|^2+|y_1-y_2|^2),\tag{2.2}$$

where $|.|$ denotes the norm of $\mathbb{R}^n$, $x \vee y = \max\{x, y\}$, $\sup\limits_{0\leq t\leq T} \varphi(t) = k$ and $t \in [0, T]$.

Assumption 2.2 For all $T_1 \in [0, T]$, $x, y \in \mathbb{R}^n$, there exist several positive bounded functions $\lambda_i(T_1) \leq C_i$ such that

$$\frac{1}{T_1}\int_0^{T_1} |f(s, E_s, x, y) - \overline{f}(x, y)|\mathrm{d}E_s \leq \lambda_1(T_1)(|x| + |y|), \tag{2.3}$$

$$\frac{1}{T_1}\int_0^{T_1} |g(s, E_s, x, y) - \overline{g}(x, y)|^2\mathrm{d}E_s \leq \lambda_2(T_1)(|x|^2 + |y|^2), \tag{2.4}$$

and

$$\frac{1}{T_1}\int_0^{T_1}\int_{|z|<c} |h(s, E_s, x, y, z) - \overline{h}(x, y, z)|^2 v(\mathrm{d}z)\mathrm{d}E_s \leq \lambda_3(T_1)(|x|^2 + |y|^2), \tag{2.5}$$

where $\lim\limits_{T_1\to\infty} \lambda_i(T_1) = 0$, $i = 1, 2, 3$. $\overline{f} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\overline{g} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $\overline{h} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{Z} \to \mathbb{R}^n$ are measurable functions.

Lemma 2.3 (Time-changed Gronwall's inequality [20]) Suppose that $D(t)$ is a β-stable subordinator and that E_t is the associated inverse stable subordinator. Let $T > 0$ and $x, v : \Omega \times [0, T] \to \mathbb{R}_+$ be $\mathcal{F}_t$-measurable functions which are integrable with respect to E_t. Assume that $u_0 \geq 0$ is a constant. Then, the inequality

$$x(t) \leq u_0 + \int_0^t v(s)x(s)\mathrm{d}E_s, \quad 0 \leq t \leq T \tag{2.6}$$

implies, almost surely, that $x(t) \leq u_0 \exp(\int_0^t v(s)\mathrm{d}E_s)$, $0 \leq t \leq T$.

3 Main Results

In this section, we will study the averaging principle for stochastic differential equations driven by time-changed Lévy noise with variable delays. The standard form of equation (1.1) is

$$\begin{aligned} x^\epsilon(t) = \xi(0) &+ \int_0^t f(\frac{s}{\epsilon}, E_{\frac{s}{\epsilon}}, x^\epsilon(s-), x^\epsilon(s - \delta(s)))\mathrm{d}E_s + \int_0^t g(\frac{s}{\epsilon}, E_{\frac{s}{\epsilon}}, x^\epsilon(s-), x^\epsilon(s - \delta(s)))\mathrm{d}B_{E_s} \\ &+ \int_0^t\int_{|z|<c} h(\frac{s}{\epsilon}, E_{\frac{s}{\epsilon}}, x^\epsilon(s-), x^\epsilon(s - \delta(s)), z)\widetilde{N}(\mathrm{d}E_s, \mathrm{d}z), \end{aligned} \tag{3.1}$$

with initial value $x^\epsilon(0) = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\} \in C([-\tau, 0]; \mathbb{R}^n)$; the coefficients have the same definitions and conditions as in Equation (1.1), and $\epsilon \in (0, \epsilon_0]$ is a positive parameter with ϵ_0 being a fixed number.

According to Khasminskii type averaging principle, we consider the following averaged SDEs which correspond to the original standard form (3.1):

$$\begin{aligned} \widehat{x}(t) = \xi(0) &+ \int_0^t \overline{f}(\widehat{x}(s-), \widehat{x}(s - \delta(s)))\mathrm{d}E_s + \int_0^t \overline{g}(\widehat{x}(s-), \widehat{x}(s - \delta(s)))\mathrm{d}B_{E_s} \\ &+ \int_0^t\int_{|z|<c} \overline{h}(\widehat{x}(s-), \widehat{x}(s - \delta(s)), z)\widetilde{N}(\mathrm{d}E_s, \mathrm{d}z). \end{aligned} \tag{3.2}$$

Here the measurable functions $\overline{f}$, $\overline{g}$, $\overline{h}$ satisfy Assumption 2.2.

Theorem 3.1 Suppose that Assumptions 2.1 and 2.2 hold. Then, for a given arbitrarily small number $\delta_1 > 0$, there exist $L > 0$, $\epsilon_1 \in (0, \epsilon_0]$ and $\beta \in (0, \alpha - 1)$ such that, for any $\epsilon \in (0, \epsilon_1]$,

$$\mathbb{E}(\sup_{t\in[-\tau, L\epsilon^{-\beta}]} |x^\epsilon(t) - \widehat{x}(t)|^2) \le \delta_1.$$

Proof For any $t' \in [0, T]$, we have

$$\begin{aligned} & x^\epsilon(t') - \widehat{x}(t') \\ = & \int_0^{t'} [f(\frac{s'}{\epsilon}, E_{\frac{s'}{\epsilon}}, x^\epsilon(s'-), x^\epsilon(s' - \delta(s'))) - \overline{f}(\widehat{x}(s'-), \widehat{x}(s' - \delta(s')))]\mathrm{d}E_{s'} \\ & + \int_0^{t'} [g(\frac{s'}{\epsilon}, E_{\frac{s'}{\epsilon}}, x^\epsilon(s'-), x^\epsilon(s' - \delta(s'))) - \overline{g}(\widehat{x}(s'-), \widehat{x}(s' - \delta(s')))]\mathrm{d}B_{E_{s'}} \\ & + \int_0^{t'} \int_{|z|<c} [h(\frac{s'}{\epsilon}, E_{\frac{s'}{\epsilon}}, x^\epsilon(s'-), x^\epsilon(s' - \delta(s')), z) - \overline{h}(\widehat{x}(s'-), \widehat{x}(s' - \delta(s')), z)]\widetilde{N}(\mathrm{d}E_{s'}, \mathrm{d}z). \end{aligned} \tag{3.3}$$

Letting $s = \frac{s'}{\epsilon}, t = \frac{t'}{\epsilon}$, we can rewrite (3.3) as

$$\begin{aligned} & x^\epsilon(\epsilon t) - \widehat{x}(\epsilon t) \\ = & \epsilon^\alpha \int_0^t [f(s, E_s, x^\epsilon(s\epsilon-), x^\epsilon(s\epsilon - \delta(s\epsilon))) - \overline{f}(\widehat{x}(s\epsilon-), \widehat{x}(s\epsilon - \delta(s\epsilon)))]\mathrm{d}E_s \\ & + \epsilon^{\frac{\alpha}{2}} \int_0^t [g(s, E_s, x^\epsilon(s\epsilon-), x^\epsilon(s\epsilon - \delta(s\epsilon))) - \overline{g}(\widehat{x}(s\epsilon-), \widehat{x}(s\epsilon - \delta(s\epsilon)))]\mathrm{d}B_{E_s} \\ & + \epsilon^{\frac{\alpha}{2}} \int_0^t \int_{|z|<c} [h(s, E_s, x^\epsilon(s\epsilon-), x^\epsilon(s\epsilon - \delta(s\epsilon)), z) - \overline{h}(\widehat{x}(s\epsilon-), \widehat{x}(s\epsilon - \delta(s\epsilon)), z)]\widetilde{N}(\mathrm{d}E_s, \mathrm{d}z). \end{aligned} \tag{3.4}$$

It follows from Jensen's inequality that for any $0 < u < T$, we have

$$\begin{aligned} & \mathbb{E}(\sup_{0\le t\epsilon\le u} |x^\epsilon(\epsilon t) - \widehat{x}(\epsilon t)|^2) \\ \le & 3\epsilon^{2\alpha}\mathbb{E}(\sup_{0\le t\epsilon\le u} |\int_0^t [f(s, E_s, x^\epsilon(s\epsilon-), x^\epsilon(s\epsilon - \delta(s\epsilon))) - \overline{f}(\widehat{x}(s\epsilon-), \widehat{x}(s\epsilon - \delta(s\epsilon)))]\mathrm{d}E_s|^2) \\ & + 3\epsilon^\alpha\mathbb{E}(\sup_{0\le t\epsilon\le u} |\int_0^t [g(s, E_s, x^\epsilon(s\epsilon-), x^\epsilon(s\epsilon - \delta(s\epsilon))) - \overline{g}(\widehat{x}(s\epsilon-), \widehat{x}(s\epsilon - \delta(s\epsilon)))]\mathrm{d}B_{E_s}|^2) \\ & + 3\epsilon^\alpha\mathbb{E}(\sup_{0\le t\epsilon\le u} |\int_0^t \int_{|z|<c} [h(s, E_s, x^\epsilon(s\epsilon-), x^\epsilon(s\epsilon - \delta(s\epsilon)), z) \\ & - \overline{h}(\widehat{x}(s\epsilon-), \widehat{x}(s\epsilon - \delta(s\epsilon)), z)]\widetilde{N}(\mathrm{d}E_s, \mathrm{d}z)|^2) \\ =: & I_1 + I_2 + I_3. \end{aligned} \tag{3.5}$$

Now we present some useful estimates for $I_i, i = 1, 2, 3$. First, for the term I_1, we have

$$\begin{aligned} I_1 \le & 6\epsilon^{2\alpha}\mathbb{E}(\sup_{0\le t\epsilon\le u} |\int_0^t (f(s, E_s, x^\epsilon(s\epsilon-), x^\epsilon(s\epsilon-\delta(s\epsilon))) - f(s, E_s, \widehat{x}(s\epsilon-), \widehat{x}(s\epsilon - \delta(s\epsilon))))\mathrm{d}E_s|^2) \\ & + 6\epsilon^{2\alpha}\mathbb{E}(\sup_{0\le t\epsilon\le u} |\int_0^t (f(s, E_s, \widehat{x}(s\epsilon-), \widehat{x}(s\epsilon - \delta(s\epsilon))) - \overline{f}(\widehat{x}(s\epsilon-), \widehat{x}(s\epsilon - \delta(s\epsilon))))\mathrm{d}E_s|^2) \\ =: & I_{11} + I_{12}. \end{aligned}$$

By Assumption 2.1, Jensen's inequality and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
I_{11} = & 6\epsilon^{2\alpha}\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}\Big|\int_0^t (f(s,E_s,x^\epsilon(s\epsilon-),x^\epsilon(s\epsilon-\delta(s\epsilon)))\\
& - f(s,E_s,\widehat{x}(s\epsilon-),\widehat{x}(s\epsilon-\delta(s\epsilon))))\mathrm{d}E_s\Big|^2\Big)\\
\le & 6\epsilon^{2\alpha}\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}\Big|\int_0^t \varphi(s)(|x^\epsilon(s\epsilon-)-\widehat{x}(s\epsilon-)|+|x^\epsilon(s\epsilon-\delta(s\epsilon))-\widehat{x}(s\epsilon-\delta(s\epsilon))|)\mathrm{d}E_s\Big|^2\Big)\\
\le & 12\epsilon^{2\alpha}\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}\Big(\Big|\int_0^t \varphi(s)|x^\epsilon(s\epsilon-)-\widehat{x}(s\epsilon-)|\mathrm{d}E_s\Big|^2\\
& + \Big|\int_0^t \varphi(s)|x^\epsilon(s\epsilon-\delta(s\epsilon))-\widehat{x}(s\epsilon-\delta(s\epsilon))|\mathrm{d}E_s\Big|^2\Big)\Big)\\
\le & 12\epsilon^{2\alpha}k^2E_T\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}\Big(\int_0^t |x^\epsilon(s\epsilon-)-\widehat{x}(s\epsilon-)|^2\mathrm{d}E_s\\
& + \int_0^t |x^\epsilon(s\epsilon-\delta(s\epsilon))-\widehat{x}(s\epsilon-\delta(s\epsilon))|^2\mathrm{d}E_s\Big)\Big)\\
\le & 12\epsilon^{2\alpha}k^2E_T\Big(\int_0^{\frac{u}{\epsilon}}\mathbb{E}\Big(\sup_{0\le r\le s}|x^\epsilon(r\epsilon)-\widehat{x}(\epsilon r)|^2\Big)\mathrm{d}E_s\\
& + \int_0^{\frac{u}{\epsilon}}\mathbb{E}\Big(\sup_{0\le r\le s}|x^\epsilon(r\epsilon-\delta(r\epsilon))-\widehat{x}(r\epsilon-\delta(r\epsilon))|^2\Big)\mathrm{d}E_s\Big).
\end{aligned}\tag{3.6}
$$

By Assumption 2.2, we can get

$$
\begin{aligned}
I_{12} = & 6\epsilon^{2\alpha}\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}\Big|\int_0^t (f(s,E_s,\widehat{x}(s\epsilon-),\widehat{x}(s\epsilon-\delta(s\epsilon)))-\overline{f}(\widehat{x}(s\epsilon-),\widehat{x}(s\epsilon-\delta(s\epsilon))))\mathrm{d}E_s\Big|^2\Big)\\
\le & 6\epsilon^{2\alpha}\sup_{0\le t\epsilon\le u}\Big\{t^2\lambda_1^2(t)\mathbb{E}\Big((\sup_{0\le s\le t}|\widehat{x}(s\epsilon)|+\sup_{0\le s\le t}|\widehat{x}(s\epsilon-\delta(s\epsilon))|)^2\Big)\Big\}\\
\le & 12\epsilon^{2\alpha}\sup_{0\le t\epsilon\le u}\Big\{t^2\lambda_1^2(t)\mathbb{E}\Big(\sup_{0\le s\le t}|\widehat{x}(s\epsilon)|^2+\sup_{0\le s\le t}|\widehat{x}(s\epsilon-\delta(s\epsilon))|^2\Big)\Big\}\\
\le & 12\epsilon^{2\alpha-2}u^2C_1^2\mathbb{E}\Big\{\Big(\sup_{0\le s\le \frac{u}{\epsilon}}|\widehat{x}(\epsilon s)|^2+\sup_{0\le s\le \frac{u}{\epsilon}}|\widehat{x}(s\epsilon-\delta(s\epsilon))|^2\Big)\Big\}.
\end{aligned}\tag{3.7}
$$

Second, for the term I_2, we have

$$
\begin{aligned}
I_2 = & 3\epsilon^{\alpha}\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}\Big|\int_0^t [(g(s,E_s,x^\epsilon(s\epsilon-),x^\epsilon(s\epsilon-\delta(s\epsilon)))-g(s,E_s,\widehat{x}(s\epsilon-),\widehat{x}(s\epsilon-\delta(s\epsilon))))\\
& + (g(s,E_s,\widehat{x}(s\epsilon-),\widehat{x}(s\epsilon-\delta(s\epsilon)))-\overline{g}(\widehat{x}(s\epsilon-),\widehat{x}(s\epsilon-\delta(s\epsilon))))]\mathrm{d}B_{E_s}\Big|^2\Big)\\
\le & 6\epsilon^{\alpha}\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}\Big|\int_0^t (g(s,E_s,x^\epsilon(s\epsilon-),x^\epsilon(s\epsilon-\delta(s\epsilon)))\\
& - g(s,E_s,\widehat{x}(s\epsilon-),\widehat{x}(s\epsilon-\delta(s\epsilon))))\mathrm{d}B_{E_s}\Big|^2\Big)\\
& + 6\epsilon^{\alpha}\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}\Big|\int_0^t (g(s,E_s,\widehat{x}(s\epsilon-),\widehat{x}(s\epsilon-\delta(s\epsilon)))-\overline{g}(\widehat{x}(s\epsilon-),\widehat{x}(s\epsilon-\delta(s\epsilon))))\mathrm{d}B_{E_s}\Big|^2\Big)\\
=: & I_{21}+I_{22}.
\end{aligned}
$$

By Assumption 2.1 and the Burkholder-Davis-Gundy inequality (Jin and Kobayashi [6]), we have

$$\begin{aligned}I_{21}&=6\epsilon^{\alpha}\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}\Big|\int_0^t(g(s,E_s,x^{\epsilon}(s\epsilon-),x^{\epsilon}(s\epsilon-\delta(s\epsilon)))\\&\quad-g(s,E_s,\widehat{x}(s\epsilon-),\widehat{x}(s\epsilon-\delta(s\epsilon))))\mathrm{d}B_{E_s}\Big|^2\Big)\\&\le 6\epsilon^{\alpha}k^2b_2\mathbb{E}\Big(\int_0^{\frac{u}{\epsilon}}(|x^{\epsilon}(\epsilon t-)-\widehat{x}(t\epsilon-)|+|x^{\epsilon}(t\epsilon-\delta(t\epsilon))-\widehat{x}(t\epsilon-\delta(t\epsilon))|)^2\mathrm{d}E_t\Big)\\&\le 12\epsilon^{\alpha}k^2b_2\Big(\int_0^{\frac{u}{\epsilon}}\mathbb{E}\Big(\sup_{0\le r\le s}|x^{\epsilon}(r\epsilon)-\widehat{x}(r\epsilon)|^2\Big)\mathrm{d}E_s\\&\quad+\int_0^{\frac{u}{\epsilon}}\mathbb{E}\Big(\sup_{0\le r\le s}|x^{\epsilon}(r\epsilon-\delta(r\epsilon))-\widehat{x}(r-\delta(r))|^2\Big)\mathrm{d}E_s\Big),\end{aligned}\tag{3.8}$$

where the positive constant b_2 comes from [6]. According to Assumption 2.2 and the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}I_{22}&=6\epsilon^{\alpha}\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}\Big|\int_0^t(g(s,E_s,\widehat{x}(s\epsilon-),\widehat{x}(s-\delta(s)))-\overline{g}(\widehat{x}(s\epsilon-),\widehat{x}(s-\delta(s))))\mathrm{d}B_{E_s}\Big|^2\Big)\\&\le 6\epsilon^{\alpha}b_2\mathbb{E}\Big(\int_0^{\frac{u}{\epsilon}}|g(s,E_s,\widehat{x}(s\epsilon-),\widehat{x}(s-\delta(s)))-\overline{g}(\widehat{x}(s\epsilon-),\widehat{x}(s-\delta(s)))|^2\mathrm{d}E_s\Big)\\&\le 6\epsilon^{\alpha-1}b_2C_2\mathbb{E}\Big(\sup_{0\le s\le\frac{u}{\epsilon}}|\widehat{x}(s\epsilon)|^2+\sup_{0\le s\le\frac{u}{\epsilon}}|\widehat{x}(s\epsilon-\delta(s\epsilon))|^2\Big).\end{aligned}\tag{3.9}$$

Finally, for the term I_3, by Doob's martingale inequality and Itô isometry, we have

$$\begin{aligned}I_3&=3\epsilon^{\alpha}\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}\Big|\int_0^t\int_{|z|<c}[h(s,E_s,x^{\epsilon}(s\epsilon-),x^{\epsilon}(s\epsilon-\delta(s\epsilon)),z)\\&\quad-\overline{h}(\widehat{x}(s\epsilon-),\widehat{x}(s-\delta(s)),z)]\widetilde{N}(\mathrm{d}E_s,\mathrm{d}z)\Big|^2\Big)\\&\le 12\epsilon^{\alpha}\mathbb{E}\Big|\int_0^{\frac{u}{\epsilon}}\int_{|z|<c}[h(s,E_s,x^{\epsilon}(s\epsilon-),x^{\epsilon}(s\epsilon-\delta(s\epsilon)),z)\\&\quad-\overline{h}(\widehat{x}(s\epsilon-),\widehat{x}(s-\delta(s)),z)]\widetilde{N}(\mathrm{d}E_s,\mathrm{d}z)\Big|^2\\&\le 24\epsilon^{\alpha}\mathbb{E}\int_0^{\frac{u}{\epsilon}}\int_{|z|<c}|h(s,E_s,x^{\epsilon}(s\epsilon-),x^{\epsilon}(s\epsilon-\delta(s\epsilon)),z)\\&\quad-h(s,E_s,\widehat{x}(s\epsilon-),\widehat{x}(s-\delta(s)),z)|^2v(\mathrm{d}z)\mathrm{d}E_s\\&\quad+24\epsilon^{\alpha}\mathbb{E}\int_0^{\frac{u}{\epsilon}}\int_{|z|<c}|h(s,E_s,\widehat{x}(s\epsilon-),\widehat{x}(s-\delta(s)),z)\\&\quad-\overline{h}(\widehat{x}(s\epsilon-),\widehat{x}(s-\delta(s)),z)|^2v(\mathrm{d}z)\mathrm{d}E_s\\&=:I_{31}+I_{32}.\end{aligned}$$

By Assumption 2.1, we have

$$\begin{aligned}I_{31}&\le 24\epsilon^{\alpha}\mathbb{E}\int_0^{\frac{u}{\epsilon}}\varphi(s)(|x^{\epsilon}(s\epsilon-)-\widehat{x}(s\epsilon-)|^2+|x^{\epsilon}(s\epsilon-\delta(s\epsilon))-\widehat{x}(s-\delta(s))|^2)\mathrm{d}E_s\\&\le 24\epsilon^{\alpha}k\Big(\int_0^{\frac{u}{\epsilon}}\mathbb{E}\Big(\sup_{0\le r\le s}|x^{\epsilon}(r\epsilon)-\widehat{x}(r\epsilon)|^2\Big)\mathrm{d}E_s\\&\quad+\int_0^{\frac{u}{\epsilon}}\mathbb{E}\Big(\sup_{0\le r\le s}|x^{\epsilon}(r\epsilon-\delta(r\epsilon))-\widehat{x}(r-\delta(r))|^2\Big)\mathrm{d}E_s\Big).\end{aligned}\tag{3.10}$$

By Assumption 2.2, we have

$$\begin{aligned} I_{32} &= 24\epsilon^{\alpha}\mathbb{E}\int_0^{\frac{u}{\epsilon}}\int_{|z|<c}|h(s,E_s,\widehat{x}(s\epsilon-),\widehat{x}(s\epsilon-\delta(s\epsilon)),z) \\ &\quad -\overline{h}(\widehat{x}(s\epsilon-),\widehat{x}(s\epsilon-\delta(s\epsilon)),z)|^2 v(\mathrm{d}z)\mathrm{d}E_s \\ &\le 24\epsilon^{\alpha-1}uC_3\mathbb{E}\Big(\sup_{0\le s\le\frac{u}{\epsilon}}|\widehat{x}(s\epsilon)|^2+\sup_{0\le s\le\frac{u}{\epsilon}}|\widehat{x}(s\epsilon-\delta(s\epsilon))|^2\Big). \end{aligned} \tag{3.11}$$

Consequently, combining (3.6)-(3.11), we have

$$\begin{aligned} &\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}|x^{\epsilon}(t\epsilon)-\widehat{x}(t\epsilon)|^2\Big) \\ &\le\Big(12\epsilon^{2\alpha-2}u^2C_1^2+6\epsilon^{\alpha-1}b_2uC_2+24\epsilon^{\alpha-1}uC_3\Big)\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}|\widehat{x}(\epsilon t)|^2+\sup_{0\le t\epsilon\le u}|\widehat{x}(t\epsilon-\delta(t\epsilon))|^2\Big) \\ &\quad+(12\epsilon^{2\alpha}k^2E_T+12\epsilon^{2\alpha}k^2b_2+24\epsilon^{\alpha}k)\Big(\int_0^{\frac{u}{\epsilon}}\mathbb{E}\Big(\sup_{0\le r\le s}|x^{\epsilon}(\epsilon r)-\widehat{x}(\epsilon r)|^2\Big)\mathrm{d}E_s \\ &\quad+\int_0^{\frac{u}{\epsilon}}\mathbb{E}\Big(\sup_{0\le r\le s}|x^{\epsilon}(r\epsilon-\delta(r\epsilon))-\widehat{x}(r\epsilon-\delta(r\epsilon))|^2\Big)\mathrm{d}E_s\Big). \end{aligned} \tag{3.12}$$

Set

$$\Lambda\Big(\frac{u}{\epsilon}\Big):=\mathbb{E}\Big(\sup_{0\le t\le\frac{u}{\epsilon}}|x^{\epsilon}(t\epsilon)-\widehat{x}(t\epsilon)|^2\Big).$$

Observe that $\mathbb{E}(\sup\limits_{-\tau\le t\le 0}|x_{\epsilon}(t)-\widehat{x}(t)|^2)=0$. Then, we have

$$\mathbb{E}\Big(\sup_{0\le r\le s}|x^{\epsilon}(r\epsilon-\delta(r\epsilon))-\widehat{x}(r\epsilon-\delta(r\epsilon))|^2\Big)=\Lambda(s-\delta(s)). \tag{3.13}$$

Thus, inequality (3.12) can be reformulated as follows:

$$\begin{aligned} \Lambda(\frac{u}{\epsilon}) &\le\Big(12\epsilon^{2\alpha-2}u^2C_1^2+6\epsilon^{\alpha-1}b_2uC_2+24\epsilon^{\alpha-1}uC_3\Big) \\ &\quad\times\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}|\widehat{x}(t\epsilon)|^2+\sup_{0\le t\epsilon\le u}|\widehat{x}(t\epsilon-\delta(t\epsilon))|^2\Big) \\ &\quad+(12\epsilon^{2\alpha}k^2E_T+12\epsilon^{2\alpha}k^2b_2+24\epsilon^{\alpha}k)\Big(\int_0^{\frac{u}{\epsilon}}\Lambda(s)\mathrm{d}E_s+\int_0^{\frac{u}{\epsilon}}\Lambda(s-\delta(s))\mathrm{d}E_s\Big). \end{aligned} \tag{3.14}$$

Next, we let $\Theta(u):=\sup\limits_{\theta\in[-\tau,u]}\Lambda(\theta)$, for every $u\in[0,T]$, then $\Lambda(s)\le\Theta(s)$ and $\Lambda(s-\delta(s))\le\Theta(s)$. Thus,

$$\begin{aligned} \Lambda(\frac{u}{\epsilon}) &\le\Big(12\epsilon^{2\alpha-2}u^2C_1^2+6\epsilon^{\alpha-1}b_2uC_2+24\epsilon^{\alpha-1}uC_3\Big) \\ &\quad\times\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}|\widehat{x}(t\epsilon)|^2+\sup_{0\le t\epsilon\le u}|\widehat{x}(t\epsilon-\delta(t\epsilon))|^2\Big) \\ &\quad+2(12\epsilon^{2\alpha}k^2E_T+12\epsilon^{2\alpha}k^2b_2+24\epsilon^{\alpha}k)\int_0^{\frac{u}{\epsilon}}\Theta(s)\mathrm{d}E_s. \end{aligned} \tag{3.15}$$

Then,

$$\begin{aligned} \Theta(\frac{u}{\epsilon}) &=\sup_{\theta\in[-\tau,\frac{u}{\epsilon}]}\Lambda(\theta)\le\max\Big\{\sup_{\theta\in[-\tau,0]}\Lambda(\theta),\sup_{\theta\in[0,\frac{u}{\epsilon}]}\Lambda(\theta)\Big\} \\ &\le\Big(12\epsilon^{2\alpha-2}u^2C_1^2+6\epsilon^{\alpha-1}b_2uC_2+24\epsilon^{\alpha-1}uC_3\Big) \\ &\quad\times\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}|\widehat{x}(t\epsilon)|^2+\sup_{0\le t\epsilon\le u}|\widehat{x}(t\epsilon-\delta(t\epsilon))|^2\Big) \end{aligned}$$

$$+ 2(12\epsilon^{2\alpha}k^2E_T + 12\epsilon^{2\alpha}k^2b_2 + 24\epsilon^{\alpha}k)\int_0^{\frac{u}{\epsilon}}\Theta(s)\mathrm{d}E_s. \tag{3.16}$$

By using the time-changed Gronwall's inequality, we get

$$\Theta(\frac{u}{\epsilon}) \le \Big(12\epsilon^{2\alpha-2}u^2C_1^2 + 6\epsilon^{\alpha-1}b_2uC_2 + 24\epsilon^{\alpha-1}uC_3\Big) \times \mathbb{E}\Big(\sup_{0\le t\epsilon\le u}|\widehat{x}(t\epsilon)|^2 + \sup_{0\le t\epsilon\le u}|\widehat{x}(t\epsilon-\delta(t\epsilon))|^2\Big)e^{2(12\epsilon^{2\alpha}k^2E_T+12\epsilon^{2\alpha}k^2b_2+24\epsilon^{\alpha}k)E_{\frac{u}{\epsilon}}}. \tag{3.17}$$

Furthermore, we have

$$\mathbb{E}\Big(\sup_{0\le t\epsilon\le u}|x^{\epsilon}(t\epsilon)-\widehat{x}(t\epsilon)|^2\Big) \le \Big(12\epsilon^{2\alpha-2}u^2C_1^2 + 6\epsilon^{\alpha-1}b_2uC_2 + 24\epsilon^{\alpha-1}uC_3\Big) \times \mathbb{E}\Big(\sup_{0\le t\epsilon\le u}|\widehat{x}(t\epsilon)|^2 + \sup_{0\le t\epsilon\le u}|\widehat{x}(t\epsilon-\delta(t\epsilon))|^2\Big) \times e^{2(12\epsilon^{\alpha}k^2E_T+12\epsilon^{\alpha}k^2b_2+24k)E_T}. \tag{3.18}$$

Select $\beta\in(0,\alpha-1)$ and $L>0$ such that, for any $t\in[0,L\epsilon^{-\beta-1}]\subseteq[0,\frac{T}{\epsilon}]$, we have

$$\mathbb{E}\Big(\sup_{0\le t\epsilon\le L\epsilon^{-\beta}}|x^{\epsilon}(t\epsilon)-\widehat{x}(t\epsilon)|^2\Big) \le \xi\epsilon^{\alpha-\beta-1}, \tag{3.19}$$

where we have the constant

$$\xi := \Big(12L^2\epsilon^{\alpha-\beta-1}C_1^2 + 6b_2LC_2 + 24LC_3\Big) \times \mathbb{E}\Big(\sup_{0\le t\epsilon\le L\epsilon^{-\beta}}|\widehat{x}(t\epsilon)|^2 + \sup_{0\le t\epsilon\le L\epsilon^{-\beta}}|\widehat{x}(t\epsilon-\delta(t\epsilon))|^2\Big)e^{2(12\epsilon^{\alpha}k^2E_T+12\epsilon^{\alpha}k^2b_2+24k)E_T}.$$

Consequently, for any given $\delta_1>0$, there exists a $\epsilon_1\in(0,\epsilon_0]$ such that, for each $\epsilon\in(0,\epsilon_1]$ and $t\in[-\tau,L\epsilon^{-\beta}]$,

$$\mathbb{E}\Big(\sup_{-\tau\le t\le L\epsilon^{-\beta}}|x^{\epsilon}(t)-\widehat{x}(t)|^2\Big) \le \delta_1. \tag{3.20}$$

This completes the proof. □

Remark 3.2 We would like to point out that the classical stochastic averaging principle for SDEs driven by Brownian motion deals with the time interval $[0,\epsilon^{-1}]$, while what we have discussed here was a strictly shorter time horizon $[0,\epsilon^{-\beta}]\subset[0,\epsilon^{-1}]$ for $\beta\in(0,\alpha-1)$. In other words, the order of convergence here is $\epsilon^{-\beta}$, which is weaker than the classical order of convergence ϵ^{-1}. Thus, our averaging principle is a weaker averaging principle. This weaker type averaging principle has been examined for various SDEs by many authors. Essentially, this is due to the fact that the regularity of trajectories of the solutions of SDEs with more general noises is weaker than that of the solutions of SDEs driven by Brownian motion. It is clear that the classical averaging principle for our equation cannot be derived by the method we used here. Of course, to establish a classical averaging principle for our equation is interesting but challenging, so one needs to seek an entirely new approach. We postpone this task to a future work.

4 Example

We consider the stochastic differential equations driven by time-changed Lévy noise with time-delays:

$$\mathrm{d}x_\epsilon(t) = \epsilon^\alpha(x_\epsilon \cos^2(E_t) - E_t x_\epsilon \sin(E_t - 1))\mathrm{d}E_t + \epsilon^{\frac{\alpha}{2}}\lambda \mathrm{d}B_{E_t} + \epsilon^{\frac{\alpha}{2}}\int_{|z|<c} 1\widetilde{N}(\mathrm{d}E_t, \mathrm{d}z) \tag{4.1}$$

for $t \in [0, T]$, with initial value $x_\epsilon(t) = 1 + t$, $t \in [-1, 0]$, $v(z)\mathrm{d}z = |z|^{-2}$ and $\lambda \in \mathbb{R}$; here

$$f(t, E_t, x_\epsilon(t), x_\epsilon(t-\tau)) = x_\epsilon \cos^2(E_t) - E_t x_\epsilon \sin(E_t - 1),$$
$$g(t, E_t, x_\epsilon(t), x_\epsilon(t-\tau)) = \lambda, \quad h(t, E_t, x_\epsilon(t), x_\epsilon(t-\tau), z) = 1.$$

Let

$$\begin{aligned}\overline{f}(\widehat{x}(s), \widehat{x}(s-\tau)) &= \int_0^1 f(t, E_t, x_\epsilon(t), x_\epsilon(t-\tau))\mathrm{d}E_t \\ &= \Big(\frac{1}{2}E_1 + \frac{\sin 2E_1}{4} + E_1\cos(E_1 - 1) - \sin(E_1 - 1)\Big)x_\epsilon,\end{aligned}$$

and

$$\overline{g}(\widehat{x}(s), \widehat{x}(s-\tau)) = \lambda, \quad \overline{h}(\widehat{x}(s), \widehat{x}(s-\tau), z) = 1.$$

We have the following corresponding averaged stochastic differential equations driven by time-changed Lévy noise with variable delays:

$$\begin{aligned}\mathrm{d}\widehat{x}(t) = \epsilon^\alpha\Big(\frac{1}{2}E_1 + \frac{\sin 2E_1}{4} + E_1\cos(E_1 - 1) - \sin(E_1 - 1)\Big)\widehat{x}\mathrm{d}E_t \\ + \epsilon^{\frac{\alpha}{2}}\lambda \mathrm{d}B_{E_t} + \epsilon^{\frac{\alpha}{2}}\int_{|z|<c} 1\widetilde{N}(\mathrm{d}E_t, \mathrm{d}z).\end{aligned} \tag{4.2}$$

Define the error $E_{rr} = [|x_\epsilon(t) - \overline{x}_\epsilon(t)|^2]^{\frac{1}{2}}$. We carry out the numerical simulation to get the solutions (4.1) and (4.2) under the conditions that $\alpha = 1.2, \epsilon = 0.001$, $\lambda = 1$ and $\alpha = 1.2$, $\epsilon = 0.001$, and $\lambda = -1$ (Figure 1 and Figure 2). One can see a good agreement between the solutions of the original equation and the averaged equation.

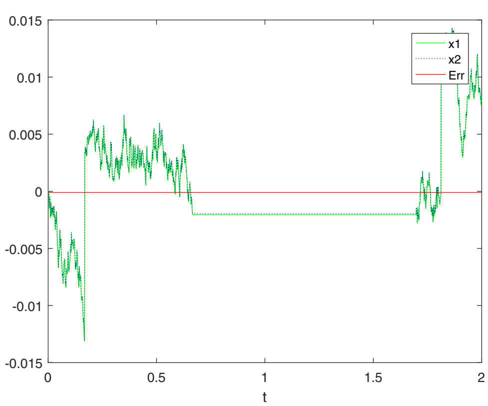

Figure 1 Comparison of the original solution $x_\epsilon(t)$ with the averaged solution $\widehat{x}(t)$ with $\epsilon = 0.001$, $\lambda = 1$

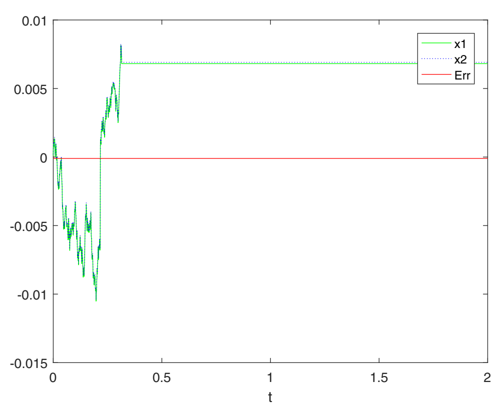

Figure 2 Comparison of the original solution $x_\epsilon(t)$ with the averaged solution $\widehat{x}(t)$ with $\epsilon = 0.001$, $\lambda = -1$

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