Acta Mathematica Scientia, 2021, **41B**(6): 2086–2106
https://doi.org/10.1007/s10473-021-0617-z

Mathematica Acta Scientia
数学物理学报
http://actams.apm.ac.cn

ON THETA-TYPE FUNCTIONS IN THE FORM $(x;q)_\infty$*

Dedicated to the memory of Professor Jiarong YU

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Abstract As in our previous work [14], a function is said to be of theta-type when its asymptotic behavior near any root of unity is similar to what happened for Jacobi theta functions. It is shown that only four Euler infinite products have this property. That this is the case is obtained by investigating the analyticity obstacle of a Laplace-type integral of the exponential generating function of Bernoulli numbers.

Key words q-series; Mock theta-functions; Stokes phenomenon; continued fractions

2010 MR Subject Classification 34M30; 33E30; 30E15

1 Introduction

In his last letter to Hardy, Ramanujan wrote that he had discovered very interesting functions that he called mock ϑ-functions. As was said in Watson's L.M.S. presidential address [10], the first three pages, where Ramanujan explained what he meant by "mock ϑ-functions", are very obscure. Therefore, Watson quoted the following comment of Hardy:

A mock ϑ-function is a function defined by a q-series convergent when $|q| < 1$, for which we can calculate asymptotic formulae, when q tends to a "rational point" $\mathrm{e}^{2r\pi\mathrm{i}/s}$ of the unit circle, of the same degree of precision as those furnished for the ordinary ϑ-functions by the theory of linear transformation.

In our previous work [14], we proposed definitions of what we call theta-type, false theta-type and mock theta-type functions, following directly from the above-mentioned comment of Hardy. The main goal of this paper is to determine the possible values of x for which the Euler q-exponential function $(x;q)_\infty$ is of theta-type, where

$$(x;q)_\infty = \prod_{n\geq 0}(1 - x\,q^n)\,. \tag{1.1}$$

*Received April 6, 2021; revised September 24, 2021. The author was supported by Labex CEMPI (Centre Européen pour les Mathématiques, la Physique et leurs Interaction).

1.1 Statement of Main Theorem

Let $\zeta = \mathrm{e}^{2\pi \mathrm{i} r}= e(r)$ be any root of unity, with $r \in \mathbb{Q}$. As in [14], a function $f(q)$ is said to be of theta-type as $q \xrightarrow{a.r.} \zeta$, and one writes $f \in \mathfrak{T}_\zeta$, if there exists a quadruplet $(\upsilon, \lambda, I, \gamma)$, composed of a couple $(\upsilon, \lambda) \in \mathbb{Q}\times\mathbb{R}$, a strictly increasing sequence $I \subset \mathbb{R}$ and a $\mathbb{C}^*$-valued map γ defined on I, such that the following relation holds for any $N \in \mathbb{Z}_{\geq 0}$ as $\tau \xrightarrow{a.v.} r$:

$$f(q) = \left(\frac{\mathrm{i}}{\hat\tau}\right)^{\upsilon} e(\lambda\hat\tau) \left(\sum_{k\in I\cap(-\infty,N]} \gamma(k)\, q_1^k + o(q_1^N)\right). \tag{1.2}$$

Here and in what follows, $q = \mathrm{e}^{2\pi\mathrm{i}\tau} = e(\tau)$, $\Im\tau > 0$, $\hat\tau = \tau - r$ and $q_1 = e(-\frac{1}{\hat\tau})$. The symbol "$q \xrightarrow{a.r.} \zeta$" could be read as "$q$ almost radially converges to the root of unity ζ". At the same time, "$\tau \xrightarrow{a.v.} r$" means that "$\tau$ almost vertically converges to the rational point r"; see §1.2 (ii), below.

The above-named variable q_1 may be considered as the modular variable with respect to the root ζ. By considering the respective modular formulae, one can easily see that the ordinary ϑ-functions, as well as the famous Dedekind η-function, satisfy the condition in (1.2) for any root of unity ζ.

Let $\mathbb{U}$ denote the set of the roots of unity. One remembers that $\eta(\tau) = q^{1/24}\,(q;q)_\infty$, where $q = e(\tau)$. Thus, one can observe that $(q;q)_\infty \in \mathfrak{T}_\zeta$ for any $\zeta \in \mathbb{U}$. Furthermore, an elementary calculation shows that the following identities hold:

$$(-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q)_\infty}, \quad (\sqrt{q};q)_\infty = \frac{(\sqrt{q};\sqrt{q})_\infty}{(q;q)_\infty}$$

and

$$(-\sqrt{q};q)_\infty = \frac{(q;q)_\infty^2}{(\sqrt{q};\sqrt{q})_\infty\,(q^2;q^2)_\infty}.$$

As one may check directly from (1.2), the set $\cap_{\zeta\in\mathbb{U}}(\mathfrak{T}_\zeta \setminus \{0\})$ constitutes a multiplicative group which is stable by the ramification operator $q \mapsto q^\nu$ for all $\nu \in \mathbb{Q}_{>0}$. Therefore, one obtains from the above the following property:

Remark 1.1 Given $x \in \{q, -q, \sqrt{q}, -\sqrt{q}\}$, one has $(x;q)_\infty \in \mathfrak{T}_\zeta$ for all $\zeta \in \mathbb{U}$.

The following result can be viewed as being converse to the above statement:

Theorem 1.2 (Main theorem) Let $(x_0, \beta) \in \mathbb{C}\times\mathbb{R}$ be such that $|x_0| = 1$ and $\beta \neq 0$, and consider $x = x_0\,q^\beta$. Then, the following conditions are equivalent:

(1) one has $(x;q)_\infty \in \mathfrak{T}_1$, with $\zeta = 1 = e(0)$;

(2) there exists a root of unity $\zeta = \mathrm{e}^{2\pi\mathrm{i}r}= e(r)$ such that $(x;q) \in \mathfrak{T}_\zeta$;

(3) one has $x \in \{q, -q, \sqrt{q}, -\sqrt{q}\}$.

It should be noted that the four Euler q-exponential functions obtained in Theorem 1.2 are intimately linked to the quadruplet $(\phi(q), \psi(q), f(-q), \chi(q))$, situated at the heart of the Ramanujan's theory about theta-functions and modular equations [2, Ch. 16]. Indeed, one can observe that $f(-q) = (q;q)_\infty$, $\chi(q) = (-q;q^2)_\infty$ and, furthermore, that

$$\phi(q) = \frac{(-q;q^2)_\infty\,(q^2;q^2)_\infty}{(q;q^2)_\infty\,(-q^2;q^2)_\infty}, \quad \psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

In particular, these relations imply that

$$(q;q^2)_\infty = \frac{f(-q^2)}{\psi(q)}, \quad (-q^2;q^2)_\infty = \frac{\chi(q)\,\psi(q)}{\phi(q)}.$$

1.2 Some Ideas for the Proof of Main Theorem

The following notational conventions will be used through the whole paper:

(i) For simplicity, we will write $e(z) = \mathrm{e}^{2\pi \mathrm{i} z}$ for all $z \in \mathbb{C}$, and this gives, in particular, a map $\tau \mapsto q = e(\tau)$ from the Poincaré half-plane $\mathcal{H}$ onto the unit disc $\mathbb{D}$. Furthermore, one has $e(\mathbb{Q} \cap [0,1)) = \mathbb{U}$, where $\mathbb{U}$ denotes the set of the roots of unity.

(ii) Let $\zeta = e(r) \in \mathbb{U}$ and $r \in \mathbb{Q} \cap [0,1)$. One writes $q \xrightarrow{a.r.} \zeta$ if there exists $\epsilon \in (0, \frac{\pi}{2})$ such that $q \to \zeta$ in the sector $|\arg(q - \zeta) + r| < \epsilon$ inside the unit disc $\mathbb{D}$. Similarly, one writes $\tau \xrightarrow{a.v.} r$ if there exists $\epsilon \in (0, \frac{\pi}{2})$ such that $\tau \to r$ in the sector $|\arg(\tau - r) - \frac{\pi}{2}| < \epsilon$ in the upper half-plane $\mathcal{H}$.

(iii) Given $\zeta = e(r) \in \mathbb{U}$, one says that f is exponentially small as $q \xrightarrow{a.r.} \zeta$ or $\tau \xrightarrow{a.v.} r$, and one writes $f \in \mathcal{A}_\zeta^{\leq -1}$ if there exists $(C, \kappa) \in \mathbb{R}^2_{>0}$ such that $|f(q)| \leq C\,\mathrm{e}^{-\kappa/|\tau - r|}$ for all $q = e(\tau)$ in some sector $\{|\tau - r| < \rho, |\arg(\tau - r) - \frac{\pi}{2}| < \epsilon\}$, where $\epsilon \in (0, \frac{\pi}{2})$ and $\rho > 0$.

(iv) For any given $(z, \tau) \in \mathbb{C} \times \mathcal{H}$, we set $(z\,|\,\tau)_0 = (x;q)_0 = 1$, $(z\,|\,\tau)_\infty = (x;q)_\infty$ and, for $N \in \mathbb{Z}_{>0}$,

$$(z\,|\,\tau)_N = (x;q)_N = \prod_{n=0}^{N-1} (1 - x\,q^n)\,, \tag{1.3}$$

where $x = e(z)$ and $q = e(\tau)$. This is in line with $(x;q)_\infty$ given in (1.1).

Letting $k = \min I$ and $c = \gamma(k)$ in (1.2) implies that

$$f(q) = c\left(\frac{\mathrm{i}}{\hat{\tau}}\right)^{\upsilon} e\left(\lambda\,\hat{\tau} - \frac{k}{\hat{\tau}}\right)(1 + f_1(q))\,, \quad f_1 \in \mathcal{A}_\zeta^{\leq -1}\,.$$

If one takes the principal branch of the logarithm for both members of the above equation, one can observe the following fact:

Remark 1.3 (Asymptotic form of the theta-type functions) Given $\zeta \in \mathbb{U}$ and $f(q) \in \mathfrak{T}_\zeta$, there exists a quadruplet $(\upsilon, c_\infty, c_0, c_1) \in \mathbb{Q} \times (\mathrm{i}\mathbb{R}) \times \mathbb{C} \times (\mathrm{i}\mathbb{R})$ such that

$$\log f(q) = \upsilon\,\log\frac{\mathrm{i}}{\hat{\tau}} + \frac{c_\infty}{\hat{\tau}} + c_0 + c_1\,\hat{\tau} \mod \mathcal{A}_\zeta^{\leq -1}\,. \tag{1.4}$$

The formula stated in (1.4) can be viewed as a necessary condition for any function to be of theta-type. Furthermore, let $\mathbb{C}\{z\}$ be the set of analytic functions at $z = 0$. One remembers that $\mathcal{A}_\zeta^{\leq -1} \cap \mathbb{C}\{\hat{\tau}\} = \{0\}$. By replacing $c_0 + c_1\hat{\tau}$ with any convergent power series of $\hat{\tau}$ in the relation in (1.4), we will introduce a larger class of functions as follows:

Definition 1.4 Let $\zeta = e(r) \in \mathbb{U}$ and $r \in \mathbb{Q} \cap [0,1)$. One says that $f(q)$ admits an exponential–convergent expansion as $q \xrightarrow{a.r.} \zeta$ or $\tau \xrightarrow{a.v.} r$ and one writes $f \in \mathfrak{C}_\zeta$ if there exists $(\upsilon, c_\infty) \in \mathbb{Q} \times (i\mathbb{R})$ such that

$$\log f(q) = \upsilon\,\log\frac{\mathrm{i}}{\hat{\tau}} + \frac{c_\infty}{\hat{\tau}} \mod \mathbb{C}\{\hat{\tau}\} \oplus \mathcal{A}_\zeta^{\leq -1}\,. \tag{1.5}$$

It is obvious that $\mathfrak{T}_\zeta \subset \mathfrak{C}_\zeta$. In this way, one will see that Theorem 1.2 can be easily deduced from

Theorem 1.5 Let $(x_0, \beta) \in \mathbb{C} \times \mathbb{R}$ be such that $|x_0| = 1$ and $\beta \neq 0$, and consider $x = x_0\,q^\beta$. Then, the following conditions are equivalent:

(1) one has $(x;q) \in \mathfrak{C}_1$, with $\zeta = 1 = \mathrm{e}^{2\pi \mathrm{i} 0}$;

(2) there exists a root of unity $\zeta = \mathrm{e}^{2\pi i r}$ such that $(x;q) \in \mathfrak{C}_\zeta$;

(3) one has $x_0 \in \{1, -1\}$ and $\beta \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$.

A modular-like formula has been found for $(x;q)_\infty$ in [13, Th. 3.2] and [15, Th. 2.9], by means of one certain perturbed factor named $P(z,\tau)$, where $x=e(z)$ and $q=e(\tau)$. Thus, it suffices to understand the analyticity obstacle of $P(\alpha+\beta\,\tau,\tau)$ around each given rational point $\tau=r\in\mathbb{Q}\cap[0,1)$. We shall obtain the condition for this function to be analytically continued at $\tau=0$ by a Stokes analysis, with the help of the Ramis-Sibuya theorem [6, 8]. This analysis will be generalized for every $r\in\mathbb{Q}\cap(0,1)$ by means of a series of transformations associated to the continued fraction of r; transformations often used in the classical theory of the modular functions.

1.3 Plan for the Paper

The rest of the paper is divided into three sections. In Section 2, we define a family of integrals involving the exponential generating function associated with the Bernoulli numbers. These integrals can be seen as being of Laplace type, and they will be used for stating an equivalent version of the above-mentioned result on $(x;q)_\infty$; see Theorems 2.1 and 2.4.

Section 3 is essentially devoted to the part $\zeta=1$ of Theorem 1.5; see Theorem 3.1. By means of Theorems 2.1 and 2.4, we will see that the fact that a Euler q-exponential function, modulo some exponentially small term, can be analytically continued at $\tau=0$ and may be interpreted as one problem of the analytic continuation inside the theory of the Gevrey asymptotic expansions; see Theorem 3.8 and the proof of Theorem 1.5 given in Subsection 3.3.

Section 4 aims to obtain Theorem 1.5 for an arbitrary root ζ of unity; see Theorem 4.1. Lemma 4.8 will play a key role, especially in terms of permitting us to make use of both continued fractions and modular transforms. Finally, a complete scheme for proving our main result, Theorem 1.2, will be outlined at the end of the paper.

2 A Laplace-type Integral Involving Bernoulli Exponential Generating Function

The goal of this section is to develop appropriate means for properly understanding the following result obtained in [13] and [15]:

Theorem 2.1 ([13, Th. 3.2], [15, Th. 2.9]) Let $(z,\tau)\in\mathcal{U}$ and let $s=z/\tau$. If $s\notin(-\infty,0]$, then

$$(z\,|\,\tau)_\infty=\frac{\sqrt{2\pi s(1-e(z))}}{\Gamma(s+1)}\,e(-\frac{\tau}{24})\,\mathrm{e}^{s(\log s-1)+\frac{\mathrm{Li}_2(e(z))}{2\pi i\tau}+P(z,\tau)}\,(\frac{z-1}{\tau}\,|\,-\frac{1}{\tau})_\infty\,, \tag{2.1}$$

where $P(z,\tau)$ denotes the analytic function in $\mathcal{U}$ defined by the following integral:

$$P^d(z,\tau)=\int_0^{\infty\mathrm{e}^{\mathrm{i}d}}\frac{\sin(\frac{zt}{\tau})}{\mathrm{e}^{\mathrm{i}t/\tau}-1}\left(\cot\frac{t}{2}-\frac{2}{t}\right)\frac{\mathrm{d}t}{t}\quad(-\pi<d<0)\,. \tag{2.2}$$

In the above, $\mathcal{U}$ denotes the domain defined in $\mathbb{C}\times\mathcal{H}$ by the relation

$$\mathcal{U}=\cup_{\delta\in(0,\pi)}\mathbb{C}_\delta\times\mathcal{H}_\delta\,, \tag{2.3}$$

where $\mathcal{H}_\delta=\{\tau\in\mathcal{H}:\arg\tau\in(0,\delta)\}$ and $\mathbb{C}_\delta=\mathbb{C}\setminus\Big(\overline{(1+\mathcal{H}_\delta)}\cup\overline{(-1-\mathcal{H}_\delta)}\Big)$. The functions Γ, log and Li_2 are the Euler Gamma function, the principal branch of the complex logarithm function and the dilogarithm function, respectively.

In Subsection 2.1, we will introduce a family of Laplace-like integrals denoted as b^d, involving the exponential generating function of Bernoulli numbers. It will be shown that the term $((z-1)/\tau\,|\,-1/\tau)_\infty$ in the right-hand side of (2.1) can be obtained from comparing these integrals in different directions; see Theorem 2.4. One will also see that the same integrals are closely linked to the function $P(z,\tau)$ used in Theorem 2.1, and an equivalent version of this theorem will come from this comparison; see Theorem 2.8 in Subsection 2.2.

2.1 Bernoulli Integrals

Through the whole paper, we will denote by $\mathbf{B}(t)$ the function defined by

$$\mathbf{B}(t)=\frac{1}{\mathrm{e}^t-1}-\frac{1}{t}+\frac{1}{2} \tag{2.4}$$

for $t\in\mathbb{C}\setminus\{\pm2\pi\mathrm{i}k:k\in\mathbb{Z}_{>0}\}$. Indeed, one knows that

$$\mathbf{B}(t)=\sum_{n\geq1}\frac{B_{2n}}{(2n)!}\,t^{2n-1}\,, \tag{2.5}$$

where B_{2n} denotes the Bernoulli numbers for $n\in\mathbb{Z}_{>0}$; see [1, p. 12]. In addition, by Binet's formula [1, p. 28] on $\log\Gamma(x)$, it follows that

$$I(x):=\int_0^\infty\mathbf{B}(t)\,\mathrm{e}^{-xt}\,\frac{\mathrm{d}t}{t}=\log\Gamma(x)-(x-\frac{1}{2})\log x+x-\frac{1}{2}\,\log2\pi\,. \tag{2.6}$$

Here, first of all, one supposes that $\Re x>0$, so the integration path is the half-axis $(0,+\infty)$. By using an open interval $(0,\infty\mathrm{e}^{\mathrm{i}d})$ in the half-plane $\Re t>0$, this integral representation can then be valid for all $x\in\mathbb{C}\setminus(-\infty,0]$.

The integral $I(x)$ stated in (2.6) is the Laplace transform of the function $t\mapsto\mathbf{B}(t)/t$. In what follows, we shall consider a modified Laplace-type integral $b^d(z,\tau)$ associated to each $d\in(-\frac{\pi}{2},\frac{\pi}{2})$:

$$b^d(z,\tau)=\int_0^{\infty\mathrm{e}^{\mathrm{i}d}}\frac{\mathrm{e}^{-zu}-1}{\mathrm{e}^u-1}\,\mathbf{B}(\tau u)\,\frac{\mathrm{d}u}{u}\,. \tag{2.7}$$

To be brief, b^d will be called a Bernoulli integral.

Let us determine the values $(z,\tau)\in\mathbb{C}^2$ where the integral $b^d(z,\tau)$ is well-defined. From (2.5), it follows that $\mathbf{B}(t)=O(t)$ at $t=0$ in $\mathbb{C}$. Thus, the above integral in (2.7) converges at $u=0$ for all $(z,\tau)\in\mathbb{C}\times\mathbb{C}$. With regard to the convergence at infinity, we define

$$V^{d,+}=\{\tau\in\mathbb{C}:\Re(\tau\mathrm{e}^{\mathrm{i}d})>0\}\,,\quad V^{d,-}=\{\tau\in\mathbb{C}:\Re(\tau\mathrm{e}^{\mathrm{i}d})<0\} \tag{2.8}$$

and

$$U^d=\{z\in\mathbb{C}:z+1\in V^{d,+}\}=V^{d,+}-1\,. \tag{2.9}$$

By using (2.4), one finds that $\mathbf{B}(t)\to\pm1/2$ when $\Re t\to\pm\infty$. Therefore, $b^d(z,\tau)$ is defined in two separated domains $U^d\times V^{d,+}$ and $U^d\times V^{d,-}$ in $\mathbb{C}^2$.

Geometrically, U^d represents the half-plane containing the point at origin and delimited by the straight-line $-1+\mathrm{e}^{\mathrm{i}(-d+\frac{\pi}{2})}\mathbb{R}$, while $V^{d,\pm}$ are half-planes separated by the straight line $\mathrm{e}^{\mathrm{i}(-d+\frac{\pi}{2})}\mathbb{R}$. One can find that the interval $(-1,\infty)$ belongs to U^d for every argument $d\in(-\frac{\pi}{2},\frac{\pi}{2})$; see Figure 1.

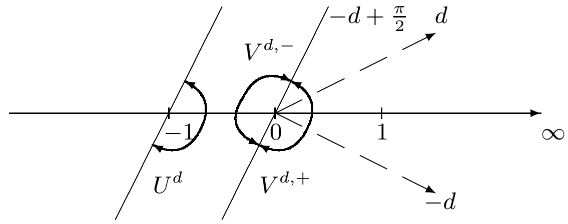

Figure 1 Half-planes U^d, $V^{d,-}$ and $V^{d,+}$

Let

$$\mathcal{W}^+ = \cup_{d\in(-\frac{\pi}{2},\frac{\pi}{2})} U^d \times V^{d,+}, \quad \mathcal{W}^- = \cup_{d\in(-\frac{\pi}{2},\frac{\pi}{2})} U^d \times V^{d,-}\,. \tag{2.10}$$

Since $\cup_{d\in(-\frac{\pi}{2},\frac{\pi}{2})} V^{d,+} = \mathbb{C}\setminus(-\infty,0]$, $\cup_{d\in(-\frac{\pi}{2},\frac{\pi}{2})} V^{d,-} = \mathbb{C}\setminus[0,\infty)$, it follows that

$$(-1,\infty)\times(\mathbb{C}\setminus(-\infty,0])\subset\mathcal{W}^+\,, \quad (-1,\infty)\times(\mathbb{C}\setminus[0,\infty))\subset\mathcal{W}^-\,. \tag{2.11}$$

Definition 2.2 We define $b^+(z,\tau)$ and $b^-(z,\tau)$ in $\mathcal{W}^+$ and $\mathcal{W}^-$, respectively, by applying the analytic continuation procedure to $b^d(z,\tau)$ from $U^d\times V^{d,+}$ and $U^d\times V^{d,-}$ as d runs through $(-\frac{\pi}{2},\frac{\pi}{2})$.

We shall make use of the following result to express the difference between $b^+(z,\tau)$ and $b^-(z,\tau)$ in their common domain $\mathcal{W}^+\cap\mathcal{W}^-$:

Lemma 2.3 If $\tau\in\mathcal{H}$ and $z\in\mathcal{H}$, then

$$\sum_{n\geq 1}\frac{1}{n}\,\frac{e(nz)}{1-e(n\tau)} = -\log\left((z\,|\,\tau)_\infty\right)\,. \tag{2.12}$$

Proof This follows from [2, p. 36, (21.1)]. □

By (2.11), one finds that $((-1,\infty)\times(\mathbb{C}\setminus\mathbb{R}))\subset\mathcal{W}^+\cap\mathcal{W}^-$. In what follows, we will write $\mathbb{C}\setminus\mathbb{R}=\mathcal{H}\cup\mathcal{H}^-$, where $\mathcal{H}^-=-\mathcal{H}=\{\tau\in\mathbb{C}:\Im\tau<0\}$.

Theorem 2.4 Let $(z,\tau)\in\mathcal{W}^+\cap\mathcal{W}^-$. The following assertions hold:

(1) if $\tau\in\mathcal{H}$, then

$$b^+(z,\tau)-b^-(z,\tau) = -\log\frac{(-(z+1)/\tau\,|\,-1/\tau)_\infty}{(-1/\tau\,|\,-1/\tau)_\infty}\,; \tag{2.13}$$

(2) if $\tau\in\mathcal{H}^-$, then

$$b^+(z,\tau)-b^-(z,\tau) = -\log\frac{((z+1)/\tau\,|\,1/\tau)_\infty}{(1/\tau\,|\,1/\tau)_\infty}\,. \tag{2.14}$$

Proof (1) By the standard argument of analytical continuation, it suffices to prove (2.13) for $(z,\tau)\in(-1,\infty)\times\mathcal{H}$. Thus, one chooses $d_1\in(-\frac{\pi}{2},0)$ and $d_2\in(0,\frac{\pi}{2})$ such that $\tau\in\left(V^{d_1,+}\cap V^{d_2,-}\right)$. The contour integral in (2.7) allows one to write that

$$\begin{aligned} b^+(z,\tau)-b^-(z,\tau) &= b^{d_1,+}(z,\tau)-b^{d_2,-}(z,\tau)\\ &= \left(\int_0^{\infty e^{id_1}}-\int_0^{\infty e^{id_2}}\right)\frac{e^{-zu}-1}{e^u-1}\,\mathbf{B}(\tau u)\,\frac{du}{u}\,. \end{aligned} \tag{2.15}$$

Since both d_1 and d_2 belong to $(-\frac{\pi}{2},\frac{\pi}{2})$, the two half straight-lines used in the contour-integral (2.15) are separated in the u-plane by the half straight-line ℓ_τ defined by the relation $\ell_\tau=\{u\in\mathbb{C}^*:\Re(\tau u)=0,\Re u>0\}$; see Figure 2.

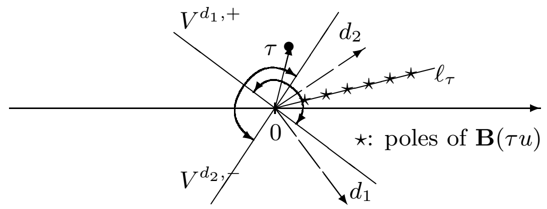

Figure 2 τ belongs to the common domain $V^{d_1,+} \cap V^{d_2,-}$ while the directions d_1 and d_2 are separated by the half-line ℓ_τ

By observing that the function $\mathbf{B}(\tau u)$ admits simple poles $u = 2n\pi\mathrm{i}/\tau$ ($n \in \mathbb{Z}_{>0}$) on the line ℓ_τ, applying the Residues Theorem to (2.15) yields that

$$\begin{aligned} b^+(z,\tau) - b^-(z,\tau) &= \sum_{n\geq 1} \frac{1}{n} \frac{e(-nz/\tau) - 1}{e(n/\tau) - 1} \\ &= \sum_{n\geq 1} \frac{1}{n} \frac{e(-n(z+1)/\tau)}{1 - e(-n/\tau)} - \sum_{n\geq 1} \frac{1}{n} \frac{e(-n/\tau)}{1 - e(-n/\tau)}\,. \end{aligned} \tag{2.16}$$

By using the relation in (2.12), the above expression in (2.16) implies that

$$b^+(z,\tau) - b^-(z,\tau) = -\log\left((-\frac{z+1}{\tau} \mid -\frac{1}{\tau})_\infty \right) + \log\left((-\frac{1}{\tau} \mid -\frac{1}{\tau})_\infty \right)\,,$$

so that one obtains (2.13).

(2) When $\tau \in \mathcal{H}^-$, the above proof can be adopted as follows: choose $d_1 \in (0, \frac{\pi}{2})$ and $d_2 \in (-\frac{\pi}{2}, 0)$, and observe that the simple poles of $\mathbf{B}(\tau u)$ to which the Residues Theorem is applied become $u = -2n\pi\mathrm{i}/\tau$ ($n \in \mathbb{Z}_{>0}$). A direct calculation implies (2.14), which ends the proof of Theorem 2.4. □

Now, consider $\tau \in \mathcal{H}$, with $\arg\tau = \delta \in (0,\pi)$. By (2.10), it follows that $(z,\tau) \in \mathcal{W}^+$ if and only if $z \in V^{d,+}$ for some suitable $d \in (-\frac{\pi}{2}, \frac{\pi}{2} - \delta)$. Thus, one obtains the equivalence

$$(z,\tau) \in \mathcal{W}^+ \quad \Longleftrightarrow \quad z \in \mathcal{H} \cup Z_\tau\,, \tag{2.17}$$

where Z_τ is the half-plane associated with τ in the following manner:

$$Z_\tau = \{z \in \mathbb{C} : \Im \frac{z+1}{\tau} < 0\}\,.$$

One may see that if $z \in Z_\tau$, then $z + z_0 \in Z_\tau$ for all $z_0 \in \mathbb{R}_{>0}$; see Figure 3.

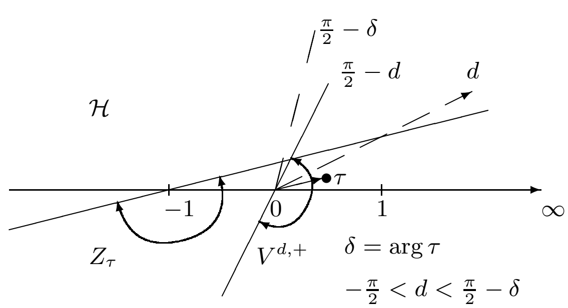

Figure 3 The half-plane Z_τ contains both the point τ and the segment $(-1,\infty)$

Lemma 2.5 Let $b^+(z,\tau)$ be as in Definition 2.2, and let $s = z/\tau$. One supposes that $\tau \in \mathcal{H}$. If $(z-1,\tau) \in \mathcal{W}^+$, then $s \notin (-\infty, 0]$ and

$$b^+(z-1,\tau) - b^+(z,\tau) = I(s)\,, \tag{2.18}$$

where $I(s)$ is the Laplace integral stated by (2.6).

Proof Since $(z-1,\tau) \in \mathcal{W}^+$, relation (2.17) implies that either $z \in \mathcal{H} \cup (0,\infty)$ or $z \in \mathcal{H}^-$ but $\Im(z/\tau) < 0$. This implies that $s \notin (-\infty, 0]$. Also, relation (2.18) follows immediately, by comparing (2.7) with (2.6). □

2.2 Functions related with Bernoulli Integrals

First, let $P(z,\tau)$ be as in (2.1) and consider how to express it by using $b^+(z,\tau)$. In view of the fact that

$$\cot\frac{t}{2} - \frac{2}{t} = \frac{\mathrm{e}^{\mathrm{i}t/2} + \mathrm{e}^{-\mathrm{i}t/2}}{\mathrm{e}^{\mathrm{i}t/2} - \mathrm{e}^{-\mathrm{i}t/2}}\,\mathrm{i} - \frac{2}{t} = 2\mathrm{i}\left(\frac{1}{\mathrm{e}^{\mathrm{i}t}-1} + \frac{1}{2} - \frac{1}{\mathrm{i}t}\right),$$

it follows that $\cot\frac{t}{2} - \frac{2}{t} = 2\mathrm{i}\,\mathbf{B}(\mathrm{i}t)$; see (2.4) for $\mathbf{B}(t)$. Thus, replacing the integration path $(0, \infty\mathrm{e}^{\mathrm{i}d})$ with $(0, \mathrm{i}\infty\mathrm{e}^{\mathrm{i}d})$ in (2.2) yields that

$$P^d(z,\tau) = \int_0^{\infty\mathrm{e}^{\mathrm{i}d'}} \frac{\mathrm{e}^{zt/\tau} - \mathrm{e}^{-zt/\tau}}{\mathrm{e}^{t/\tau}-1}\,\mathbf{B}(t)\,\frac{\mathrm{d}t}{t}\,, \tag{2.19}$$

where we write $d' = d + \frac{\pi}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Let $\mathcal{W}^+$ be as in (2.10), and let $\mathcal{U}$ be as in (2.3). A simple computation shows that

$$\mathcal{U} = \{(z,\tau) \in \mathcal{W}^+ : \tau \in \mathcal{H}, (-z,\tau) \in \mathcal{W}^+\}\,.$$

Furthermore, comparing the integrals in (2.19) and (2.7) allows one to immediately observe the following:

Remark 2.6 Let $b^+(z,\tau)$ be as in Definition 2.2. The function $P(z,\tau)$ can be expressed as

$$P(z,\tau) = -b^+(z,\tau) + b^+(-z,\tau)\,. \tag{2.20}$$

Furthermore, by gathering together (2.20) with (2.18), it follows that

$$P(z+1,\tau) - P(z,\tau) = I(-s) + I(s + \frac{1}{\tau})\,, \tag{2.21}$$

where $I(s)$ denotes the function defined by (2.6) with $s = z/\tau$.

Next, given $d \in (-\frac{\pi}{2}, \frac{\pi}{2})$, let $V^{d,\pm}$ be as in (2.8), and define

$$H^d = \left(V^{d,+} - \frac{1}{2}\right) \cap \left(-V^{d,+} + \frac{1}{2}\right).$$

It is easy to see that the integral

$$B^d(z,\tau) = \int_0^{\infty\mathrm{e}^{\mathrm{i}d}} \frac{\mathrm{e}^{zu} - \mathrm{e}^{-zu}}{\mathrm{e}^{u/2} - \mathrm{e}^{-u/2}}\,\mathbf{B}(\tau u)\,\frac{\mathrm{d}u}{u} \tag{2.22}$$

is well-defined for any $(z,\tau) \in H^d \times (V^{d,+} \cup V^{d,-})$. Furthermore, by noticing that

$$\frac{\mathrm{e}^{zu} - \mathrm{e}^{-zu}}{\mathrm{e}^{u/2} - \mathrm{e}^{-u/2}} = \frac{\mathrm{e}^{(z+\frac{1}{2})u} - 1}{\mathrm{e}^u - 1} - \frac{\mathrm{e}^{-(z-\frac{1}{2})} - 1}{\mathrm{e}^u - 1}\,,$$

comparing (2.22) with (2.7) yields that

$$B^d(z,\tau) = b^d(-z - \frac{1}{2}, \tau) - b^d(z - \frac{1}{2}, \tau)\,. \tag{2.23}$$

Definition 2.7 Let

$$\Omega^+ = \cup_{d\in(-\frac{\pi}{2},\frac{\pi}{2})} H^d \times V^{d,+}\,, \quad \Omega^- = \cup_{d\in(-\frac{\pi}{2},\frac{\pi}{2})} H^d \times V^{d,-}\,. \tag{2.24}$$

We will denote by $B^+(z,\tau)$ and $B^-(z,\tau)$ the respective functions defined in Ω^+ and Ω^- by the integral (2.22).

If there is no possible confusion, we will simply write Ω and $B(z,\tau)$ instead of Ω^+ and $B^+(z,\tau)$, respectively.

From (2.23), it follows that

$$B^\pm(z,\tau) = b^\pm(-z-\frac{1}{2},\tau) - b^\pm(z-\frac{1}{2},\tau)\,. \tag{2.25}$$

Furthermore, combining this last equality with (2.20) and (2.18) yields that

$$B(z,\tau) = P(z+\frac{1}{2},\tau) - I(\frac{z+1/2}{\tau})\,. \tag{2.26}$$

Theorem 2.8 The following relation holds for all $(z,\tau)\in\Omega$ with $\tau\in\mathcal{H}$:

$$(z+\frac{1}{2}\,|\,\tau)_\infty = e(-\frac{\tau}{24})\,\sqrt{1+e(z)}\,\mathrm{e}^{\frac{\mathrm{Li}_2(-e(z))}{2\pi\mathrm{i}\tau}+B(z,\tau)}\,(\frac{z-1/2}{\tau}\,|\,-\frac{1}{\tau})_\infty\,. \tag{2.27}$$

Proof By considering (2.6), the formula in (2.1) can be put into the following form:

$$(z\,|\,\tau)_\infty = \sqrt{1-e(z)}\,e(-\frac{\tau}{24})\,\mathrm{e}^{\frac{\mathrm{Li}_2(e(z))}{2\pi\mathrm{i}\tau}-I(\frac{z}{\tau})+P(z,\tau)}\,(\frac{z-1}{\tau}\,|\,-\frac{1}{\tau})_\infty\,.$$

Thus, one obtains (2.27), with the help of (2.26). □

3 Conditions for a Euler q-exponential Function to be of Theta-type at One

Let $x_0 = e(\alpha) = \mathrm{e}^{2\pi\mathrm{i}\alpha}$ with $\alpha\in\mathbb{R}$, and let $\beta\in\mathbb{R}$. By using (1.3) with $N=\infty$, we write $(x_0\,q^\beta;q)_\infty = (\alpha+\beta\,\tau\,|\,\tau)_\infty$, where $\tau\in\mathcal{H}$ and $q=e(\tau)$. The goal of this section is to establish the next result, which will be useful for the proof of Theorem 1.5.

Theorem 3.1 Let $(\alpha,\beta)\in[0,1)\times(0,1]$, and consider $f(q)=(\alpha+\beta\,\tau\,|\,\tau)_\infty$. Then, $f\in\mathfrak{C}_1$ if and only if $\alpha\in\{0,\frac{1}{2}\}$ and $\beta\in\{\frac{1}{2},1\}$.

The main idea here will consist of using Theorems 2.1 and 2.8 to rewrite $f(q)$ in such a way that $\log f(q) = B(\cdots)$ or $= P(\cdots)$ mod $\mathbb{C}\{\tau\}\oplus\mathcal{A}_1^{\le -1}$, the exponential small term being furnished by an infinite product of $e(-1/\tau)$. Thus, we will be led to consider the analytic continuation of B or P around $\tau=0$ in the complex plane; see Theorem 3.8 in Subsection 3.2. In this way, we will obtain the condition for (α,β) required by Theorem 3.1, whose proof will be completed in Subsection 3.3.

We shall make use of the Gevrey asymptotic expansions for understanding the analytic obstacle at $\tau=0$ of the above-mentioned functions B and P. This is linked to the so-called Stokes' phenomenon. One tool to treat this problem may be Ramis-Sibuya Theorem, which will be briefly explained in Subsection 3.1 in what follows.

3.1 Ramis-Sibuya's Theorem on Gevrey Asymptotic Expansions

Let $x_0\in\mathbb{C}$ and let $\tilde{\mathbb{C}}_{x_0}$ be the Riemann surface of the function $x\mapsto\log(x-x_0)$; let $I=(\alpha_1,\alpha_2)\subset\mathbb{R}$ and let $R>0$. We let $V_{x_0}(I;R)$ denote the sector of a vertex at x_0 in $\tilde{\mathbb{C}}_{x_0}$, with an opening in I and a radius R; that is to say,

$$V_{x_0}(I;R)=\{x_0+r\mathrm{e}^{\mathrm{i}\alpha}:\alpha\in I, r\in(0,R)\}. \tag{3.1}$$

By definition, a proper sub-sector of $V_{x_0}(I;R)$ will be any domain of the form $V_{x_0}(J;\rho)$ such that $\bar{J}\subset I$ and $\rho<R$.

If the length of the open interval I is smaller than or equal to 2π, any sector $V_{x_0}(I;R)$ is not overlapped in $\tilde{\mathbb{C}}_{x_0}$; in this case, one will consider $V_{x_0}(I;R)$ as a sector in $\mathbb{C}$. When $x_0=0$, we will remove the sub-index 0 and simply write $V(I;R)$ instead of $V_0(I;R)$.

Let $V=V(I;R)$ be a sector in $\mathbb{C}$ at 0. By definition ([6], [8], $\cdots$), a given function f defined and analytic in V is said to have a power series $\sum\limits_{n\geq 0}a_nx^n$, $a_n\in\mathbb{C}$, as a Gevrey asymptotic expansion at 0 in V, if, for any proper sub-sector $U=V(J;\rho)$, one can find $C>0$ and $A>0$ such that the following estimates hold for all $n\in\mathbb{Z}_{\geq 0}$:

$$\sup_{x\in U}\left|(f(x)-\sum_{m=0}^{n-1}a_mx^m)x^{-n}\right|\leq C\,A^n\,n!\,. \tag{3.2}$$

As a typical example, the Borel-sum function of a given divergent series, if it exists, admits this series as a Gevrey asymptotic expansion. A Gevrey type asymptotic expansion is also called an exponential asymptotic expansion, due to the following fact:

Remark 3.2 ([6, p. 175, Th. 1.2.4.1 1)]) A function f admits the identically null series as a Gevrey asymptotic expansion at 0 in V if and only if f is exponentially small there, which means that, for all proper sub-sectors U in V, there exists $C>0$ and $\kappa>0$ such that, for all $x\in U$, $|f(x)|\leq C\,\mathrm{e}^{-\kappa/|x|}$.

In what follows, we will denote by $\mathcal{A}^{\leq -1}(V)$ the space of all functions that are exponentially small in V as indicated in Remark 3.2. More generally, when $V=V_{x_0}(I;R)$, we will say that $f\in\mathcal{A}^{\leq -1}(V)$ when f is exponentially small as $x\to x_0$ in V.

Theorem 3.3 ([6, p. 176, Th. 1.3.2.1]) Let $V_1,\cdots,V_m$, V_{m+1} be a family of open sectors at 0 in $\mathbb{C}$ such that $V_{m+1}=V_1$, $V_j\cap V_{j+1}\neq\emptyset$ for $1\leq j\leq m$ and that the whole union $\cup_{j=1}^m V_j$ contains a neighborhood of 0 in $\mathbb{C}$. For every j, let f_j be a given analytic and bounded function in V_j. If

$$f_{j+1}=f_j \mod \mathcal{A}^{\leq -1}(V_j\cap V_{j+1}),$$

then all f_j's admit the same Gevrey asymptotic expansion at 0.

The above result is currently called Ramis-Sibuya's theorem. We shall make use of the following statement deduced from Theorem 3.3:

Corollary 3.4 Let $R>0$, and let I_1 and I_2 be open intervals such that

$$[-\epsilon,\pi-\epsilon]\subset I_1\subset(-\pi,\pi),\quad [\pi-\epsilon,2\pi-\epsilon]\subset I_2\subset(0,2\pi)$$

for some $\epsilon\in(0,\pi)$. Let $V_1=V(I_1;R)$, $V_2=V(I_2;R)$, and consider two analytic and bounded functions f_1 and f_2 defined, respectively, in V_1 and V_2. If $f_1-f_2\in\mathcal{A}^{\leq -1}(V_1\cap V_2)$, then f_1 and f_2 have the same Gevrey asymptotic expansion and, moreover, the following conditions are equivalent:

(1) one of the functions f_1 and f_2 can be continued into an analytic function at 0 in $\mathbb{C}$;
(2) both f_1 and f_2 can be continued into an analytic function at 0 in $\mathbb{C}$;
(3) $f_1\equiv f_2$ in $V_1\cap V_2$.

Proof The existence of a Gevrey asymptotic expansion for f_1 and f_2 follows immediately from Theorem 3.3.

Let $\hat{f} = \sum\limits_{n\geq 0} a_n x^n$ be the common asymptotic expansion of f_1 and f_2. Since the length of I_1 and that of I_2 are larger that π, one finds that f_1 and f_2 are the respective Borel-sum functions of $\hat{f}$ in V_1 and V_2. Thus, the above statement in (1) implies that $\hat{f}$ is really a convergent series, so that their two Borel-sums are equal to each other. In this way, one obtains that (1) implies all other statements.

On the other hand, if the statement in (3) is true, then both f_1 and f_2 equal to a same analytic and bounded function in the punctuated disc $\{0 < |x| < R\}$. By the Riemann removable singularities Theorem, one finds the statements (1) and (2). □

3.2 Asymptotic expansion of Bernoulli integrals

From now on, we will identify the upper half-plane $\mathcal{H}$ as the sector $V(I;R)$ with $I = (0,\pi)$ and $R=\infty$. Thus, $\mathcal{A}^{\leq -1}(\mathcal{H})$ will be the space of all analytic functions in $\mathcal{H}$ that are exponentially small as $\tau\to 0$. It is easy to see that $\mathcal{A}^{\leq -1}(\mathcal{H}) \subset \mathcal{A}_1^{\leq -1}$, where $\zeta = 1 = \mathrm{e}^{2\pi i 0}$ with $r=0$; see Subsection 1.2 (iii).

Proposition 3.5 Let $(\alpha,\beta)\in\mathbb{R}^2$, and consider $f(\tau) = \log\left((-\frac{\alpha+\beta\,\tau}{\tau}\,|\,-\frac{1}{\tau})_\infty\right)$ for $\tau\to 0$ in $\mathcal{H}$. If $\alpha>0$, then $f\in\mathcal{A}^{\leq -1}(\mathcal{H})$.

Proof Thanks to Euler [1, p. 490, Corollary 10.2.2 (b)], one can write that, for all $x\in\mathbb{C}$,

$$(x;q)_\infty = \sum_{n\geq 0} \frac{q^{n(n-1)/2}}{(q;q)_n}\,(-x)^n\,. \tag{3.3}$$

Letting $x = e(-(\alpha+\beta\,\tau)/\tau)$ and $q = e(\,-1/\tau)$ into (3.3), one gets that

$$(-\frac{\alpha+\beta\,\tau}{\tau}\,|\,-\frac{1}{\tau})_\infty = 1 + \sum_{n\geq 1} \frac{(-1)^n e(-n(\alpha+\beta\,\tau)/\tau)}{(\,-1/\tau\,|\,-1/\tau)_n}\, e\left(\frac{n(n-1)}{2}(\,-\frac{1}{\tau})\right)\,,$$

where $(.\,|\,.)_n$ is defined as in (1.3). Since $e(\,-\frac{\nu}{\tau}) \in \mathcal{A}^{\leq -1}(\mathcal{H})$ for any $\nu>0$, it follows that, when $\alpha>0$,

$$\begin{aligned}(-\frac{\alpha+\beta\,\tau}{\tau}\,|\,-\frac{1}{\tau})_\infty &= 1 - \frac{e(-(\alpha+\beta\,\tau)/\tau)}{1-e(\,-1/\tau)} \mod \mathcal{A}^{\leq -1}(\mathcal{H}) \\ &= 1 \mod \mathcal{A}^{\leq -1}(\mathcal{H})\,.\end{aligned}$$

This finishes the proof. □

Proposition 3.6 Let $(\alpha,\beta)\in\mathbb{R}^2$ and let $I=(-\pi,\pi)$. If $\alpha>-1$, then $b^+(\alpha+\beta\,\tau,\tau)$ is well-defined and analytic in $V(I;R)$, and is bounded in every proper sub-sector of $V(I;R)$ with $R>0$.

Proof For all $\tau\in\mathbb{C}^*$, let D_τ be the sector containing 0 that is bounded by $(-\infty,-1]\cup[-1,-1-\infty\tau)$, where $[-1,-1-\infty\tau)$ denotes the half straight-line starting from -1 to ∞ with the direction $-\tau$. By combining (2.8) together with (2.9), one can find that, for all fixed $\tau\in\mathbb{C}\setminus(-\infty,0]$, the function $b^+(z,\tau)$ is defined and analytic for $z\in D_\tau$.

If $\alpha>-1$, one can easily see that $\alpha+\beta\,\tau$ belongs to this half-plane D_τ when $\tau\notin\mathbb{R}^-$. This implies that $b^+(\alpha+\beta\,\tau,\tau)$ is well-defined and analytic in any sector $V(I;R)$.

The boundedness of this function over any proper sub-sector comes from direct estimates done for (2.7). □

In a similar way, one can find that the statement of Proposition 3.6 remains true if $b^+(z,\tau)$ and I are replaced with $b^-(z,\tau)$ and $(0,2\pi)$, respectively. Thus, one obtains the following:

Theorem 3.7 Let $(\alpha,\beta)\in\mathbb{R}^2$ and let $b^+(z,\tau)$ as in Definition 2.2. If $\alpha>-1$, then $b^+(\alpha+\beta\,\tau,\tau)$ admits a Gevrey asymptotic expansion in any sector $V(I;R)$ with $I=(-\pi,\pi)$ and $R>0$.

Moreover, $b^+(\alpha+\beta\,\tau,\tau)$ can be continued into an analytic function at $\tau=0$ if and only if $\alpha=0$ and $\beta\in\mathbb{Z}$.

Proof Fix $R>0$, and let

$$V_1=V((-\pi,\pi);R),\quad V_2=V((0,2\pi);R).$$

Define

$$f_1(\tau)=b^+(\alpha+\beta\,\tau,\tau)=b^+(\alpha+\beta\,\tau,\tau),\quad f_2(\tau)=b^-(\alpha+\beta\,\tau,\tau)$$

for $\tau\in V_1$ and V_2, respectively. By putting $z=\alpha+\beta\,\tau$ into both relations (2.13) and (2.14) of Theorem 2.4, it follows that

$$f_1(\tau)-f_2(\tau)=-\log\frac{(-(\alpha+\beta\,\tau+1)/\tau\,|\,-1/\tau)_\infty}{(-1/\tau\,|\,-1/\tau)_\infty}\,, \tag{3.4}$$

if $\tau\in\mathcal{H}\cap V_1\cap V_2$, and that

$$f_1(\tau)-f_2(\tau)=-\log\frac{((\alpha+\beta\,\tau+1)/\tau\,|\,1/\tau)_\infty}{(1/\tau\,|\,1/\tau)_\infty}\,, \tag{3.5}$$

if $\tau\in\mathcal{H}^-\cap V_1\cap V_2$.

One observes that $V_1\cap V_2\cap\mathbb{R}=\emptyset$. Therefore, by considering Proposition 3.5, relation (3.4) together with (3.5) imply that $f_1(\tau)-f_2(\tau)$ is exponentially small in the common domain $V_1\cap V_2$. This allows us to apply Corollary 3.4 to get a particularly the common Gevrey asymptotic expansion of both f_1 and f_2.

Furthermore, Corollary 3.4 implies that f_1 can be extended into an analytic function at $\tau=0$ in $\mathbb{C}$ if and only if

$$(-\frac{\alpha+\beta\,\tau+1}{\tau}\,|\,-\frac{1}{\tau})_\infty=(-\frac{1}{\tau}\,|\,-\frac{1}{\tau})_\infty$$

for all $\tau\in\mathcal{H}$, or, equivalently,

$$-\frac{\alpha+\beta\,\tau+1}{\tau}=-\frac{1}{\tau}\quad\bmod\mathbb{Z}\,.$$

In this way, one finds the necessary and sufficient condition $\alpha+\beta\,\tau\in\tau\mathbb{Z}$ in order to have an analytic function $b^+(\alpha+\beta\,\tau,\tau)$ at $\tau=0$ in $\mathbb{C}$. This ends the proof of Theorem 3.7. □

Now, consider the functions $P(z,\tau)$ and $B(z,\tau)$ appearing in Theorems 2.1 and 2.8. In keeping with the spirit of Theorem 3.7, one finds the following result:

Theorem 3.8 Let $(\alpha,\beta)\in\mathbb{R}^2$, and let $V=V(I;R)$ with $I=(-\pi,\pi)$ and $R>0$. Then

(1) if $\alpha\in(-1,1)$, the function $P(\alpha+\beta\,\tau,\tau)$ admits a Gevrey asymptotic expansion as $\tau\to0$ in V;

(2) if $\alpha\in(-\frac12,\frac12)$, the function $B(\alpha+\beta\,\tau,\tau)$ admits a Gevrey asymptotic expansion as $\tau\to0$ in V.

Furthermore, $P(\alpha+\beta\,\tau,\tau)$ or $B(\alpha+\beta\,\tau,\tau)$ can be continued into an analytic function at $\tau=0$ in $\mathbb{C}$ if and only if $\alpha=0$ and $\beta\in\frac12\mathbb{Z}$.

Proof We shall consider only the function $P(z,\tau)$, and the case of $B(z,\tau)$ is very similar. By using (2.20), $P(\alpha+\beta\tau,\tau)$ can be expressed in terms of $b^+(\alpha+\beta\tau,\tau)$ as follows:

$$P(\alpha+\beta\tau,\tau) = -b^+(\alpha+\beta\tau,\tau) + b^+(-\alpha-\beta\tau,\tau)\,.$$

Thus, Theorem 3.7 implies that $P(\alpha+\beta\tau,\tau)$ remains analytic and has a Gevrey asymptotic expansion as $\tau\to 0$ in V when $\alpha\in(-1,1)$.

Furthermore, combining the expression of $P(z,\tau)$ in (2.20) with Stokes's relations (2.13) and (2.14) allows one to obtain the following equation: for all $\tau\in\mathcal{H}$,

$$P(z,\tau) - P^-(z,\tau) = \log\frac{(-(z+1)/\tau\,|\,-1/\tau)_\infty}{(-(-z+1)/\tau\,|\,-1/\tau)_\infty}\,,$$

where $P^-(z,\tau)$ denotes the function defined by (2.19) with $d\in(0,\pi)$. Thus, $P(\alpha+\beta\tau,\tau)$ can be continued into an analytic function at $\tau=0$ if and only if

$$-(\alpha+\beta\tau+1)/\tau = (\alpha+\beta\tau-1)/\tau \mod \mathbb{Z}\,.$$

This achieves the proof of Theorem 3.8. □

3.3 Proof of Theorem 3.1

In what follows, we will denote by $\mathbb{C}\{\tau\}$ the space of the germs of analytic functions at $\tau=0$ in $\mathcal{H}$. One knows that the dilogarithm is well-defined and analytic in the universal covering of $\mathbb{C}\setminus\{1\}$. Thus, $u\mapsto \mathrm{Li}_2\,(e(u))$ represents an analytic function on the Riemann surface of logarithm, i.e, the universal covering $\tilde{\mathbb{C}}_0$ of $\mathbb{C}\setminus\{0\}$.

Lemma 3.9 The following relation holds for all $u\in\tilde{\mathbb{C}}_0$:

$$\mathrm{Li}_2\left(e(u\mathrm{e}^{2\pi\mathrm{i}})\right) - \mathrm{Li}_2\left(e(u)\right) = 4\pi^2\,u\,. \tag{3.6}$$

Proof Let $x=e(u)$ for $u\in\tilde{\mathbb{C}}_0$. When u makes a complete rotation along a circle around $u=0$, the corresponding x forms a circle around $x=1$. By using a relation between $\mathrm{Li}_2(x)$ and $\mathrm{Li}_2(1-x)$ [12, §2],

$$\mathrm{Li}_2(1-x) = -\mathrm{Li}_2(x) + \frac{\pi^2}{6} - \log x\,\log(1-x),$$

one finds that the monodromy of Li_2 around $x=1$ can be expressed as follows:

$$\mathrm{Li}_2(1+x\,\mathrm{e}^{2\pi\mathrm{i}}) = \mathrm{Li}_2(1+x) - 2\pi\mathrm{i}\,\log(1+x)\,.$$

Therefore, one gets that

$$\mathrm{Li}_2\left(e\left(u\mathrm{e}^{2\pi\mathrm{i}}\right)\right) - \mathrm{Li}_2\left(e\left(u\right)\right) = -2\pi\mathrm{i}\log\left(e\left(u\right)\right)\,,$$

which implies the desired relation (3.6). □

Proof of Theorem 3.1 First of all, suppose that $(\alpha,\beta)\in\{0,\frac12\}\times\{\frac12,1\}$. It follows from Remark 1.1 that $f\in\mathfrak{T}_1$, so $f\in\mathfrak{C}_1$, also.

Now, consider the " only if" part, and suppose that $f\in\mathfrak{C}_1$. The rest of the proof will be divided into two parts, according whether α may be null or not.

• Case 1: $\alpha\in(0,1)$. Let $\alpha'=\alpha-\frac12$, and observe that $\alpha'\in(-\frac12,\frac12)$. By putting $z=\alpha'+\beta\tau$ into (2.27) of Theorem 2.8, it follows that

$$\log f(q) = A(\tau) + L(\tau) + B(\alpha'+\beta\tau,\tau) + R(\tau), \tag{3.7}$$

where one introduces the following notation:

$$A(\tau) = \log\left(e\left(-\frac{\tau}{24}\right)\sqrt{1+e(\alpha'+\beta\,\tau)}\right) = \frac{\pi}{12}\,\frac{\tau}{\mathrm{i}} + \frac{1}{2}\,\log\left(1-e(\alpha+\beta\,\tau)\right)\,,$$

$$L(\tau) = \frac{\mathrm{Li}_2(e(\alpha+\beta\,\tau))}{2\pi\mathrm{i}\tau}\,,\quad R(\tau) = \log\Big(\frac{\alpha+\beta\,\tau-1}{\tau}\mid -\frac{1}{\tau}\Big)_\infty\,.$$

On the one hand, as $\alpha'-1/2<0$, Proposition 3.5 implies that $R(\tau)\in\mathcal{A}^{\le -1}(\mathcal{H})\subset\mathcal{A}_1^{\le -1}$. On the other hand, it is easy to see that

$$A(\tau)\in\mathbb{C}\{\tau\},\quad L(\tau) = \frac{\mathrm{Li}_2(e(\alpha))}{2\pi\mathrm{i}\tau}\mod\mathbb{C}\{\tau\}\,.$$

Thus, comparing (1.5) ($\zeta=1$, $r=0$, $\hat\tau=\tau$) with (3.7) yields that

$$B(\alpha'+\beta\,\tau,\tau) = \nu\log\frac{\tau}{\mathrm{i}} + \frac{\lambda}{\tau}\mod\mathbb{C}\{\tau\}\oplus\mathcal{A}_1^{\le -1}\,, \tag{3.8}$$

where $\nu\in\mathbb{Q}$ and $\lambda\in\mathbb{C}$.

By Theorem 3.8, it follows that $B(\alpha'+\beta\,\tau,\tau)$ has a Gevrey asymptotic expansion as $\tau\to 0$ in any sector $V=V(I;R)$, where $I=(-\pi,\pi)$ and $R>0$. This implies that $\nu=0$ and $\lambda=0$ in (3.8). Furthermore, the exponentially small term used in (3.8) will be bounded in any proper sub-sector of V. As the openness of V is larger than π, a classical argument such as the Phragemen-Lindeloff Theorem implies that this term is identically null; see [5] for more on this matter. Thus, one gets that $B(\alpha'+\beta\,\tau,\tau)$ can be really continued into an analytic function at $\tau=0$. Applying Theorem 3.8 (2) implies that $\alpha'=0$ and $\beta\in\frac{1}{2}\mathbb{Z}$, so it follows that $\alpha=\frac{1}{2}$ and $\beta\in\{\frac{1}{2},1\}$.

• Case 2: $\alpha=0$. Putting $z=\beta\,\tau$ and $s=z/\tau=\beta>0$ into (2.1) of Theorem 2.1 gives that

$$f(q) = \frac{\sqrt{2\pi\beta(1-e(\beta\,\tau))}}{\Gamma(\beta+1)}\,e\Big(-\frac{\tau}{24}\Big)\,\mathrm{e}^{\beta(\log\beta-1)+L(\tau)+P(\beta\,\tau,\tau)}\,R_1(\tau)\,,$$

so that

$$\log f(q) = \frac{1}{2}\,\log\frac{\tau}{\mathrm{i}} - I(\beta) + A(\tau) + L(\tau) + P(\beta\,\tau,\tau) + R(\tau)\,. \tag{3.9}$$

In the above, I denotes the function given by (2.6),

$$A(\tau) = \frac{\pi}{12}\,\frac{\tau}{\mathrm{i}} + \frac{1}{2}\,\log\frac{e\,(\beta\,\tau)-1}{\mathrm{i}\,\beta\,\tau}\,,\quad L(\tau) = \frac{\mathrm{Li}_2(e(\beta\,\tau))}{2\pi\mathrm{i}\tau}\,,$$

and

$$R(\tau) = \log R_1(\tau),\quad R_1(\tau) = \Big(\frac{\beta\,\tau-1}{\tau}\mid -\frac{1}{\tau}\Big)_\infty\,.$$

One can easily see that $A(\tau)\in\mathbb{C}\{\tau\}$. Letting $u=\beta\,\tau$ in (3.6) gives that $L(\tau\,\mathrm{e}^{2\pi\mathrm{i}})-L(\tau) = -2\beta\pi\mathrm{i}$. In view of the equality $\mathrm{Li}_2(1)=\frac{\pi^2}{6}$, the function L can be put into the following form:

$$L(\tau) = \frac{c_\infty}{\tau} + \beta\log\frac{\tau}{\mathrm{i}}\mod\mathbb{C}\{\tau\}\,,\quad c_\infty = -\frac{\pi}{12}\,\mathrm{i}\,.$$

In addition, by Proposition 3.5, one gets that $R\in\mathcal{A}_1^{\le -1}$. Thus, it follows from (3.9) that

$$\log f(q) = \frac{c_\infty}{\tau} + \Big(\frac{1}{2}+\beta\Big)\,\log\frac{\tau}{\mathrm{i}} + P(\beta\,\tau,\tau)\mod\mathbb{C}\{\tau\}\oplus\mathcal{A}_1^{\le -1}\,. \tag{3.10}$$

One knows that $P(\beta\,\tau,\tau)$ admits a Gevrey asymptotic expansion as $\tau\to 0$ in $\mathbb{C}\setminus(-\infty,0]$. As in Case 1 for $B(\cdots,\tau)$, comparing (3.10) with (1.5) gives that $P(\beta\,\tau,\tau)$ can be continued into an analytic function at $\tau = 0$. Thus, applying Theorem 3.8 (1) implies, finally, that $\beta \in \frac{1}{2}\mathbb{Z}$, whis gives that $\beta \in \{\frac{1}{2}, 1\}$.

• In summary, one finds that $f \in \mathfrak{C}_1$ implies that $\alpha \in \{0, \frac{1}{2}\}$ and $\beta \in \{\frac{1}{2}, 1\}$. This ends the proof of Theorem 3.1. □

4 Asymptotic Behavior at an Arbitrary Root via Continued Fractions

With regard to an arbitrary root ζ of unity, we shall establish the following result, which, together with Theorem 3.1, will imply Theorem 1.5:

Theorem 4.1 Let $r \in \mathbb{Q} \cap (0, 1)$, $\zeta = e(r)$ and $(\alpha, \beta) \in [0, 1) \times (0, 1]$, and consider $f(q) = (\alpha + \beta\,\tau \,|\, \tau)_\infty$. Then $f \in \mathfrak{C}_\zeta$ if and only if $\alpha \in \{0, \frac{1}{2}\}$ and $\beta \in \{\frac{1}{2}, 1\}$.

First, one will observe, in §4.1, that the corresponding functions B and P used in Theorems 2.8 and 2.1 are analytic at each non-zero rational point $\tau = r$. This allows us to establish one key lemma, Lemma 4.8, in Subsection 4.2, that permits us to pass an arbitrary rational value r to an other r_1. By iterating this procedure, one arrives at the case of $r = 0$, to which case Theorem 3.1 can be applied. This is realized in terms of the continued fractions relative to r and related modular transforms; see Theorem 4.9 in Subsection 4.3. We complete the proofs of Theorems 4.1, 1.5 and 1.2.

4.1 Bernoulli Integral and Associated Functions on a Real Axis

We will discuss the degenerate case $\tau \in \mathbb{R}_{>0}$ for the functions $b^+(z, \tau)$, $B(z, \tau)$ and $P(z, \tau)$. In what follows, we will make use of the notational convention

$$\epsilon \in (0, \frac{\pi}{2}), \quad W_\epsilon = V((-\epsilon.\epsilon); \infty), \quad W_\epsilon^c = \mathbb{C} \setminus \bar{W}_\epsilon\,, \tag{4.1}$$

and the letter r always denotes a given positive number.

First of all, we consider the function $b^+(z, \tau)$. It should be noted that the relation stated in (2.18) is valid for any $(z, \tau) \in \mathcal{H} \times \mathcal{H}$. If $z \notin \mathcal{H}$, we have to avoid the poles of the Gamma function, and the right-hand side of (2.18) continues to be well-defined over the Riemann surface of log while $s \notin \mathbb{Z}_{\leq 0}$. Thus, Lemma 2.5 allows one to make the analytic continuation of the function b^+ at (z, τ) provided that $(z + n)/\tau \notin \mathbb{Z}_{\leq 0}$ for all $n \in \mathbb{Z}_{\geq 1}$. This yields the following observation:

Remark 4.2 For any fixed $\tau \in \mathcal{H}$, $z \mapsto b^+(z, \tau)$ can be continued into an analytic function on the universal covering of $\mathbb{C} \setminus \Delta_\tau$, where

$$\Delta_\tau = \mathbb{Z}_{\leq -1} \oplus \tau\mathbb{Z}_{\leq 0}\,. \tag{4.2}$$

By Definition 2.2 and Remark 4.2, $b^+(z, \tau)$ is well-defined and analytic in the domain $(-W_\epsilon^c - 1) \times W_\epsilon$, where $-W_\epsilon^c - 1 = \mathbb{C} \setminus \left(-\bar{W}_\epsilon - 1\right)$; see Figure 4 below. In particular, we note the following fact:

Remark 4.3 Given $r > 0$, $\alpha \in (-1, 0)$ and $\beta \in \mathbb{R}$, there exists $\rho \in (0, r)$ such that the function $\tau \mapsto b^+(\alpha+\beta(\tau-r), \tau)$ is well-defined and analytic in the open disc $\{\tau \in \mathbb{C} : |\tau-r| < \rho\}$.

When $\epsilon \to 0^+$, W_ϵ becomes $(0, \infty)$ and $-W_\epsilon^c - 1$ is reduced into $\mathbb{C} \setminus (-\infty, -1]$. By replacing τ with r in the partial lattice Δ_τ given by (4.2) for all $\tau \in \mathcal{H}$, we will continue to write

$\Delta_r = \{n + mr : n \in \mathbb{Z}_{\le -1}, m \in \mathbb{Z}_{\le 0}\}$. It is easy to see that Δ_r is discrete on the real axis if and only if r is a rational number. In this way, we shall make use of the following remark:

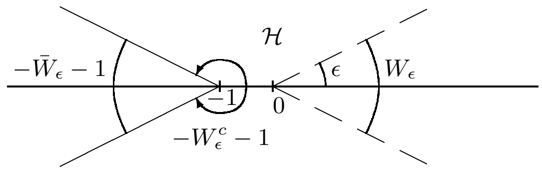

Figure 4 $b^+(z,\tau)$ is analytic for $z \in (-W_\epsilon^c - 1)$ and $\tau \in W_\epsilon$

Remark 4.4 If n and d are co-prime positive integers such that $r = \frac{n}{d} \in (0,1)$, then

$$\Delta_r = -1 + \frac{1}{d}\,\mathbb{Z}_{\le 0} \subset (-\infty, -1]\,. \tag{4.3}$$

For any $\alpha \in \mathbb{R}$, let $[\alpha]$ denote the integral part of α and $\{\alpha\}$ denote the corresponding fractional part, that is $\alpha - [\alpha]$. If $z_0 \in (-\infty, -1] \setminus \Delta_r$, we define

$$\breve{b}(z_0, r) = \left([-z_0] - 1 - 2\sum_{k=1}^{[-z_0]-1}\left\{\frac{z_0 + k}{r}\right\}\right)\pi\mathrm{i}\,. \tag{4.4}$$

Theorem 4.5 Let $r \in (0,1)\cap\mathbb{Q}$ and let Δ_r, $\breve{b}$ be as in the above. Then $b^+(z,r)$ is analytic in $\mathbb{C}\setminus(-\infty,-1]$ and can be continued to be an analytic function over the universal covering of $\mathbb{C}\setminus\Delta_r$ in such a way that the following relation holds for all $z_0 \in (-\infty,-1]\setminus\Delta_r$:

$$\lim_{\epsilon\to 0^+}\big(b^+(z_0 + \mathrm{i}\epsilon, r) - b^+(z_0 - \mathrm{i}\epsilon, r)\big) = \breve{b}(z_0, r)\,. \tag{4.5}$$

Proof As $\tau \to r$ in $\mathcal{H}$, the limit set Δ_r of the singularities of $b^+(z,\tau)$ is discrete, as stated in (4.3). By considering Remark 4.2, one obtains that $b^+(z,r)$ is analytic over the universal covering of $\mathbb{C}\setminus\Delta_r$.

Now, let $D(z_0)$ denote the expression in the left-hand side of (4.5). By putting $\tau = r$ into (2.18), one finds that, if $z \notin (-\infty, 0]$, then $b^+(z-1,r) = b^+(z,r) + I(s)$, where $s = z/r$ and $I(s)$ is given in (2.6). Thus, one can write

$$D(z_0 - 1) = D(z_0) + \lim_{\epsilon\to 0^+}\Big(I(\frac{z_0}{r} + \epsilon\mathrm{i}) - I(\frac{z_0}{r} - \epsilon\mathrm{i})\Big)$$

for all $z_0 \in (-\infty, 0]\setminus\Delta_r$. By using (2.6), one gets that

$$D(z_0 - 1) = D(z_0) + (1 - \frac{2z_0}{r})\pi\mathrm{i} + \lim_{\epsilon\to 0^+}\Big(\log\Gamma(\frac{z_0}{r} + \epsilon\mathrm{i}) - \log\Gamma(\frac{z_0}{r} - \epsilon\mathrm{i})\Big)\,. \tag{4.6}$$

Let n be any negative integer, say, $n = -m$, $m > 0$, and let $s \in (n, n+1)$. From the relation $\Gamma(x) = \dfrac{\Gamma(x+m)}{(x)_m}$ and the fact that $\log\Gamma(x+m)$ is well-defined and analytic for $\Re(x) > n$, it follows that

$$\lim_{\epsilon\to 0^+}\left(\log\Gamma(s+\epsilon\mathrm{i}) - \log\Gamma(s-\epsilon\mathrm{i})\right) = -\lim_{\epsilon\to 0^+}\left(\log(s+\epsilon\mathrm{i})_m - \log(s-\epsilon\mathrm{i})_m\right) = 2\pi\mathrm{i}n\,.$$

Therefore, (4.6) yields that

$$D(z_0 - 1) = D(z_0) + 2\left(n - \frac{z_0}{r}\right)\pi\mathrm{i} + \pi\mathrm{i}\,. \tag{4.7}$$

Letting $z_0 = (n+\delta_0)r$ with $n_0 \in \mathbb{Z}_{<0}$ and $\delta_0 \in (0,1)$, the above expression in (4.7) becomes $D(z_0-1) - D(z_0) = -2\pi \mathrm{i}\delta_0 + \pi \mathrm{i}$. By replacing z_0 with z_0+1 and iterating this process, one obtains the finite sequences (n_k) and (δ_k) associated with the pair (z_0, r) in the following manner: $z_0+k = (n_k+\delta_k)r$, where $n_k \in \mathbb{Z}_{<0}$ and $\delta_k = \delta_k(z_0,r) = \{\frac{z_0+k}{r}\}$. Since $D(z_0+k)=0$ for $k > -1-z_0$, one finds that

$$D(z_0-1) = -2\left(\delta_0 + \cdots + \delta_{[-z_0]-1}\right)\pi \mathrm{i} + [-z_0]\,\pi \mathrm{i}\,.$$

Replacing z_0 with z_0+1 in this last relation gives $D(z_0) = \breve{b}(z_0, r)$, where $\breve{b}$ is as given in (4.4), so one obtains the expected relation (4.5) and Theorem 4.5. □

By using (2.25), one finds that $B(z,\tau)$ is analytic in the domain

$$((1/2 + W_\epsilon^c) \cap (-1/2 - W_\epsilon^c)) \times W_\epsilon\,.$$

In addition, from (2.20) one obtains that $P(z,\tau)$ can be continued to be analytic in the domain $((-1-W_\epsilon^c)\cap(1+W_\epsilon^c)) \times W_\epsilon$; see Figure 5 below, where W_ϵ and W_ϵ^c are given in (4.1).

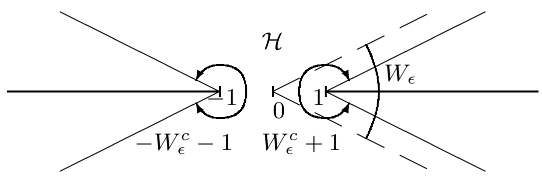

Figure 5 $P(z,\tau)$ is analytic for $z \in (-W_\epsilon^c - 1) \cap (W_\epsilon^c + 1)$ and $\tau \in W_\epsilon$

Similarly to Remark 4.3, one can observe the following property:

Remark 4.6 Let $r>0$ and $(\alpha,\beta) \in \mathbb{R}^2$. If $\alpha + \beta r \in (-\frac{1}{2}, \frac{1}{2})$, then $\tau \mapsto B(\alpha+\beta\,\tau, \tau)$ is well-defined and analytic inside some open disc centered at $\tau = r$.

Moreover, things are the same for $\tau \mapsto P(\alpha + \beta\,\tau, \tau)$ when $\alpha + \beta r \in (-1, 1)$.

By letting $\epsilon \to 0^+$, one sees that, for any $r>0$, $B(z,r)$ is analytic for all $z \in \mathbb{C} \setminus (-\infty, -1/2) \cup (1/2, \infty)$, while $P(z,r)$ is analytic for $z \in \mathbb{C} \setminus (-\infty, -1] \cup [1, \infty)$.

Theorem 4.7 Let $r \in (0,1) \cap \mathbb{Q}$, and let Δ_r and $\breve{b}$ be as in (4.3) and (4.4). Then

(1) the function $B(z,r)$ can be continued to be analytic in the universal covering of $\mathbb{C} \setminus ((1/2+\Delta_r) \cup (-1/2-\Delta_r))$, and the following relations hold for all $z_0 \in \mathbb{R} \setminus \big((1/2+\Delta_r) \cup (-1/2-\Delta_r)\big)$:

$$\lim_{\epsilon \to 0^+} \left(B(z_0 + i\epsilon, r) - B(z_0 - i\epsilon, r)\right) = -\breve{b}(-|z_0| - \frac{1}{2}, r)\,; \tag{4.8}$$

(2) the function $P(z,r)$ can be continued to be analytic in the universal covering of $\mathbb{C} \setminus (\Delta_r \cup (-\Delta_r))$ in such a way that, for all $z_0 \in \mathbb{R} \setminus (\Delta_r \cup (-\Delta_r))$,

$$\lim_{\epsilon \to 0^+} \left(P(z_0 + \mathrm{i}\epsilon, r) - P(z_0 - \mathrm{i}\epsilon, r)\right) = -\breve{b}(-|z_0|, r)\,. \tag{4.9}$$

Proof This follows directly from Theorem 4.5 together with relations (2.25) and (2.20). □

4.2 One Key Lemma

As in the definition of $\breve{b}(z,r)$ in (4.4), we will let $[a]$ and $\{a\}$ denote the integral and fractional part, respectively, of any given real a. Given each non-zero real r, consider the associated one-to-one map T_r defined on $[0,1)\times(0,1]$ as follows:

$$T_r\ :(\alpha,\beta)\mapsto(\{(1-\alpha)\ \left[-\frac{1}{r}\right]+\beta\},1-\alpha)\,. \tag{4.10}$$

One finds easily that

$$T_r(\{0,\frac{1}{2}\}\times\{\frac{1}{2},1\})=\{0,\frac{1}{2}\}\times\{\frac{1}{2},1\}\,. \tag{4.11}$$

Lemma 4.8 Let $r\in(0,1)\cap\mathbb{Q}$, $\zeta=e(r)$, $\zeta_1=e(\{-\frac{1}{r}\})$, and let $z(\tau)=\alpha+\beta\tau$ with $(\alpha,\beta)\in[0,1)\times(0,1]$. Consider

$$f(q)=(z(\tau)\,|\,\tau)_\infty\,,\quad g(q_1)=(z_1(\tau_1)\,|\,\tau_1)_\infty\,,$$

where $q=e(\tau)$, $\tau_1=-\frac{1}{\tau}-\left[-\frac{1}{r}\right]$, $q_1=e(\tau_1)$ and

$$z_1(\tau_1)=\alpha_1+\beta_1\,\tau_1\,, \tag{4.12}$$

(α_1,β_1) being the transform of (α,β) defined by (4.10). Then $f\in\mathfrak{C}_\zeta$ if and only if $g\in\mathfrak{C}_{\zeta_1}$.

Proof For simplicity, write $r_1=\{-\frac{1}{r}\}$ and $N=[-\frac{1}{r}]$. As $-\frac{1}{r}=r_1+N$, the following equivalence holds on the upper half-planes $\Im\tau>0$ and $\Im\tau_1>0$:

$$\tau\overset{a.v.}{\longrightarrow}r\quad\Longleftrightarrow\quad\tau_1\overset{a.v.}{\longrightarrow}r_1\,. \tag{4.13}$$

By observing that $\tau=-1/(\tau_1+N)$, one gets that $(z(\tau)-1)/\tau=(1-\alpha)(\tau_1+N)+\beta$, so it follows from (4.12) that $z_1(\tau_1)=(z(\tau)-1)/\tau \bmod \mathbb{Z}$. One remembers that $(z\,|\,\tau)_\infty=(z'\,|\,\tau')$ if $(z,\tau)=(z',\tau') \bmod \mathbb{Z}^2$; see (1.3). Thus, one finds that $g(q_1)=\tilde g(q)$ if one defines

$$\tilde g(q)=\left(\frac{z(\tau)-1}{\tau}\mid-\frac{1}{\tau}\right)_\infty\,. \tag{4.14}$$

By noticing the relation $\tau_1-r_1=(\tau-r)/(r\tau)$, it follows from (4.13) that $g(q_1)\in\mathfrak{C}_{\zeta_1}$ if and only if $\tilde g(q)\in\mathfrak{C}_\zeta$. Thus, we shall use Theorems 2.8 and 2.1 to link $f(q)$ with $\tilde g(q)$ in the following fashion:

$$f(q)=H(q)\,\tilde g(q)\,. \tag{4.15}$$

By hypothesis, $(\alpha,\beta)\notin\mathbb{Z}\times\{0\}$, so $f(q)$ is not identically null. As $\mathfrak{C}_\zeta\setminus\{0\}$ constitutes a multiplicative group, Lemma 4.8 says exactly that $H\in\mathfrak{C}_\zeta$, which is what we need to establish.

As before, write $\hat\tau=\tau-r$. As in the proof of Theorem 3.1, we shall distinguish two cases: $\alpha+\beta r\notin\mathbb{Z}$ and $\alpha+\beta r\in\mathbb{Z}$.

• Case 1: $\alpha+\beta r-n\in(0,1)$ with some $n\in\mathbb{Z}$. Let $\alpha'=\alpha-n-\frac{1}{2}$, and observe that $\alpha'+\beta r\in(-\frac{1}{2},\frac{1}{2})$. By applying Theorem 2.8 to $z=z(\tau)-n-\frac{1}{2}$, the factor $H(q)$ defined by (4.15) can be written as follows: $H(q)=H_1(q)\,H_2(q)\,H_3(q)$, where

$$H_1(q)=\sqrt{1-e(z(\tau))}\,e(-\frac{\tau}{24})\,\mathrm{e}^{\frac{\mathrm{Li}_2(e(z(\tau)))}{2\pi\mathrm{i}\tau}}\,,\quad H_2(q)=\mathrm{e}^{B(\alpha'+\beta\,\tau,\tau)}$$

and

$$H_3(q)=\frac{((z(\tau)-n-1)/\tau\mid-1/\tau)_\infty}{((z(\tau)-1)/\tau\mid-1/\tau)_\infty}\,. \tag{4.16}$$

When $\tau\to r$, it follows that $e\,(z(\tau))\to e(\alpha+\beta r)\neq1$ for $\alpha+\beta r\notin\mathbb{Z}$. Thus $\mathrm{Li}_2(e(z(\tau)))$ is really analytic at $\tau=r$ in $\mathbb{C}$. As $r\neq0$, one finds, finally, that $H_1\in\mathbb{C}\{\hat\tau\}$. In addition, Remark 4.6 implies that $B(\alpha'+\beta\,\tau,\tau)$ is analytic at $\tau=r$. Furthermore, one can express H_3 as

$$H_3(q)=\left(\frac{z(\tau)-n-1}{\tau}\mid-\frac{1}{\tau}\right)_{-n}\quad\text{or}\quad H_3(q)=\frac{1}{((z(\tau)-1)/\tau\mid-1/\tau)_n}$$

for $n \in \mathbb{Z}_{\le 0}$ or $\mathbb{Z}_{\ge 0}$, respectively; see (1.3). This also shows that $H_3 \in \mathbb{C}\{\hat{\tau}\}$. Thus, one gets that $H \in (\mathbb{C}\{\hat{\tau}\} \setminus \{0\}) \subset \mathfrak{C}_\zeta$.

• Case 2: $\alpha + \beta r = n \in \mathbb{Z}$. Let $s = (z(\tau) - n)/\tau$, and notice that $z(\tau) - n = \beta\,\hat{\tau}$, so $s = \beta\,\hat{\tau}/\tau$. As $\beta \ne 0$, one gets that $s \in \mathbb{C} \setminus \mathbb{R}$ for all $\tau \in \mathcal{H}$. Moreover, one has $\mathbb{C}\{s\} = \mathbb{C}\{\hat{\tau}\}$ for $r \ne 0$.

Putting $z = z(\tau) - n = \beta\,\hat{\tau}$ into Theorem 2.1 gives that the factor $H(q)$ defined by (4.15) can be read as follows: $H(q) = H_0(q)\,H_1(q)\,H_2(q)\,H_3(q)$, where

$$H_0(q) = \frac{\sqrt{2\pi s(1 - e(\tau s))}}{\Gamma(s+1)}\, e(-\frac{\tau}{24}), \quad H_1(q) = \mathrm{e}^{s(\log s - 1) + \frac{\mathrm{Li}_2(e(\tau s))}{2\pi \mathrm{i}\tau}},$$

$H_2(q) = \mathrm{e}^{P(\beta\,(\tau - r), \tau)}$, and where $H_3(q)$ is given by (4.16). It is immediately apparent that $H_0 \in \mathbb{C}\{s\}$, so $H_0 \in \mathbb{C}\{\hat{\tau}\}$ also. Also, from Remark 4.6, one knows that $H_2 \in \mathbb{C}\{\hat{\tau}\}$; it is the same for H_3, as explained in the above. Furthermore, replacing u with τs in (3.6) of Lemma 3.9 gives that $\mathrm{Li}_2(e(\tau s \mathrm{e}^{2\pi \mathrm{i}})) - \mathrm{Li}_2(e(\tau s)) = 4\pi^2 \tau s$, so one gets the identity

$$s\,\log(s\mathrm{e}^{2\pi\mathrm{i}}) + \frac{\mathrm{Li}_2(e(\tau s \mathrm{e}^{2\pi\mathrm{i}}))}{2\pi\mathrm{i}\tau} = s\,\log s + \frac{\mathrm{Li}_2(e(\tau s))}{2\pi\mathrm{i}\tau},$$

which implies that $H_2 \in \mathbb{C}\{s\} = \mathbb{C}\{\hat{\tau}\}$. Finally, one finds that $H \in (\mathbb{C}\{\hat{\tau}\} \setminus \{0\}) \subset \mathfrak{C}_\zeta$, which ends the proof. □

4.3 Continued Fractions and Modular Transforms

Let us consider the asymptotic behavior of the Euler function $(z\,|\,\tau)_\infty$ when $\tau \xrightarrow{a.v.} r \in (0,1) \cap \mathbb{Q}$ or, equivalently, when $q \xrightarrow{a.r.} \zeta = e(r)$. Our strategy is to use continued fractions in order to reduce the general case $\tau \xrightarrow{a.v.} r$ to the known case $\tau \xrightarrow{a.v.} 0$.

Indeed, the above operation $(r, \tau, z) \mapsto (r_1, \tau_1, z_1)$, considered in Lemma 4.8, allows one to link two rational numbers: r and r_1. By iterating this process, one arrives at the case where $\tau \xrightarrow{a.v.} 0$. This iteration procedure requires one to write r into continued fraction. Thus, to any given $r = \frac{p}{m} \in \mathbb{Q} \cap (0,1)$ will be associated the sequences $r_j \in \mathbb{Q} \cap [0,1)$ and $d_j \in \mathbb{Z}_{>0}$ in the following manner:

$$r_0 = \frac{p}{m}, \ d_0 = 0; \quad r_j = \left\{-\frac{1}{r_{j-1}}\right\}, \ d_j = \left[-\frac{1}{r_{j-1}}\right] \ (\, 1 \le j \le \nu), \tag{4.17}$$

where ν denotes the smallest index such that $r_\nu = 0$, i.e., $1/r_{\nu-1} \in \mathbb{Z}_{>0}$. With the standard notation for the continued fractions, one can notice that

$$r = [0, d_1, -d_2 \cdots, (-1)^{\nu-1} d_\nu] = \frac{1}{d_1 -|}\ \frac{1}{d_2 -|}\ \cdots\ \frac{1}{|d_\nu}. \tag{4.18}$$

Now, given $(\alpha_0, \beta_0) \in [0,1) \times (0,1]$ and $r \in (0,1) \cap \mathbb{Q}$ as in (4.18), define the r-depending sequence $(z_j, \tau_j)_{0 \le j \le \nu}$ as follows: $\tau_0 = \tau$, $z_0 = z_0(\tau_0) = \alpha_0 + \beta_0\,\tau$,

$$\tau_j = -\frac{1}{\tau_{j-1}} - d_j, \quad z_j = z_j(\tau_j) = \alpha_j + \beta_j\,\tau_j, \tag{4.19}$$

where $(\alpha_j, \beta_j) = T_{r_{j-1}}(\alpha_{j-1}, \beta_{j-1})$, $T_{r_{j-1}}$ being the transform obtained by substituting r_{j-1} to r in (4.10). If $\tau = \tau_0 \xrightarrow{a.v.} r$ in $\mathcal{H}$, then $\tau_j \xrightarrow{a.v.} r_j$ in $\mathcal{H}$, particularly with $\tau_\nu \xrightarrow{a.v.} 0$. Furthermore, it is easy to see that $\tau_j \in \mathcal{H}$, with

$$\tau_j = M_j\,\tau_{j-1}, \quad M_j = \begin{pmatrix} -d_j & -1 \\ 1 & 0 \end{pmatrix} \in SL(2;\mathbb{Z}),$$

where $\tau \mapsto M\,\tau$ denotes the classic modular transform associated with a modular matrix M. Thus, one can find that $\tau_\nu = M\,\tau$ with $M = M_\nu \cdots M_1 \in SL(2;\mathbb{Z})$.

Theorem 4.9 Let (ν, r), τ_j and $z_j(\tau_j)$ be given as in (4.18) and (4.19), with $(\alpha,\beta) \in [0,1) \times (0,1]$. Let $\zeta_j = e(r_j)$ and $q_j = e(\tau_j)$ for j from 0 to ν. Consider

$$f(q) = (z_0(\tau_0)|\tau)_\infty, \quad f_j(q_j) = (z_j(\tau_j)|\tau_j)_\infty\,.$$

Then, the following conditions are equivalent:

(1) $f \in \mathfrak{C}_\zeta$;

(2) $f_\nu \in \mathfrak{C}_1$;

(3) $f_j \in \mathfrak{C}_{\zeta_j}$ for all j from 1 until ν;

(4) $\alpha \in \{0, \frac{1}{2}\}$ and $\beta \in \{1, \frac{1}{2}\}$.

Proof For simplicity, write $\Delta = \{0, \frac{1}{2}\} \times \{1, \frac{1}{2}\}$. By (4.11), it follows that $(\alpha,\beta) \in \Delta$ if and only if $(\alpha_j, \beta_j) \in \Delta$ for (one of) all indices j from 0 to ν. In addition, applying Theorem 3.1 to f_ν implies that $(\alpha_\nu, \beta_\nu) \in \Delta$ if and only if $f_\nu \in \mathfrak{C}_1$. Thus, by considering Lemma 4.8, one finds that all conditions (1)–(4) stated in Theorem 4.9 are equivalent. □

Now, we are ready to finish, successively, the proofs for Theorems 4.1 and 1.5 and, therefore, the proof for the main theorem.

Proof of Theorem 4.1 This follows directly from Theorem 4.9. □

Proof of Theorem 1.5 In view of Theorems 3.1 and 4.1, it suffices to notice that, given $\zeta \in \mathbb{U}$, one has $(x_0\, q^\beta; q)_\infty \in \mathfrak{C}_\zeta$ if and only if the same holds by replacing β with $\beta + 1$. This last equivalence can be deduced from the relation $(x_0\, q^\beta; q)_\infty = (1 - x_0\, q^\beta)\,(x_0\, q^{\beta+1}; q)_\infty$ and the fact that $(1 - x_0\, q^\beta) \in \mathfrak{C}_\zeta$, for $\mathfrak{C}_\zeta \setminus \{0\}$ constitutes a multiplicative group. □

Proof of Theorem 1.2 By taking into account Remark 1.1 and Theorem 1.5, one needs only to observe that, for any positive integer $n \in \mathbb{Z}_{>0}$ and any root $\zeta \in \mathbb{U}$, any finite product of the form $(x_0\, q^\beta; q)_n$ does not belong to $\mathfrak{T}_\zeta$, although the same function belongs to the larger class $\mathfrak{C}_\zeta$. □

Addendum After having finished a first version of our paper, we learned that the interesting work [4] is closely related to the present paper. Indeed, let $\alpha > 0$ and $\mu \in [0,1)$ be as in [4, Theorem 1]. By combining [4, (3.2) & (3.3)] together with (1.2) and (1.4), one can observe the following result:

Remark 4.10 One has $(e(\mu)q^\alpha; q)_\infty \in \mathfrak{T}_1$ only if the following conditions are satisfied for all integers $k \geq 2$:

$$\mathcal{B}_k(0, e(\mu))\, B_{k+1}(\alpha) = 0\,. \tag{4.20}$$

In the above, $B_k(\alpha)$ denotes the usual Bernoulli polynomials, and $\mathcal{B}_k(\alpha, y)$ are the rational functions defined by the Taylor series expansion

$$\frac{z\,\mathrm{e}^{\alpha z}}{y\,\mathrm{e}^z - 1} = \sum_{k=0}^\infty \frac{\mathcal{B}_k(\alpha, y)}{k!}\, z^k \tag{4.21}$$

near $z = 0$. One can notice that $B_k(\alpha) = \mathcal{B}_k(\alpha, 1)$ and $B_k = B_k(0) = \mathcal{B}_k(0,1)$, where B_k are the Bernoulli numbers. Furthermore, one knows from [1, p. 55, 44 (c)] or [7, p. 6, (2.71)] that

$$B_k(\alpha) = \sum_{j=0}^k \binom{k}{j} B_j\, \alpha^{k-j}\,. \tag{4.22}$$

Proposition 4.11 The following conditions are equivalent for any $(\mu,\alpha)\in[0,1)\times(0,+\infty)$:

(1) relation (4.20) holds simultaneously for all integers $k\geq 2$;

(2) relation (4.20) holds simultaneously for $k=2$ and $k=3$;

(3) $(\mu,\alpha)\in\{(0,\frac{1}{2}),(0,1),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},1)\}$.

Proof It is obvious that (1) implies (2). To see (2) $\Leftrightarrow$ (3), firstly let $\mu=0$ and $y=e(0)=1$. Relation (4.20) becomes $B_k\,B_{k+1}(\alpha)=0$. By [1, p. 12] or [7, p. 6], it follows that $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$ and $B_3=0$. Using (4.22) gives that $B_3(\alpha)=\alpha(\alpha-\frac{1}{2})(\alpha-1)$. Thus, (4.20) holds simultaneously for $k=2$ and $k=3$ if and only if $\alpha\in\{\frac{1}{2},1\}$. Next, let $\mu\in(0,1)$ and $y=e(\mu)\neq 1$; putting $\alpha=0$ into (4.21), one can get that $\mathcal{B}_0(0,y)=0$, $\mathcal{B}_1(0,y)=\frac{1}{y-1}$, $\mathcal{B}_2(0,y)=-\frac{2\,y}{(y-1)^2}$ and $\mathcal{B}_3(0,y)=\frac{3y(y+1)}{(y-1)^3}$. Again, one can notice that $\mathcal{B}_2(0,y)\neq 0$, so relation (4.20) holds for $k=2$ only if $\alpha\in\{\frac{1}{2},1\}$. Since $B_4=-\frac{1}{30}$, one obtains, applying (4.22), that $B_4(\alpha)=\alpha^4-2\alpha^3+\alpha^2-\frac{1}{30}$. One can see that $B_4(\frac{1}{2})\neq 0$ and $B_4(1)\neq 0$. Therefore, (4.20) holds simultaneously for both $k=2$ and $k=3$ if and only if $\alpha\in\{\frac{1}{2},1\}$ and $e(\mu)=-1$. This implies the equivalence between the conditions (2) and (3) stated in Proposition 4.11. Finally, suppose that condition (3) holds. It follows from Theorem 1.2 that $(e(\mu)\,q^\alpha;q)_\infty\in\mathfrak{T}_1$, which, together with Remark 4.10, implies (1). In this way, we complete the proof of Proposition 4.11. □

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