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THE GROWTH OF DIFFERENCE EQUATIONS AND DIFFERENTIAL EQUATIONS*

Dedicated to the memory of Professor Jiarong YU

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Abstract In this paper, we mainly apply a new, asymptotic method to investigate the growth of meromorphic solutions of linear higher order difference equations and differential equations. We delete the condition (1.6) of Theorems E and F, yet obtain the same results for Theorems E and F. We also weaken the condition (1.4) of Theorems C and D.

Key words asymptotic Method; difference equations; differential equations

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1 Introduction and Main Results

In this paper, we apply a new, asymptotic method to investigate the growth of transcendental meromorphic solutions of linear higher order difference equations and differential equations.

We assume that the reader is familiar with the basic idea of Nevanlinna's value distribution theory. The Nevanlinna theory is an important tool in this paper, and its usual notations and basic results come mainly from [13, 16, 20, 22]. Here we let $f(z)$ be a nonconstant meromorphic function in the complex plane.

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We use $\sigma(f)$ to denote the order of $f(z)$, $\lambda(f)$ to denote the convergence exponent of zeros of $f(z)$, and $\overline{\lambda}(f)$ to denote the convergence exponent of distinct zeros of $f(z)$.

For $n \in \mathbb{N}$, we define

$$\Delta f(z) = f(z+1) - f(z), \quad \Delta^n f(z) = \Delta(\Delta^{n-1} f(z)).$$

Over the course of the last 15 years, many authors have paid great attention to complex difference equations and to the difference analogues of Nevanlinna's theory, and have obtained many interesting results, including [2–6, 8, 10, 11, 15, 18, 19, 21, 23].

Ishizaki and Yanagihara [19] considered the growth of transcendental entire solutions of difference equations

$$Q_n(z)\Delta^n f(z) + \cdots + Q_1(z)\Delta f(z) + Q_0(z) f(z) = 0, \tag{1.1}$$

where $Q_n, \cdots, Q_0$ are polynomials, and obtained the following theorem:

Theorem A Let $f(z)$ be a transcendental entire solution of (1.1), and let its order $\chi < 1/2$. Then

$$\log M(r, f) = Lr^{\chi}(1 + o(1)),$$

where a rational number χ is the slope of a Newton polygon for the equation (1.1), and $L > 0$ is a constant. In particular, we have that $\chi > 0$.

Note that the equation (1.1) can be rewritten as

$$P_n(z) f(z+n) + \cdots + P_1(z) f(z+1) + P_0(z) f(z) = 0. \tag{1.2}$$

Example 1.1 (see [19]) Suppose that $f(z)$ is a transcendental entire solution of the difference equation

$$(6z^2 + 19z + 15)\Delta^3 f(z) + (z+3)\Delta^2 f(z) - \Delta f(z) - f(z) = 0.$$

In fact, in [19], Ishizaki and Yanagihara proved that this difference equation admits an entire solution of order $\frac{1}{3}$ by using the method of a Newton polygon.

Chiang and Feng [10] proved

Theorem B Let $P_n(z), \cdots, P_0(z)$ be polynomials such that there exists an integer, l, $0 \le l \le n$, such that

$$\deg(P_l) > \max_{0 \le j \le n, j \ne l} \{\deg(P_j)\} \tag{1.3}$$

holds. Supposing that $f(z)$ is a meromorphic solution of (1.2), we then have that $\sigma(f) \ge 1$.

Chen weakened the condition (1.3) of Theorem B and obtained

Theorem C (see [4, 5]) Let $F(z), P_n(z), \cdots, P_0(z)$ be polynomials such that $FP_nP_0 \not\equiv 0$ and

$$\deg(P_n + \cdots + P_0) = \max\{\deg(P_j) : j = 0, \cdots, n\} \ge 1. \tag{1.4}$$

Then every finite order transcendental meromorphic solution $f(z)$ of

$$P_n(z) f(z+n) + \cdots + P_1(z) f(z+1) + P_0(z) f(z) = F(z) \tag{1.5}$$

satisfies $\sigma(f) \ge 1$ and $\lambda(f) = \sigma(f)$.

Theorem D (see [4, 5]) Let $P_n(z), \cdots, P_0(z)$ be polynomials such that $P_nP_0 \not\equiv 0$ and (1.4) is satisfied. Then every finite order transcendental meromorphic solution $f(z)(\not\equiv 0)$ of equation (1.2) satisfies $\sigma(f) \geq 1$, and $f(z)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often and $\lambda(f-a) = \sigma(f)$.

Chen considered difference equations with constant coefficients, and obtained the following two theorems:

Theorem E (see [4]) Let $C_n, \cdots, C_0$ be constants such that $C_nC_0 \neq 0$ and such that they satisfy

$$C_n + \cdots + C_0 \neq 0. \tag{1.6}$$

Then every finite order transcendental meromorphic solution $f(z)(\not\equiv 0)$ of the equation

$$C_n f(z+n) + \cdots + C_1 f(z+1) + C_0 f(z) = 0 \tag{1.7}$$

satisfies $\sigma(f) \geq 1$, $f(z)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often, and $\lambda(f-a) = \sigma(f)$.

Theorem F Let $C_n, \cdots, C_0$ be constants and let $F(z)$ be a polynomial such that $FC_nC_0 \not\equiv 0$ and (1.6) is satisfied. Then every finite order transcendental meromorphic solution $f(z)$ of the equation

$$C_n f(z+n) + \cdots + C_1 f(z+1) + C_0 f(z) = F(z) \tag{1.8}$$

satisfies that $\lambda(f) = \sigma(f) \geq 1$.

Question 1.1 Can the condition (1.6) be deleted from Theorems E and F?

In this paper, we answer this question in the affirmative and delete condition (1.6) from Theorems E and F, and obtain the following theorems:

Theorem 1.1 Let $C_n, \cdots, C_0$ be constants such that $C_nC_0 \neq 0$. Then every finite order transcendental meromorphic solution $f(z)(\not\equiv 0)$ of the equation (1.7) satisfies that $\sigma(f) \geq 1$.

Theorem 1.2 Let $C_n, \cdots, C_0$ be constants, and let $F(z)$ be a polynomial such that $FC_nC_0 \not\equiv 0$. Then every finite order transcendental meromorphic solution $f(z)$ of the equation (1.8) satisfies that $\lambda(f) = \sigma(f) \geq 1$.

Remark 1.1 In Theorems 1.1 and 1.2, we have deleted condition (1.6) of Theorems E and F.

In Theorem 1.1, we cannot give the result that every finite order transcendental meromorphic solution $f(z)$ of (1.7) assumes every nonzero value $a \in \mathbb{C}$ infinitely often and that $\lambda(f-a) = \sigma(f)$.

In Theorem 1.2, we obtain the same results as for Theorem F.

Remark 1.2 By Theorems 1.1 and 1.2, we see that in Theorems C and D, the condition (1.4) can be weakened as

$$\deg(P_n + \cdots + P_0) = \max\{\deg(P_j) : j = 0, \cdots, n\}, \tag{1.4$'$}$$

where "≥ 1" of (1.4) is deleted. Thus, we can obtain the following corollaries:

Corollary 1.1 Let $F(z), P_n(z), \cdots, P_0(z)$ be polynomials such that $FP_nP_0 \not\equiv 0$ and such that they satisfy $(1.4)'$. Then every finite order transcendental meromorphic solution $f(z)$ of (1.5) satisfies that $\sigma(f) \geq 1$ and $\lambda(f) = \sigma(f)$.

Corollary 1.2 Let $P_n(z), \cdots, P_0(z)$ be polynomials such that $P_nP_0 \not\equiv 0$ and such that they satisfy $(1.4)'$. Then every finite order transcendental meromorphic solution $f(z)(\not\equiv 0)$ of (1.2) satisfies that $\sigma(f) \geq 1$.

Example 1.2 The difference equation

$$e^{-1}f(z+2)+2f(z+1)-3ef(z)=0$$

has a solution such that $f(z)=e^{z}$ and $\sigma(f)=1$.

Example 1.3 The difference equation

$$f(z+2)+2f(z+1)+f(z)=4z^2+8z+6$$

has a polynomial solution $f_1(z)=z^2$ and a transcendental entire solution $f_2(z)=\cos\pi z+z^2$ with $\sigma(f_2)=1$.

In what follows, we discuss the growth of linear differential equations with constant coefficients. In 1982, Bank and Laine proved the following result:

Theorem H (see [1]) Suppose that A_0 is a polynomial with $\deg A_0(z)\geq 1$, and that $f(z)\not\equiv 0$ is a meromorphic solution of a differential equation

$$f''(z)+A_0(z)f(z)=0. \tag{1.9}$$

Then $\sigma(f)=\frac{n+1}{2}$.

Since 1982, many authors have studied the growth of solutions of linear differential equations and obtained many good results, see [7, 9, 12, 14].

Now, we consider the growth of solutions of homogeneous and non-homogeneous linear differential equations with constant coefficients, and obtain the following results:

Theorem 1.3 Let $C_n,\cdots,\ C_0$ be constants such that $C_nC_0\neq 0$. Then every meromorphic solution $f(z)(\not\equiv 0)$ of the homogeneous differential equation

$$C_nf^{(n)}(z)+C_{n-1}f^{(n-1)}(z)+\cdots+C_1f'(z)+C_0f(z)=0 \tag{1.10}$$

satisfies that $\sigma(f)=1$.

Theorem 1.4 Let $C_n,\cdots,\ C_0$ be constants, and let $F(z)$ be a polynomial such that $FC_nC_0\not\equiv 0$. Then every transcendental meromorphic solution $f(z)$ of the non-homogeneous differential equation

$$C_nf^{(n)}(z)+C_{n-1}f^{(n-1)}(z)+\cdots+C_1f'(z)+C_0f(z)=F(z) \tag{1.11}$$

satisfies that $\overline{\lambda}(f)=\lambda(f)=\sigma(f)=1$.

Remark 1.3 From Theorems 1.1 and 1.3, we see that for homogeneous equations (1.7) or (1.10), we can only obtain that $\sigma(f)\geq 1$ or $\sigma(f)=1$.

From Theorems 1.2 and 1.4, we see that for non-homogeneous equations (1.8) or (1.11), we can obtain that $\lambda(f)=\sigma(f)\geq 1$ or $\lambda(f)=\sigma(f)=1$.

Remark 1.4 From Theorem 1.3, we see that homogeneous equation (1.10) does not have a polynomial solution. From Theorem 1.4, we see that non-homogeneous equation (1.11) may have a polynomial solution.

Example 1.4 The differential equation

$$2f^{(3)}+3f''(z)+2f'(z)+3f(z)=0$$

has solutions $f(z)=\cos z$ and $\sigma(f)=1$.

Example 1.5 The differential equation

$$3f''(z)+2f'(z)-4f(z)=-4z^2+4$$

has a polynomial solution $f(z)=z^2+z+1$.

2 The Asymptotic Method

Theorem G (see [17(p.30), 20]; the Wiman-Valiron Theory) Suppose that f is a transcendental entire function, and that for any given $0<\delta<1/8$, there exists a set H of finite logarithmic measure such that

$$\frac{f^{(n)}(z)}{f(z)}=\Big(\frac{\nu(r,f)}{z}\Big)^n(1+o(1)),\ |z|=r\notin H, \tag{2.1}$$

whenever

$$|f(z)|\geq M(r,f)\nu(r,f)^{-\frac{1}{8}+\delta},$$

$\nu(r,f)$ is the central index of f.

Remark 2.1 Suppose that $f(z)=\sum\limits_{n=0}^{\infty}a_nz^n$. The maximum term of f, $\mu_f(r)=\mu(r,\ f)$, is defined as

$$\mu_f(r)=\mu(r,\ f):=\max\{|a_n|r^n:\ n\geq 0\}.$$

The central index of f, $\nu_f(r)=\nu(r,\ f)$ is defined as

$$\nu_f(r)=\nu(r,\ f)=\max\{n:\ \ |a_n|r^n\leq\mu(r,\ f)\ \text{ for all } n\geq 0\}.$$

Asymptotic Method (see [13 (P.183-184), 17 (P. 227-229)]) Suppose that $a_j(z)$ ($j=0,1,\cdots,n$), $F(z)$ are polynomials, and consider the linear differential equation

$$a_n(z)f^{(n)}(z)+a_{n-1}(z)f^{(n-1)}(z)+\cdots+a_0(z)f(z)=F(z) \tag{2.2}$$

and the corresponding homogeneous linear differential equation

$$a_n(z)f^{(n)}(z)+a_{n-1}(z)f^{(n-1)}(z)+\cdots+a_0(z)f(z)=0. \tag{2.3}$$

If a solution $f(z)$ of equation (2.2) (or (2.3)) is a transcendental entire function, then, from Theorem G (the Wiman-Valiron theory), there is a set $H\subset(1,+\infty)$ having logarithmic measure $lmH<\infty$, we can choose z satisfying $|z|=r\notin[0,1]\cup H$ and $|f(z)|=M(r,f)$ to such that get that

$$\frac{f^{(j)}(z)}{f(z)}=\Big(\frac{\nu_f(r)}{z}\Big)^j(1+o(1))\ \ (j=1,2,\cdots,n), \tag{2.4}$$

where $\nu_f(r)$ is the central index of $f(z)$. Substituting (2.4) into (2.2) and (2.3), respectively, we obtain

$$\begin{aligned}&a_n(z)\Big(\frac{\nu_f(r,f)}{z}\Big)^n(1+o(1))+a_{n-1}(z)\Big(\frac{\nu_f(r,f)}{z}\Big)^{n-1}(1+o(1))\\&+\cdots+a_0(z)=o(1),\quad r\notin[0,\ 1]\cup H,\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}&a_n(z)\Big(\frac{\nu_f(r,f)}{z}\Big)^n(1+o(1))+a_{n-1}(z)\Big(\frac{\nu_f(r,f)}{z}\Big)^{n-1}(1+o(1))\\&+\cdots+a_0(z)=0,\quad r\notin[0,\ 1]\cup H.\end{aligned} \tag{2.6}$$

Suppose that $a_j(z)=A_jz^{m_j}(1+o(1))$ ($j=0,1,\cdots,n$) as $r\to\infty$, and $A_n\neq 0$. Set

$$\nu_f(r,f)=\nu_f(r).$$

Thus, $\nu_f(r) \to \infty$ as $r \to \infty$ and $r \notin [0,\ 1] \cup H$, so by (2.5) and (2.6), we obtain

$$A_n(\nu_f)^n z^{m_n-n}(1+o(1)) + A_{n-1}(\nu_f)^{n-1} z^{m_{n-1}-(n-1)}(1+o(1)) + \cdots + A_0 z^{m_0}(1+o(1)) = o(1), \quad r \notin [0,\ 1] \cup H, \tag{2.7}$$

and

$$A_n(\nu_f)^n z^{m_n-n}(1+o(1)) + A_{n-1}(\nu_f)^{n-1} z^{m_{n-1}-(n-1)}(1+o(1)) + \cdots + A_0 z^{m_0}(1+o(1)) = 0, \quad r \notin [0,\ 1] \cup H. \tag{2.8}$$

Since solutions of algebraic equations (2.7) and (2.8) are continuous functions of coefficients, solutions $\nu_f(r)$ of equations (2.7) and (2.8) must asymptotically be equal to the solution of equation

$$A_n(\nu_f)^n z^{m_n-n} + A_{n-1}(\nu_f)^{n-1} z^{m_{n-1}-(n-1)} + \cdots + A_0 z^{m_0} = 0. \tag{2.9}$$

Since the solution ν_f of (2.9) is an algebraic function of z, setting the principal part of ν_f as $a(\rho)z^{\rho}$ (a, ρ are nonzero real numbers) in the neighborhood of $z=\infty$, we get that,

$$\nu_f(r) = a(\rho)z^{\rho}(1+o(1), \text{ in the neighborhood of } z=\infty. \tag{2.10}$$

By (2.9) and (2.10), it is easy to see that the degrees of all of the terms of the left of (2.9) are

$$n\rho + m_n - n,\ (n-1)\rho + m_{n-1} - (n-1),\ \cdots,\ m_0. \tag{2.11}$$

Since $\nu_f(r)$ is the solution of (2.9), we see that in (2.11), at least, there are two terms that are both the largest numbers and equal, and that the sum of coefficients of their corresponding terms in (2.9) is zero. Hence, ρ satisfies that we have i and j such that

$$i\rho + m_i - i = j\rho + m_j - j\ (i<j,\ i=0,\ 1,\ \cdots,\ n-1). \tag{2.12}$$

Thus, we see that ρ is a rational number, and we have at most n such rational numbers that are not less than $1/n$.

3 Proofs of Theorems 1.1 and 1.2

We need the following lemmas to prove Theorems 1.1 and 1.2:

Lemma 3.1 (see [2, 4]) Let $n \in \mathbb{N}$, and let f be a transcendental meromorphic function of an order less than 1. Then there exists an ε-set E_n such that

$$\Delta^n f(z) \sim f^{(n)}(z) \quad (n=1,\ \cdots) \ \text{ as } z \to \infty \text{ in } \mathbb{C} \setminus E_n.$$

Lemma 3.2 (see [3, 4]) Let $f(z)$ be a non-constant finite-order meromorphic solution of

$$P(z,\ f) = 0,$$

where $P(z,\ f)$ is a difference polynomial in $f(z)$, and let $\delta < 1$. If $P(z,\ a) \not\equiv 0$ for a slowly moving target a, then

$$m\left(r,\ \frac{1}{f-a}\right) = o\left(\frac{T(r+|c|,\ f)}{r^{\delta}}\right) + o(T(r,\ f))$$

for all r outside of a possible exceptional set with finite logarithmic measure. Moreover, the Nevanlinna deficiency satisfies that

$$\delta(a,\ f) =: \varliminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r,f)} = 0.$$

Proof of Theorem 1.1 We divide this proof into the following two cases:

Case 1 Suppose that $f(z)$ is a transcendental meromorphic solution of (1.7) and has a pole z_0. Thus, by (1.7), we see that there is j_1 $(1 \leq j_1 \leq n)$ such that $f(z_0+j_1)=\infty$. Again by (1.7), we see that there are j_s $(1 \leq j_s \leq n,\ s=2,\cdots,\infty)$ such that $f(z_0+j_1+\cdots+j_s)=\infty$. Thus, we see that $f(z)$ has poles $z_0,\quad z_0+j_1,\quad \cdots, z_0+j_1+\cdots+j_s, \cdots$, such that $\lambda(\frac{1}{f}) \geq 1$.

Hence, $\sigma(f) \geq \lambda(\frac{1}{f}) \geq 1$.

Case 2 Suppose that $f(z)$ is an entire function, and that $\sigma(f)<1$. We can rewrite (1.7) as

$$A_n\Delta^n f(z)+A_{n-1}\Delta^{n-1}f(z)+\cdots+A_0f(z)=0, \tag{3.1}$$

where A_j $(j=0,\ \cdots,\ n)$ are constants.

Since $f(z)$ is an entire function, and $\sigma(f)<1$, by Lemma 3.1 we see that there exist ε-set E_j $(j=1,\ \cdots,\ n)$ such that

$$\Delta^j f(z) \sim f^{(j)}(z) \quad (j=1,\ \cdots) \ \text{ as } z \to \infty \text{ in } \mathbb{C}\setminus E_j. \tag{3.2}$$

Set

$$H_1=\Big\{|z|=r,\ z\in\bigcup_{j=1}^{n}E_j\Big\}.$$

Since E_j are ε-set, we see that a set H_1 is of finite logarithmic measure. By (3.1) and (3.2), we obtain that

$$A_nf^{(n)}(z)(1+o(1))+\cdots+A_1f'(z)(1+o(1))+A_0f(z)=0. \tag{3.3}$$

If $A_0=0$, then we may suppose that $A_1\neq 0$ and $f'(z)=g(z), f^{(j)}(z)=g^{(j-1)},\ j=2,\cdots,n$. Thus, without loss of generality, we may suppose that $A_0\neq 0$. By the Wiman-Valiron theory (see Theorem G) we see that there exists a set H_2 of finite logarithmic measure such that

$$\frac{f^{(j)}(z)}{f(z)}=\Big(\frac{\nu(r,f)}{z}\Big)^j(1+o(1)), \quad j=1,\ \cdots,\ n, \tag{3.4}$$

where $|f(z)|=M(r,f), |z|=r\notin[0,1]\cup H_1\cup H_2$, $\nu(r,f)$ is the central index of $f(z)$. Combining (3.3) and (3.4), we obtain that

$$\begin{aligned}&A_n\Big(\frac{\nu(r,f)}{z}\Big)^n(1+o(1))+A_{n-1}\Big(\frac{\nu(r,f)}{z}\Big)^{(n-1)}(1+o(1))\\&+\cdots+A_1\frac{\nu(r,f)}{z}(1+o(1))+A_0=0,\end{aligned} \tag{3.5}$$

where $|f(z)|=M(r,\quad f)$, $|z|=r\notin[0,1]\cup H_1\cup H_2$.

Thus, applying the Asymptotic Method to (3.5), we see that the solution $\nu(r,f)$ of (3.5) is asymptotic and equal to the solution $\nu(r)$ of algebraic equation

$$A_nz^{-n}\nu(r)^n+A_{n-1}z^{-(n-1)}\nu(r)^{(n-1)}+\cdots+A_1z^{-1}\nu(r)+A_0=0. \tag{3.6}$$

By supposition $\sigma(f)=\alpha<1$, we then have that $\nu(r)\sim ar^{\alpha}$, where a is a nonzero real number, and α is a rational number no less than $\frac{1}{n}$.

Thus, $n+1$ terms in the left hand side of (3.6) are equal to

$$A_na^nr^{n(\alpha-1)}, \quad A_{n-1}a^{n-1}r^{(n-1)(\alpha-1)}, \quad \cdots, \quad A_1ar^{(\alpha-1)}, \quad A_0. \tag{3.7}$$

Since $\frac{1}{n}\leq\alpha<1$, we see that the degrees of $n+1$ terms in (3.7) satisfy

$$n(\alpha-1)<(n-1)(\alpha-1)<\cdots<\alpha-1<0. \tag{3.8}$$

By (3.7) and (3.8), we see that (3.6) is a contradiction.

Hence, every finite order transcendental meromorphic solution $f(z)(\not\equiv 0)$ of the equation (1.7) satisfies that $\sigma(f) \geq 1$. □

Proof of Theorem 1.2 Using the same method as in the proof of Theorem 1.1, we can prove that every finite order transcendental meromorphic solution $f(z)$ of the equation (1.8) satisfies $\sigma(f) \geq 1$.

Now, we prove that every finite order transcendental meromorphic solution $f(z)$ of the equation (1.8) satisfies $\lambda(f) = \sigma(f)$.

By (1.8), we set

$$E(z,\ f) := C_n f(z+n) + \cdots + C_0 f(z) - F(z).$$

Thus,

$$E(z,\ 0) = -F(z) \not\equiv 0.$$

By Lemma 3.2, we have that $m\left(r,\ \frac{1}{f}\right) = S(r,f)$, so

$$N\left(r,\ \frac{1}{f}\right) = T(r,f) + S(r,f).$$

Hence, $\lambda(f) = \sigma(f)$. □

4 Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3 It is well known that all meromorphic solutions of equations (1.10) are entire functions.

Suppose that $f(z)$ is a solution of (1.10).

First, we prove that $f(z)$ cannot be a polynomial. If $f(z)$ is a nonzero constant, then $f'(z) = \cdots = f^{(z)}(z) = 0$, and this is not possible. If $f(z)$ is a polynomial with $\deg f(z) \geq 1$, then $\deg f^{(j)}(z) < \deg f(z)$ $(j = 1, \cdots, n)$ which is also not possible.

Now, we suppose that $f(z)$ is a transcendental entire function with $\sigma(f) = \sigma$.

By the Wiman-Valiron theory, we see that there exists a set H_2 of finite logarithmic measure such that (3.4) holds, where $|f(z)| = M(r,f), |z| = r \notin [0,1] \cup H_1 \cup H_2$, $\nu(r,f)$ is the central index of $f(z)$. By (3.4) and (1.10), we obtain

$$\begin{aligned} &C_n\Big(\frac{\nu(r,f)}{z}\Big)^n(1+o(1)) + C_{n-1}\Big(\frac{\nu(r,f)}{z}\Big)^{(n-1)}(1+o(1)) \\ &+\cdots + C_1\frac{\nu(r,f)}{z}(1+o(1)) + C_0 = 0, \end{aligned} \tag{4.1}$$

where $|f(z)| = M(r,\ \ f)$, $|z| = r \notin [0,1] \cup H_1 \cup H_2$, and

$$C_n\Big(\frac{\nu(r,f)}{z}\Big)^n + C_{n-1}\Big(\frac{\nu(r,f)}{z}\Big)^{(n-1)} + \cdots + C_1\frac{\nu(r,f)}{z} + C_0 = 0. \tag{4.2}$$

Using the same method as in the proof of Theorem 1.1, we see that $\nu(r) \sim ar^{\alpha}$, where a is nonzero real number, and α is a rational number no less than $\frac{1}{n}$.

Thus, we see that the degrees of $n+1$ terms in (4.2) are, respectively,

$$n(\sigma-1), \quad (n-1)(\sigma-1), \quad \cdots, \quad \sigma-1, \quad 0. \tag{4.3}$$

If $0 \leq \sigma < 1$, then

$$n(\sigma - 1) < (n-1)(\sigma - 1) < \cdots < \sigma - 1 < 0. \tag{4.4}$$

If $\sigma > 1$, then

$$n(\sigma - 1) > (n-1)(\sigma - 1) > \cdots > \sigma - 1 > 0. \tag{4.5}$$

Thus, by (4.4) and (4.5), we see that (4.1) is a contradiction.

Hence, by (4.4) and (4.5), we see that $\sigma = 1$, that is, every solution $f(z)(\not\equiv 0)$ of the equation (1.10) satisfies $\sigma(f) = 1$.

Theorem 1.3 is thus proved. □

Proof of Theorem 1.4

We need the following lemma:

Lemma 4.1 (see [13, pp. 168]) Suppose that $a_0,\ a_1,\ \cdots,\ a_{k-1},\ \ F \not\equiv 0$ are entire functions, that f satisfies the differential equation

$$f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_0 f \ = \ F, \tag{4.6}$$

and that

$$\max\{\sigma(F),\ \sigma(a_j);\ j = 0,\ 1,\ \cdots,\ k-1\} < \sigma(f) = \sigma\ (0 < \sigma \leq \infty). \tag{4.6}$$

Then,

$$\overline{\lambda}(f) = \lambda(f) = \sigma(f).$$

Proof of Theorem 1.4 Using the same method as in the proof of Theorem 1.3, we see that all meromorphic solutions of equations (1.11) are entire functions, and if $f(z)$ is a transcendental entire solution of (1.11), then $\sigma(f) = 1$.

Since $C_j (j = 0, \cdots, n)$ are constants, $F(z) \not\equiv 0$ is a polynomial, and thus $C_j, F(z)$ and $f(z)$ satisfy condition (4.6) of Lemma 4.1. By Lemma 4.1, we obtain that $\overline{\lambda}(f) = \lambda(f) = \sigma(f) = 1$.

Theorem 1.4 is thus proved. □

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