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NORMAL CRITERIA FOR A FAMILY OF HOLOMORPHIC CURVES*

Dedicated to the memory of Professor Jiarong YU

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Abstract In this paper, we extend the concept of holomorphic curves sharing hyperplanes and introduce definitions of restricted hyperplanes and partial shared hypersurfaces. Then, we prove several normal criteria of the family of holomorphic curves and holomorphic mappings that concern restricted hyperplanes and partial shared hypersurfaces. These results generalize the Montel-type normal criterion of holomorphic curves.

Key words complex projective spaces; holomorphic curves; holomorphic mappings; normal families; restricted hyperplanes

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1 Introduction

In the theory of a normal family of meromorphic functions, the following Montel theorem plays an important role:

Theorem A Let $\mathfrak{F}$ be a family of meromorphic functions in a plane domain D. If there exist three distinct points, $a_1,\ a_2,\ a_3$, on the Riemann sphere such that for each $f \in \mathfrak{F}$, $f(z)-a_j$ (for $j=1,2,3$) has no zeros in D, then $\mathfrak{F}$ is normal.

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The Montel theorem is also very useful in valued distribution. For example, the Montel theorem yields the Picard theorem and the Julia theorem. More generally, the three exception values in Theorem A can be extended to shared values.

Theorem B Let $\mathfrak{F}$ be a family of meromorphic functions in a plane domain D. If there are three distinct points, $a_1,\ a_2,\ a_3$, in the Riemann sphere such that any two functions $f, g \in \mathfrak{F}$ share $a_1,\ a_2$ and a_3 in D, then $\mathfrak{F}$ is normal.

This Montel-type theorem raises an interesting question about two families of meromorphic functions sharing some values. In 2013, Liu, Li and Pang ([7]) proved the following result:

Theorem C Suppose that $\mathfrak{F}$ and $\mathfrak{G}$ are two families of meromophic functions in a plane domain D. Let a_1, a_2, a_3, a_4 be four fixed distinct points on the Riemann sphere. If for each $f \in \mathfrak{F}$ there exists $g \in \mathfrak{G}$ such that f and g share a_1, a_2, a_3, a_4, and $\mathfrak{G}$ is normal in D, then $\mathfrak{F}$ is normal.

In the 1950s, Wu ([14]) and Fujimot ([4]) began to study the normal family for holomorphic mappings and extended some classical results for meromorphic functions to holomorphic mappings. The notion of a normal family has proved its importance in geometric function theory in several complex variables. In recent years, as more and more attention has been paid to high dimensional complex analysis, the study of normal families of holomorphic curves and holomorphic mappings has become well developed. Many researchers, such as Aladro, Krantz, Ru, Tu and Pang, etc. have done much work on the normal family of holomorphic mappings and holomorphic curves (see, e.g. refs. [1, 8, 13, 15, 16]). In this paper, we will prove several theorems on the normality of holomorphic curves and holomorphic mappings.

We first give some definitions. Let $\mathbb{P}^N(\mathbb{C})$ be the complex projective space of dimension N; that is, $\mathbb{P}^N(\mathbb{C}) = \mathbb{C}^{N+1} - \{0\}/\sim$, where $(z_0, z_1, \cdots, z_N) \sim (w_0, w_1, \cdots, w_N)$ if and only if $(z_0, z_1, \cdots, z_N) = \lambda(w_0, w_1, \cdots, w_N)$ for some nonzero complex number λ. We denote by $[\mathbf{z}]$ the equivalent class of $[z_0 : z_1 : \cdots : z_N]$. Suppose that

$$f = [f_0 : f_1 : \cdots : f_N] : \mathbb{C}^m \to \mathbb{P}^N(\mathbb{C})$$

is a holomorphic mapping, where $f_0, f_1, \cdots, f_N$ are holomorphic functions of m variables. Denote by $\mathbf{f} = (f_0, f_1, \cdots, f_N)$ a reduced representation of f if $f_0, f_1, \cdots, f_N$ have no common zeros. The holomorphic mapping f is called a holomorphic curve when $m = 1$. Letting D be a domain in $\mathbb{C}^m$, we denote by $\mathscr{H}(D, \mathbb{P}^N(\mathbb{C}))$ the set of all holomorphic mappings $f : D \to \mathbb{P}^N(\mathbb{C})$.

Definition 1.1 A family $\mathfrak{F} \subset \mathscr{H}(D, \mathbb{P}^N(\mathbb{C}))$ is said to be normal on D if any sequence in $\mathfrak{F}$ contains a subsequence which is relatively compact; that is, if any sequence $\{f_n\} \subset \mathfrak{F}$ contains a subsequence which converges to $f \in \mathscr{H}(D, \mathbb{P}^N(\mathbb{C}))$ uniformly on every compact subset of D. A family $\mathfrak{F}$ is said to be normal at $a \in D$ if any sequence in $\mathfrak{F}$ contains a subsequence which is relatively compact; that is, if any sequence of $\mathfrak{F}$ contains a subsequence which converges compactly to $f \in \mathscr{H}(D, \mathbb{P}^N(\mathbb{C}))$ on some neighborhood U_a of a.

If $f \in \mathscr{H}(D, \mathbb{P}^N(\mathbb{C}))$ is representable as $\mathbf{f} = (f_0, f_1, \cdots, f_N)$ with a polynomial (or constant) f_j, we say that f is rational (or constant, respectively). In particular, letting M be a domain in the complex plane, $f : M \to \mathbb{P}^N(\mathbb{C})$ is a holomorphic curve which we denote by $f \in \mathscr{H}(M, \mathbb{P}^N(\mathbb{C}))$.

A hypersurface $Q \in \mathbb{C}[z_0, z_1, \cdots, z_N]_d$ is

$$Q = \{[z] \in \mathbb{P}^N(\mathbb{C});\quad Q(z_0, z_1, \cdots, z_N) = 0\},$$

where Q is a homogeneous polynomial of degree d. Hypersurfaces $Q_1, Q_2, \cdots, Q_q$ in $\mathbb{P}^N(\mathbb{C})$ are said to be in t-subgeneral position if, for any $0 \leqslant j_0 \leqslant j_1 \leqslant \cdots \leqslant j_t \leqslant q$, we have

$$\bigcap_{j=j_0}^{j_t} Q_j = \emptyset,$$

and they are said to be in general position if $t = N$. Let f be a holomorphic mapping, where $\mathbf{f} = (f_0, f_1, \cdots, f_N)$ is the reduced representation. Then

$$\langle \mathbf{f}, Q\rangle := Q \circ \mathbf{f} = Q(f_0, f_1, \cdots, f_N)$$

is a holomorphic function.

Definition 1.2 The multiplicity of holomorphic function $\langle \mathbf{f}, Q\rangle$ at a point a is said to be the multiplicity of holomorphic mapping f intersecting Q at a.

Remark 1.3 It is easy to verify that the zeros in $\langle \mathbf{f}, Q\rangle$ are independent of the choice of f_j. Hence, Definition 1.2 is well defined.

We denote $f^{-1}(Q)$ by

$$f^{-1}(Q) := \{z \in D; f(z) \in Q\} = \{z \in D; \langle \mathbf{f}(z), Q\rangle = 0\}.$$

Furthermore, for a family of holomorphic mappings $\mathfrak{F} \subset \mathscr{H}(D, \mathbb{P}^N(\mathbb{C}))$, we denote

$$\mathfrak{F}^{-1}(Q) := \bigcup_{f\in\mathfrak{F}} \{z \in D;\ f(z) \in Q\}.$$

We then introduce the definition of a shared hypersurface, which is the extension of shared values for meromorphic functions.

Definition 1.4 Suppose that $f,\ g \in \mathscr{H}(D, \mathbb{P}^N(\mathbb{C}))$ and Q is a hypersurface in $\mathbb{P}^N(\mathbb{C})$. f and g are said to be sharing the hypersurface Q in D if

$$f^{-1}(Q) = g^{-1}(Q),$$

and is denoted by

$$f \in Q \Leftrightarrow g \in Q.$$

Furthermore, a family of holomorphic mappings $\mathfrak{F} \subset \mathscr{H}(D, \mathbb{P}^N(\mathbb{C}))$ is said to be sharing a hypersurface Q in $D \subset \mathbb{C}^m$ if, for any $f \in \mathfrak{F}$, we have

$$\mathfrak{F}^{-1}(Q) = f^{-1}(Q).$$

If H is a homogeneous polynomial of degree 1, we say that H is a hyperplane; that is,

$$H = \{[z_0 : z_1 : \cdots : z_N] \in \mathbb{P}^N(\mathbb{C});\quad a_0z_0 + a_1z_1 + \cdots + a_Nz_N = 0\},$$

where $a_j \in \mathbb{C},\ 0 \le j \le N$. We also denote by $H = \{\langle z, \alpha\rangle = 0\}$, or $\alpha = (a_0, a_1, a_2, \cdots, a_N)$, the hyperplane where $\langle z, \alpha\rangle$ is the inner product of z and α. Similarly, we can define a family of holomorphic curves sharing hyperplanes. In particular, we say that holomorphic curve f is linearly degenerate if

$$\langle \mathbf{f}(z), H\rangle \equiv 0.$$

Otherwise, it is linearly nondegenerate. Throughout this paper, we use M, D, H and Q as a complex domain in $\mathbb{C}$, a domain in $\mathbb{C}^m$, a hyperplane and a hypersurface, respectively.

In 2014, Yang, Fang and Pang generalized Theorem B on a family of holomorphic curves sharing some hyperlanes; see [15].

Theorem D Let M be a domain in the complex plane. Suppose that $\mathfrak{F} \subset \mathscr{H}(M, \mathbb{P}^N(\mathbb{C}))$ and $H_1, H_2, \cdots, H_{2t+1}$ are the $2t+1$ hyperplanes located in t-subgeneral position, where $t \geqslant N$. If $\mathfrak{F}$ shares $H_1, H_2, \cdots, H_{2t+1}$ on M, then $\mathfrak{F}$ is normal.

In this paper, we will improve Theorem D. First, we will prove

Theorem 1.5 Let f be a holomorphic curve in a plane domain M, and let Q be a hypersurface in $\mathbb{P}^N(\mathbb{C})$. Then, either $f^{-1}(Q)$ is a discrete set in M, or $f^{-1}(Q) = M$.

By Theorem 1.5, we can introduce the definition of a restricted hyperplane.

Definition 1.6 Let $\mathfrak{F} \in \mathscr{H}(M, \mathbb{P}^N(\mathbb{C}))$. A hyperplane H in $\mathbb{P}^N(\mathbb{C})$ is said to be restricted for $\mathfrak{F}$ if, for any closed subset $G \subset M$,

$$\#\{G \cap \mathfrak{F}^{-1}(H)\} < \infty,$$

where # is the number of elements. H is said to be a general restricted hyperplane for $\mathfrak{F}$ if there is a discrete set E_H such that, for any closed set $G \subset M$, we have

$$\#\{G \cap E_H\} < \infty,$$

and for any $f \in \mathfrak{F}$, we have either $f^{-1}(H) \subset E_H$ or $f^{-1}(H) = M$.

Remark 1.7 i) If H is a restricted hyperplane for $\mathfrak{F}$, then there is no linearly degenerate holomorphic curve in $\mathfrak{F}$.

ii) If a family of holomorphic curves $\mathfrak{F}$ shares a hyperplane H, then it follows from Theorem 1.5 that for any $f \in \mathfrak{F}$, either $f^{-1}(H) = M$ or $f^{-1}(H) = \mathfrak{F}^{-1}(H)$ is a discrete set.

For the first case, H is obviously generally restricted. For the second case, if we let

$$E_H = \mathfrak{F}^{-1}(H),$$

then H is restricted. Hence, in either of the two cases, H is generally restricted.

iii) A restricted hyperplane for $\mathfrak{F}$ may not be a shared hyperplane and a shared hyperplane for $\mathfrak{F}$ may not be a restricted hyperplane.

Based on the definition of a general restricted hyperplane, we prove the following, which is an extension of Theorem D:

Theorem 1.8 Suppose that $\mathfrak{F} \subset \mathscr{H}(M, \mathbb{P}^N(\mathbb{C}))$, where M is a domain in the complex plane. If for $\mathfrak{F}$ there exist $2t+1$ general restricted hyperplanes $H_1, H_2, \cdots, H_{2t+1}$ located in t-subgeneral positions, where $t \geqslant N$, then $\mathfrak{F}$ is normal.

It follows from Definition 1.6 that a restricted hyperplane is also a general restricted hyperplane. Hence, we can also obtain the following result:

Corollary 1.9 Let $\mathfrak{F} \subset \mathscr{H}(M, \mathbb{P}^N(\mathbb{C}))$, where M is a plane domain. If for $\mathfrak{F}$ there exist $2t+1$ restricted hyperplanes $H_1, H_2, \cdots, H_{2t+1}$ located in t-subgeneral position, where $t \geqslant N$, then $\mathfrak{F}$ is normal.

For the normality of two families of holomorphic curves, Yang, Fang and Pang generalized Theorem C and proved the following (see [16]):

Theorem E Suppose that $\mathfrak{F}$, $\mathfrak{G} \subset \mathscr{H}(D, \mathbb{P}^N(\mathbb{C}))$, where D is a domain in $\mathbb{C}^m$ and $Q_1, Q_2, \cdots, Q_{3t+1}$ are hypersurfaces located in t-subgeneral position, where $t \geqslant N$. If, for any $f \in \mathfrak{F}$, there exists $g \in \mathfrak{G}$ such that f and g share Q_j $(1 \leq i \leq 3t+1)$ and $\mathfrak{G}$ is normal, then $\mathfrak{F}$ is normal.

We now introduce the definition of partial sharing.

Definition 1.10 Suppose that $f,\ g \subset \mathscr{H}(D, \mathbb{P}^N(\mathbb{C}))$, where D is a domain in $\mathbb{C}^m$. For a hypersurface Q in $\mathbb{P}^N(\mathbb{C})$, f is said to be left sharing Q with g in D if $f^{-1}(Q) \subset g^{-1}(Q)$, denoted by

$$f \in Q \Rightarrow g \in Q.$$

Remark 1.11 It follows from the definition that $f \in Q \Leftrightarrow g \in Q$ if and only if

$$f \in Q \Rightarrow g \in Q \text{ and } g \in Q \Rightarrow f \in Q.$$

We will improve Theorem E and obtain the following result for holomorphic mappings:

Theorem 1.12 Suppose that $Q_1, Q_2, \cdots, Q_{3t+1} \subset \mathbb{P}^N(\mathbb{C})$ are hypersurfaces located in t-subgeneral position, where $t \geqslant N$. Let $\mathfrak{F}$, $\mathfrak{G} \subset \mathscr{H}(D, \mathbb{P}^N(\mathbb{C}))$, where D is a domain in $\mathbb{C}^m$. If

i) for any $f \in \mathfrak{F}$, there exists $g \in \mathfrak{G}$ such that for any $j = 1, 2, \cdots, 3t+1$, we have

$$f \in Q_j \Rightarrow g \in Q_j,$$

ii) $\mathfrak{G}$ is normal,

then $\mathfrak{F}$ is also normal.

2 Lemmas

Lemma 2.1 ([1]) Suppose that D is a domain in $\mathbb{C}^m$ and that $\mathfrak{F} \subset \mathscr{H}(D, \mathbb{P}^N(\mathbb{C}))$. The family $\mathfrak{F}$ is not normal on D if and only if there exist sequences $\{z_n\}_{n=1}^{\infty} \subset D$, $\{\rho_n\}_{n=1}^{\infty} \subset \mathbb{R}^+$, $\{u_n\}_{n=1}^{\infty} \subset \mathbb{C}^m$ and $\{f_n\}_{n=1}^{\infty} \subset \mathfrak{F}$ such that $\lim\limits_{n\to\infty} z_n = z_0$, $\lim\limits_{n\to\infty} \rho_n = 0$, $\|u_n\| = 1$, and

$$h_n(\zeta) := f_n(z_n + \rho_n u_n \zeta)$$

converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping $h \in \mathscr{H}(\mathbb{C}, \mathbb{P}^N(\mathbb{C}))$, where $\|u_n\|$ is the Euclidean length, and $\zeta \in \mathbb{C}$ satisfies $z_n + \rho_n u_n \zeta \in D$.

Lemma 2.2 ([15]) Suppose that f is a holomorphic curve, and that $H_1,\ H_2,\ \cdots,\ H_{2t+1}$ are hyperplanes in $\mathbb{P}^N(\mathbb{C})$ located in t-subgeneral position, where $t \geqslant N$. If, for each hyperplane $H_j,\ j \in \{1, 2, \cdots, 2t+1\}$, and either $\langle \mathbf{f}, H_j \rangle \equiv 0$ or $\langle \mathbf{f}, H_j \rangle$ has finitely many zeros in $\mathbb{C}$ at most (no zero is allowed), then the map f is rational.

Lemma 2.3 Let f be a holomorphic curve. If there exist $2t+1$ hyperplanes $H_1,\ H_2,\ \cdots,$ H_{2t+1} in $\mathbb{P}^N(\mathbb{C})$ located in t-subgeneral position, where $t \geqslant N$, satisfying that

i) for each $j \in \{1, 2, \cdots, s\}$, either $\langle \mathbf{f}, H_j \rangle \equiv 0$ or $\langle \mathbf{f}, H_j \rangle \neq 0$, where $s \geqslant t+1$,

ii) for each $j \in \{s+1, s+2, \cdots, 2t+1\}$, $\langle \mathbf{f}, H_j \rangle$ has finitely many zeros in $\mathbb{C}$ at most,

then f is a constant mapping.

Proof It follows from Lemma 2.2 that f is a rational mapping; that is, f is representable as $\mathbf{f} = (f_0, f_1, \cdots, f_N)$ with polynomial f_j. Hence, $\langle \mathbf{f}, H_j \rangle$ is a polynomial for each $j = 1, 2, \cdots, 2t+1$.

If, for each $j \in \{1, 2, \cdots, s\}$,

$$\langle \mathbf{f}, H_j \rangle \neq 0,$$

then $\langle \mathbf{f}, H_j \rangle$ is a nonzero constant, denoted by $\langle \mathbf{f}, H_j \rangle \equiv c_j$. Hence, let

$$\langle \mathbf{f}, H_j \rangle \equiv c_j, \quad j = 1, 2, \cdots, s,$$

where $c_j = 0$ if $\langle \mathbf{f}, H_j \rangle \equiv 0$, and $c_j \neq 0$ if $\langle \mathbf{f}, H_j \rangle \neq 0$.

For any $j = 1, 2, \cdots, N+1$, let

$$H_j = \{[z_0 : z_1 : \cdots : z_N] \in \mathbb{P}^N(\mathbb{C});\ a_0^j z_0 + a_1^j z_1 + \cdots + a_N^j z_N = 0\}.$$

Then we have

$$\begin{cases} a_0^1 f_0 + a_1^1 f_1 + \cdots + a_N^1 f_N = c_1, \\ a_0^2 f_0 + a_1^2 f_1 + \cdots + a_N^2 f_N = c_2, \\ \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_0^{N+1} f_0 + a_1^{N+1} f_1 + \cdots + a_N^{N+1} f_N = c_{N+1}. \end{cases}$$

Since $H_1, H_2, \cdots, H_{2t+1}$ are located in t-subgeneral position, vectors $(a_0^j,\ a_1^j,\ \cdots,\ a_N^j)$, $j = 1, 2, \cdots, N+1$ are linearly independent, so the linear system has a unique solution $f_j \equiv d_j$, $j = 1, 2, \cdots, N+1$.

We know that $s \geq N+1$, hence if we choose other $N+1$ linear equations, we will have the same solution for the linear system. Anything else would contradict the uniqueness theorem of holomorphic functions. □

Lemma 2.4 ([16]) Let $f \subset \mathscr{H}(\mathbb{C}, X)$, where X is a closed set in $\mathbb{P}^N(\mathbb{C})$. If there exist $2t+1$ hypersurfaces $Q_1, Q_2, \cdots, Q_{2t+1}$ in $\mathbb{P}^N(\mathbb{C})$ located in t-subgeneral position, where $t \geqslant N$, such that either $f(\mathbb{C}) \subset Q_j$ or $f(\mathbb{C}) \cap Q_j = \emptyset$, then f is a constant mapping.

3 Proof of Theorems

3.1 Proof of Theorem 1.5

Let $\mathbf{f} = (f_0, f_1, \cdots, f_n)$ be a reduced representation of f. Noticing that Q is a polynomial and that f_j is a holomorphic function, we can obtain that

$$\langle \mathbf{f}, Q \rangle(z) = Q(f_0, f_1, \cdots, f_N)$$

is also a holomorphic function. Hence, either $\langle \mathbf{f}, Q \rangle(z) = Q(f_0, f_1, \cdots, f_N) \equiv 0$, or its zeros are isolated. For the former, we have $f^{-1}(Q) = M$. For the latter, we can conclude that $f^{-1}(Q)$ is a discrete set in M.

3.2 Proof of Theorem 1.8

Let $J = \{1, 2, \cdots, 2t+1\}$ and

$$\mathfrak{F}_j = \{f \in \mathfrak{F};\ f^{-1}(H_j) \neq M, \}, \quad E_j = \bigcup_{f \in \mathfrak{F}_j} f^{-1}(H_j), \quad E := \bigcup_{j=1}^{2t+1} E_j \quad (j \in J).$$

It follows from Theorem 1.5 that for each $f \in \mathfrak{F}_j$, $f^{-1}(H_j)$ is a discrete set. Since for all $j \in J$ H_j is a restricted hyperplane, we have that E_j is a discrete set, and there have been E.

Furthermore, for any closed subset $G \subset M$, we have

$$\#(E \cap G) < \infty.$$

Hence, for any fixed point $a \in M$, there exists a neighborhood U_a of a such that the radius of U_a is sufficiently small and

$$\#(E \cap U_a) \leq 1.$$

Then, for each $f \in \mathfrak{F}$ and $H_j (1 \leq j \leq 2t+1)$, there are only three possibilities:

1) $f(U_a) \subset H_j$; that is, for any $z \in U_a$, we have $\langle \mathbf{f}, H_j \rangle \equiv 0$;

2) $f(a) \notin H_j$; that is, for any $z \in U_a$, we have $\langle \mathbf{f}, H_j \rangle \neq 0$;

3) $f(a) \in H_j$; that is, $f^{-1}(H_j) \cap U_a = \{a\}$.

If we suppose that $\mathfrak{F}$ is not normal at some $a \in M$, then $\mathfrak{F}$ is not normal in U_a. It follows from Lemma 2.1 that there exist sequences $\{z_n\} \subset U_a$, $\{\rho_n\} \subset \mathbb{R}^+$, and $\{f_n\} \subset \mathfrak{F}$ such that

$$\lim_{n\to\infty} z_n = a, \quad \lim_{n\to\infty} \rho_n = 0$$

and

$$h_n(\zeta) := f_n(z_n + \rho_n\zeta), \quad (z_n + \rho_n\zeta \in U_a)$$

converges uniformly to a nonconstant holomorphic curve h in any compact subset of $\mathbb{C}$.

Next we will prove that for any $j \in J$ there are three possibilities: $\langle \mathbf{h}, H_j \rangle \equiv 0$, $\langle \mathbf{h}, H_j \rangle \neq 0$, or $\langle \mathbf{h}, H_j \rangle$ has only one zero.

Let H_j satisfy $\langle \mathbf{h}, H_j \rangle \not\equiv 0$ and $\{z \in U_a;\ \langle \mathbf{h}, H_j \rangle = 0\} \neq \emptyset$. If holomorphic function $\langle \mathbf{h}, H_j \rangle$ has two distinct zeros, then it follows from the Hurwitz theorem that when n is sufficiently large, holomorphic function $\langle \mathbf{h}_n, H_j \rangle = \langle \mathbf{f}_n(z_n + \rho_n\zeta), H_j \rangle$ has two distinct zeros. In other words, there are two distinct intersection points of f_n and H_j, which contradicts the definition of U_a. Therefore, $\langle \mathbf{h}, H_j \rangle$ has only one zero.

Let

$$\begin{aligned} J_0 &= \{j \in J;\ \langle \mathbf{h}, H_j \rangle \neq 0\}, \\ J_1 &= \{j \in J;\ \langle \mathbf{h}, H_j \rangle \text{ has only one zero}\}, \\ J_2 &= \{j \in J;\ \langle \mathbf{h}, H_j \rangle \equiv 0\}. \end{aligned}$$

Then $J = J_0 \cup J_1 \cup J_2$.

We claim that for any $j \in J_1$, we have $\{\langle \mathbf{h}, H_j \rangle = 0\} = \bigcap\limits_{j\in J_1} \{\langle \mathbf{h}, H_j \rangle = 0\}$. In fact, since

$$h_n(\zeta) = f_n(z_n + \rho_n\zeta) \to h(\zeta),$$

then we have

$$\langle \mathbf{h}_n, H_j \rangle \to \langle \mathbf{h}, H_j \rangle.$$

For any fixed $j \in J_1$, if $\zeta_0 \in \mathbb{C}$ is a zero of $\langle \mathbf{h}, H_j \rangle$ with multiplicity μ, where $\mu \geq 1$, then it follows from the Hurwitz Theorem that for $\varepsilon > 0$, when n is sufficiently large, holomorphic function $\langle \mathbf{h}_n, H_j \rangle$ has μ zeros in $|\zeta - \zeta_0| < \varepsilon$, counting its multiplicity. Therefore, we can choose a sequence $\zeta_n \to \zeta_0$ such that when n is sufficiently large, we have $z_n + \rho_n\zeta_n \in U_a$ and

$$\langle \mathbf{h}_n(\zeta_n), H_j \rangle = \langle \mathbf{f}_n(z_n + \rho_n\zeta_n), H_j \rangle = 0;$$

that is,

$$z_n + \rho_n\zeta_n \in f_n^{-1}(H_j).$$

Since $\langle \mathbf{f}_n, H_j\rangle$ has at most one zero a in U_a, when n is sufficiently large, we have

$$z_n + \rho_n\zeta_n = a.$$

Hence,

$$\zeta_n = \frac{a - z_n}{\rho_n}.$$

It follows from the fact that $\zeta_n \to \zeta_0$ that

$$\zeta_0 = \lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} \frac{a - z_n}{\rho_n}.$$

Hence, ζ_0 is unique and is independent of $H_j (j \in J_1)$. Then we prove our claim.

Finally, since $H_1, H_2, \cdots, H_{2t+1}$ are located in t-subgeneral position, we have that

$$\# J_1 \leqslant t.$$

Otherwise, there would be $t + 1$ hyperplanes, say, $\{H_{j_k}\}_{k=1}^{t+1}$, such that

$$\bigcap_{k=1}^{t+1} H_{j_k} \neq \emptyset.$$

It follows from Lemma 2.3 that h is a constant mapping, which is a contradiction. By the arbitrariness of a and Definition 1.1, we can obtain that $\mathfrak{F}$ is normal in D.

3.3 Proof of Theorem 1.12

We suppose that the result is false, that is, that $\mathfrak{F}$ is not normal in D. Then it follows from Lemma 2.1 that there exist $\{z_n\}_{n=1}^{\infty} \subset D$, $\{\rho_n\}_{n=1}^{\infty} \subset \mathbb{R}^+$, $\{u_n\}_{n=1}^{\infty} \subset \mathbb{C}^m$ and $\{f_n\}_{n=1}^{\infty} \subset \mathfrak{F}$ such that $\lim\limits_{n\to\infty} z_n = z_0$, $\lim\limits_{n\to\infty} \rho_n = 0$, $\|u_n\| = 1$ and the sequence of holomorphic mappings

$$h_n(\zeta) := f_n(z_n + \rho_n u_n \zeta)$$

converges compactly to a nonconstant holomorphic mapping h in $\mathbb{C}$.

By condition i), for the sequence of holomorphic mappings $\{f_n\}_{n=1}^{\infty}$, there exists a sequence $\{g_n\}_{n=1}^{\infty} \subset \mathfrak{G}$ such that, for any $n \in \mathbb{Z}^+$ and $1 \le j \le 3t + 1$, we have

$$f_n \in Q_j \Rightarrow g_n \in Q_j.$$

Noticing that $\mathfrak{G}$ is normal, we can suppose, without loss of generality, that $\{g_n\}_{n=1}^{\infty}$ converges compactly in D; that is,

$$g_n \to g \in \mathscr{H}(D, \mathbb{P}^N(\mathbb{C})).$$

Since h is not a constant mapping, it follows from Lemma 2.4 that there exist at least $t+1$ hypersurfaces, say, $Q_1, Q_2, \cdots, Q_{t+1}$ satisfying $\langle \mathbf{h}, Q_j\rangle \not\equiv 0$ and let $\{\zeta \in D;\ \langle \mathbf{h}(\zeta), Q_j\rangle = 0\} \neq \emptyset$, where $1 \le j \le t + 1$. Let $Q_{j_0} \in \{Q_j\}_{j=1}^{t+1}$ and let $\zeta_0 \in \{\zeta \in D;\ \langle \mathbf{h}(\zeta), Q_j\rangle = 0\}$. Then there exists a neighborhood U_0 of ζ_0, a reduced representation of h

$$\mathbf{h} = (h_0, h_1, \cdots, h_N),$$

and h_n

$$\begin{aligned}\mathbf{h}_n(\zeta) &= (h_{n0}(\zeta), h_{n1}(\zeta), \cdots, h_{nN}(\zeta))\\ &= (f_{n0}(z_n + \rho_n u_n\zeta), f_{n1}(z_n + \rho_n u_n\zeta), \cdots, f_{nN}(z_n + \rho_n u_n\zeta))\end{aligned}$$

such that the sequences of holomorphic functions $\{h_{nm}\}_{n=1}^{\infty}$ converge compactly to h_m in U_0, where $0 \le m \le N$. Therefore, $\langle \mathbf{h}_n(\zeta), Q_{j_0}\rangle$ converges compactly to $\langle \mathbf{h}(\zeta), Q_{j_0}\rangle$ in U_0. Since $\langle \mathbf{h}(\zeta), Q_{j_0}\rangle \not\equiv 0$, it follows from the Hurwitz theorem that there exists $\{\zeta_n\} \subset U_0$ such that $\zeta_n \to \zeta_0$ and

$$\langle \mathbf{h}_n(\zeta_n), Q_{j_0}\rangle = 0, \quad n \in \mathbb{Z}^+;$$

that is,

$$\langle \mathbf{f}_n(z_n + \rho_n u_n \zeta_n), Q_{j_0}\rangle = 0, \quad n \in \mathbb{Z}^+.$$

Knowing that $f_n \in Q_j \Rightarrow g_n \in Q_j$, we can obtain

$$\langle \mathbf{g}_n(z_n + \rho_n u_n \zeta_n), Q_{j_0}\rangle = 0, \quad n \in \mathbb{Z}^+.$$

Letting $n \to \infty$, we have

$$\langle \mathbf{g}(z_0), Q_{j_0}\rangle = 0.$$

Since j_0 is arbitrary, there exist $t+1$ hypersurfaces $Q_1,\ Q_2,\ Q_{t+1}$ such that $\langle \mathbf{g}(z_0), Q_j\rangle = 0$, which contradicts the hypothesis that $Q_1, Q_2, \cdots, Q_{t+1}$ is located in t-subgeneral position in X. Hence, Theorem 1.12 is true.

References

[1] Aladro G, Krantz S G. A criterion for normality in $\mathbb{C}^n$. J Math Anal Appl, 1991, **161**: 1–8

[2] Bergweiler W. Bloch's principle. Comput Methods Funct Theory, 2006, **6**(1): 77–108

[3] Chai F J, Gao Z S. The normality of algebroid multifunctions and their coefficient functions. Acta Math Sci, 2015, **35B**(1): 121–132

[4] Fujimoto H. On families of meromorphic maps into the complex projective space. Nagoya Math J, 1974, **54**: 21–51

[5] Hayman W K. Meromorphic Functions. Oxford: Clarendon Press, 1964

[6] Liu X Y. Normal families and shared values of meromorphic functions. Bull Malays Math Sci Soc, 2016, **39**: 155–171

[7] Liu X J, Li L S, Pang X C. A normal criterion about two families of meromorphic functions concerning shared values. Acta Math Sin-Engl Series, 2013, **29**(1): 151–158

[8] Ru M. Nevanlinna Theory and its Relation to Diophantine Approximation. Singapore: World Scientific, 2001

[9] Sun D C. Normality criteria for families of meromorphic algebroid functions. Acta Math Sci, 2010, **30B**(1): 166–172

[10] Sun D C, Yang L. Normal families of algebroidal functions. Sci China Ser A, 2001, **44**: 1271–1277

[11] Sun D C, Liu H F. Normal criteria on the family of meromorphic functions with shared set. Isr J Math, 2011, **184**: 403–412

[12] Shabat B V. Distribution of Values of Holomorphic Mappings. Translations of Mathematical Monographs 61. Providence, RI: Amer Math Soc, 1985

[13] Tu Z H. Normality criteria for families of holomorphic mappings of several complex variables into $\mathbb{P}^n(\mathbb{C})$. Proc Amer Math Soc, 1999, **127**(4): 1039–1049

[14] Wu H. Normal families of holomorphic mappings. Acta Math, 1967, **119**: 193–233

[15] Yang L, Fang C Y, Pang X C. Normal families of holomorphic mappings into complex projective space concerning shared hyperplanes. Pac J Math, 2014, **272**(1): 245–256

[16] Yang L, Liu X J, Pang X C. On families of meromorphic map into the complex projective space. Housten J Math, 2016, **42**(3): 775–789

[17] Zalcman L. Normal families: new perspectives. Bull Amer Math Soc, 1998, **35**: 215–230