Acta Mathematica Scientia, 2021, **41B**(5): 1809–1826
https://doi.org/10.1007/s10473-021-0525-2

Mathematica Acta Scientia 数学物理学报
http://actams.apm.ac.cn

HYERS–ULAM STABILITY OF SECOND-ORDER LINEAR DYNAMIC EQUATIONS ON TIME SCALES*

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Abstract We investigate the Hyers–Ulam stability (HUS) of certain second-order linear constant coefficient dynamic equations on time scales, building on recent results for first-order constant coefficient time-scale equations. In particular, for the case where the roots of the characteristic equation are non-zero real numbers that are positively regressive on the time scale, we establish that the best HUS constant in this case is the reciprocal of the absolute product of these two roots. Conditions for instability are also given.

Key words stability; second order; Hyers–Ulam; time scales

2010 MR Subject Classification 34N05; 34A30; 34A05; 34D20

1 Introduction

The story of Ulam and Hyers–Ulam stability (HUS) is recounted in many papers dealing with the subject, as has the development of dynamic equations on time scales. In this work, we continue the connection between those two areas by extending recent results on HUS for first-order time scale equations with constant coefficient [1], to second-order dynamic equations with constant coefficients. Other papers exploring HUS for dynamic equations on time scales include [2–6]. With a more general view, [7] established Ulam–type stability for a first-order equation using measure theory.

First, a brief review of time scales. Any closed, nonempty subset of the real line $\mathbb{R}$ is a time scale [11]. For example, $\mathbb{R}$, $h\mathbb{Z}$, and $\mathbb{N}$ are common examples of time scales. In this paper, we denote a time scale by $\mathbb{T}$. We define the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} :\ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} :\ s < t\}.$$

Set $\mu(t) = \sigma(t) - t$. The point $t \in \mathbb{T}$ is called right-scattered, right-dense, left-dense, left-scattered if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) = t$, $\rho(t) < t$, respectively. For example if $\mathbb{T} = \mathbb{R}$, then

*Received May 3, 2019; revised October 5, 2020. The second author was supported by JSPS KAKENHI Grant Number JP20K03668.

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$\sigma(t)=t=\rho(t)$ and $\mu(t)=0$ hold for any $t\in\mathbb{R}$; that is, all $t\in\mathbb{R}$ are left and right-dense. If $\mathbb{T}=h\mathbb{Z}:=\{hk:\ k\in\mathbb{Z}\}$ for $h>0$, then $\sigma(t)=t+h$, $\rho(t)=t-h$, and $\mu(t)=h$ hold for any $t\in h\mathbb{Z}$; that is, all $t\in h\mathbb{Z}$ are right and left-scattered. Define the set $\mathbb{T}^\kappa$ by $\mathbb{T}^\kappa=\mathbb{T}-\{M\}$ if $\mathbb{T}$ has a left-scattered maximum M; if not, then $\mathbb{T}^\kappa=\mathbb{T}$. Let $t\in\mathbb{T}^\kappa$ and $f:\mathbb{T}\to\mathbb{R}$. The delta derivative at t of f is defined by the following: For any $\varepsilon>0$, there is a neighborhood B of t for which

$$|f(\sigma(t))-f(\xi)-f^\Delta(t)[\sigma(t)-\xi]|\le\varepsilon|\sigma(t)-\xi|$$

for all $\xi\in B$. For example, $f^\Delta(t)=f'(t)$ when $\mathbb{T}=\mathbb{R}$; $f^\Delta(t)=\Delta_h f(t)=\frac{f(t+h)-f(t)}{h}$ when $\mathbb{T}=h\mathbb{Z}$. It is clear that if a function f exists on $\mathbb{T}$ and is delta differentiable, then f^Δ exists on $\mathbb{T}^\kappa$. The following product rule [11, Theorem 1.20] hold for the differentiable functions f and g:

$$(fg)^\Delta(t)=f^\Delta(t)g(t)+f(\sigma(t))g^\Delta(t)=f(t)g^\Delta(t)+f^\Delta(t)g(\sigma(t)).$$

In this study, we call it the Δ-derivative product rule.

Initially, we focus on the first-order linear dynamic equation

$$x^\Delta(t)-ax(t)=0 \tag{1.1}$$

on $\mathbb{T}^\kappa$, where $a\in\mathbb{R}$. By [11, Theorem 2.62], solutions of (1.1) are guaranteed to have global existence and uniqueness for the initial-value problem if a is regressive; that is, $1+a\mu(t)\ne0$ for all $t\in\mathbb{T}^\kappa$. In 2018, Anderson and Onitsuka [1] dealt with the Hyers–Ulam stability of (1.1), assuming that $1+a\mu(t)>0$ for all $t\in\mathbb{T}^\kappa$; that is to say, $a\in\mathcal{R}^+$, where $a\in\mathcal{R}^+$ is called positively regressive. We say that (1.1) has Hyers–Ulam stability (HUS) on $\mathbb{T}$ if and only if there exists a constant $K>0$ with the following property. For arbitrary $\varepsilon>0$, if a function $\phi:\mathbb{T}\to\mathbb{R}$ satisfies $|\phi^\Delta(t)-a\phi(t)|\le\varepsilon$ for all $t\in\mathbb{T}^\kappa$, then there exists a solution $x:\mathbb{T}\to\mathbb{R}$ of (1.1) such that $|x(t)-\phi(t)|\le K\varepsilon$ for all $t\in\mathbb{T}$. Any such constant K is known as an HUS constant on $\mathbb{T}$ for (1.1). For example, see the related references [8–10] for HUS of the first-order differential equations, difference equations, functional differential equations. It is known that (1.1) is not Hyers–Ulam stable on $\mathbb{T}$ when $a=0$ and $\sup\mathbb{T}=\infty$ (see, [1, Remark 3.3]). In this paper, the unique solution of the initial value problem (1.1) with $x(t_0)=1$ is denoted by $e_a(t,t_0)$. Note that if $a\in\mathcal{R}^+$, then $e_a(t,t_0)>0$ for all $t\in\mathbb{T}$ (see, [11, Theorem 2.44]). Anderson and Onitsuka gave the following result.

Theorem 1.1 ([1, Theorem 3.7]) Let $t_0\in\mathbb{T}$ and $\varepsilon>0$ be given. Suppose a function $\phi:\mathbb{T}\to\mathbb{R}$ that is delta-differentiable on $\mathbb{T}^\kappa$ satisfies

$$|\phi^\Delta(t)-a\phi(t)|\le\varepsilon,\qquad t\in\mathbb{T}^\kappa,$$

where $a\ne0$ and $a\in\mathcal{R}^+$. Then, one of the following holds.

(i) If $a>0$ and $\sup\mathbb{T}<\infty$, then any solution x of (1.1) with $|\phi(\sup\mathbb{T})-x(\sup\mathbb{T})|<\frac{\varepsilon}{a}$ satisfies $|\phi(t)-x(t)|<\frac{\varepsilon}{a}$ for all $t\in\mathbb{T}$.

(ii) If $a>0$ and $\sup\mathbb{T}=\infty$, then $\lim\limits_{t\to\infty}\frac{\phi(t)}{e_a(t,t_0)}$ exists, and there exists a unique solution

$$x(t):=\left(\lim_{t\to\infty}\frac{\phi(t)}{e_a(t,t_0)}\right)e_a(t,t_0)$$

of (1.1) such that $|\phi(t)-x(t)|\le\frac{\varepsilon}{a}$ for all $t\in\mathbb{T}$.

(iii) If $a<0$ and $\inf\mathbb{T}>-\infty$, then any solution x of (1.1) with $|\phi(\inf\mathbb{T})-x(\inf\mathbb{T})|<\frac{\varepsilon}{|a|}$ satisfies $|\phi(t)-x(t)|<\frac{\varepsilon}{|a|}$ for all $t\in\mathbb{T}$.

(iv) If $a<0$ and $\inf\mathbb{T}=-\infty$, then $\lim_{t\to-\infty}\frac{\phi(t)}{e_a(t,t_0)}$ exists, and there exists a unique solution

$$x(t):=\left(\lim_{t\to-\infty}\frac{\phi(t)}{e_a(t,t_0)}\right)e_a(t,t_0)$$

of (1.1) such that $|\phi(t)-x(t)|\le\frac{\varepsilon}{|a|}$ for all $t\in\mathbb{T}$.

Remark 1.2 Since $\mathbb{T}$ is any closed, nonempty subset of $\mathbb{R}$, we have the following facts: $\sup\mathbb{T}<\infty$ if and only if $\max\mathbb{T}$ exists, and thus, $\sup\mathbb{T}=\max\mathbb{T}$ holds; $\sup\mathbb{T}=\infty$ if and only if $\max\mathbb{T}$ does not exist; $\inf\mathbb{T}>-\infty$ if and only if $\min\mathbb{T}$ exists, and thus, $\inf\mathbb{T}=\min\mathbb{T}$ holds; $\inf\mathbb{T}=-\infty$ if and only if $\min\mathbb{T}$ does not exist.

2 First-order Non-homogeneous Linear Dynamic Equations

A function $f:\mathbb{T}\to\mathbb{R}$ is said to be rd-continuous if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limit exists (finite) at left-dense points of $\mathbb{T}$. If f is rd-continuous, then there is a function F such that $F^\Delta=f$ (see, [11]). We define

$$\int f(t)\Delta t=F(t)+C,\qquad \int_a^b f(t)\Delta t=F(b)-F(a),$$

where C is an arbitrary constant of integration. Next, consider the first-order non-homogeneous linear dynamic equation

$$x^\Delta(t)-ax(t)=f(t),\tag{2.1}$$

where the function f is rd-continuous on $\mathbb{T}$. Theorem 1.1 is improved as follows.

Theorem 2.1 Let $t_0\in\mathbb{T}$ and $\varepsilon>0$ be given. Suppose a function $\phi:\mathbb{T}\to\mathbb{R}$ that is delta-differentiable on $\mathbb{T}^\kappa$ satisfies

$$|\phi^\Delta(t)-a\phi(t)-f(t)|\le\varepsilon,\qquad t\in\mathbb{T}^\kappa,$$

where $a\in\mathcal{R}^+$ and $a\neq0$. Then, one of the following holds.

(i) If $a>0$ and $\sup\mathbb{T}<\infty$, then any solution x of (2.1) with $|\phi(\sup\mathbb{T})-x(\sup\mathbb{T})|<\frac{\varepsilon}{a}$ satisfies $|\phi(t)-x(t)|<\frac{\varepsilon}{a}$ for all $t\in\mathbb{T}$.

(ii) If $a>0$ and $\sup\mathbb{T}=\infty$, then $\lim_{t\to\infty}\left(\frac{\phi(t)}{e_a(t,t_0)}-\int\frac{f(t)}{e_a(\sigma(t),t_0)}\Delta t\right)$ exists, and there exists a unique solution

$$x(t):=\left[\int\frac{f(t)}{e_a(\sigma(t),t_0)}\Delta t+\lim_{t\to\infty}\left(\frac{\phi(t)}{e_a(t,t_0)}-\int\frac{f(t)}{e_a(\sigma(t),t_0)}\Delta t\right)\right]e_a(t,t_0)$$

of (2.1) such that $|\phi(t)-x(t)|\le\frac{\varepsilon}{a}$ for all $t\in\mathbb{T}$.

(iii) If $a<0$ and $\inf\mathbb{T}>-\infty$, then any solution x of (2.1) with $|\phi(\inf\mathbb{T})-x(\inf\mathbb{T})|<\frac{\varepsilon}{|a|}$ satisfies $|\phi(t)-x(t)|<\frac{\varepsilon}{|a|}$ for all $t\in\mathbb{T}$.

(iv) If $a<0$ and $\inf\mathbb{T}=-\infty$, then $\lim_{t\to-\infty}\left(\frac{\phi(t)}{e_a(t,t_0)}-\int\frac{f(t)}{e_a(\sigma(t),t_0)}\Delta t\right)$ exists, and there exists a unique solution

$$x(t):=\left[\int\frac{f(t)}{e_a(\sigma(t),t_0)}\Delta t+\lim_{t\to-\infty}\left(\frac{\phi(t)}{e_a(t,t_0)}-\int\frac{f(t)}{e_a(\sigma(t),t_0)}\Delta t\right)\right]e_a(t,t_0)$$

of (2.1) such that $|\phi(t)-x(t)|\le\frac{\varepsilon}{|a|}$ for all $t\in\mathbb{T}$.

Before proving the theorem, we give two lemmas as follows.

Lemma 2.2 ([11, Theorem 6.2]) Let $t_0 \in \mathbb{T}$. If $a \in \mathcal{R}^+$, then the inequality

$$e_a(t,t_0) \geq 1 + a(t-t_0)$$

holds for all $t \in [t_0, \infty)_{\mathbb{T}}$.

Lemma 2.3 ([1, Lemma 3.5]) Let $t_0 \in \mathbb{T}$. If $a \in \mathcal{R}^+$ and $a < 0$, then the inequality

$$e_a(t,t_0) \geq 1 + a(t-t_0)$$

holds for all $t \in (-\infty, t_0]_{\mathbb{T}}$.

Proof of Theorem 2.1 Given any $\varepsilon > 0$, let $\phi : \mathbb{T} \to \mathbb{R}$ satisfy

$$|\phi^{\Delta}(t) - a\phi(t) - f(t)| \leq \varepsilon, \qquad t \in \mathbb{T}^{\kappa}.$$

Fix $t_0 \in \mathbb{T}$, and let

$$\varphi_p(t) := e_a(t,t_0) \int \frac{f(t)}{e_a(\sigma(t),t_0)} \Delta t, \qquad t \in \mathbb{T}.$$

Then, φ_p is a particular solution of (2.1). We see, moreover, that

$$\varphi_p^{\Delta}(t) = e_a(\sigma(t),t_0)\frac{f(t)}{e_a(\sigma(t),t_0)} + a e_a(t,t_0) \int \frac{f(t)}{e_a(\sigma(t),t_0)} \Delta t = f(t) + a\varphi_p(t)$$

follows from the Δ-derivative product rule. Let $y(t) = \phi(t) - \varphi_p(t)$. Then,

$$|y^{\Delta}(t) - ay(t)| = |\phi^{\Delta}(t) - a\phi(t) - f(t)| \leq \varepsilon \tag{2.2}$$

holds for all $t \in \mathbb{T}^{\kappa}$.

First, we consider case (i), that is, assume that $a > 0$ and $\sup \mathbb{T} < \infty$. By Remark 1.2, $\max \mathbb{T}$ exists and $\sup \mathbb{T} = \max \mathbb{T}$. Now, we consider any solution x of (2.1) with $|\phi(\sup \mathbb{T}) - x(\sup \mathbb{T})| < \frac{\varepsilon}{a}$. Let $z(t) := x(t) - \varphi_p(t)$. Then, we see that z is a solution of (1.1) with

$$|y(\sup \mathbb{T}) - z(\sup \mathbb{T})| = |y(\sup \mathbb{T}) + \varphi_p(\sup \mathbb{T}) - x(\sup \mathbb{T})| < \frac{\varepsilon}{a}.$$

So that this together with Theorem 1.1 (i) and (2.2) implies that $|\phi(t) - x(t)| = |y(t) - z(t)| < \frac{\varepsilon}{a}$ for all $t \in \mathbb{T}$.

Next, we consider case (ii), that is, assume that $a > 0$ and $\sup \mathbb{T} = \infty$. Using Theorem 1.1 (ii) with (2.2), we can conclude that $\lim\limits_{t\to\infty} \frac{y(t)}{e_a(t,t_0)}$ exists, and there exists a unique solution

$$w(t) := \left(\lim_{t\to\infty} \frac{y(t)}{e_a(t,t_0)}\right) e_a(t,t_0)$$

of (1.1) satisfying $|y(t) - w(t)| \leq \frac{\varepsilon}{a}$ for all $t \in \mathbb{T}$. Recalling $y(t) = \phi(t) - \varphi_p(t)$, we see that

$$c := \lim_{t\to\infty} \left(\frac{\phi(t)}{e_a(t,t_0)} - \int \frac{f(t)}{e_a(\sigma(t),t_0)} \Delta t\right)$$

exists. Set

$$x(t) := w(t) + \varphi_p(t) = \left(\int \frac{f(t)}{e_a(\sigma(t),t_0)} \Delta t + c\right) e_a(t,t_0).$$

Then, x is a solution of (2.1) and satisfies $|\phi(t) - x(t)| = |y(t) - w(t)| \leq \frac{\varepsilon}{a}$ for all $t \in \mathbb{T}$.

Now, we will show that the above mentioned x is the unique solution satisfying $|\phi(t) - x(t)| \le \frac{\varepsilon}{a}$ for all $t \in \mathbb{T}$. Assume that there exists a solution $\tilde{x}$ of (2.1) with $\tilde{x}(t) \ne x(t)$ and $|\phi(t) - \tilde{x}(t)| \le \frac{\varepsilon}{a}$ for all $t \in \mathbb{T}$. From the uniqueness of solutions of (2.1), $\tilde{x}$ is rewritten by

$$\tilde{x}(t) := \left(\int \frac{f(t)}{e_a(\sigma(t), t_0)} \Delta t + \tilde{c}\right) e_a(t, t_0),$$

where $\tilde{c} \ne c$. Thus,

$$|c - \tilde{c}| e_a(t, t_0) = |x(t) - \tilde{x}(t)| \le |\phi(t) - x(t)| + |\phi(t) - \tilde{x}(t)| \le \frac{2\varepsilon}{a}$$

for all $t \in \mathbb{T}$. This is a contradiction of the fact that $e_a(t, t_0) \to \infty$ as $t \to \infty$ by Lemma 2.2.

The arguments given above for (i) and (ii) can be modified to establish the validity of (iii) and (iv); the details are omitted. Note here that the uniqueness of the solution x in (iv) is shown by using Lemma 2.3. This completes the proof. □

In a definition analogous to that given for the homogeneous equation (1.1), equation (2.1) is Hyers–Ulam stable (HUS) on $\mathbb{T}$ if and only if there exists a constant $K > 0$ satisfying the following property. For any $\varepsilon > 0$, if some function $\phi : \mathbb{T} \to \mathbb{R}$ satisfies $|\phi^{\Delta}(t) - a\phi(t) - f(t)| \le \varepsilon$ for all $t \in \mathbb{T}^{\kappa}$, then there exists some solution $x : \mathbb{T} \to \mathbb{R}$ of (2.1) such that $|\phi(t) - x(t)| \le K\varepsilon$ for all $t \in \mathbb{T}$. By Theorem 2.1, we get a simple result, immediately.

Corollary 2.4 If $a \in \mathcal{R}^+$ with $a \ne 0$, then (2.1) is Hyers–Ulam stable, with an HUS constant $\frac{1}{|a|}$ on $\mathbb{T}$.

Proof Let $\varepsilon > 0$ be given. Assume that $|\phi^{\Delta}(t) - a\phi(t) - f(t)| \le \varepsilon$ for all $t \in \mathbb{T}^{\kappa}$. We now consider the case $a > 0$. If we suppose that $\sup \mathbb{T} < \infty$, then there exists a solution x of (2.1) such that $|\phi(t) - x(t)| < \frac{\varepsilon}{a}$ for all $t \in \mathbb{T}$ by Theorem 2.1 (i). That is, (2.1) has HUS on $\mathbb{T}$. Note that (i) in Theorem 2.1 says that any solution satisfying the appropriate initial condition will satisfy $|\phi(t) - x(t)| < \frac{\varepsilon}{a}$ for all $t \in \mathbb{T}$. We just have to choose one of them. If $\sup \mathbb{T} = \infty$, then by Theorem 2.1 (ii), we can find an exact solution x of (2.1) satisfying $|\phi(t) - x(t)| < \frac{\varepsilon}{a}$ for all $t \in \mathbb{T}$, so that, (2.1) has HUS on $\mathbb{T}$.

For the cases $a < 0$ with $\inf \mathbb{T} > -\infty$ or $\inf \mathbb{T} = -\infty$, we can use Theorem 2.1 (iii) and (iv), so that, (2.1) has HUS on $\mathbb{T}$ when $a < 0$ as well. □

Theorem 2.5 Suppose that $\sup \mathbb{T} = \infty$ and $\inf \mathbb{T} = -\infty$, and that $a \in \mathcal{R}^+$ with $a \ne 0$. If (2.1) is Hyers–Ulam stable, then the minimal HUS constant is at least $\frac{1}{|a|}$ on $\mathbb{T}$.

Proof Let $t_0 \in \mathbb{T}$. For arbitrary $\varepsilon > 0$, let $\phi : \mathbb{T} \to \mathbb{R}$ be given by

$$\phi(t) := e_a(t, t_0) \int \frac{f(t)}{e_a(\sigma(t), t_0)} \Delta t - \frac{\varepsilon}{a}.$$

Then, we have

$$\phi^{\Delta}(t) = e_a(\sigma(t), t_0) \frac{f(t)}{e_a(\sigma(t), t_0)} + a e_a(t, t_0) \int \frac{f(t)}{e_a(\sigma(t), t_0)} \Delta t = f(t) + a\phi(t) + \varepsilon$$

by using the Δ-derivative product rule, so that $|\phi^{\Delta}(t) - a\phi(t) - f(t)| = \varepsilon$ holds for all $t \in \mathbb{T}$. Let

$$x(t) := \left(\int \frac{f(t)}{e_a(\sigma(t), t_0)} \Delta t + c\right) e_a(t, t_0),$$

where $c \in \mathbb{R}$ is an arbitrary constant. Then, x is the general solution of (2.1). Since the following facts hold, namely: if $a > 0$, then $\lim_{t \to \infty} e_a(t, t_0) = \infty$ holds by Lemma 2.2, and if

$a < 0$, then $\lim\limits_{t\to-\infty} e_a(t,t_0) = \infty$ by Lemma 2.3, we see that $|\phi(t) - x(t)| < \infty$ whenever $c = 0$; $c = 0$ implies that $|\phi(t) - x(t)| = \frac{\varepsilon}{|a|}$ for all $t \in \mathbb{T}$; that is, the minimum (best) HU-stability constant for (2.1) on $\mathbb{T}$ is at least $\frac{1}{|a|}$ on $\mathbb{T}$. $\square$

Corollary 2.4 and Theorem 2.5 imply the following theorem, immediately.

Theorem 2.6 Suppose that $\sup \mathbb{T} = \infty$ and $\inf \mathbb{T} = -\infty$. If $a \in \mathcal{R}^+$ with $a \neq 0$, then (2.1) is Hyers–Ulam stable, with best (minimum) HU-stability constant $\frac{1}{|a|}$ on $\mathbb{T}$.

Remark 2.7 Almost at the same time as the start of this study, Shen and Li [12] gave a result similar to Theorem 2.1. The difference from their result is that the integral that appears in Theorem 2.1 is the indefinite integral. Therefore, our result is represented by any primitive function. Furthermore, in this study, we deal with the best (minimum) value of the HU-stability constant. Shen and Li [12] do not study the minimum HUS constant for (2.1) on $\mathbb{T}$. For this reason, the originality of this section is guaranteed. If we can obtain the minimum HUS constant, then it is often called the best HUS constant or the best constant. For the best constant for functional equations and some positive linear operators, see Popa and Raşa [13, 14]. For linear differential equations and linear difference equations, the papers [15–18] are representative of recent results.

3 Second-Order Linear Dynamic Equations

Using the previous results on first-order linear dynamic equations, in this section we focus on the second-order linear constant coefficient dynamic equation

$$x^{\Delta\Delta}(t) + \alpha x^{\Delta}(t) + \beta x(t) = f(t), \qquad t \in \mathbb{T}^{\kappa\kappa}, \tag{3.1}$$

where α and β are real numbers and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous. Here, the set $\mathbb{T}^{\kappa\kappa}$ is defined by $\mathbb{T}^{\kappa\kappa} = \mathbb{T} - \{M, M^{\kappa}\}$ if $\mathbb{T}$ and $\mathbb{T}^{\kappa}$ have left-scattered maximums M and M^{κ}, respectively; $\mathbb{T}^{\kappa\kappa} = \mathbb{T}^{\kappa}$ if $\mathbb{T}$ has a left-scattered maximum M, but $\mathbb{T}^{\kappa}$ does not have a left-scattered maximum; $\mathbb{T}^{\kappa\kappa} = \mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum. Note here that if $\mathbb{T}$ does not have a left-scattered maximum, then $\mathbb{T}^{\kappa} = \mathbb{T}$, so that, $\mathbb{T}^{\kappa}$ does not have a left-scattered maximum. We say that (3.1) is Hyers–Ulam stable (HUS) on $\mathbb{T}$ if and only if there exists some constant $K > 0$ with the following property. For arbitrary $\varepsilon > 0$, if some function $\phi : \mathbb{T} \to \mathbb{R}$ satisfies

$$|\phi^{\Delta\Delta}(t) + \alpha\phi^{\Delta}(t) + \beta\phi(t) - f(t)| \le \varepsilon$$

for all $t \in \mathbb{T}^{\kappa\kappa}$, then there exists some solution $x : \mathbb{T} \to \mathbb{R}$ of (3.1) such that $|\phi(t) - x(t)| \le K\varepsilon$ for all $t \in \mathbb{T}$. Any such constant K is called an HUS constant for (3.1) on $\mathbb{T}$. Hyers–Ulam stability for second-order linear constant coefficient difference equations, differential equations, and dynamic equations on times scales were studied in [19–26]. In addition, Hyers–Ulam stability for second-order non-constant coefficient differential equations, functional differential equations, and dynamic equations were discussed in [27–31]. The first result for (3.1) is as follows.

Theorem 3.1 Suppose that the characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$ for (3.1) has non-zero real roots λ_1 and λ_2 with λ_1, $\lambda_2 \in \mathcal{R}^+$. Then, (3.1) is Hyers–Ulam stable, with an HUS constant $\frac{1}{|\lambda_1\lambda_2|}$ on $\mathbb{T}$.

Proof Set $\psi(t) := \phi^{\Delta}(t) - \lambda_1\phi(t)$ for $t \in \mathbb{T}^{\kappa}$. Since ϕ^{Δ} and ϕ are delta-differentiable, we get

$$\begin{aligned}|\psi^\Delta(t)-\lambda_2\psi(t)-f(t)| &= |(\phi^\Delta(t)-\lambda_1\phi(t))^\Delta-\lambda_2(\phi^\Delta(t)-\lambda_1\phi(t))-f(t)|\\ &= |\phi^{\Delta\Delta}(t)-(\lambda_1+\lambda_2)\phi^\Delta(t)+\lambda_1\lambda_2\phi(t)-f(t)|\\ &= |\phi^{\Delta\Delta}(t)+\alpha\phi^\Delta(t)+\beta\phi(t)-f(t)|\le\varepsilon \end{aligned}\tag{3.2}$$

for all $t\in\mathbb{T}^{\kappa\kappa}$. Using Corollary 2.4 with (3.2), a solution $y:\mathbb{T}\to\mathbb{R}$ of

$$y^\Delta(t)-\lambda_2 y(t)=f(t)\tag{3.3}$$

exists, such that $|\psi(t)-y(t)|\le\frac{\varepsilon}{|\lambda_2|}$ for all $t\in\mathbb{T}^\kappa$; that is,

$$|\phi^\Delta(t)-\lambda_1\phi(t)-y(t)|=|\psi(t)-y(t)|\le\frac{\varepsilon}{|\lambda_2|}\tag{3.4}$$

for all $t\in\mathbb{T}^\kappa$. Since y is a solution of (3.3), y is delta-differentiable, and thus y is rd-continuous on $\mathbb{T}^\kappa$. Using Corollary 2.4 with (3.4), a solution $x:\mathbb{T}\to\mathbb{R}$ of

$$x^\Delta(t)-\lambda_1 x(t)=y(t)\tag{3.5}$$

exists, such that $|\phi(t)-x(t)|\le\frac{\varepsilon}{|\lambda_1\lambda_2|}$ for all $t\in\mathbb{T}$. Now, we will check that x is a solution of (3.1). Since y is delta-differentiable, x^Δ is also delta-differentiable. From (3.3) and (3.5), we obtain

$$\begin{aligned}x^{\Delta\Delta}(t)+\alpha x^\Delta(t)+\beta x(t) &= x^{\Delta\Delta}(t)-(\lambda_1+\lambda_2)x^\Delta(t)+\lambda_1\lambda_2 x(t)\\ &= (x^\Delta(t)-\lambda_1 x(t))^\Delta-\lambda_2(x^\Delta(t)-\lambda_1 x(t))\\ &= y^\Delta(t)-\lambda_2 y(t)=f(t)\end{aligned}\tag{3.6}$$

for $t\in\mathbb{T}^{\kappa\kappa}$. This completes the proof. □

Theorem 3.2 Suppose that $\sup\mathbb{T}=\infty$ and $\inf\mathbb{T}=-\infty$, and that the characteristic equation $\lambda^2+\alpha\lambda+\beta=0$ for (3.1) has non-zero real roots λ_1 and λ_2 with $\lambda_1,\ \lambda_2\in\mathcal{R}^+$. Let $t_0\in\mathbb{T}$ and $\varepsilon>0$ be given. If a twice Δ-differentiable function $\phi:\mathbb{T}\to\mathbb{R}$ satisfies

$$|\phi^{\Delta\Delta}(t)+\alpha\phi^\Delta(t)+\beta\phi(t)-f(t)|\le\varepsilon,\qquad t\in\mathbb{T}^{\kappa\kappa},$$

then the following hold.

(i) If $0<\lambda_1\le\lambda_2$ and $\lim\limits_{t\to\infty}\int_{t_0}^t\frac{\Delta s}{1+\lambda_1\mu(s)}=\infty$, then

$$c_1:=\lim_{t\to\infty}\left(\frac{\phi^\Delta(t)-\lambda_1\phi(t)}{e_{\lambda_2}(t,t_0)}-\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)$$

and

$$c_2:=\lim_{t\to\infty}\left[\frac{\phi(t)}{e_{\lambda_1}(t,t_0)}-\int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t-c_1\int\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t\right]$$

exist, and there exists the unique solution

$$x(t):=\left[\int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t+c_1\int\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t+c_2\right]e_{\lambda_1}(t,t_0)$$

of (3.1) such that $|\phi(t)-x(t)|\le\frac{\varepsilon}{\lambda_1\lambda_2}$ for all $t\in\mathbb{T}$;

(ii) If $\lambda_1\le\lambda_2<0$ and $\lim\limits_{t\to-\infty}\int_{t_0}^t\frac{\Delta s}{1+\lambda_2\mu(s)}=-\infty$, then

$$c_3:=\lim_{t\to-\infty}\left(\frac{\phi^\Delta(t)-\lambda_1\phi(t)}{e_{\lambda_2}(t,t_0)}-\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)$$

and

$$c_4 := \lim_{t\to-\infty}\left[\frac{\phi(t)}{e_{\lambda_1}(t,t_0)} - \int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t - c_3\int\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t\right]$$

exist, and there exists the unique solution

$$x(t) := \left[\int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t + c_3\int\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t + c_4\right]e_{\lambda_1}(t,t_0)$$

of (3.1) such that $|\phi(t)-x(t)| \le \frac{\varepsilon}{\lambda_1\lambda_2}$ for all $t\in\mathbb{T}$;

(iii) If $\lambda_1 < 0 < \lambda_2$, then

$$c_1 := \lim_{t\to\infty}\left(\frac{\phi^\Delta(t)-\lambda_1\phi(t)}{e_{\lambda_2}(t,t_0)} - \int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)$$

and

$$c_5 := \lim_{t\to-\infty}\left[\frac{\phi(t)}{e_{\lambda_1}(t,t_0)} - \int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t - c_1\int\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t\right]$$

exist, and there exists the unique solution

$$x(t) := \left[\int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t + c_1\int\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t + c_5\right]e_{\lambda_1}(t,t_0)$$

of (3.1) such that $|\phi(t)-x(t)| \le \frac{\varepsilon}{|\lambda_1\lambda_2|}$ for all $t\in\mathbb{T}$.

Before proving the theorem, we state and establish the following lemmas.

Lemma 3.3 Let $t_0\in\mathbb{T}$. If $\lambda_1,\ \lambda_2\in\mathcal{R}^+$ with $\lambda_1\ne 0\ne\lambda_2$, then

$$\int\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t = \begin{cases}\dfrac{e_{\lambda_2}(t,t_0)}{(\lambda_2-\lambda_1)e_{\lambda_1}(t,t_0)} + C & \text{if } \lambda_1\ne\lambda_2,\\ \displaystyle\int_{t_0}^t\frac{\Delta s}{1+\lambda_1\mu(s)} + C & \text{if } \lambda_1=\lambda_2,\end{cases}$$

where C is any constant.

Proof First, consider the case $\lambda_1\ne\lambda_2$. By the Δ-derivative quotient rule, we have

$$\begin{aligned}\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(t,t_0)}\right)^\Delta &= \frac{(e_{\lambda_2}(t,t_0))^\Delta e_{\lambda_1}(t,t_0) - e_{\lambda_2}(t,t_0)(e_{\lambda_1}(t,t_0))^\Delta}{e_{\lambda_1}(t,t_0)e_{\lambda_1}(\sigma(t),t_0)}\\ &= (\lambda_2-\lambda_1)\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)},\end{aligned} \tag{3.7}$$

so that the assertion is established. Next, consider the case $\lambda_1=\lambda_2$. By [11, Theorem 2.36], it is known that

$$e_{\lambda_1}(\sigma(t),t_0) = (1+\lambda_1\mu(t))e_{\lambda_1}(t,t_0). \tag{3.8}$$

Using this, the assertion is true. □

From [11, Theorems 3.16 and 3.34] the following lemma is immediately true.

Lemma 3.4 For (3.1), assume the characteristic equation $\lambda^2+\alpha\lambda+\beta=0$ has non-zero real roots λ_1 and λ_2 with $\lambda_1,\ \lambda_2\in\mathcal{R}^+$. Let $t_0\in\mathbb{T}$. Then, the general solution of (3.1) is given by:

$$x(t) := \begin{cases} \left[\int \left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)} \int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)} \Delta t \right) \Delta t + C_1 \right] e_{\lambda_1}(t,t_0) + C_2 e_{\lambda_2}(t,t_0) \\ \hfill \text{if } \lambda_1 \neq \lambda_2, \\ \left[\int \left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)} \int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)} \Delta t \right) \Delta t + C_1 + C_2 \int_{t_0}^{t} \frac{\Delta s}{1+\lambda_1 \mu(s)} \right] e_{\lambda_1}(t,t_0) \\ \hfill \text{if } \lambda_1 = \lambda_2, \end{cases}$$

where C_1 and C_2 are arbitrary constants.

Lemma 3.5 ([1, Lemma 5.2]) Assume $a \in \mathcal{R}^+$ with $a < 0$, and let $t_0 \in \mathbb{T}$. Then, the inequality

$$e_a(t,t_0) \leq \frac{1}{1-a(t-t_0)}$$

holds for all $t \in [t_0,\infty)_{\mathbb{T}}$.

Lemma 3.6 Assume $a > 0$, and let $t_0 \in \mathbb{T}$. Then, the inequality

$$e_a(t,t_0) \leq \frac{1}{1-a(t-t_0)}$$

holds for all $t \in (-\infty,t_0]_{\mathbb{T}}$.

Proof Let

$$y(t) := \frac{1}{e_a(t,t_0)} + a(t-t_0) - 1$$

for all $t \in (-\infty,t_0]_{\mathbb{T}}$, then $y(t_0) = 0$. We will show that $y(t) \geq 0$ for all $t \in (-\infty,t_0]_{\mathbb{T}}$. Using the Δ-derivative quotient rule, we have

$$y^{\Delta}(t) = \frac{-(e_a(t,t_0))^{\Delta}}{e_a(t,t_0)e_a(\sigma(t),t_0)} + a = a\left(1 - \frac{1}{e_a(\sigma(t),t_0)}\right)$$

for all $t \in (-\infty,t_0)_{\mathbb{T}}$. Note here that $t \leq \sigma(t) \leq t_0$ from $t < t_0$ and the definition of $\sigma(t)$. This together with the monotonicity of $e_a(t,t_0)$ with $a > 0$ and $e_a(t_0,t_0) = 1$ implies that

$$0 < e_a(\sigma(t),t_0) \leq e_a(t_0,t_0) = 1,$$

and thus, $y^{\Delta}(t) \leq 0$ for all $t \in (-\infty,t_0)_{\mathbb{T}}$. Consequently, we obtain $y(t) \geq 0$ for all $t \in (-\infty,t_0]_{\mathbb{T}}$, completing the proof. □

Lemmas 2.2, 2.3, 3.5 and 3.6 imply the following lemma.

Lemma 3.7 Assume $a \in \mathcal{R}^+$ with $a \neq 0$, and let $t_0 \in \mathbb{T}$. Then, the following hold.

(i) If $\sup \mathbb{T} = \infty$, then

$$\lim_{t\to\infty} e_a(t,t_0) = \begin{cases} \infty & \text{if } a > 0, \\ 0 & \text{if } a < 0. \end{cases}$$

(ii) If $\inf \mathbb{T} = -\infty$, then

$$\lim_{t\to-\infty} e_a(t,t_0) = \begin{cases} 0 & \text{if } a > 0, \\ \infty & \text{if } a < 0. \end{cases}$$

Lemma 3.8 Suppose that $\lambda_1, \lambda_2 \in \mathcal{R}^+$ and $\lambda_1 \neq 0 \neq \lambda_2$. Let $t_0 \in \mathbb{T}$. Then, the following hold.

(i) If $\sup \mathbb{T} = \infty$, $0 < \lambda_1 < \lambda_2$ and $\lim\limits_{t\to\infty} \int_{t_0}^t \frac{\Delta s}{1+\lambda_1\mu(s)} = \infty$, then

$$\lim_{t\to\infty} \frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(t,t_0)} = \infty.$$

(ii) If $\inf \mathbb{T} = -\infty$, $\lambda_1 < \lambda_2 < 0$ and $\lim\limits_{t\to-\infty} \int_{t_0}^t \frac{\Delta s}{1+\lambda_2\mu(s)} = -\infty$, then

$$\lim_{t\to-\infty} \frac{e_{\lambda_1}(t,t_0)}{e_{\lambda_2}(t,t_0)} = \infty.$$

Proof First, we consider case (i). By (3.7), (3.8) and $\lambda_1, \lambda_2 \in \mathcal{R}^+$, we have

$$\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(t,t_0)}\right)^{\Delta} = \frac{\lambda_2-\lambda_1}{1+\lambda_1\mu(t)} \frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(t,t_0)} > 0 \tag{3.9}$$

for all $t \in \mathbb{T}$. Using this inequality, we get

$$\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(t,t_0)} \geq \frac{e_{\lambda_2}(t_0,t_0)}{e_{\lambda_1}(t_0,t_0)} = 1$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$. This together with (3.9) implies that

$$\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(t,t_0)}\right)^{\Delta} \geq \frac{\lambda_2-\lambda_1}{1+\lambda_1\mu(t)}$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$, and thus,

$$\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(t,t_0)} - 1 \geq \int_{t_0}^t \frac{\lambda_2-\lambda_1}{1+\lambda_1\mu(s)}\Delta s$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$. Therefore, we get

$$\lim_{t\to\infty} \frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(t,t_0)} = \infty.$$

Next, we consider case (ii). In the same way, we have

$$\left(\frac{e_{\lambda_1}(t,t_0)}{e_{\lambda_2}(t,t_0)}\right)^{\Delta} = \frac{\lambda_1-\lambda_2}{1+\lambda_2\mu(t)} \frac{e_{\lambda_1}(t,t_0)}{e_{\lambda_2}(t,t_0)} < 0 \tag{3.10}$$

for all $t \in \mathbb{T}$. From this, we get

$$\frac{e_{\lambda_1}(t,t_0)}{e_{\lambda_2}(t,t_0)} \geq \frac{e_{\lambda_1}(t_0,t_0)}{e_{\lambda_2}(t_0,t_0)} = 1$$

for all $t \in (-\infty,t_0]_{\mathbb{T}}$. This together with (3.10) implies that

$$\left(\frac{e_{\lambda_1}(t,t_0)}{e_{\lambda_2}(t,t_0)}\right)^{\Delta} \leq \frac{\lambda_1-\lambda_2}{1+\lambda_2\mu(t)}$$

for all $t \in (-\infty,t_0]_{\mathbb{T}}$, and thus,

$$1 - \frac{e_{\lambda_1}(t,t_0)}{e_{\lambda_2}(t,t_0)} \leq \int_t^{t_0} \frac{\lambda_1-\lambda_2}{1+\lambda_2\mu(s)}\Delta s = \int_{t_0}^t \frac{\lambda_2-\lambda_1}{1+\lambda_2\mu(s)}\Delta s$$

for all $t \in (-\infty,t_0]_{\mathbb{T}}$. Therefore, we get

$$\lim_{t\to-\infty} \frac{e_{\lambda_1}(t,t_0)}{e_{\lambda_2}(t,t_0)} = \infty.$$

□

Remark 3.9 Let $\lambda \in \mathcal{R}^+$. If $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$, then $\mu(t) = 0$, so that we have

$$\int_{t_0}^{t} \frac{\Delta s}{1+\lambda\mu(s)} = t \to \pm\infty \quad \text{as} \ \ t \to \pm\infty.$$

If $\mathbb{T} = h\mathbb{Z}$ and $t_0 = 0$, then $\mu(t) = h$, so that we have

$$\int_{t_0}^{t} \frac{\Delta s}{1+\lambda\mu(s)} = \frac{t}{1+\lambda h} \to \pm\infty \quad \text{as} \ \ t \to \pm\infty.$$

From these facts, we say that the assumptions

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{\Delta s}{1+\lambda_1\mu(s)} = \infty \quad \text{and} \quad \lim_{t\to-\infty} \int_{t_0}^{t} \frac{\Delta s}{1+\lambda_2\mu(s)} = -\infty$$

in Theorem 3.2 and Lemma 3.8 are natural conditions.

Proof of Theorem 3.2 Set $\psi(t) := \phi^{\Delta}(t) - \lambda_1\phi(t)$ for $t \in \mathbb{T}^{\kappa}$. Then, we get (3.2) for all $t \in \mathbb{T}^{\kappa\kappa}$.

First, we prove case (i). Using Theorem 2.1 (ii) with (3.2), we see that

$$\lim_{t\to\infty}\left(\frac{\psi(t)}{e_{\lambda_2}(t,t_0)} - \int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right) = \lim_{t\to\infty}\left(\frac{\phi^{\Delta}(t)-\lambda_1\phi(t)}{e_{\lambda_2}(t,t_0)} - \int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right) =: c_1$$

exists, and there exists a unique solution

$$y(t) := \left(\int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t + c_1\right) e_{\lambda_2}(t,t_0)$$

of (3.3) such that $|\psi(t) - y(t)| \le \frac{\varepsilon}{\lambda_2}$ for all $t \in \mathbb{T}^{\kappa}$; that is, (3.4) holds. Using Theorem 2.1 (ii) with (3.4), we see that

$$\begin{aligned}
&\lim_{t\to\infty}\left(\frac{\phi(t)}{e_{\lambda_1}(t,t_0)} - \int \frac{y(t)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t\right)\\
&= \lim_{t\to\infty}\left[\frac{\phi(t)}{e_{\lambda_1}(t,t_0)} - \int\left\{\left(\int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t + c_1\right)\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\right\}\Delta t\right]\\
&= \lim_{t\to\infty}\left[\frac{\phi(t)}{e_{\lambda_1}(t,t_0)} - \int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t - c_1\int \frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t\right]\\
&=: c_2
\end{aligned}$$

exists, and there exists a unique solution

$$\begin{aligned}
x(t) :&= \left(\int \frac{y(t)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t + c_2\right) e_{\lambda_1}(t,t_0)\\
&= \left[\int\left\{\left(\int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t + c_1\right)\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\right\}\Delta t + c_2\right] e_{\lambda_1}(t,t_0)\\
&= \left[\int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t + c_1\int \frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t + c_2\right] e_{\lambda_1}(t,t_0)
\end{aligned}$$

of (3.5) such that $|\phi(t) - x(t)| \le \frac{\varepsilon}{\lambda_1\lambda_2}$ for all $t \in \mathbb{T}$. Clearly, x satisfies (3.6) for $t \in \mathbb{T}^{\kappa\kappa}$, and thus, we see that x is a solution of (3.1). To simplify matters, we write

$$F(t) := \int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t. \tag{3.11}$$

Note here that x is rewritten as

$$x(t) := \begin{cases} (F(t) + c_1C + c_2)\, e_{\lambda_1}(t,t_0) + \dfrac{c_1}{\lambda_2-\lambda_1} e_{\lambda_2}(t,t_0) & \text{if} \ \ \lambda_1 \ne \lambda_2,\\[2ex] \left(F(t) + c_1C + c_2 + c_1\displaystyle\int_{t_0}^{t} \frac{\Delta s}{1+\lambda_1\mu(s)}\right) e_{\lambda_1}(t,t_0) & \text{if} \ \ \lambda_1 = \lambda_2, \end{cases}$$

from Lemma 3.3, where C is an arbitrary constant.

Now, we will show that the above mentioned x is the unique solution satisfying $|\phi(t)-x(t)| \leq \frac{\varepsilon}{\lambda_1\lambda_2}$ for all $t \in \mathbb{T}$. Assume that there exists a solution of (3.1), call it $\tilde{x}$, with $\tilde{x}(t) \neq x(t)$ and $|\phi(t)-\tilde{x}(t)| \leq \frac{\varepsilon}{\lambda_1\lambda_2}$ for all $t \in \mathbb{T}$. This implies that

$$|x(t)-\tilde{x}(t)| \leq |\phi(t)-x(t)| + |\phi(t)-\tilde{x}(t)| \leq \frac{2\varepsilon}{|\lambda_1\lambda_2|} \tag{3.12}$$

for all $t \in \mathbb{T}$. From Lemma 3.4 and the uniqueness of solutions of (3.1), $\tilde{x}$ is given by

$$\tilde{x}(t) := \begin{cases} \left(F(t)+\tilde{C}_1\right) e_{\lambda_1}(t,t_0) + \dfrac{\tilde{C}_2}{\lambda_2-\lambda_1} e_{\lambda_2}(t,t_0) & \text{if } \lambda_1 \neq \lambda_2, \\ \left(F(t)+\tilde{C}_1+\tilde{C}_2\displaystyle\int_{t_0}^{t}\frac{\Delta s}{1+\lambda_1\mu(s)}\right) e_{\lambda_1}(t,t_0) & \text{if } \lambda_1 = \lambda_2, \end{cases}$$

where $(\tilde{C}_1,\tilde{C}_2) \neq (c_1C+c_2,c_1)$. We consider the case $0<\lambda_1<\lambda_2$. Using Lemmas 3.7 (i), 3.8 (i) and $(\tilde{C}_1,\tilde{C}_2) \neq (c_1C+c_2,c_1)$, we see that

$$|x(t)-\tilde{x}(t)| = \left|c_1C+c_2-\tilde{C}_1+\frac{c_1-\tilde{C}_2}{\lambda_2-\lambda_1}\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(t,t_0)}\right| e_{\lambda_1}(t,t_0) \to \infty \quad \text{as } t\to\infty.$$

This contradicts (3.12). Next, we consider the case $\lambda_1=\lambda_2$. Using Lemma 3.7 (i) and

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{\Delta s}{1+\lambda_1\mu(s)} = \infty,$$

we can conclude that

$$|x(t)-\tilde{x}(t)| = \left|\left(c_1C+c_2-\tilde{C}_1+(c_1-\tilde{C}_2)\int_{t_0}^{t}\frac{\Delta s}{1+\lambda_1\mu(s)}\right)\right| e_{\lambda_1}(t,t_0) \to \infty \quad \text{as } t\to\infty.$$

This contradicts (3.12). Ergo, x is unique solution satisfying $|\phi(t)-x(t)| \leq \frac{\varepsilon}{\lambda_1\lambda_2}$ for all $t \in \mathbb{T}$.

Next, we consider case (ii). Using the same technique above with Theorem 2.1 (iv), (3.2) and (3.4), we see that

$$\lim_{t\to-\infty}\left(\frac{\psi(t)}{e_{\lambda_2}(t,t_0)} - \int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right) = \lim_{t\to-\infty}\left(\frac{\phi^{\Delta}(t)-\lambda_1\phi(t)}{e_{\lambda_2}(t,t_0)} - \int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right) =: c_3$$

and

$$\lim_{t\to-\infty}\left[\frac{\phi(t)}{e_{\lambda_1}(t,t_0)} - \int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t - c_3\int\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t\right] =: c_4$$

exist, and there exists a solution

$$x(t) := \left[\int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t + c_3\int\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t + c_4\right] e_{\lambda_1}(t,t_0)$$

of (3.1) such that $|\phi(t)-x(t)| \leq \frac{\varepsilon}{\lambda_1\lambda_2}$ for all $t \in \mathbb{T}$. Note here that x is rewritten as

$$x(t) := \begin{cases} \left(F(t)+c_3C+c_4\right) e_{\lambda_1}(t,t_0) + \dfrac{c_3}{\lambda_2-\lambda_1} e_{\lambda_2}(t,t_0) & \text{if } \lambda_1 \neq \lambda_2, \\ \left(F(t)+c_3C+c_4+c_3\displaystyle\int_{t_0}^{t}\frac{\Delta s}{1+\lambda_1\mu(s)}\right) e_{\lambda_1}(t,t_0) & \text{if } \lambda_1 = \lambda_2, \end{cases}$$

from Lemma 3.3, where C is any constant, and F is defined by (3.11).

Now, we will show that the above mentioned x is the unique solution satisfying $|\phi(t)-x(t)| \le \frac{\varepsilon}{\lambda_1\lambda_2}$ for all $t \in \mathbb{T}$. Consider the function

$$\tilde{x}(t) := \begin{cases} \left(F(t) + \tilde{C}_1\right) e_{\lambda_1}(t,t_0) + \dfrac{\tilde{C}_2}{\lambda_2 - \lambda_1} e_{\lambda_2}(t,t_0) & \text{if } \lambda_1 \neq \lambda_2, \\ \left(F(t) + \tilde{C}_1 + \tilde{C}_2 \displaystyle\int_{t_0}^{t} \frac{\Delta s}{1+\lambda_1\mu(s)}\right) e_{\lambda_1}(t,t_0) & \text{if } \lambda_1 = \lambda_2, \end{cases}$$

where $(\tilde{C}_1, \tilde{C}_2) \neq (c_3C + c_4, c_3)$. From Lemma 3.4 and the uniqueness of solutions of (3.1), $\tilde{x}$ is a solution of (3.1) satisfying $\tilde{x}(t) \neq x(t)$ for all $t \in \mathbb{T}$. Assume that $|\phi(t) - \tilde{x}(t)| \le \frac{\varepsilon}{\lambda_1\lambda_2}$ for all $t \in \mathbb{T}$. Then, (3.12) holds for all $t \in \mathbb{T}$. We consider the case $\lambda_1 < \lambda_2 < 0$. Using Lemmas 3.7 (ii), 3.8 (ii) and $(\tilde{C}_1, \tilde{C}_2) \neq (c_3C + c_4, c_3)$, we see that

$$|x(t) - \tilde{x}(t)| = \left|(c_3C + c_4 - \tilde{C}_1)\frac{e_{\lambda_1}(t,t_0)}{e_{\lambda_2}(t,t_0)} + \frac{c_3 - \tilde{C}_2}{\lambda_2 - \lambda_1}\right| e_{\lambda_2}(t,t_0) \to \infty \quad \text{as } t \to -\infty.$$

This contradicts (3.12). Next, we consider the case $\lambda_1 = \lambda_2$. Using Lemma 3.7 (ii) and

$$\lim_{t\to-\infty} \int_{t_0}^{t} \frac{\Delta s}{1+\lambda_2\mu(s)} = -\infty,$$

we can conclude that

$$|x(t) - \tilde{x}(t)| = \left|\left(c_3C + c_4 - \tilde{C}_1 + (c_3 - \tilde{C}_2)\int_{t_0}^{t} \frac{\Delta s}{1+\lambda_2\mu(s)}\right)\right| e_{\lambda_2}(t,t_0) \to \infty \quad \text{as } t \to -\infty.$$

This contradicts (3.12). Ergo, x is the unique solution satisfying $|\phi(t) - x(t)| \le \frac{\varepsilon}{\lambda_1\lambda_2}$ for all $t \in \mathbb{T}$.

Finally, we consider case (iii). Using the same technique as above with Theorem 2.1 (ii), (iv), (3.2) and (3.4), we see that

$$\lim_{t\to\infty}\left(\frac{\psi(t)}{e_{\lambda_2}(t,t_0)} - \int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right) = \lim_{t\to\infty}\left(\frac{\phi^{\Delta}(t) - \lambda_1\phi(t)}{e_{\lambda_2}(t,t_0)} - \int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right) =: c_1$$

and

$$\lim_{t\to-\infty}\left[\frac{\phi(t)}{e_{\lambda_1}(t,t_0)} - \int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t - c_1\int \frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t\right] =: c_5$$

exist, and there exists a solution

$$x(t) := \left[\int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int \frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t + c_1\int \frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t + c_5\right] e_{\lambda_1}(t,t_0)$$

of (3.1) such that $|\phi(t) - x(t)| \le \frac{\varepsilon}{|\lambda_1\lambda_2|}$ for all $t \in \mathbb{T}$. Note here that x is rewritten as

$$x(t) := (F(t) + c_1C + c_5)\, e_{\lambda_1}(t,t_0) + \frac{c_1}{\lambda_2 - \lambda_1} e_{\lambda_2}(t,t_0)$$

from Lemma 3.3, where C is any constant, and F is defined by (3.11).

Now, we will show that the above mentioned x is the unique solution satisfying $|\phi(t)-x(t)| \le \frac{\varepsilon}{|\lambda_1\lambda_2|}$ for all $t \in \mathbb{T}$. Consider the function

$$\tilde{x}(t) := \left(F(t) + \tilde{C}_1\right) e_{\lambda_1}(t,t_0) + \frac{\tilde{C}_2}{\lambda_2 - \lambda_1} e_{\lambda_2}(t,t_0)$$

where $(\tilde{C}_1, \tilde{C}_2) \neq (c_1C + c_5, c_1)$. From Lemma 3.4 and the uniqueness of solutions of (3.1), $\tilde{x}$ is a solution of (3.1) satisfying $\tilde{x}(t) \neq x(t)$ for all $t \in \mathbb{T}$. Assume that $|\phi(t) - \tilde{x}(t)| \le \frac{\varepsilon}{|\lambda_1\lambda_2|}$ for all

$t \in \mathbb{T}$. Then, (3.12) holds for all $t \in \mathbb{T}$. We consider the case $\tilde{C}_2 = c_1$. Using Lemma 3.7 (ii) and $\tilde{C}_1 \neq c_1 C + c_5$, we have

$$|x(t) - \tilde{x}(t)| = |c_1 C + c_5 - \tilde{C}_1| e_{\lambda_1}(t, t_0) \to \infty \quad \text{as } \ t \to -\infty.$$

This contradicts (3.12). On the other hand, consider the case $\tilde{C}_2 \neq c_1$. Using Lemma 3.7 (i) and (ii), we have

$$|x(t) - \tilde{x}(t)| = \left| (c_1 C + c_5 - \tilde{C}_1) \frac{e_{\lambda_1}(t, t_0)}{e_{\lambda_2}(t, t_0)} + \frac{c_1 - \tilde{C}_2}{\lambda_2 - \lambda_1} \right| e_{\lambda_2}(t, t_0) \to \infty \quad \text{as } \ t \to \infty.$$

This contradicts (3.12). Hence, x is the unique solution satisfying $|\phi(t) - x(t)| \leq \frac{\varepsilon}{|\lambda_1 \lambda_2|}$ for all $t \in \mathbb{T}$. This ends the proof of Theorem 3.2. □

To end this section, we give instability results for the homogeneous version of (3.1), under certain assumptions on the characteristic roots and their corresponding dynamic exponential functions.

Theorem 3.10 Assume $\sup \mathbb{T} = \infty$. The second-order homogeneous dynamic equation

$$x^{\Delta\Delta}(t) - \lambda x^{\Delta}(t) = 0 \tag{3.13}$$

is unstable in the Hyers–Ulam sense, for any $\lambda \in \mathcal{R} \backslash \{0\}$ satisfying $\lim_{t \to \infty} \left| \frac{e_\lambda(t, t_0)}{t} \right| = 0$ or ∞.

Proof Given any $\varepsilon > 0$, $t_0 \in \mathbb{T}$, and $\lambda \in \mathcal{R} \backslash \{0\}$, set

$$\phi(t) := -\frac{t\varepsilon}{\lambda}$$

for all $t \in \mathbb{T}$. Then, $\phi^{\Delta}(t) = -\frac{\varepsilon}{\lambda}$ and $\phi^{\Delta\Delta}(t) = 0$ imply that $|\phi^{\Delta\Delta}(t) - \lambda \phi^{\Delta}(t)| = \varepsilon$ for all $t \in \mathbb{T}^{\kappa\kappa}$. Since the general solution of (3.13) is $x(t) = c_1 + c_2 e_\lambda(t, t_0)$, we see that

$$\limsup_{t \to \infty} |\phi(t) - x(t)| = \limsup_{t \to \infty} \left| \frac{\varepsilon}{\lambda} + \frac{c_1}{t} + c_2 \frac{e_\lambda(t, t_0)}{t} \right| t = \infty$$

for any choice of c_1 and c_2. This ends the proof. □

Theorem 3.11 Assume $\sup \mathbb{T} = \infty$, and fix $t_0 \in \mathbb{T}$. If $\lambda_2 \in \mathcal{R}$ but not positively regressive, with $m \leq |e_{\lambda_2}(t, t_0)| \leq M$ for all $t \in \mathbb{T}$, for some $0 < m < M < \infty$, then the second-order homogeneous dynamic equation

$$x^{\Delta\Delta}(t) - (\lambda_1 + \lambda_2) x^{\Delta}(t) + (\lambda_1 \lambda_2) x(t) = 0 \tag{3.14}$$

is unstable in the Hyers–Ulam sense, for any $\lambda_1 \in \mathcal{R}$ satisfying $\lim_{t \to \infty} \frac{e_{\lambda_1}(t, t_0)}{t} = 0$ and

$$\limsup_{t \to \infty} |e_{\lambda_1}(t, t_0)| > 0.$$

Proof Given any $\varepsilon > 0$, $t_0 \in \mathbb{T}$, and $\lambda_1 \in \mathcal{R}$, suppose $\lambda_2 \in \mathcal{R}$ but not positively regressive, with $m \leq |e_{\lambda_2}(t, t_0)| \leq M$ for all $t \in \mathbb{T}$, for some $0 < m < M < \infty$. Set

$$\phi(t) := \frac{\varepsilon}{M} e_{\lambda_1}(t, t_0) \int \frac{t e_{\lambda_2}(t, t_0)}{e_{\lambda_1}(\sigma(t), t_0)} \Delta t$$

for all $t \in \mathbb{T}$. Then,

$$\phi^{\Delta}(t) = \frac{\varepsilon}{M} t e_{\lambda_2}(t, t_0) + \lambda_1 \phi(t)$$

and

$$\phi^{\Delta\Delta}(t) = \frac{\varepsilon}{M} e_{\lambda_2}(\sigma(t), t_0) + \frac{\varepsilon \lambda_2}{M} t e_{\lambda_2}(t, t_0) + \lambda_1 \phi^{\Delta}(t)$$

imply that
$$|\phi^{\Delta\Delta}(t) - (\lambda_1 + \lambda_2)\phi^{\Delta}(t) + (\lambda_1\lambda_2)\phi(t)| = \frac{\varepsilon}{M}|e_{\lambda_2}(\sigma(t), t_0)| \le \varepsilon$$
for all $t \in \mathbb{T}^{\kappa\kappa}$. Since the general solution of (3.14) is $x(t) = c_1 e_{\lambda_1}(t, t_0) + c_2 e_{\lambda_2}(t, t_0)$, we see that
$$\begin{aligned}\limsup_{t\to\infty} |\phi(t) - x(t)| &= \limsup_{t\to\infty} \left|\frac{\varepsilon}{M} e_{\lambda_1}(t, t_0) \int \frac{t e_{\lambda_2}(t, t_0)}{e_{\lambda_1}(\sigma(t), t_0)}\Delta t - c_1 e_{\lambda_1}(t, t_0) - c_2 e_{\lambda_2}(t, t_0)\right| \\ &= \infty\end{aligned}$$
for any choice of c_1 and c_2. We now show this fact. The assumption $\lim\limits_{t\to\infty} \frac{e_{\lambda_1}(t,t_0)}{t} = 0$ says that there exists $l > 0$ and $t_1 \ge t_0$ such that
$$\frac{e_{\lambda_1}(t, t_0)}{t} \le l$$
for $t \ge t_1$. Hence,
$$\begin{aligned}\left|\int \frac{t e_{\lambda_2}(t, t_0)}{e_{\lambda_1}(\sigma(t), t_0)}\Delta t\right| &= \left|\int_{t_0}^{t} \frac{s e_{\lambda_2}(s, t_0)}{e_{\lambda_1}(\sigma(s), t_0)}\Delta s + c_3\right| \ge \int_{t_0}^{t} \left|\frac{s e_{\lambda_2}(s, t_0)}{e_{\lambda_1}(\sigma(s), t_0)}\right| \Delta s - |c_3| \\ &\ge \frac{m}{l}(t - t_0) - |c_3|\end{aligned}$$
for some $c_3 \in \mathbb{R}$, and so that $\lim\limits_{t\to\infty} \left|\int \frac{t e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\Delta t\right| = \infty$. This, together with
$$\limsup_{t\to\infty} |e_{\lambda_1}(t, t_0)| > 0,$$
implies that
$$\limsup_{t\to\infty} |\phi(t) - x(t)| \ge \limsup_{t\to\infty} \left(\left|\frac{\varepsilon}{M} \int \frac{t e_{\lambda_2}(t, t_0)}{e_{\lambda_1}(\sigma(t), t_0)}\Delta t - c_1\right| |e_{\lambda_1}(t, t_0)| - M|c_2|\right) = \infty.$$
This ends the proof. □

4 Minimum HUS Constant

Theorem 4.1 Let $t_0 \in \mathbb{T}$. Suppose that $\sup \mathbb{T} = \infty$ and $\inf \mathbb{T} = -\infty$, and that the characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$ for (3.1) has non-zero real roots λ_1 and λ_2 with $\lambda_1, \lambda_2 \in \mathcal{R}^+$. If $0 < \lambda_1 \le \lambda_2$ and $\lim\limits_{t\to\infty} \int_{t_0}^{t} \frac{\Delta s}{1+\lambda_1\mu(s)} = \infty$, or if $\lambda_1 \le \lambda_2 < 0$ and $\lim\limits_{t\to-\infty} \int_{t_0}^{t} \frac{\Delta s}{1+\lambda_2\mu(s)} = -\infty$, or if $\lambda_1 < 0 < \lambda_2$, then (3.1) is Hyers–Ulam stable, with minimum HUS constant $\frac{1}{|\lambda_1\lambda_2|} = \frac{1}{|\beta|}$ on $\mathbb{T}$.

Proof From Theorem 3.1, we know that (3.1) has HUS with HUS constant $\frac{1}{|\lambda_1\lambda_2|} = \frac{1}{|\beta|}$ on $\mathbb{T}$.

Next, we will show that the minimal HUS constant is at least $\frac{1}{|\beta|}$. Fix $t_0 \in \mathbb{T}$, and for arbitrary $\varepsilon > 0$ let $Y : \mathbb{T} \to \mathbb{R}$ be given by
$$Y(t) \equiv Y(t; \lambda_1, \lambda_2) := e_{\lambda_1}(t, t_0) \int \left(\frac{e_{\lambda_2}(t, t_0)}{e_{\lambda_1}(\sigma(t), t_0)} \int \frac{f(t)}{e_{\lambda_2}(\sigma(t), t_0)}\Delta t\right) \Delta t,$$
$\varphi_p : \mathbb{T} \to \mathbb{R}$ be given by
$$\varphi_p(t) := \frac{\varepsilon}{\lambda_1\lambda_2},$$
and
$$\phi(t) := Y(t) + \varphi_p(t). \tag{4.1}$$

Then

$$Y^{\Delta}(t)=e_{\lambda_2}(t,t_0)\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t+\lambda_1 e_{\lambda_1}(t,t_0)\int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t$$

and

$$\begin{aligned}Y^{\Delta\Delta}(t)&=f(t)+(\lambda_1+\lambda_2)e_{\lambda_2}(t,t_0)\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\\&\quad+\lambda_1^2 e_{\lambda_1}(t,t_0)\int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t,\end{aligned}$$

yielding

$$\begin{aligned}Y^{\Delta\Delta}(t)-\lambda_1 Y^{\Delta}(t)&=f(t)+\lambda_2 e_{\lambda_2}(t,t_0)\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\\&=f(t)+\lambda_2 Y^{\Delta}(t)\\&\quad-\lambda_1\lambda_2 e_{\lambda_1}(t,t_0)\int\left(\frac{e_{\lambda_2}(t,t_0)}{e_{\lambda_1}(\sigma(t),t_0)}\int\frac{f(t)}{e_{\lambda_2}(\sigma(t),t_0)}\Delta t\right)\Delta t\\&=f(t)+\lambda_2 Y^{\Delta}(t)-\lambda_1\lambda_2 Y(t).\end{aligned}$$

We see that

$$Y^{\Delta\Delta}(t)-(\lambda_1+\lambda_2)Y^{\Delta}(t)+\lambda_1\lambda_2 Y(t)=f(t),$$

that is, Y solves (3.1). Moreover,

$$\varphi_p^{\Delta\Delta}(t)+\alpha\varphi_p^{\Delta}(t)+\beta\varphi_p(t)=\frac{\beta\varepsilon}{\lambda_1\lambda_2}=\varepsilon,$$

making φ_p a solution of (3.1) for $f\equiv\varepsilon$. Consequently,

$$\left|\phi^{\Delta\Delta}(t)+\alpha\phi^{\Delta}(t)+\beta\phi(t)-f(t)\right|=\varepsilon,$$

and

$$|\phi(t)-Y(t)|=|\varphi_p(t)|=\frac{\varepsilon}{|\lambda_1\lambda_2|}=\frac{1}{|\beta|}.$$

for all $\mathbb{T}$. Note here that by Theorem 3.2, Y is the unique solution satisfying $|\phi(t)-Y(t)|\le\frac{\varepsilon}{|\beta|}$ for all $t\in\mathbb{T}$. Thus, the minimum HUS constant is at least $\frac{1}{|\beta|}$, completing the proof. □

From Remark 3.9, we get the following corollaries.

Corollary 4.2 Let $\mathbb{T}=\mathbb{R}$. Suppose that the characteristic equation $\lambda^2+\alpha\lambda+\beta=0$ for (3.1) has non-zero real roots λ_1 and λ_2. Then, (3.1) is Hyers–Ulam stable, with minimum HUS constant $\frac{1}{|\lambda_1\lambda_2|}=\frac{1}{|\beta|}$ on $\mathbb{T}$.

Remark 4.3 In [17], Baias and Popa studied the Hyers–Ulam stability of the second order linear differential operator. When restricted to the case where the characteristic equation has non-zero real roots, a result given in [17] matches Corollary 4.2. Note that they are also considering the case of complex roots.

Corollary 4.4 Let $\mathbb{T}=h\mathbb{Z}$. Suppose that the characteristic equation $\lambda^2+\alpha\lambda+\beta=0$ for (3.1) has non-zero real roots λ_1 and λ_2 with $\lambda_1,\ \lambda_2\in\mathcal{R}^+$. Then, (3.1) is Hyers–Ulam stable, with minimum HUS constant $\frac{1}{|\lambda_1\lambda_2|}=\frac{1}{|\beta|}$ on $\mathbb{T}$.

Remark 4.5 When $\mathbb{T}=h\mathbb{Z}$, the condition $\lambda_1,\ \lambda_2\in\mathcal{R}^+$ means that $1+\lambda_1 h>0$ and $1+\lambda_2 h>0$. Since λ_1 and λ_2 are non-zero real roots of (3.1), we have

$$-\frac{1}{h}<\lambda_i<0\quad\text{or}\quad 0<\lambda_i$$

for $i \in \{1, 2\}$. Under this condition, using a result in [18], we can obtain the same best HUS constant given in Corollary 4.4.

Example 4.6 Consider a time scale with discrete step size $s > 0$ and continuous interval length $\xi > 0$, and let $a \in \mathbb{R} \backslash \left\{-\frac{1}{s}\right\}$. When

$$\mathbb{T} = \mathbb{P}_{\xi,s} := \bigcup_{j=0}^{\infty} [j(\xi + s), \xi + j(\xi + s)],$$

the HUS classification of (1.1) for not positively regressive a is as follows. Suppose $0 < s < \frac{\xi}{W_0(1/\mathrm{e})}$, where W_0 is the principal branch of the Lambert W function. If $a \in \left(-\infty, -\frac{1}{s}\right)$, then $a \notin \mathcal{R}^+$, and (1.1) is Hyers–Ulam stable, with best HUS constant

$$K = \frac{1}{-a}\left(\frac{\mathrm{e}^{a\xi}(1 + as) - (1 + 2as)}{1 + \mathrm{e}^{a\xi}(1 + as)}\right);$$

see [2, Theorem 2.1]. We conjecture that the best HUS constant for (3.1) on $\mathbb{P}_{\xi,s}$ is no greater than

$$\frac{1}{\lambda_1\lambda_2}\left(\frac{\mathrm{e}^{\xi\lambda_1}(1 + s\lambda_1) - (1 + 2s\lambda_1)}{1 + \mathrm{e}^{\xi\lambda_1}(1 + s\lambda_1)}\right)\left(\frac{\mathrm{e}^{\xi\lambda_2}(1 + s\lambda_2) - (1 + 2s\lambda_2)}{1 + \mathrm{e}^{\xi\lambda_2}(1 + s\lambda_2)}\right),$$

for distinct characteristic roots $\lambda_1, \lambda_2 \in \left(-\infty, -\frac{1}{s}\right)$, assuming $0 < s < \frac{\xi}{W_0(\mathrm{e}^{-1})}$. We leave the details for a future work.

5 Conclusion and Future Direction

This study deals with conditions under which second-order linear dynamic equations on time scales with constant coefficients are Hyers–Ulam stable (HUS), but also unstable in other cases. To achieve this goal, HUS for first-order non-homogeneous linear dynamic equations is established first. Moreover, the best HUS constant is obtained in some cases. By using the results, a sufficient condition for HUS and the main theorem related to HUS are obtained. It is also shown that an HUS constant obtained here is the best one. Finally, the results for several time scales are introduced.

The stability results presented here cover only the case where the characteristic equation has non-zero real roots, with instability in the zero-root case, because these results depend on the results of the first-order dynamic equation. If one can obtain HUS results for the complex-valued dynamic equations of first-order, one will obtain some results for the case where the characteristic equation has complex roots, by using the same methods.

References

[1] Anderson D R, Onitsuka M. Hyers–Ulam stability of first-order homogeneous linear dynamic equations on time scales. Demonstr Math, 2018, **51**(1): 198–210

[2] Anderson D R, Jennissen A J, Montplaisir C J. Hyers–Ulam stability for a continuous time scale with discrete uniform jumps. Int J Difference Equ, 2020, **15**(2): 1–21

[3] Anderson D R, Onitsuka M. Hyers–Ulam stability for a discrete time scale with two step sizes. Appl Math Comput, 2019, **344**: 128–140

[4] András S, Mészáros A R. Ulam–Hyers stability of dynamic equations on time scales via Picard operators. Appl Math Comput, 2013, **219**: 4853–4864

[5] Hua L, Li Y, Feng J. On Hyers–Ulam stability of dynamic integral equation on time scales. Mathematica Aeterna, 2014, **4**(6): 559–571

[6] Shen Y H. The Ulam stability of first order linear dynamic equations on time scales. Results Math, 2017, **72**(4): 1881–1895

[7] Satco B-R. Ulam–type stability for differential equations driven by measures. Math Nachr, 2020, **293**(1): 147–157

[8] Anderson D R, Onitsuka M. Hyers–Ulam stability for quantum equations. Aequat Math, 2021, **95**(2): 201–214

[9] Onitsuka M. Hyers-Ulam stability of first order linear differential equations of Carathéodory type and its application. Appl Math Lett, 2019, **90**: 61–68

[10] Tunç C, Biçer E. Hyers–Ulam–Rassias stability for a first order functional differential equation. J Math Fundam Sci, 2015, **47**(2): 143–153

[11] Bohner M, Peterson A. Dynamic Equations on Time Scales, An Introduction with Applications. Boston: Birkhäuser, 2001

[12] Shen Y, Li Y. Hyers–Ulam stability of first order nonhomogeneous linear dynamic equations on time scales. Commun Math Res, 2019, **35**(2): 139–148

[13] Popa D, Raşa I. Best constant in Hyers–Ulam stability of some functional equations. Carpathian J Math, 2014, **30**(3): 383–386

[14] Popa D, Raşa I. On the best constant in Hyers–Ulam stability of some positive linear operators. J Math Anal Appl, 2014, **412**(1): 103–108

[15] Anderson D R, Onitsuka M. Best constant for Hyers–Ulam stability of second-order h-difference equations with constant coefficients. Results Math, 2019, **74**(4): 151

[16] Anderson D R, Onitsuka M. Hyers–Ulam stability and best constant for Cayley h-difference equations. Bull Malays Math Sci Soc, 2020, **43**(6): 4207–4222

[17] Baias A R, Popa D. On the best Ulam constant of the second order linear differential operator. Rev R Acad Cienc Exactas Fís Nat Ser A Mat, 2020, **114**(1): 23

[18] Onitsuka M. Hyers–Ulam stability of second-order nonhomogeneous linear difference equations with a constant stepsize. J Comput Anal Appl, 2020, **28**(1): 152–165

[19] Hamza A E, Yaseen N A. Hyers–Ulam stability of abstract second order linear dynamic equations on time scales. Int J Math Anal, 2014, **8**(29): 1421–1432

[20] Li Y. Hyers–Ulam stability of linear differential equations $y'' = \lambda^2 y$. Thai J Math, 2010, **8**(2): 215–219

[21] Li Y, Huang J. Hyers–Ulam stability of linear second-order differential equations in complex Banach spaces. Electron J Differential Equations, 2013, **2013**(184): 1–7

[22] Li Y, Shen Y. Hyers–Ulam stability of linear differential equations of second order. Appl Math Lett, 2010, **23**(3): 306–309

[23] Najati A, Abdollahpour M, Park C. On the stability of linear differential equations of second order. Int J Nonlinear Anal Appl, 2017, **8**(2): 65–70

[24] Onitsuka M. Hyers–Ulam stability of second-order nonhomogeneous linear difference equations with a constant stepsize. J Comput Anal Appl, 2020, **28**(1): 152–165

[25] Tripathy A K. Hyers–Ulam stability of second order linear difference equations. Int J Diff Equations Appl, 2017, **16**(1): 53–65

[26] Xue J. Hyers–Ulam stability of linear differential equations of second order with constant coefficient. Ital J Pure Appl Math, 2014, **32**: 419–424

[27] Anderson D R. Hyers–Ulam stability of higher-order Cauchy-Euler dynamic equations on time scales. Dyn Sys Appl, 2014, **23**: 653–664

[28] Anderson D R, Gates B, Heuer D. Hyers–Ulam stability of second-order linear dynamic equations on time scales. Comm Appl Anal, 2012, **16**(3): 281–292

[29] Ghaemi M B, Gordji M E, Madjid E, Alizadeh B, Park C. Hyers–Ulam stability of exact second-order linear differential equations. Adv Difference Equ, 2012, **2012**(36): 1–7

[30] Javadian A, Sorouri E, Kim G H, Gordji M E. Generalized Hyers–Ulam stability of the second-order linear differential equations. J Appl Math, 2011, **2011**: 813137

[31] Murali R, Selvan A P. Ulam stability of a differential equation of second order: A fixed point approach. J Phys: Conf Ser, 2018, **1139**: 012051 doi:10.1088/1742-6596/1139/1/012051