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# NON-INSTANTANEOUS IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS WITH STATE DEPENDENT DELAY AND PRACTICAL STABILITY*

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**Abstract** Nonlinear delay Caputo fractional differential equations with non-instantaneous impulses are studied and we consider the general case of delay, depending on both the time and the state variable. The case when the lower limit of the Caputo fractional derivative is fixed at the initial time, and the case when the lower limit of the fractional derivative is changed at the end of each interval of action of the impulse are studied. Practical stability properties, based on the modified Razumikhin method are investigated. Several examples are given in this paper to illustrate the results.

**Key words** non-instantaneous impulses; Caputo fractional differential equations; practical stability

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## 1 Introduction

Many evolution processes are characterized by abrupt state changes and these are modeled by impulsive differential equations. In the literature there are two popular types of impulses: instantaneous impulses (whose duration is short compared to the overall duration of the whole process) and non-instantaneous impulses (here the action starts at some points and remain active on a finite time interval). Additionally, when fractional derivatives with their memory property are involved in the equations, the impulses cause some problems connected with the lower limit of the fractional derivative. There are mainly two types of fractional differential equations with impulses in the literature: ones with fixed lower limit at the initial time and the second with a changeable lower limit at each each time of impulse. Caputo fractional differential equations with changeable lower limit at the impulsive time are studied in [16] and the explicit formulas for the solutions are given. Also, the non-instantaneous impulsive differential equations are natural generalizations of impulsive differential equations (see, for example, [3, 6, 7, 12]). An overview of the main properties of the presence of non-instantaneous impulses to differential equations with ordinary derivatives as well as Caputo fractional derivatives is given in the book [2].

An important qualitative problem for differential equations is stability. Often Lyapunov functions and different modifications of the Lyapunov direct method are applied to study stability properties of solutions ([4, 5]). The application of Lyapunov functions to fractional differential equations requires appropriate definition of their derivatives among the solutions of the studied fractional equations. Note there many different types of stability defined and used to various kinds of differential equations. In [14, 15] the stability properties of fractional difference equations are investigated. One type of stability useful in real world problems is the so called practical stability problem, introduced by LaSalle and Lefschetz [11], and it considers the question of whether the system state evolves within certain subsets of the state-space. For example, an equilibrium point may not be stable in the sense of Lyapunov and yet the system response may be acceptable in the vicinity of this equilibrium.

In this paper we study nonlinear delay Caputo fractional differential equations with non-instantaneous impulses and we consider delays depending on both the time and the state variable. Some applications of state dependent delays are given in [8, 13]. We study two cases, one when the lower limit of the Caputo fractional derivative is fixed at the initial time, and the other when the lower limit of the fractional derivative is changed at the end of each interval of action of the impulse. Practical stability of the solutions is investigated and our arguments are based on the application of Lyapunov like functions and the modified Razumikhin method. We will need appropriate definitions of the derivative of Lyapunov functions among the studied fractional equations. In our paper we use three different types of fractional derivatives of Lyapunov functions. Comparison results for nonlinear non-instantaneous impulsive fractional differential equations without any delay are used. Some sufficient conditions for practical stability and quasi practical stability are obtained. Also several examples are given to illustrate our results.

The main contributions in this paper could be summarized:

- non-linear Caputo fractional differential equations with non-instantaneous impulses and general delay depending on both the time and the state variable are set up in both cases: the case of a fractional derivative with fixed lower limit at the initial time and the case of a fractional derivative with changed lower limit at the end of each interval of acting of the impulse;

- practical and quasi practical stability for studied equations are defined ;
- both the initial conditions and the impulsive conditions are set up in appropriate way;
- three types of the derivatives of the applied Lyapunov functions among the solutions of the studied equation are used: Caputo fractional derivative; Dini fractional derivative and Caputo fractional Dini derivative;
- fractional modification of Razumikhin method is suggested and applied in the case of any of the mentioned above three types of derivatives of Lyapunov functions;
- several sufficient conditions for practical stability are obtained for both types of the fractional derivatives- fixed lower limit as well as changeable one.

## 2 Preliminaries

Let two increasing sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ be given such that $s_0 = 0$, $0 < s_i < t_i < s_{i+1}, i = 1, 2, \cdots$, and $\lim\limits_{k\to\infty} t_k = \infty$.

Let $t_0 \in [0, s_1) \bigcup \bigcup\limits_{k=1}^{\infty} [t_k, s_{k+1})$ be a given arbitrary point. Without loss of generality we will assume that $t_0 \in [0, s_1)$.

The intervals $(t_i, s_{i+1}), i = 0, 1, 2, \cdots, k$, will be the domain of the fractional differential equations, while in the intervals $(s_i, t_i), i = 1, 2, \cdots, k$, the impulsive conditions are given.

In the paper we will use the Caputo fractional derivative of order $q \in (0, 1)$ for a function $m \in C^1([t_0, t_0 + T], \mathbb{R})$, and is given by

$$ {}_{t_0}^{c}D^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q}\, m'(s)\mathrm{d}s, \qquad t \in (t_0, t_0+T], $$

and Riemann - Liouville fractional derivative of order $q \in (0, 1)$

$$ {}_{t_0}^{RL}D_t^q m(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big( {}_{t_0}I_t^{1-q} m(t)\big) = \frac{1}{\Gamma(1-q)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_0}^{t} (t-s)^{-q}\, m(s)\mathrm{d}s, \qquad t \in (t_0, t_0+T], $$

and Grünwald−Letnikov fractional derivative

$$ {}_{t_0}^{GL}D^q m(t) = \lim_{h\to 0} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \, {}_qC_r m(t-rh), \qquad t \in (t_0, t_0+T], $$

where ${}_qC_r = \frac{q(q-1)\cdots(q-r+1)}{r!}$, $r \geq 0$ is an integer and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$ and $T \leq \infty$.

The point $t_0$ is the lower limit of the fractional derivative. Note that, for vector valued functions, the Caputo fractional derivative is taken component-wise.

Consider the space $PC_0$ of all piecewise continuous functions $\phi : [-r, 0] \to \mathbb{R}^n$ with finite number of points of discontinuity $\tau \in (-r, 0)$ at which

$$ \phi(\tau) = \lim_{t\to\tau-0} \phi(t), $$

endowed with the norm

$$ \|\phi\|_0 = \sup_{t\in[-r,0]} \{\|\phi(t)\| : \ \phi \in PC_0\}, $$

where $\|.\|$ is a norm in $\mathbb{R}^n$.

In the case of the presence of any kind of impulses to the fractional differential equations, the memory of the fractional derivative leads to considering two different types of equations:

- fractional derivative with fixed lower limit at the initial time (NIFrDDE):

$$\begin{aligned}
&{}_{t_0}^{c}D^q x(t) = f(t, x(t), x_{\rho(t,x_t)}), \text{ for } t \in \bigcup_{k=0}^{\infty}(t_k, s_{k+1}], \\
&x(t) = \Phi_k(t, x(s_k - 0)), \text{ for } t \in (s_k, t_k], k = 1, 2, \cdots, \\
&x(t_0 + t) = \phi(t), \quad t \in [-r, 0],
\end{aligned} \tag{2.1}$$

where the function $f : \mathbb{R}_+ \times \mathbb{R}^n \times PC_0 \to \mathbb{R}^n$;

- fractional derivative with changed lower limit at the end of each interval of acting of the impulse (NIFrDDE):

$$\begin{aligned}
&{}_{t_k}^{c}D^q x(t) = f(t, x(t), x_{\rho(t,x_t)}), \text{ for } t \in (t_k, s_{k+1}], k = 0, 1, 2, \cdots, \\
&x(t) = \Phi_k(t, x(s_k - 0)), \text{ for } t \in (s_k, t_k], k = 1, 2, \cdots, \\
&x(t_0 + t) = \phi(t), \quad t \in [-r, 0],
\end{aligned} \tag{2.2}$$

where the function $f : [0, s_1] \bigcup \bigcup\limits_{i=1}^{\infty}[t_i, s_{i+1}] \times \mathbb{R}^n \times PC_0 \to \mathbb{R}^n$.

In both cases, $\Phi_k : [s_k, t_k] \times \mathbb{R}^n \to \mathbb{R}^n$ $(k = 1, 2, 3, \cdots)$, $\rho : [0, s_1] \bigcup \bigcup\limits_{i=1}^{\infty}[t_i, s_{i+1}] \times PC_0 \to [0, \infty)$, $\phi \in PC_0$, $r > 0$ is a given number and the notation $x_t(s) = x(t + s), s \in [-r, 0]$ is used, i.e., $x_t \in PC_0$ represents the history of the state $x$ from time $t - r$ up to the present time $t$. Also, for any $t \geq 0$, we let $x_{\rho(t,x_t)}(s) = x(\rho(t, x(t + s))), s \in [-r, 0]$.

**Remark 2.1** The functions $\Phi_k$ are called impulsive functions and the intervals $(s_k, t_k]$, $k = 1, 2, \cdots$ are called intervals of non-instantaneous impulses.

We introduce the following assumptions:

**A1.1** The function $f \in C([0, \infty) \times \mathbb{R}^n \times PC_0, \mathbb{R}^n)$ and $f(t, 0, 0) = 0$, for $t \geq 0$.

**A1.2** The function $f \in C([0, s_1] \bigcup \bigcup\limits_{k=1}^{\infty}[t_k, s_{k+1}] \times \mathbb{R}^n \times PC_0, \mathbb{R}^n)$ and $f(t, 0, 0) = 0$ for $t \in [0, s_1] \bigcup\limits_{k=1}^{\infty}[t_k, s_{k+1}]$.

**A2.1** The function $\rho \in C([0, \infty) \times PC_0, \mathbb{R})$ and $t - r \leq \rho(t, u) \leq t$, for $u \in PC_0$ and $t \in [0, \infty)$.

**A2.2** The function $\rho \in C([0, s_1] \bigcup \bigcup\limits_{k=1}^{\infty}[t_k, s_{k+1}] \times PC_0, \mathbb{R})$ and $t - r \leq \rho(t, u) \leq t$, for $u \in PC_0$ and $t \in [0, s_1] \bigcup\limits_{i=1}^{\infty}[t_i, s_{i+1}]$.

**A3.1** The functions $\Phi_k \in C^1([s_k, t_k] \times \mathbb{R}^n, \mathbb{R}^n)$, $(k = 1, 2, 3, \cdots)$ and $\Phi_k(t, 0) = 0$, for $t \in [s_k, t_k]$.

**A3.2** The functions $\Phi_k \in C([s_k, t_k] \times \mathbb{R}^n, \mathbb{R}^n)$, $(k = 1, 2, 3, \cdots)$ and $\Phi_k(t, 0) = 0$, for $t \in [s_k, t_k]$.

**A4** The function $\phi \in PC_0$.

**Remark 2.2** Any of the assumptions A2.1 or A2.2 guarantee the delay in the argument of the unknown function in (2.2), i.e., the function $\rho$ determines the state-dependent delay.

**Example 2.3** The function $\rho(t,u) = t - \cos^2(u)$ satisfies the assumptions A2.1 and A2.2 with $r = 1$, i.e., $t - 1 \le t - \cos^2(u) \le t$.

**Remark 2.4** Let $x(t)$, with $t \ge t_0$, be a solution of (2.2) and $t \in \bigcup\limits_{k=0}^{\infty} [t_k, s_{k+1}]$ be a fixed number. Define the function $\psi(s) = x(t+s),\ s \in [-r, 0]$. Then, $\psi_0 = x_t \in PC_0$ and

$$x_{\rho(t,x_t)} = x(\rho(t, x(t+s))) = x(t + (\rho(t, x(t+s)) - t)) = \psi((\rho(t, \psi(s)) - t)) = \psi_{(\rho(t,\psi_0)-t)}.$$

If any of the assumptions A2.1 or A2.1 is satisfied, then $\psi \in PC_0$ and $\rho(t, \psi_0) - t \in [-r, 0]$.

**Remark 2.5** Note that, for the NIFrDDE (2.1), the functions $f$ and $\rho$ have to be defined for all $t \ge 0$, in spite of the fact they only appear in the fractional differential equation (see assumptions A1.1 and A2.1. With respect to the NIFrDDE (2.2), both functions are defined only on the intervals of fractional differential equations (see assumptions A1.2 and A2.2).

Let $J \subset \mathbb{R}^+$ be a given interval. We will use the following classes of functions

$$\begin{aligned} PC(J) = \{u : J \to \mathbb{R}^n : u \in C(J \setminus \bigcup_{k=1}^{\infty} \{s_k\}, \mathbb{R}^n) :& \\ u(s_k) = u(s_k - 0) = \lim_{t \uparrow s_k} u(t) < \infty,\quad u(s_k + 0) = \lim_{t \downarrow s_k} u(t) < \infty,\quad k : s_k \in J\},& \\ NPC^1(J) = \{u : J \to \mathbb{R}^n : u \in PC(J),\ u \in C^1(J \cap \Big([0, s_1] \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1}]\Big), \mathbb{R}^n) :& \\ u'(s_k) = u'(s_k - 0) = \lim_{t \uparrow s_k} u'(t) < \infty,\quad k : s_k \in J\}.& \end{aligned}$$

Define the sets:

$$\begin{aligned} \mathcal{K} &= \{\sigma \in C(\mathbb{R}_+, \mathbb{R}_+) :\ \text{strictly increasing and } \sigma(0) = 0\}, \\ S_A &= \{x \in \mathbb{R}^n :\ \|x\| \le A\}, \quad \text{where} \quad A > 0. \end{aligned}$$

## 3 Definition of Practical Stability and Lyapunov Functions

We will consider the cases of fractional differential equations with non-instantaneous impulses (2.1) and (2.2). Following the classical concept of the idea of practical stability (see [10]), we will give a definition for various types of practical stability of the zero solution of NIFrDDE (2.1) (respectively (2.2)). In the definition below, we denote by $x(t; t_0, \phi)$ any solution of the IVP for NIFrDDE (2.1) (respectively (2.2)).

**Definition 3.1** The zero solution of the system of NIFrDDE (2.1) (respectively (2.2)) is said to be

(S1) practically stable with respect to $(\lambda, A)$ if there exists $t_0 \in [0, s_0) \bigcup \bigcup\limits_{k=1}^{\infty} [t_k, s_k)$ such that, for any $\phi \in PC_0$, inequality $\|\phi\|_0 < \lambda$ implies $\|x(t; t_0, \phi)\| < A$, for $t \ge t_0$, where the positive constants $\lambda, A$ $(\lambda < A)$ are given;

(S2) practically quasi stable with respect to $(\lambda, B, T)$ if there exists an initial time $t_0 \in [0, s_0) \bigcup \bigcup\limits_{k=1}^{\infty} [t_k, s_k)$ such that, for any $\phi \in PC_0$, inequality $\|\phi\|_0 < \lambda$ implies $\|x(t; t_0, \phi)\| < B$, for $t \ge t_0 + T$, where the positive constants $\lambda, B$ $(\lambda < B)$ and $T$ are given.

In connection with our stability study, we will use Lyapunov type functions:

**Definition 3.2** ([2]) Let $\alpha < \beta \le \infty$ be given numbers and $\Delta \subset \mathbb{R}^n$ be a given set. We say that the function $V : [\alpha - r, \beta] \times \Delta \to \mathbb{R}_+$ belongs to the class $\Lambda([\alpha - r.\beta], \Delta)$ if

- The function $V$ is continuous on $[\alpha,\beta)/\bigcup\limits_{k=1}^{\infty}\{s_k\}\times\Delta$ and it is locally Lipschitz with respect to its second argument; - For each $s_k\in(\alpha,\beta)$ and $x\in\Delta$, there exist finite limits

$$V(s_k,x)=V(s_k-0,x)=\lim_{t\uparrow s_k}V(t,x)\ \text{and}\ V(s_k+0,x)=\lim_{t\downarrow s_k}V(t,x).$$

In this paper we will use three main type derivatives of Lyapunov functions $V\in\Lambda([t_0-r,t_0+T),\Delta)$, $0<T\le\infty$, among the solutions of NIFrDDE (2.1) (respectively (2.2)):

- Caputo fractional derivative given $t\in(\bigcup\limits_{k=0}^{\infty}(t_k,s_{k+1}])\cap[t_0,t_0+T)$,

$$ {}_{t_0}^{c}D^qV(t,x(t))=\frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}\frac{\mathrm{d}}{\mathrm{d}s}\Big(V(s,x(s))\Big)\mathrm{d}s, \tag{3.1}$$

where $x(t)\in\Delta,\ t\in[t_0-r,t_0+T)$, is a solution of NIFrDDE (2.1).

- Dini fractional derivative given $t\in(t_k,s_{k+1}]\cap[t_0,t_0+T),\ k=0,1,2,\cdots$,

$$\begin{aligned}{}_{t_k}D^q_{(2.2)}V(t,\psi(0),\psi)=\limsup_{h\to0}\frac{1}{h^q}\Big[V(t,\psi(0))-\sum_{r=1}^{[\frac{t-t_k}{h}]}(-1)^{r+1}\ {}_qC_rV(t-rh,\psi(0)\\ -h^qf(t,\psi(0),\psi_{(\rho(t,\psi_0)-t)}))\Big],\end{aligned}\tag{3.2}$$

where $\psi\in PC([-r,0],\Delta)$, $\psi_0(s)=\psi(s)$ and $\psi_{(\rho(t,\psi_0)-t)}=\psi(\rho(t,\psi(s))-t)$, for $s\in[-r,0]$.

Note that, if condition (A2.2) is satisfied, then $\rho(t,\psi(s))-t\in[-r,0]$ and $\psi_{\rho(t,\psi(s))-t}$ is well defined.

- Caputo fractional Dini derivative given $t\in(t_k,s_{k+1}]\cap[t_0,t_0+T),\ k=0,1,2,\cdots$,

$$\begin{aligned}&{}_{t_k}^{c}D^q_{(2.2)}V(t,\psi(0),\psi;t_k,\phi(0))=\limsup_{h\to0^+}\frac{1}{h^q}\Big\{V(t,\psi(0))-V(t_k,\phi(0))\\&-\sum_{r=1}^{[\frac{t-t_k}{h}]}(-1)^{r+1}\ {}_qC_r\Big(V(t-rh,\psi(0)-h^qf(t,\psi(0),\psi_{\rho(t,\psi_0)-t}))-V(t_k,\phi(0))\Big)\Big\},\end{aligned}\tag{3.3}$$

where $\phi,\psi\in PC([-r,0],\Delta)$.

Expression (3.3) is equivalent to

$$\begin{aligned}&{}_{t_k}^{c}D^q_{(2.2)}V(t,\psi(0),\psi;t_k,\phi(0))\\&=\limsup_{h\to0^+}\frac{1}{h^q}\Big\{V(t,\psi(0))+\sum_{r=1}^{[\frac{t-t_k}{h}]}(-1)^{r}\ {}_qC_rV(t-rh,\psi(0)-h^qf(t,\psi(0),\psi_{\rho(t,\psi_0)-t})\Big\}\\&\qquad-\frac{V(t_k,\phi(0))}{(t-t_k)^q\Gamma(1-q)}.\end{aligned}\tag{3.4}$$

**Remark 3.3** The relation between the Dini fractional derivative defined by (3.2) and the Caputo fractional Dini derivative defined by (3.4) is given by

$${}_{t_k}^{c}D^q_{(2.2)}V(t,\psi(0),\psi;t_k,\phi(0))={}_{t_k}D^q_{(2.2)}V(t,\psi(0),\psi)-\frac{V(t_k,\phi(0))}{(t-t_k)^q\Gamma(1-q)}.$$

**Example 3.4** Let $n=1$ and $V(t,x)=m(t)\ x^2$, where $m\in C^1(\mathbb{R}_+,\mathbb{R}_+)$.

**Case 1** Caputo fractional derivative:

$${}_{t_k}^{c}D^qV(t,x(t))=\frac{1}{\Gamma(1-q)}\int_{t_k}^{t}\frac{m'(s)x^2(s)+2m(s)x(s)x'(s)}{(t-s)^q}\mathrm{d}s,\quad t\in(t_k,s_{k+1}],\ k=0,1,\cdots,$$

where $x(t)=x(t;t_0,\phi),\ t\geq t_0$ is a solution of (2.1).

**Case 2** Dini fractional derivative.

$$ {}_{t_k}D^q_{(2.2)}V(t,\psi(0),\psi)=2\psi(0)\ m(t)f(t,\psi(0),\psi_{\rho(t,\psi_0)-t})+(\psi(0))^2\ {}^{GL}_{t_k}D^q m(t), \tag{3.5}$$

where $\psi\in PC([-r,0],R^n)$.

**Case 3** Caputo fractional Dini derivative: Let $\psi\in PC([-r,0],\mathbb{R}_n)$. Then

$$\begin{aligned}
&{}^{c}_{t_k}D^q_{(2.2)}V(t,\psi(0),\psi;t_k,\phi(0))\\
&=\limsup_{h\to0^+}\frac{1}{h^q}\Big\{m(t)\psi^2(0))-m(t_k)\phi^2(0))\\
&\quad-\sum_{r=1}^{[\frac{t-t_k}{h}]}(-1)^{r+1}\binom{q}{r}\Big[\Big(m(t-rh)\Big(\psi(0)-h^qf(t,\psi(0),\psi_{\rho(t,\psi_0)-t})\Big)^2-m(t_k)\phi^2(0)\Big]\Big\}\\
&=\limsup_{h\to0^+}\frac{1}{h^q}\Big\{(\psi(0))^2\sum_{r=0}^{[\frac{t-t_k}{h}]}(-1)^r\ {}_qC_r m(t-rh)-m(t_k)\phi^2(0))\sum_{r=0}^{[\frac{t-t_k}{h}]}(-1)^r\ {}_qC_r\\
&\quad-2\psi(0)h^qf(t,\psi(0),\psi_{\rho(t,\psi_0)-t})\sum_{r=0}^{[\frac{t-t_k}{h}]}(-1)^r\ {}_qC_r m(t-rh)\\
&\quad+2\psi(0)h^qf(t,\psi(0),\psi_{\rho(t,\psi_0)-t})m(t)\\
&\quad+(\psi(0))^2h^{2q}f^2(t,\psi(0),\psi_{\rho(t,\psi_0)-t})\sum_{r=1}^{[\frac{t-t_k}{h}]}(-1)^{r+1}\ {}_qC_r\Big(m(t-rh)\Big\}\\
&=2\psi(0)m(t)f(t,\psi(0),\psi_{\rho(t,\psi_0)-t})+(\psi(0))^2\ {}^{GL}_{t_k}D^q m(t)-\frac{(\phi(0))^2m(t_k)}{(t-t_k)^q\Gamma(1-q)}.
\end{aligned}\tag{3.6}$$

# 4 Main Results About Practical Stability

## 4.1 Fixed lower limit of the fractional derivative

Consider the initial value problem for the nonlinear system of non-instantaneous impulsive Caputo fractional differential equations with state dependent delay (2.1). We will study practical stability by the fractional extension of the Razumikhin method. In [5], some stability results for delay fractional differential equations (no impulses of any kind) are obtained, by applying the Caputo fractional derivative of the Lyapunov function and the generalized Razumikhin condition

$$\sup_{\Theta\in[-r,t]}V(\Theta,x(\Theta))=V(t,x(t)). \tag{4.1}$$

**Remark 4.1** Note that condition (4.1) is restrictive, but it is necessary because of the application of Caputo fractional derivative of the Lyapunov function and the fractional derivative with fixed lower limit in (2.1).

We will give sufficient conditions for practical stability of the zero solution of NFrDDE (2.1) by applying the Caputo fractional derivative of the Lyapunov function.

**Theorem 4.2** (Practical stability for the Caputo fractional derivative) Let assumptions A1.1, A2.1, A3.1, and A4 be satisfied. Assume that there exist a number $t_0\in[0,s_1)$ and a continuously differentiable Lyapunov function $V \in \Lambda([t_0 - r, \infty), \mathbb{R}^n)$, with $V(t, 0) = 0$, such that

(i) the inequalities

$$\alpha_1(\|x\|) \le V(t,x), \text{ for } t \ge t_0, x \in \mathbb{R}^n,$$
$$V(t,x) \le \alpha_2(\|x\|), \text{ for } t \ge t_0, x \in S_\lambda$$

hold, where $\alpha_i \in \mathcal{K}$, $i = 1, 2$, and $\lambda > 0$ is a given number.

(ii) for any $t \in \bigcup_{k=0}^{\infty}(t_k, s_{k+1}]$ and for any solution $x(t) = x(t; t_0, \phi)$ of (2.1), with $\phi \in PC_0$ such that

$$\sup_{\Theta \in [t_0 - r, t]} V(\Theta, x(\Theta)) = V(t, x(t)),$$

the inequality ${}^{C}_{t_0}D^q_t V(t, x(t)) \le 0$ holds.

(iii) for any $k = 1, 2, \cdots$, the inequality

$$V(t, \Phi_k(t, y)) \le V(s_k - 0, y), \quad t \in (s_k, t_{k+1}], \quad y \in S_\beta$$

holds with $\beta = \alpha_2(\lambda)$.

Then, the zero solution of (2.1) is practically stable w.r.t. $(\lambda, A)$, with $A = \alpha_1^{-1}(\alpha_2(\lambda))$.

**Proof** Let $x(t) = x(t; t_0, \phi)$ be a solution of NIFrDDE (2.1), with $\|\phi\|_0 < \lambda$. Define the function

$$v(t) = \sup_{s \in [t_0 - r, t]} V(s, x(s)), \ t \ge t_0.$$

The function $v$ is nondecreasing. According to condition (i), the inequalities

$$V(t_0 + s, x(t_0 + s)) = V(t_0 + s, \phi(s)) \le \alpha_2(\|\phi(s)\|) \le \alpha_2(\lambda)$$

hold for $s \in [-r, 0]$, i.e., $v(t_0) \in S_\beta$. We will prove that

$$v(t) = v(t_0), \quad \text{for } t \ge t_0. \tag{4.2}$$

Assume that (4.2) is not true.

**Case 1** There exists a natural number $p$ such that $v(t) = v(t_0) \in S_\beta$, for $t \in [t_0, s_p]$, but $v(t) > v(t_0)$, for $t \in (s_p, s_p + \varepsilon]$, where $\varepsilon > 0$ is a small enough number. Then, given $t^* \in (s_p, s_p + \varepsilon]$, we get $v(t^*) > v_0$ and $V(s_p, x(s_p - 0)) \in S_\beta$. According to condition (iii) we have

$$v(t^*) = \sup_{s \in [t_0 - r, t^*]} V(s, x(s)) = \sup_{s \in [t_0 - r, s_p]} V(s, x(s)) = v(s_p) = v(t_0).$$

This proves this case is impossible.

**Case 2** There exists a point $T > t_0$, $T \in \bigcup_{k=0}^{\infty}(t_k, s_{k+1})$, such that $v(t) = v(t_0)$, for $t \in [t_0, T]$, but $v(t) > v(t_0)$ and $v$ is strictly increasing for $t \in (T, T + \varepsilon]$, where $\varepsilon > 0$ is a small enough number. Then, $v(s) = v(t_0) \ge V(s, x(s))$, for $s \in [t_0, T]$, and $v(t) = V(t, x(t))$, for $t \in (T, T + \varepsilon]$. Therefore, for $t \in (T, T + \varepsilon]$, the equality

$$\sup_{\Theta \in [t_0 - r, t]} V(\Theta, x(\Theta)) = v(t) = V(t, x(t))$$

holds, and according to condition (ii), the inequality ${}^{C}_{t_0}D^q_t V(t, x(t)) \le 0$ holds. Then, for any $t \in (T, T + \varepsilon]$, we obtain

$${}^{c}_{t_0}D^q v(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{v'(s)}{(t-s)^q} \mathrm{d}s \le \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{V'(s, x(s))}{(t-s)^q} \mathrm{d}s = {}^{C}_{t_0} D^q_t V(t, x(t)) \le 0,$$

because $v(s) \geq V(s,x(s))$, for $s \in [t_0, T+\varepsilon]$, and

$$\begin{aligned}\int_{t_0}^{t} \frac{v'(s)}{(t-s)^q}\mathrm{d}s &= \int_{t_0}^{t} \frac{v'(s)-V'(s,x(s))}{(t-s)^q}\mathrm{d}s + \int_{t_0}^{t} \frac{V'(s,x(s))}{(t-s)^q}\mathrm{d}s \\ &= -\frac{v(t_0)-V(t_0,x(t_0))}{(t-t_0)^q} - q\int_{t_0}^{t} \frac{v(s)-V(s,x(s))}{(t-s)^{q+1}}\mathrm{d}s + \int_{t_0}^{t} \frac{V'(s,x(s))}{(t-s)^q}\mathrm{d}s. \quad (4.3)\end{aligned}$$

According to the assumption, we get $v'(t) = 0$, for $t \in [t_0, T]$, and $v'(t) > 0$, for $t \in (T, T+\varepsilon]$. Then, for any $t \in (T, T+\varepsilon]$, we obtain

$$ {}_{t_0}^{c}D^q v(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{v'(s)}{(t-s)^q}\mathrm{d}s = \frac{1}{\Gamma(1-q)} \int_{T}^{t} \frac{v'(s)}{(t-s)^q}\mathrm{d}s > 0, \quad t \in (T, T+\varepsilon],$$

a contradiction. This proves (4.2).

From (4.2) and condition (i), we get

$$\begin{aligned}\alpha_1(\|x(t)\|) &\leq V(t,x(t)) \leq v(t) = v(t_0) = \sup_{s\in[t_0-r,t_0]} V(s,\phi(t_0+s)) \\ &\leq \sup_{s\in[t_0-r,t_0]} \alpha_2(\|\phi(t_0+s)\|) \leq \alpha_2(\lambda), \quad t \geq t_0.\end{aligned}$$

□

### 4.2 Fractional equations with changed lower limit of the derivative

Consider the initial value problem for a nonlinear system of non-instantaneous impulsive Caputo fractional differential equations with state dependent delay (2.2). First we give comparison results (known in the literature ) by Lyapunov functions for systems of fractional differential equations with state dependent delays (no impulses)

$$\begin{aligned} &{}_{a}^{c}D^q x(t) = f(t, x(t), x_{\rho(t,x_t)}), \text{ for } t \in (a, a+\Theta], \\ &x(a+t) = \phi(t), \quad t \in [-r, 0], \end{aligned} \quad (4.4)$$

where $0 < \Theta \leq \infty$.

**Lemma 4.3** ([1] Caputo fractional Dini derivative) Assume that

1. The function $x^*(t) = x(t; a, \phi) \in \Delta$, $\Delta \subset \mathbb{R}^n$ is a solution of (4.4), defined for $t \in [a, a+\Theta]$, $\Theta > 0$.

2. The scalar fractional differential equation ${}_{a}^{c}D^q u = G(t,u)$, for $t \in (a, a+\Theta]$, with $u(a) = u_0$, where $G \in C([a, a+\Theta] \times \mathbb{R}, \mathbb{R}_+)$, has a solution $u(t; a, u_0)$.

3. The function $V \in \Lambda([a-r, a+\Theta], \Delta)$ and, for any point $t \in [a, a+\Theta]$ such that

$$V(t, x^*(t)) = \sup_{s\in[t-r,t]} V(s, x^*(s)),$$

the inequality

$$ {}_{a}^{c}D^q_{(4.4)} V(t, x^*(t), x^*; a, \phi(0)) \leq G(t, V(t, x^*(t))) \quad (4.5)$$

holds.

Then, the inequality $\sup\limits_{s\in[-r,0]} V(a+s, \phi(s)) \leq u_0$ implies $V(t, x^*(t)) \leq u^*(t)$, for $t \in [a, a+\Theta]$.

**Lemma 4.4** ([1] Dini fractional derivative) Assume the conditions of Lemma 4.3 are satisfied, where inequality (4.5) is replaced by

$$ {}_{a}^{c}D^q_{(4.4)} V(t, x^*(t), x^*) \leq G(t, V(t, x^*(t))). \quad (4.6)$$

Then, the inequality $\sup\limits_{s\in[-r,0]} V(a+s,\phi(s)) \le u_0$ implies $V(t,x^*(t)) \le u^*(t)$, for $t\in[a,a+\Theta]$.

We study the practical stability using the following scalar comparison differential equation with non-instantaneous impulses (NIFrDE):

$$\begin{aligned} &{}_{t_k}^{c}D^q u(t) = g(t,u),\ \text{for } t\in(t_k,s_{k+1}], k=0,1,2,\cdots,\\ &u(t)=\Psi_k(t,u(s_k-0)),\ \text{for } t\in(s_k,t_k], k=1,2,\cdots, \end{aligned} \tag{4.7}$$

where $u\in\mathbb{R}, g:[0,s_1]\bigcup\limits_{k=1}^{\infty}[t_k,s_{k+1}]\times\mathbb{R}\to\mathbb{R}$, and $\Psi_k:[s_k,t_k]\times\mathbb{R}\to\mathbb{R}$ $(k=1,2,3,\cdots)$.

We introduce the following assumptions:

**H1** The function $g\in C([0,s_1]\bigcup\bigcup\limits_{k=1}^{\infty}[t_k,s_{k+1}]\times\mathbb{R},\mathbb{R}_+)$ and $g(t,0)=0$.

**H2** For all natural numbers $k$, the functions $\Psi_k\in C([s_k,t_k]\times\mathbb{R},\mathbb{R})$ are such that $\Psi_k(t,0)=0$, for $t\in[s_k,t_k]$, and $\Psi_k(t,u)\le\Psi_k(t,v)$, for $u\le v,\ t\in[s_k,t_k]$.

**H3** There exists a positive number $K$ such that, for any $k=1,2,\cdots$, the inequality $|\Psi_k(s_k,u)|<K$ holds for $|u|<K$.

We will study the connection between the practical stability properties of the system NIFrDDE (2.2) and the practical stability properties of the scalar NIFrDE (4.7).

**Theorem 4.5** (for the Caputo fractional Dini derivative) Let the following conditions be satisfied:

1. Assumptions A1.2, A2, A3.2, A4 and H1–H3 are fulfilled.
2. There exist a point $t_0\in[0,s_1)$, a function $V\in\Lambda([t_0-r,\infty),\mathbb{R}^n)$, and

(i) the inequality

$$b(\|x\|)\le V(t,x)\le a(\|x\|),\quad x\in S_A, t\in[t_0-r,\infty)$$

holds, where $a,b\in\mathcal{K}$, $A=b^{-1}(K)$, and $K$ is the number defined in condition (H3);

(ii) for any functions $\psi,\phi\in PC_0$ such that $\|\psi\|_0\in S_A$ and $\|\phi\|_0\in S_A$, and for any point $t\in(t_k,s_{k+1}),\ k=0,1,2,\cdots$, such that $V(t+\tau,\psi(\tau))\le V(t,\psi(0))$, for all $\tau\in[-r,0]$, the inequality

$${}_{t_k}^{c}D^q_{(2.2)}V(t,\psi(0),\psi;t_k,\phi(0))\le g(t,V(t,\psi(0)))$$

holds;

(iii) for any $k=1,2,\cdots$, the inequality

$$V(t,\Phi_k(t,y))\le\Psi_k(t,V(s_k-0,y)),\quad t\in(s_k,t_{k+1}],\quad y\in S_A$$

holds.

3. There exists a constant $\lambda$, with $0<\lambda<A$ and $a(\lambda)\le b(A)$, such that the solution of (4.7), with an initial value $u_0:\ |u_0|<a(\lambda)$, satisfies the inequality $|u(t;t_0,u_0)|<b(A)$, for $t\ge t_0$, where $t_0$ is defined in condition 2.

Then, the zero solution of (2.2) is practically stable w.r.t. $(\lambda,A)$.

**Proof** Choose the initial function $\phi\in PC_0$ with $\|\phi\|_0<\lambda$, and consider the solution $x(t)=x(t;t_0,\phi)$ of system (2.2) for the initial time $t_0$ defined in condition 2. Let $u_0^*=\sup\limits_{t\in[t_0-r,t_0]}V(t,\phi(t-t_0))$. From the choice of the initial function $\phi$ and the properties of the function $b(u)$, applying condition 2(i), we get

$$u_0^*=\sup_{t\in[t_0-r,t_0]}V(t,\phi(t-t_0))\le a(\|\phi\|_0)<a(\lambda).$$

Consider the solution $u^*$ of (4.7) with the initial value $u_0$. According to condition 3, the inequality

$$|u^*(t)| < b(A) \quad \text{for } t \geq t_0 \tag{4.8}$$

holds. Let $p$ be a fixed nonnegative integer and $t \in (t_p, s_{p+1})$ be a given number such that $x(t+s) \in S_A$, for all $s \in [-r, 0]$, and $V(t, x(t)) = \sup_{s\in[t-r,t]} V(s, x(s))$. Denote $\psi(s) = x(t+s)$. Then, $\|\psi\|_0 \in S_A$, and from the choice of $t$, we get

$$V(t+\tau, \psi(\tau)) = V(t+\tau, x(t+\tau)) \leq V(t, x(t)) = V(t, \psi(0)),$$

for $\tau \in [-r, 0]$. According to condition 2(ii) of Theorem 4.2, we get

$${}_{t_p}^{c}D^q_{(2.2)} V(t, \psi(0), \psi; t_p, \phi(0)) \leq g(t, V(t, \psi(0))),$$

with $\phi(0) = x(t_p)$, i.e., condition 3 of Lemma 4.3 is satisfied with $a = t_p$, $G(t, u) = g(t, u)$ and $\phi(0) = x(t_p)$, $\Theta = s_{p+1} - t_p$.

We will prove that

$$V(t, x(t)) < b(A), \qquad t \geq t_0. \tag{4.9}$$

For $t = t_0$ we get

$$V(t_0, x(t_0)) \leq \sup_{t\in[t_0-r,t_0]} V(t, \phi(t-t_0)) \leq a(\lambda) < b(A).$$

Assume (4.9) is not true and let $t^* = \inf\{t > t_0 : \; V(t, x(t)) \geq b(A)\}$.

**Case 1** Suppose that $t^* \in (t_p, s_{p+1})$, for some non-negative integer $p$. Then, the function $x$ is continuous at $t^*$ and $V(t, x(t)) < b(A)$, for $t \in [t_0, t^*)$, and $V(t^*, x(t^*)) = b(A)$.

**Case 1.1** Assume that $p = 0$. From condition 2(i) and the choice of the initial function, we have

$$b(\|x(t)\|) \leq V(t, x(t) \leq b(A),$$

i.e., $x(t) \in S_A$, for $t \in [t_0 - r, t^*]$. Also,

$$\begin{aligned} V(t^*, x(t^*)) = b(A) > a(\lambda) &\geq \sup_{t\in[t_0-r,t_0]} V(t, \phi(t-t_0)) \\ &\geq \sup_{t\in[t^*-r,t_0]} V(t, \phi(t-t_0)) > V(t, x(t)), \end{aligned}$$

for $t \in [t^*-r, t_0]$. Then, $x(t^*+s) \in S_A$, for all $s \in [-r, 0]$, and $V(t^*, x(t^*)) = \sup_{s\in[t^*-r,t^*]} V(s, x(s))$. Therefore, the conditions of Lemma 4.3 are satisfied for $a = t_0$, $G(t, u) = g(t, u)$, $\phi(0) = x(t_0)$, and $\Theta = t^* - t_0$. Then, $V(t, x(t)) \leq u^*(t)$, for $t \in [t_0, t^*]$, and according to condition 2(i) of Theorem 4.2, we get

$$b(A) = b(\|x(t^*)\| \leq V(t^*, x(t^*)) \leq u^*(t) < b(A).$$

This contradiction proves the assumption is not true.

**Case 1.2** Assume now that $p \geq 1$. From condition 2(i) we have

$$b(\|x(t)\|) \leq V(t, x(t) \leq b(A).$$

Then, $x(t^*+s) \in S_A$, for all $s \in [-r, 0]$, and $V(t^*, x(t^*)) = \sup_{s\in[t^*-r,t^*]} V(s, x(s))$. Therefore, the conditions of Lemma 4.3 are satisfied for $a = t_p$, $G(t, u) = g(t, u)$, $\phi(0) = x(t_p)$, and $\Theta = t^* - t_p$.

Then, $V(t,x(t)) \le u^(t)$, for $t \in [t_p, t^*]$. Thus we get

$$b(A) = V(t^*, x(t^*)) \le u(t^*) < b(A).$$

This contradiction proves the assumption is not true.

**Case 2** Suppose that $t^* \in (s_p, t_p)$, for some natural number $p$. From condition 2(i) it follows that

$$b(\|x(t^*)\|) \le V(t^*, x(t^*) \le b(A),$$

i.e., $x(t^*) \in S_A$. Then, from condition 2(iii), we get

$$V(t^*, x(t^*)) = V(t^*, \Phi_p(t^*, x(s_p - 0))) \le \Psi_p(t^*, V(s_p - 0, x(s_p - 0))).$$

From condition (H3), using the inequality $V(s_p - 0, x(s_p - 0)) < b(A) = K$, we get the contradiction

$$b(A) \le \Psi_p(t^*, V(s_p - 0, x(s_p - 0))) < K = b(A).$$

**Case 3** Finally, suppose that $t^* = s_p$, for some natural number $p$.

**Case 3.1** Let $V(t, x(t)) < b(A)$, for $t \in [t_0, s_p)$, and $V(s+p-0, x(s_p-0)) = b(A)$. Thus, the inclusion $x(t) \in S_A$, for $t \in [t_0, s_p]$, is valid and as in the case 1 we get a contradiction.

**Case 3.2** Let $V(t, x(t)) < b(A)$ for $t \in [t_0, s_p]$ and $V(s_p + 0, x(s_p + 0)) \ge b(A)$. Thus, from condition 2(i), we get $V(s_p - 0, x(s_p - 0)) < b(A) = K$. From condition (H3) we have $\Psi_p(s_p + 0, V(s_p - 0, x(s_p - 0))) < K$ which leads to the contradiction

$$\begin{aligned} b(A) &\le V(s_p + 0, x(s_p + 0)) = V(s_p + 0, \Phi_p(s_p + 0, x(s_p - 0))) \\ &\le \Psi_p(s_p + 0, V(s_p - 0, x(s_p - 0))) < K = b(A). \end{aligned}$$

From (4.9) and condition 3(i) we obtain the claim in Theorem 4.5. □

**Remark 4.6** Note that, in Theorem 4.5, the condition in (ii) is similar to the Razumikhin condition and it is not as restrictive as the condition used in Theorem 4.2 (we note the type of the fractional derivatives used in (2.1) and (2.2)).

**Theorem 4.7** (for the Dini fractional derivative) Let the conditions of Theorem 4.5 be satisfied but replace condition 2(ii) by:

2(ii) for any function $\psi \in PC_0$ with $\|\psi\|_0 \in S_A$, and any point $t \in (t_k, s_{k+1}),\ k = 0, 1, 2, \cdots$, such that $V(t + \tau, \psi(\tau)) \le V(t, \psi(0))$, for $\tau \in [-r, 0]$, the inequality

$${}_{t_k}^{c}D_{(2.2)}^{q} V(t, \psi(0), \psi) \le g(t, V(t, \psi(0)))$$

holds. Then, the zero solution of (2.2) is practically stable w.r.t. $(\lambda, A)$.

The proof of Theorem 4.7 is similar to that in Theorem 4.5 where instead of Lemma 4.3 we apply Lemma 4.4.

**Theorem 4.8** Let the following conditions be satisfied:

1. Assumptions A1.2, A2, A3.2, A4 and H1–H3 are fulfilled.
2. There exist a point $t_0 \in [0, s_1)$ and a function $V \in \Lambda([t_0 - r, \infty), \mathbb{R}^n)$ such that

(i) the inequality

$$b(\|x\|) \le V(t, x) \le a(\|x\|), \quad x \in \mathbb{R}^n, t \in [t_0 - r, \infty)$$

holds, where $a, b \in \mathcal{K}$;

(ii) for any function $\psi \in PC_0$ and any point $t \in (t_k, s_{k+1}),\ k = 0, 1, 2, \cdots$, such that $V(t+\tau, \psi(\tau)) \le V(t, \psi(0))$, for $\tau \in [-r, 0]$, the inequality

$$ {}_{t_k}^{c}D^q_{(2.2)}V(t, \psi(0), \psi; t_k, \phi(0)) \le g(t, V(t, \psi(0))) $$

holds;

(iii) for any $k = 1, 2, \cdots$, the inequality

$$ V(t, \Phi_k(t, y)) \le \Psi_k(t, V(s_k - 0, y)), \quad t \in (s_k, t_{k+1}], \quad y \in \mathbb{R}^n $$

holds.

3. There exist positive constants $\lambda, T$ with $0 < \lambda < B$ and $a(\lambda) \le b(B)$, where $B = b^{-1}(K)$, and $K$ is defined in condition (H3), such that the solution of (4.7), with the initial condition $u_0 \in \mathbb{R}_+ : u_0 < a(\lambda)$, satisfies the inequality $|u(t; t_0, u_0)| < b(B)$, for $t \ge t_0 + T$, where $t_0$ is defined in condition 2.

Then, the zero solution of (2.2) is practically quasi stable w.r.t. $(\lambda, B, T)$.

**Proof** Choose the initial function $\phi \in PC_0$ with $\|\phi\|_0 < \lambda$, and consider the solution $x(t) = x(t; t_0, \phi)$ of system (2.2) for the initial time $t_0$ defined in condition 2. Let

$$ u_0 = \sup_{t \in [t_0 - r, t_0]} V(t, \phi(t - t_0)). $$

From the choice of the initial function $\phi$ and the properties of the function $b$, applying condition 2(i), we get

$$ u_0 = \sup_{t \in [t_0 - r, t_0]} V(t, \phi(t - t_0)) \le a(\|\phi\|_0) < a(\lambda). $$

Consider the solution $u(t) = u(t; t_0, u_0)$ of (4.7). Therefore, the function $u$ satisfies

$$ |u(t; t_0, u_0)| \ < b(B), \quad \text{for } t \ge t_0 + T, \tag{4.10} $$

for $t \ge t_0$ with $u_0 = u_0^*$, where $u^*(t) = u(t; t_0, u_0^*)$ is a solution of (4.7). According to condition 2(ii), condition 4(i) of Lemma 4.3 is satisfied for the solution $x$ (with $\Theta = \infty$). From Lemma 4.3, with $\Theta = \infty$, it follows that the inequality $V(t, x(t)) \le u^*(t)$, for $t \ge t_0$, holds. From condition 2(i) and inequality (4.10) we get

$$ b(\|x(t)\|) \le V(t, x(t)) \le u^*(t) < b(B), $$

for $t \ge t_0 + T$. □

In the case of the application of the Dini fractional derivative we obtain:

**Theorem 4.9** Let the conditions of Theorem 4.8 be satisfied with replacing the condition 2(ii) by:

2(ii) for any function $\psi \in PC_0$ with $\|\psi\|_0 \in S_A$, and any point $t \in (t_k, s_{k+1}),\ k = 0, 1, 2, \cdots$, such that $V(t+\tau, \psi(\tau)) \le V(t, \psi(0))$, for $\tau \in [-r, 0]$, the inequality

$$ {}_{t_k}^{c}D^q_{(2.2)}V(t, \psi(0), \psi) \le g(t, V(t, \psi(0))) $$

holds.

Then, the zero solution of (2.2) is practically quasi stable w.r.t. $(\lambda, B, T)$.

The proof of Theorem 4.9 is similar to that in Theorem 4.8, where instead of Lemma 4.3, we apply Lemma 4.4.

**Remark 4.10** Note that condition (H3) of Theorems 4.5, 4.8, and 4.9, could be replaced by the condition:

For all $k = 1, 2, \cdots$, the functions $\Psi_k$ satisfy $\Psi_k(t, u) \leq u$, for $t \in [s_k, t_k]$ and $u \in \mathbb{R}$.

**Remark 4.11** The point $t_0$ in the conditions of all the above Theorems is from the interval $[0, s_1)$ but one can modify the proofs so it can be from any interval $[t_k, s_{k+1}), k = 1, 2, \cdots$ .

### 4.3 Some examples

We will consider several particular examples and apply our results to illustrate the practical stability properties.

**Example 4.12** (constant delay) Let $s_k = 2k + 1$, $t_k = 2k$, for $k = 0, 1, 2, \cdots$, and $r = 1$. Consider the IVP for the nonlinear system of non-instantaneous impulsive fractional differential equations with constant delay:

$$\begin{aligned}
&{}_{2k}^{c}D^q x(t) = y(t)\frac{t}{1+t}\Big(x(t) + y^2(t)\Big) + \mathrm{e}^{-t}y(t-1), \text{ for } t \in (2k, 2(k+1)], k = 0, 1, 2, \cdots ,\\
&{}_{2k}^{c}D^q y(t) = -0.5x(t)\frac{t}{1+t}\Big(x^2(t) + y^2(t)\Big) + \mathrm{e}^{-t}x(t-1),\\
&\qquad\qquad\qquad \text{for } t \in (2k, 2(k+1)], k = 0, 1, 2, \cdots\\
&x(t) = -\sqrt{\frac{0.5}{E_q(2)}}\sin(t)x(2k-1-0), \text{ for } t \in (2k-1, 2k], k = 1, 2, \cdots ,\\
&y(t) = \sqrt{\frac{0.5}{E_q(2)}}\sin(t)y(2k-1-0), \text{ for } t \in (2k-1, 2k], k = 1, 2, \cdots ,\\
&x(s) = \phi_1(s), \quad y(s) = \phi_2(s) \quad s \in [-1, 0],
\end{aligned} \tag{4.11}$$

where $x, y \in \mathbb{R}$, $a, b \in (-1, 1)$ are given constants. In this particular case,

$$\rho(t, x(t+s), y(t+s)) \equiv t - 1, \ s \in [-1, 0],$$

and the conditions A2.1 and A2.2 are satisfied. Therefore,

$$x_{\rho(t,x_t,y_t)}(s) = x(\rho(t, x(t+s)), y(t+s))) = x(t-1) \quad \text{and} \quad y_{\rho(t,x_t,y_t)}(s) = y(t-1),$$

for $s \in [-1, 0]$. Let $V(t, x, y) = 1.5(x^2 + 2y^2)$. Then,

$$1.5(x^2 + y^2) \leq V(t, x, y) \leq 3(x^2 + y^2),$$

i.e., $b(s) = 1.5s^2$ and $a(s) = 3s^2$. Let $t \in \bigcup_{k=0}^{\infty}(2k, 2k+1]$ and $\psi = (\psi_1, \psi_2) \in PC_0$ be such that

$$V(t, \psi_1(0), \psi_2(0)) > V(t+s, \psi_1(s), \psi_2(s)),$$

for $s \in [-1, 0)$, i.e.,

$$\psi_1^2(0) + 2\psi_2^2(0) > \psi_1^2(s) + 2\psi_2^2(s), \quad s \in [-1, 0).$$

In this case,

$$\psi_{1_{\rho(0,(\psi_1)_0,(\psi_2)_0)}}(s) = \psi_1(-1) \quad \text{and} \quad \psi_{2_{\rho(0,(\psi_1)_0,(\psi_2)_0)}}(s) = \psi_2(-1),$$

for $s \in [-1, 0]$. Then, according to Example 3.4, for $m(t) \equiv 1$, we obtain

$$\begin{aligned}
&{}_{2k}^{c}D_{(4.11)}^{q}V(t, \psi_1(0), \psi_2(0), \psi_1, \psi_2; t_k, \phi_1(0), \phi_1(0))\\
&= 1.5\ {}_{2k}^{c}D_{(4.11)}^{q}(\psi_1^2) + 3\ {}_{2k}^{c}D_{(4.11)}^{q}(\psi_2^2)
\end{aligned}$$

$$
\begin{aligned}
&= 3\mathrm{e}^{-t}\Big(\psi_1(0)(\psi_1)_{\rho(0,(\psi_1)_0,(\psi_2)_0)}(s) + 2\psi_2(0)(\psi_2)_{\rho(0,(\psi_1)_0,(\psi_2)_0)}(s)\Big)\\
&\le 1.5\mathrm{e}^{-t}\Big(\psi_1^2(0) + (\psi_1(-1))^2 + 2\psi_2^2(0) + 2(\psi_2(-1))^2\Big)\\
&\le 3\mathrm{e}^{-t}\Big(\psi_1^2(0) + 2\psi_2^2(0)\Big) = 2V(t,\psi_1(0),\psi_2(0)). \qquad (4.12)
\end{aligned}
$$

For any $t \in (2k+1, 2k+2]$, $k = 0, 1, 2, \cdots$, $x, y \in \mathbb{R}$, we have

$$
\begin{aligned}
V\left(t, -\sqrt{\frac{0.5}{E_q(2)}}\sin(t)x, \sqrt{\frac{0.5}{E_q(2)}}\sin(t)y\right) &= \frac{0.5}{E_q(2)}\sin^2(t)1.5\Big(x^2+2y^2\Big)\\
&\le \frac{0.5}{E_q(2)}\sin^2(t)V(2k+1-0,x,y)\\
&\le \frac{0.5}{E_q(2)}V(2k+1-0,x,y). \qquad (4.13)
\end{aligned}
$$

Consider the IVP for the scalar fractional differential equation with non-instantaneous impulses

$$
\begin{aligned}
&{}^{c}_{2k}D^q u(t) = 2u, \ \text{for } t \in (2k, 2k+1], \ k = 0, 1, \cdots\\
&u(t) = \frac{0.5}{E_q(2)}u(2k-1-0), \ \text{for } t \in (2k-1, 2k], k = 0, 1, 2, \cdots, \qquad (4.14)\\
&u(t_0) = u_0,
\end{aligned}
$$

It has a solution (see Figure 1)

$$
u(t) = \begin{cases} u_0 E_q(2t^q), & \text{if } t \in (0,1]\\ 0.5^k u_0 E_q(2(t-2k)^q), & \text{if } t \in (2k, 2k+1], \ k = 1, 2, \cdots\\ 0.5^k u_0, & \text{if } \ t \in (2k-1, 2k], \ k = 1, 2, \cdots \end{cases}
$$

If $\lambda = 1/\sqrt{3E_q(2)}$ then, for $|u_0| < a(\lambda) = 3\lambda^2 = \frac{1}{E_q(2)} \approx 0.00917929$, we have $|u(t)| \le 0.5^{\frac{t-1}{2}} \to_{t\to\infty} 0$.

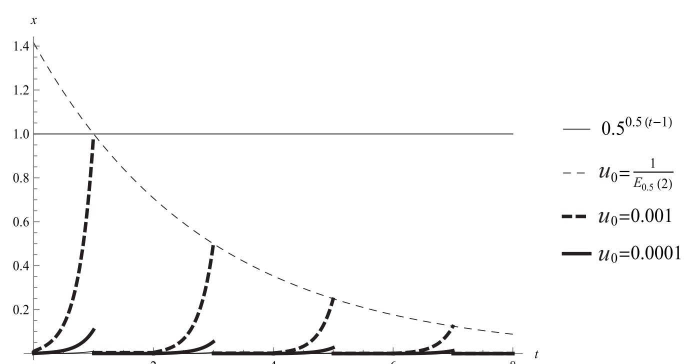


Figure 1 Example 4.12. Graph of the solution of (4.13) for different initial values

Now choose $A$ such that $A > \sqrt{2/(3E_q(2))}$. According to Theorem 4.5, the solution of (4.11) is practically stable w.r.t. the couple $(1/\sqrt{3E_q(2)}, \sqrt{2/(3E_q(2))})$, i.e., if

$$
\|\phi_1\|_0 + \|\phi_2\|_0 < \frac{1}{\sqrt{3E_q(2)}},
$$

then

$$\|(x(t), y(t))\| < \sqrt{\frac{2}{3E_q(2)}}.$$

Let, for example, $q = 0.5$ and the initial functions $\phi_1(s) = \phi_2(s) = 0.2(1 + \sin(s)),\ s \in [-1, 0]$, i.e.,

$$\|\phi_1\|_0 + \|\phi_1\|_1 < \frac{1}{\sqrt{3E_{0.5}(2)}} \approx 0.055315.$$

The solution of the system (4.11) with these particular initial functions $\phi_1(s) = \phi_2(s) = 0.2(1+\sin(s)),\ s \in [-1, 0]$ is shown in Figure 2.

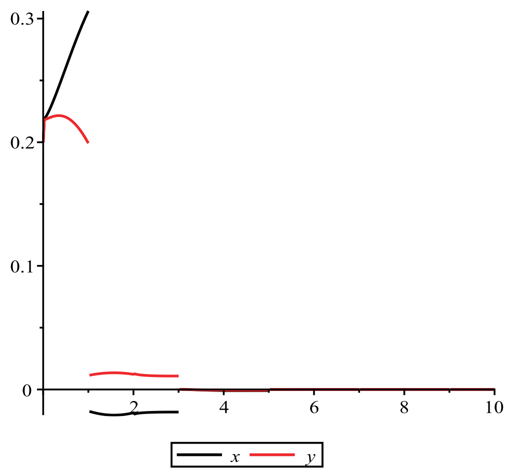


Figure 2 Example 4.12. Graph of the solution of (4.11)

Now, let change the impulsive functions in (4.11) to other ones, for example, let the impulsive conditions of the system (4.11) be changed by

$$\begin{aligned} x(t) &= -tx(2k-1-0),\ \text{for } t \in (2k-1, 2k], k = 1, 2, \cdots, \\ y(t) &= ty(2k-1-0),\ \text{for } t \in (2k-1, 2k], k = 1, 2, \cdots. \end{aligned} \tag{4.15}$$

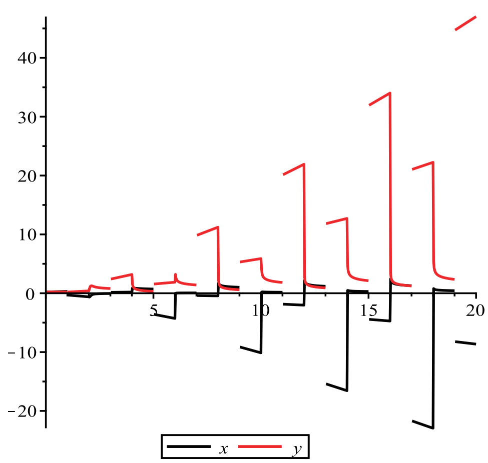


Figure 3 Example 4.12. Graph of the solution of (4.11) for large impulsive functions

Note the impulsive functions $-tx$ and $ty$ do not satisfy the conditions of Theorems 4.5, 4.7, and 4.8, and as can be seen from Figure 3, the system (4.11) is not practically stable. Thus, condition (iii) is not only sufficient but also it is necessary to assure the practical stability for the system.

**Example 4.13** (variable time delay) Let $s_k = 2k+1$, $t_k = 2k$ for $k = 0,1,2,\cdots$, and $r = 1$. Consider the initial value problem for the nonlinear system of non-instantaneous impulsive fractional differential equations with time variable delay:

$$\begin{aligned}
&{}_{2k}^{c}D^q x(t) = y(t)\frac{t}{1+t}\Big(x(t)+y^2(t)\Big) + \mathrm{e}^{-t}y(t-\sin^2(t)),\\
&\qquad\qquad \text{for } t\in(2k,2k+1],\ k=0,1,\cdots\\
&{}_{2k}^{c}D^q y(t) = \frac{-x(t)}{2}\frac{t}{1+t}\Big(x^2(t)+y^2(t)\Big) + \mathrm{e}^{-t}x(t-\sin^2(t)),\\
&\qquad\qquad \text{for } t\in(2k,2k+1],\ k=0,1,\cdots\\
&x(t) = -\sqrt{\frac{0.5}{E_q(2)}}\sin(t)x(2k-1-0),\ \text{for } t\in(2k-1,2k], k=1,2,\cdots,\\
&y(t) = \sqrt{\frac{0.5}{E_q(2)}}\sin(t)y(2k-1-0),\ \text{for } t\in(2k-1,2k], k=1,2,\cdots,\\
&x(s)=\phi_1(s),\quad y(s)=\phi_2(s)\quad s\in[-1,0],
\end{aligned}\tag{4.16}$$

where $x, y \in \mathbb{R}$.

In this partial case,

$$\rho(t, x(t+s), y(t+s)) \equiv t - \sin^2(t),\ s\in[-1,0],$$

and the assumptions A2.1 and A2.2 are satisfied, with $r = 1$ (see Figure 4).

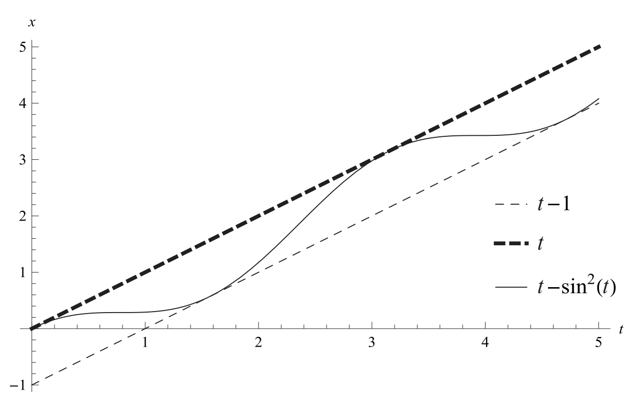


Figure 4 Example 4.13. Graph of the delay in (4.16)

Therefore,

$$x_{\rho(t,x_t,y_t)}(s) = x(\rho(t, x(t+s)), y(t+s))) = x(t-\sin^2(t))$$

and

$$y_{\rho(t,x_t,y_t)}(s) = y(t-\sin^2(t)),\ \text{for } s\in[-1,0].$$

Let $V(t,x,y)=1.5(x^2+2y^2)$. Similar to Example 4.12 and (4.12) and (4.13), we prove the validity of Conditions 2(ii) and 2(iii) of Theorem 4.5 and the comparison scalar equation is also (4.14). According to Theorem 4.5, the solution of the system (4.16) is practically stable.

Let for example $q=0.5$ and the initial functions $\phi_1(s)=\phi_2(s)=0.2(1+\sin(s)),\ s\in[-1,0]$, i.e.,

$$\|\phi_1\|_0+\|\phi_1\|_1<\frac{1}{\sqrt{3E_{0.5}(2)}}\approx 0.055315.$$

The solution of the system (4.16) for these particular initial functions is shown in Figure 5.

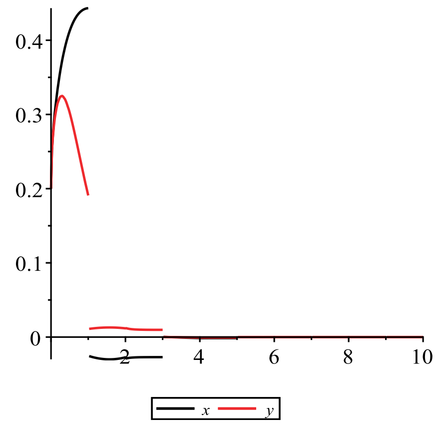


Figure 5 Example 4.13. Graph of the solution of (4.16) with time variable delay

Similar to Example 4.12, we change the impulsive functions in (4.16) by othr ones. For example, let the impulsive conditions of the system (4.16) be changed by

$$\begin{aligned}&x(t)=-tx(2k-1-0),\ \text{for } t\in(2k-1,2k],k=1,2,\cdots,\\&y(t)=ty(2k-1-0),\ \text{for } t\in(2k-1,2k],k=1,2,\cdots.\end{aligned}\tag{4.17}$$

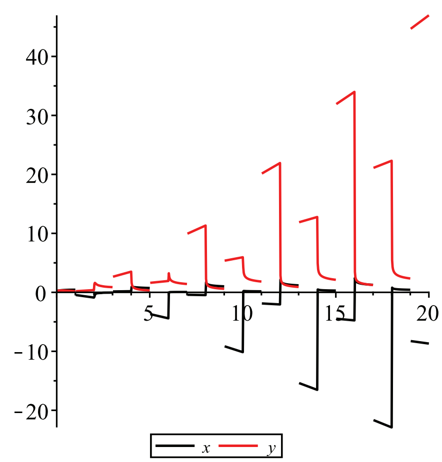


Figure 6 Example 4.13. Graph of the solution of (4.16) with large impulsive functions

The impulsive functions $-tx$ and $ty$ do not satisfy the conditions of Theorems 4.5, 4.7, and

4.8, and as can be seen from Figure 6 the system (4.16) is not practically stable. Therefore, condition (iii) for the impulsive functions is not only sufficient but also it is necessary to assure the practical stability for the system.

**Example 4.14** (state dependent delay) Let $s_k = 2k+1$, $t_k = 2k$ for $k = 0, 1, 2, \cdots$, and $r = 1$. Consider the initial value problem for the nonlinear system of non-instantaneous impulsive fractional differential equations with time variable delay

$$\begin{aligned}
&{}_{2k}^{c}D^q x(t) = y(t)\frac{t}{1+t}\Big(x(t) + y^2(t)\Big) + \mathrm{e}^{-t} y_{\rho(t,x(t),y(t))},\\
&\qquad\qquad t \in (2k, 2k+1],\ k = 0, 1, \cdots\\
&{}_{2k}^{c}D^q y(t) = \frac{-x(t)}{2}\frac{t}{1+t}\Big(x^2(t) + y^2(t)\Big) + \mathrm{e}^{-t} x_{\rho(t,x(t),y(t))},\\
&\qquad\qquad t \in (2k, 2k+1],\ k = 0, 1, \cdots\\
&x(t) = a\sin(t)x(2k+1-0), \quad y(t) = b\sin(t)y(2k+1-0),\\
&\qquad\qquad \text{for } t \in (2k+1, 2k+2], k = 0, 1, 2, \cdots,\\
&x(s) = \phi_1(s), \quad y(s) = \phi_2(s) \quad t \in [-r, 0],
\end{aligned} \tag{4.18}$$

where $x, y \in \mathbb{R}$, $r > 0$ is a small constant, $a, b \in (-1, 1)$ are given constants,

$$\rho(t, x, y) \equiv t - 0.5(\sin^2(x) + \cos^2(y)).$$

Then, the assumptions A2.1 and A2.2 are satisfied. Therefore,

$$\begin{aligned}
x_{\rho(t,x_t,y_t)}(s) &= x(\rho(t, x(t+s)) + s, y(t+s)))\\
&= x(t - 0.5(\sin^2(x(t+s)) + \cos^2(y(t+s))))
\end{aligned}$$

and

$$y_{\rho(t,x_t,y_t)}(s) = y(t - 0.5(\sin^2(x(t+s)) + \cos^2(y(t+s)))) \text{ for } s \in [-1, 0].$$

Let $V(t, x, y) = 1.5(x^2 + 2y^2)$. Also, let $t \in \bigcup\limits_{k=0}^{\infty}(2k, 2k+1]$ and $\psi = (\psi_1, \psi_2) \in PC_0$ be such that

$$V(t, \psi_1(0), \psi_2(0)) > V(t+s, \psi_1(s), \psi_2(s)) \text{ for } s \in [-r, 0),$$

or

$$\psi_1^2(0) + 2\psi_2^2(0) > \psi_1^2(s) + 2\psi_2^2(s) \text{ for } s \in [-r, 0)$$

In this case,

$$\begin{aligned}
\psi_{1_{\rho(0,(\psi_1)_0,(\psi_2)_0)}}(s) = \psi_1(-0.5(\sin^2(\psi_1(s)) + \cos^2(\psi_2(s))),\ s \in [-1, 0],\\
\psi_{2_{\rho(0,(\psi_1)_0,(\psi_2)_0)}}(s) = \psi_2(-0.5(\sin^2(\psi_1(s)) + \cos^2(\psi_2(s))),\ s \in [-1, 0].
\end{aligned}$$

Apply $-0.5(\sin^2(\psi_1(s)) + \cos^2(\psi_2(s)) \in [-1, 0]$ and we get

$$\begin{aligned}
&(\psi_1(-0.5(\sin^2(\psi_1(s)) + \cos^2(\psi_2(s))))^2 + 2(\psi_2(-0.5(\sin^2(\psi_1(s)) + \cos^2(\psi_2(s))))^2\\
&\le \psi_1^2(0) + 2\psi_2^2(0),\ s \in [-1, 0].
\end{aligned} \tag{4.19}$$

Similar as in Example 4.12, applying inequality (4.19), we get inequalities (4.12), (4.13) and the comparison scalar equation (4.14). According to Theorem 4.5, the solution of the system (4.18) is practically stable.

## References

[1] Agarwal R, Almeida R, Hristova S, O'Regan D. Caputo fractinal differential equations with state dependent delay and practical stability. Dynam Syst Appl, 2019, **28**(3): 715–742

[2] Agarwal R, Hristova S, O'Regan D. Non-Instantaneous Impulses in Differential Equations. Springer, 2017

[3] Agarwal R, O'Regan D, Hristova S. Stability by Lyapunov like functions of nonlinear differential equations with non-instantaneous impulses. Appl Math Comput, 2017, **53**(1/2): 147–168

[4] Baleanu D, Sadati S J, Ghaderi R, Ranjbar A, Abdeljawad T, Jarad F. Razumikhin stability theorem for fractional systems with delay. Abstr Appl Anal, 2010, **2010**: Article ID 124812

[5] Chen B, Chen J. Razumikhin-type stability theorems for functional fractional-order differential systems and applications. Appl Math Comput, 2015, **254**: 63–69

[6] Fečkan M, Wang J R, Zhou Y. Periodic solutions for nonlinear evolution equations with non-instantaneous impulses. Nonauton Dyn Syst, 2014, **254**: 93–101

[7] Hernandez E, O'Regan D. On a new class of abstract impulsive differential equations. Proc Amer Math Soc, 2013, **141**: 1641–1649

[8] Insperger T, Stepan G, Turi J. State-dependent delay in regenerative turning processes. Nonlinear Dyn, 2007, **47**: 275–283

[9] Lakshmikantham V, Bainov D D, Simeonov P S. Theory of Impulsive Differential Equations. Singapore: World Scientific, 1989

[10] Lakshmikantham V. Leela S, Martynyuk A A. Practical Stability of Nonlinear Systems. Singapore: World Scientific, 1990

[11] La Salle J, Lefschetz S. Stability by Lyapunovs Direct Method. New York: Academic Press, Inc, 1961

[12] Pandey D N, Das S, Sukavanam N. Existence of solutions for a second order neutral differential equation with state dependent delay and not instantaneous impulses. Intern J Nonlinear Sci, 2014, **18**(2): 145–155

[13] Walther H O. On a model for soft landing with state dependent delay. J Dyn Diff Eqns, 2003, **19**: 593–622

[14] Wu G C, Baleanu D. Stability analysis of impulsive fractional difference equations. Fract Calc Appl Anal, 2018, **21**(2): 354–375

[15] Wu G C, Baleanu D, Huang L L. Novel Mittag-Leffler stability of linear fractional delay difference equations with impulse, Appl Math Letters, 2018, **82**: 71–78

[16] Wu G C, Zeng D Q, Baleanu D. Fractional impulsive differential equations: Exact solutions, integral equations and short memory case. Fract Calc Appl Anal, 2019, **22**(1): 180–192