Acta Mathematica Scientia, 2021, **41B**(5): 1659–1669
https://doi.org/10.1007/s10473-021-0515-4

数学物理学报
http://actams.apm.ac.cn

THE $\partial\overline{\partial}$-BOCHNER FORMULAS FOR HOLOMORPHIC MAPPINGS BETWEEN HERMITIAN MANIFOLDS AND THEIR APPLICATIONS*

Kai TANG (汤凯)
College of Mathematics and Computer Science, Zhejiang Normal University, Jinhua 321004, China
E-mail: *kaitang001@zjnu.edu.cn*

Abstract In this paper, we derive some $\partial\overline{\partial}$-Bochner formulas for holomorphic maps between Hermitian manifolds. As applications, we prove some Schwarz lemma type estimates, and some rigidity and degeneracy theorems. For instance, we show that there is no non-constant holomorphic map from a compact Hermitian manifold with positive (resp. non-negative) ℓ-second Ricci curvature to a Hermitian manifold with non-positive (resp. negative) real bisectional curvature. These theorems generalize the results [5, 6] proved recently by L. Ni on Kähler manifolds to Hermitian manifolds. We also derive an integral inequality for a holomorphic map between Hermitian manifolds.

Key words Schwarz lemmas; Bochner formulas; holomorphic map; Hermitian manifolds; ℓ-second Ricci curvature

2010 MR Subject Classification 53C55; 32Q05; 32Q10

1 Introduction

There are many generalizations of the classical Schwarz Lemma on holomorphic maps between unit balls via the work of Ahlfors, Chen-Cheng-Look, Lu, Mok-Yau, Royden, Yau, etc. (see [2, 4, 8, 14]). Here, we recall in particular Yau's general Schwarz Lemma [14] that a holomorphic map from a complete Kähler manifold of Ricci curvature bounded from below to a Hermitian manifold of holomorphic bisectional curvature bounded from above by a negative constant decreases distances. Recently, there has been significant progress on this topic, which has involed relaxing either the curvature assumptions or the Kählerian condition; see [5, 6, 9, 11, 12] and references therein for more details. In particular, Ni [5, 6] proved some new estimates interpolating the Schwarz Lemmata of Royden-Yau for holomorphic mappings between Kähler manifolds. These more flexible estimates provide additional information on (algebraic) geometric aspects of compact Kähler manifolds with nonnegative holomorphic sectional curvature, nonnegative Ric_ℓ or positive S_ℓ. One wonders if the results of Ni could be extended or modified to apply to the Hermitian setting.

*Received January 3, 2020; revised April 16, 2021. The author was supported by National Natural Science Foundation of China (12001490) and Natural Science Foundation of Zhejiang Province (LQ20A010005).

A classical differential geometric approach for proving Schwarz type inequalities for a holomorphic map makes use of the Chern-Lu formula and the maximum principle arguments. Therefore, we first generalize the $\partial\overline{\partial}$-Bochner formulas derived by Ni [6] on Kähler manifolds to Hermitian manifolds. Let $f:(M,g)\to(N,h)$ be a holomorphic map between Hermitian manifolds. Assume that $\dim_{\mathbb{C}} M=m\le n=\dim_{\mathbb{C}} N$. Let $\partial f(\frac{\partial}{\partial z^\alpha})=\sum\limits_{i=1}^{n} f^i_\alpha\frac{\partial}{\partial\omega^i}$ with respect to local coordinates $(z^1,\cdots,z^m)$ and $(\omega^1,\cdots,\omega^n)$. The Hermitian form $f^*h=A_{\alpha\overline{\beta}}dz^\alpha\wedge d\overline{z}^\beta$ with $A_{\alpha\overline{\beta}}=f^i_\alpha\overline{f^j_\beta}h_{i\overline{j}}$ is the pull-back h via f. For the local Hermitian metric $A=(A_{\alpha\overline{\beta}})$ and $G=(g_{\alpha\overline{\beta}})$, we denote the A_ℓ and G_ℓ be the upper-left $\ell\times\ell$ blocks.

Theorem 1.1 Let $W_\ell=\frac{\det(A_\ell)}{\det(G_\ell)}$ be the function defined in a small neighborhood of p, $1\le\ell\le m=\dim_{\mathbb{C}} M$. We assume that $g_{\alpha\overline{\beta}}=\delta_{\alpha\beta}$ at $p\in M$, $h_{i\overline{j}}=\delta_{i\overline{j}}$ at $f(p)$. Also, we assume that $df(\frac{\partial}{\partial z^\alpha})=\sum\limits_{i=1}\lambda_\alpha\delta_{i\alpha}\frac{\partial}{\partial\omega^i}$ with $|\lambda_1|\ge\cdots\ge|\lambda_\alpha|\ge\cdots\ge|\lambda_m|$ are the singular values of $\partial f:(T'_pM,g)\to(T'_{f(p)}N,h)$. Then at p, for nonzero W_ℓ, we have

$$\begin{aligned}\frac{\partial^2}{\partial z^\gamma\partial\overline{z}^\delta}\log W_\ell=&\sum_{\alpha=1}^{\ell}R^M(\gamma,\overline{\delta},\alpha,\overline{\alpha})-\sum_{\alpha=1}^{\ell}R^N(\gamma,\overline{\delta},\alpha,\overline{\alpha})\lambda_\gamma\overline{\lambda_\delta}-\sum_{\alpha=1}^{\ell}\sum_{t=\ell+1}^{m}g_{\alpha\bar{t},\gamma}g_{t\overline{\alpha},\overline{\delta}}\\&+\sum_{\alpha=1}^{\ell}\sum_{i=\ell+1}^{n}\frac{1}{|\lambda_\alpha|^2}(f^i_{\alpha\gamma}+h_{\alpha\bar{i},\gamma}\lambda_\alpha\lambda_\gamma)(\overline{f^i_{\alpha\delta}}+h_{i\overline{\alpha},\overline{\delta}}\overline{\lambda_\alpha\lambda_\delta}),\end{aligned}\tag{1.1}$$

where R^M and R^N are the Chern curvature tensors of M and N, respectively.

Theorem 1.2 Let $U_\ell=\sum\limits_{1\le\alpha,\beta\le\ell}g^{\alpha\overline{\beta}}A_{\alpha\overline{\beta}}$ be the function defined in a small neighborhood of p, $1\le\ell\le m=\dim_{\mathbb{C}} M$. We assume that $g_{\alpha\overline{\beta}}=\delta_{\alpha\beta}$ at $p\in M$, $h_{i\overline{j}}=\delta_{i\overline{j}}$ at $f(p)$. Also, we assume that $df(\frac{\partial}{\partial z^\alpha})=\sum\limits_{i=1}\lambda_\alpha\delta_{i\alpha}\frac{\partial}{\partial\omega^i}$ with $|\lambda_1|\ge\cdots\ge|\lambda_\alpha|\ge\cdots\ge|\lambda_m|$ are the singular values of $\partial f:(T'_pM,g)\to(T'_{f(p)}N,h)$. Then at p, for nonzero U_ℓ, we have

$$\partial\overline{\partial}U_\ell=\left(\sum_{\delta=1}^{\ell}R^M(\alpha,\overline{\beta},\delta,\overline{\delta})|\lambda_\delta|^2-\sum_{\gamma=1}^{\ell}R^N(\alpha,\overline{\beta},\gamma,\overline{\gamma})|\lambda_\gamma|^2\lambda_\alpha\overline{\lambda_\beta}\right)dz^\alpha\wedge d\overline{z}^\beta+\langle\nabla V_\ell,\nabla V_\ell\rangle,\tag{1.2}$$

where ∇ is the induced connection on the bundle $E=T'^*M\otimes f^*(T'N)$, $V_\ell=\sum\limits_{\alpha=1}^{\ell}\sum\limits_{i=1}^{n}f^i_\alpha dz^\alpha\otimes e_i\in\Gamma(M,E)$, $e_i=f^*\frac{\partial}{\partial\omega^i}$.

Remark 1.3 When $\ell=m=\dim_{\mathbb{C}} M$, we get $W_m=\frac{(f^*h)^m}{g^m}$. In particular, if the domain and target manifolds have an equal dimension, namely $m=n$, then W_m involves volume forms related by a holomorphic map. Similarly, $U_m=\mathrm{tr}_g f^*h$ is the trace of f^*h with respect to g when $\ell=m=\dim_{\mathbb{C}} M$. Note that if (M,g) and (N,h) are both Kähler manifolds, we can take the normal coordinates near p and $f(p)$ such that $g_{\alpha\overline{\beta}}(p)=\delta_{\alpha\beta}$, $dg_{\alpha\overline{\beta}}(p)=0$ and $h_{i\overline{j}}(f(p))=\delta_{ij}$, $dh_{i\overline{j}}(f(p))=0$. Thus, the calculation of the above formulas is much simpler (see [5, 6]).

Before we state the applications of Theorem 1.1 and Theorem 1.2 we first recall some basic notions (also see [6]). Assume that $f:(M^m,g)\to(N^n,h)$ is a holomorphic map between two Hermitian manifolds. Let $\partial f:T'M\to T'N$ be the tangent map. Let $\wedge^\ell\partial f:\wedge^\ell T'_xM\to\wedge^\ell T'_{f(x)}N$ be the associated map defined as $\wedge^\ell\partial f(v_1\wedge\cdots\wedge v_\ell)=\partial f(v_1)\wedge\cdots\wedge\partial f(v_\ell)$. Define

$\|\cdot\|_0$ as

$$\|\wedge^\ell \partial f\|_0(x) = \sup_{\mathbf{a}=v_1\wedge\cdots\wedge v_\ell\neq 0, \mathbf{a}\in\wedge^\ell T'_x M} \frac{|\wedge^\ell \partial f(\mathbf{a})|}{|\mathbf{a}|}.$$

We assume that $g_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$ at p, $h_{i\overline{j}} = \delta_{i\overline{j}}$ at $f(p)$ such that $df(\frac{\partial}{\partial z^\alpha}) = \sum\limits_{i=1} \lambda_\alpha \delta_{i\alpha} \frac{\partial}{\partial \omega^i}$ with $|\lambda_1| \geq \cdots \geq |\lambda_\alpha| \geq \cdots \geq |\lambda_m|$, so $\|\wedge^\ell \partial f\|_0(p) = |\lambda_1 \cdots \lambda_\ell|$. It is also easy to see that $\|\partial f\|^2 = \mathrm{tr}_g f^*h = \sum\limits_{\alpha,\beta=1}^m g^{\alpha\overline{\beta}} A_{\alpha\overline{\beta}} = \sum\limits_{\alpha=1}^m |\lambda_\alpha|^2$. The second goal of this paper is to prove some estimates for holomorphic maps between Hermitian manifolds.

Theorem 1.4 Let $f:(M^m,g)\to(N^n,h)$ be a holomorphic map between Hermitian manifolds with M being compact. Let $m\leq n$ and $\ell\leq m$. Then,

(a) Assume that the scalar curvature of (M,g), $S(x)\geq -K$ and the m-first Ricci curvature of (N,h), $\mathrm{Ric}_m^{(1)}(x)\leq -\kappa$ for some $K\geq 0$, $\kappa>0$. Then

$$\frac{(f^*h)^m}{g^m}(x) \leq (\frac{K}{m\kappa})^m,$$

(b) Assume that metric g on M is Kähler and that $1\leq \ell < m$. Assume the ℓ-scalar curvature of (M,g), $S_\ell(x)\geq -K$ and the ℓ-first Ricci curvature of (N,h), $\mathrm{Ric}_\ell^{(1)}(x)\leq -\kappa$ for some $K\geq 0$, $\kappa>0$. Then

$$\|\wedge^\ell \partial f\|_0^2(x) \leq (\frac{K}{\ell\kappa})^\ell,$$

(c) Assume the ℓ-second Ricci curvature of (M,g), $\mathrm{Ric}_\ell^{(2)}\geq -K$, and the real bisectional curvature of (N,h), $\widetilde{B}(x)\leq -\kappa$ for some $K\geq 0$, $\kappa>0$. Then

$$\sigma_\ell(x) \leq \frac{\ell K}{\kappa},$$

where $\sigma_\ell(x) = \sum\limits_{\sigma=1}^\ell |\lambda_\alpha|^2(x)$.

We will give specific definitions of these curvatures in the next section. For $\ell=1$, the 1-first Ricci curvature and 1-second Ricci curvature are both holomorphic sectional curvature; If $\ell=m=\dim M$, the m-first Ricci curvature is the (first) Chern Ricci curvature and the m-second Ricci curvature is the second Ricci curavture. For $1\leq\ell\leq\dim M$, they are the same when the metric is Kähler. In an attempt to generalize Wu-Yau's Theorem ([10]) to the Hermitian case, Yang and Zheng [13] introduced the concept of real bisectional curvature for Hermitian manifolds. When the metric is Kähler, this curvature is the same as the holomorphic sectional curvature H, and when the metric is not Kähler, the curvature condition is slightly stronger than H, at least algebraically. This condition also appeared in a recent work by Lee and Streets [3], where it is referred to as a "positive (resp. negative) curvature operator".

The following are the rigidity and degeneracy results:

Theorem 1.5 Let $f:(M^m,g)\to(N^n,h)$ be a holomorphic map between Hermitian manifolds with M being a compact. Let $m\leq n$ and $\ell\leq m$. Then,

(a) If $S^M\geq 0$ and manifold (N,h) has $\mathrm{Ric}_m^{(1)}<0$, or $S^M>0$ and manifold (N,h) has $\mathrm{Ric}_m^{(1)}\leq 0$, then f must be degenerate. The same result holds if $\mathrm{Ric}^M\geq 0$ and $S_m^N<0$, or $\mathrm{Ric}^M>0$ and $S_m^N\leq 0$.

(b) Assume that metric g on M is Kähler and that $1 \leq \ell < m$. If $S_\ell^M \geq 0$ and manifold (N,h) has $\mathrm{Ric}_\ell^{(1)} < 0$, or $S_\ell^M > 0$ and manifold (N,h) has $\mathrm{Ric}_\ell^{(1)} \leq 0$, then $\mathrm{rank}(f) < \ell$. The same result holds if $\mathrm{Ric}_\ell^M \geq 0$ and $S_\ell^N < 0$, or $\mathrm{Ric}_\ell^M > 0$ and $S_\ell^N \leq 0$.

(c) If manifold M has $\mathrm{Ric}_\ell^{(2)} > 0$ and manifold N has $\widetilde{B}^N \leq 0$, or $\mathrm{Ric}_\ell^{(2)} \geq 0$ and $\widetilde{B}^N < 0$, then f must be constant.

In particular, from the proof of theorem 1.5 (c), we easily get the following result:

Corollary 1.6 There is no non-constant holomorphic map from a compact Hermitian manifold with positive (resp. non-negative) holomorphic sectional curvature to a Hermitian manifold with non-positive (resp. negative) holomorphic sectional curvature.

Note that the above Corollary 1.6 is also proved independently by Yang in [11, 12] using a different method.

As an application of Theorem 1.1, we will also give an integral inequality for non-degenerate holomorphic maps between two Hermitian manifolds without assuming any curvature condition. More precisely, we shall prove the following:

Theorem 1.7 Let (M^m, g) and (N^n, h) be two Hermitian manifolds and let M be compact. Assume that $\dim M = m \leq n = \dim N$. Then there exists a smooth real function ψ on M such that for any non-degenerate holomorphic map $f : M \to N$, it holds that

$$\int_M S_g e^{(m-1)\psi} g^m \leq m \int_M e^{(m-1)\psi} f^*(\mathrm{Ric}_m^{(1)}(h)) \wedge g^{m-1}, \tag{1.3}$$

where S_g is the Chern scalar curvature of g and $\mathrm{Ric}_m^{(1)}(h)$ is the m-first Ricci curvature of h.

Remark 1.8 The above Theorem 1.7 recovers Theorem 1.2 in [15], which is proved by Zhang when $\dim M = \dim N$. The above result can be applied to prove degeneracy theorems for holomorphic maps without assuming any pointwise curvature signs for both the domain and the target manifolds.

2 Preliminaries

2.1 Curvatures in complex geometry

Let (M, g) be a Hermitian manifold of dimension $\dim_{\mathbb{C}} M = m$, where $\omega = \omega_g$ is the metric form of a Hermitian metric g. If ω is closed, that is, if $d\omega = 0$, we call g a Kähler metric. In local holomorphic chart $(z_1, \cdots, z^m)$, we write

$$\omega = \sqrt{-1} \sum_{i=1,j=1}^{m} g_{i\overline{j}} dz^i \wedge d\overline{z}^j.$$

Recall that the curvature tensor $R = \{R_{i\overline{j}k\overline{l}}\}$ of the Chern connection is given by

$$R_{i\overline{j}k\overline{l}} = -\frac{\partial^2 g_{k\overline{l}}}{\partial z^i \partial \overline{z}^j} + g^{p\overline{q}} \frac{\partial g_{k\overline{q}}}{\partial z^i} \frac{\partial g_{p\overline{l}}}{\partial \overline{z}^j}.$$

Then the (first) Chern Ricci curvature $\mathrm{Ric}(\omega_g) = \mathrm{tr}_g R \in \Gamma(M, \wedge^{1,1} T'^* M)$ has components

$$R_{i\overline{j}} = \sum_{k=1,l=1}^{m} g^{k\overline{l}} R_{i\overline{j}k\overline{l}} = -\frac{\partial^2 \log\det(g)}{\partial z^i \partial \overline{z}^j}.$$

The second Chern Ricci curvature $\mathrm{Ric}^{(2)}(\omega_g) = \mathrm{tr}_{\omega_g} R \in \Gamma(M, \mathrm{End}(T'M))$ has components

$$\mathrm{Ric}^{(2)}_{i\bar{j}} = \sum_{i=1,j=1}^{m} g^{i\bar{j}} R_{i\bar{j}k\bar{l}} \ .$$

Note that $\mathrm{Ric}(\omega_g)$ and $\mathrm{Ric}^{(2)}(\omega_g)$ are the same when ω_g is a Kähler metric. The Chern scalar curvature S_ω is given by

$$S_\omega = \sum_{i,j,k,l=1}^{m} g^{i\bar{j}} g^{k\bar{l}} R_{i\bar{j}k\bar{l}} \ .$$

The holomorphic bisectional curvature $B(X,Y)$ for X, Y in T'_pM at $p \in M$ is given by

$$B(X,Y) = \frac{H(X,\overline{X},Y,\overline{Y})}{g(X,X)g(Y,Y)} \ .$$

The holomorphic sectional curvature $H(X)$ is denoted by

$$H(X) = B(X,X) = \frac{R(X,\overline{X},Y,\overline{Y})}{g(X,X)^2} \ .$$

Definition 2.1 Let (M^m, g) be a Hermitian manifold and let $R^g \in \Gamma(M, \wedge^{1,1}T'^*M \otimes \mathrm{End}(T'M))$ be the Chern curvature tensor. Assume the ℓ-dimensional subspace $\Sigma \subset T'_pM$, $p \in M$, $1 \leq \ell \leq m$. For any $\upsilon \in \Sigma$, we define $\mathrm{Ric}^{(1)}_\ell(p,\Sigma)(\upsilon,\overline{\upsilon}) = \sum\limits_{i=1}^{\ell} R(\upsilon,\overline{\upsilon},E_i,\overline{E}_i)$ with $\{E_i\}$ being a unitary basis of Σ. We say that $\mathrm{Ric}^{(1)}_\ell(p) < 0$ if $\mathrm{Ric}^{(1)}_\ell(p,\Sigma) < 0$ for any ℓ-dimensional subspace Σ. We call $\mathrm{Ric}^{(1)}_\ell(p)$ the ℓ-first Ricci curvature at p. Similarly, for any $\upsilon \in \Sigma$, we define $\mathrm{Ric}^{(2)}_\ell(p,\Sigma)(\upsilon,\overline{\upsilon}) = \sum\limits_{i=1}^{\ell} R(E_i,\overline{E}_i,\upsilon,\overline{\upsilon})$. We say that $\mathrm{Ric}^{(2)}_\ell(p) < 0$ if $\mathrm{Ric}^{(2)}_\ell(p,\Sigma) < 0$ for any ℓ-dimensional subspace Σ. We call $\mathrm{Ric}^{(2)}_\ell(p)$ the ℓ-second Ricci curvature at p. We define $S_\ell(p,\Sigma) = \sum\limits_{i,j=1}^{\ell} R(E_i,\overline{E}_i,E_j,\overline{E}_j)$ with E_i being a unitary basis of Σ. We say that $S_\ell(p) < 0$ if $S_\ell(p,\Sigma) < 0$ for any ℓ-dimensional subspace Σ. We call $S_\ell(p)$ the ℓ-scalar curvature at p.

Clearly, $\mathrm{Ric}^{(1)}_\ell(p) < 0$ or $\mathrm{Ric}^{(2)}_\ell(p) < 0$ implies that $S_\ell(p) < 0$. $\mathrm{Ric}^{(1)}_\ell$ and $\mathrm{Ric}^{(2)}_\ell$ are both holomorphic sectional curvature when $\ell = 1$. If $\ell = m$, they are Chern Ricci curvature and second Ricci curvature, respectively. If the metric g is Kähler, $\mathrm{Ric}^{(1)}_\ell$ and $\mathrm{Ric}^{(2)}_\ell$ will be the same. In the case of Kähler manifolds, there are many studies (see [5–7]) for Ric_ℓ and S_ℓ. We will do more research on the above new curvature condition on Hermitian manifolds in the future.

Let us recall the concept of real bisectional curvature introduced in [13]. Let (M^m, g) be a Hermitian manifold. Denote by R the curvature tensor of the Chern connection. For $p \in M$, let $e = \{e_1, \cdots, e_m\}$ be a unitary tangent frame at p, and let $a = \{a_1, \cdots, a_m\}$ be non-negative constants with $|a|^2 = a_1^2 + \cdots + a_m^2 > 0$. Define the real bisectional curvature of g by

$$\widetilde{B}_g(e,a) = \frac{1}{|a|^2} \sum_{i,j=1}^{m} R_{i\bar{i}j\bar{j}} a_i a_j. \tag{2.1}$$

We will say that a Hermitian manifold (M^m, g) has positive real bisectional curvature, denoted by $\widetilde{B}_g > 0$, if, for any $p \in M$ and any unitary frame e at p, and any nonnegative constant $a = \{a_1, \cdots, a_m\}$, it holds that $\widetilde{B}_g(e,a) > 0$.

Recall that the holomorphic sectional curvature in the direction v is defined by $H(v) = R_{v\overline{v}v\overline{v}}/|v|^4$. If we take e so that e_1 is parallel to v, and take $a_1 = 1$, $a_2 = \cdots = a_m = 0$, then $\widetilde{B}$ becomes $H(v)$. Thus, $\widetilde{B} > 0$ ($\geq 0, < 0$, or ≤ 0) implies that $H > 0$ ($\geq 0, < 0, or \leq 0$). For a more detailed discussion of this, we refer readers to [13].

2.2 Gauduchon metric

Let M^m be a compact Hermitian manifold. A Hermitian metric ω is called Gauduchon if

$$\partial\overline{\partial}(\omega^{m-1}) = 0.$$

For a Gauduchon metric ω and a smooth function u on M, we easily get

$$\int_M (\Delta_\omega u)\omega^m = 0,$$

where $\Delta_\omega u$ is the complex Laplacian defined by $\Delta_\omega u = \mathrm{tr}_\omega(\sqrt{-1}\partial\overline{\partial}u)$. A classical result of Gauduchon [1] states that, for any Hermitian metric ω, there is a $\psi \in C^\infty(M, \mathbb{R})$ (unique up to scaling) such that $e^\psi\omega$ is Gauduchon.

2.3 Non-degenerate holomorphic maps

Let $f : M^m \to N^n$ be a holomorphic map between two Hermitian manifolds ($m \leq n$). If $\dim(f(M)) = m$, then we say that f is non-degenerate. If $\dim(f(M)) < m$, we say that f is degenerate.

3 $\partial\overline{\partial}$-Bochner Formulas for Holomorphic Mappings

In this section, we will give the proof of Theorems 1.1 and 1.2. The calculation is more complicated in the Hermitian case than in the Kähler case.

Proof of Theorem 1.1 As stated in the theorem, we assume that $g_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}}$ at $p \in M$, $h_{i\overline{j}} = \delta_{i\overline{j}}$ at $f(p)$. After a constant unitary change of coordinates z and ω, at p, we have $df(\frac{\partial}{\partial z^\alpha}) = \sum\limits_{i=1}^{n} \lambda_\alpha \delta_{i\alpha} \frac{\partial}{\partial \omega^i}$ with $|\lambda_1| \geq \cdots \geq |\lambda_\alpha| \geq \cdots |\lambda_m|$. The Hermitian form is $f^*h = A_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta$ with $A_{\alpha\overline{\beta}} = \sum\limits_{i,j=1}^{n} f^i_\alpha \overline{f^j_\beta} h_{i\overline{j}}$. For the local Hermitian metrics $A = (A_{\alpha\overline{\beta}})$ and $G = (g_{\alpha\overline{\beta}})$, we denote that A_ℓ and G_ℓ be the upper-left $\ell \times \ell$ blocks. To simplify notation, we write $\frac{\partial f^i}{\partial z^\alpha}$ as f^i_α, and $\frac{\partial^2 f^i}{\partial z^\alpha \partial z^\gamma}$ as $f^i_{\alpha\gamma}$. We can perform the computation at p and $f(p)$, where

$$R^M_{\gamma\overline{\delta}\alpha\overline{\beta}} = -g_{\alpha\overline{\beta},\overline{\delta}\gamma} + \sum_{t=1}^{m} g_{\alpha\overline{t},\gamma} g_{t\overline{\beta},\overline{\delta}} \ , R^N_{i\overline{j}k\overline{l}} = -g_{k\overline{l},\overline{j}i} + \sum_{s=1}^{n} g_{k\overline{s},i} g_{s\overline{l},\overline{j}} \ .$$

Hence,

$$\frac{\partial^2}{\partial z^\gamma \partial \overline{z}^\delta} \log W_\ell = \frac{\partial^2}{\partial z^\gamma \partial \overline{z}^\delta} \log \frac{\det(A_\ell)}{\det(G_\ell)} = \frac{\partial^2}{\partial z^\gamma \partial \overline{z}^\delta} \log\det(A_\ell) - \frac{\partial^2}{\partial z^\gamma \partial \overline{z}^\delta} \log\det(G_\ell) \ .$$

Direct calculation shows that

$$\begin{aligned} \frac{\partial^2}{\partial z^\gamma \partial \overline{z}^\delta} \log\det(G_\ell) &= \frac{\partial}{\partial z^\gamma} \Big[\sum_{\alpha,\beta=1}^{\ell} (G_\ell)^{\alpha\overline{\beta}} g_{\alpha\overline{\beta},\overline{\delta}} \Big] \\ &= - \sum_{\alpha,\beta,t,\lambda=1}^{\ell} (G_\ell)^{\alpha\overline{t}} g_{\lambda\overline{t},\gamma} (G_\ell)^{\lambda\overline{\beta}} g_{\alpha\overline{\beta},\overline{\delta}} + \sum_{\alpha,\beta=1}^{\ell} (G_\ell)^{\alpha\overline{\beta}} g_{\alpha\overline{\beta},\overline{\delta}\gamma} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha=1}^{\ell}\Big(- \sum_{t=1}^{\ell} g_{\alpha\bar{t},\gamma} g_{t\overline{\alpha},\overline{\delta}} + g_{\alpha\overline{\alpha},\overline{\delta}\gamma}\Big)\\
&= \sum_{\alpha=1}^{\ell}\Big(- R^M_{\gamma\overline{\delta}\alpha\overline{\alpha}} + \sum_{t=\ell+1}^{m} g_{\alpha\bar{t},\gamma} g_{t\overline{\alpha},\overline{\delta}}\Big).
\end{aligned}
$$

The last two lines only hold at point p.

$$
\begin{aligned}
(\log\det(A_\ell))_{\overline{\delta}} &= \sum_{\alpha,\beta=1}^{\ell}\sum_{i,j=1}^{n} (A_\ell)^{\alpha\overline{\beta}} (f^i_\alpha h_{i\bar{j}} \overline{f^j_\beta})_{\overline{\delta}}\\
&= \sum_{\alpha,\beta=1}^{\ell}\sum_{i,j=1}^{n} (A_\ell)^{\alpha\overline{\beta}}\Big[\sum_{t=1}^{n} h_{i\bar{j},\bar{t}} f^i_\alpha \overline{f^j_\beta}\,\overline{f^t_\delta} + f^i_\alpha h_{i\bar{j}} \overline{f^j_{\beta\delta}}\Big]\\
&= \sum_{\alpha=1}^{\ell} \frac{1}{|\lambda_\alpha|^2}[h_{\alpha\overline{\alpha},\overline{\delta}}|\lambda_\alpha|^2\overline{\lambda_\delta} + \lambda_\alpha \overline{f^\alpha_{\alpha\delta}}]\ .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(\log\det(A_\ell))_{\gamma} &= \sum_{\alpha,\beta=1}^{\ell}\sum_{i,j=1}^{n} (A_\ell)^{\alpha\overline{\beta}}\Big[f^i_{\alpha\gamma} h_{i\bar{j}}\overline{f^j_\beta} + \sum_{k=1}^{n} f^i_\alpha h_{i\bar{j},k} f^k_\gamma \overline{f^j_\beta}\Big]\\
&= \sum_{\alpha=1}^{\ell} \frac{1}{|\lambda_\alpha|^2}[f^\alpha_{\alpha\gamma}\overline{\lambda_\alpha} + h_{\alpha\overline{\alpha},\gamma}|\lambda_\alpha|^2\lambda_\gamma]\ .
\end{aligned}
$$

Taking the second derivative and evaluating at p, we have

$$
\begin{aligned}
(\log\det(A_\ell))_{\gamma\overline{\delta}} &= \frac{\partial}{\partial z^\gamma}\Big\{\sum_{\alpha,\beta=1}^{\ell}\sum_{i,j=1}^{n} (A_\ell)^{\alpha\overline{\beta}}\Big[\sum_{t=1}^{n} h_{i\bar{j},\bar{t}} f^i_\alpha \overline{f^j_\beta}\,\overline{f^t_\delta} + f^i_\alpha h_{i\bar{j}}\overline{f^j_{\beta\delta}}\Big]\Big\}\\
&= \sum_{\alpha,\beta=1}^{\ell}\sum_{i,j=1}^{n}\Big\{ - \sum_{k,s=1}^{\ell} (A_\ell)^{\alpha\overline{s}}(A_\ell)_{k\overline{s},\gamma}(A_\ell)^{k\overline{\beta}}\Big[\sum_{t=1}^{n} h_{i\bar{j},\bar{t}} f^i_\alpha \overline{f^j_\beta}\,\overline{f^t_\delta} + f^i_\alpha h_{i\bar{j}}\overline{f^j_{\beta\delta}}\Big]\Big\}\\
&\quad + \sum_{\alpha,\beta=1}^{\ell}\sum_{i,j=1}^{n} (A_\ell)^{\alpha\overline{\beta}}\Big[\sum_{t,p=1}^{n} h_{i\bar{j},\bar{t}p} f^p_\gamma f^i_\alpha \overline{f^j_\beta}\,\overline{f^t_\delta} + \sum_{t=1}^{n} h_{i\bar{j},\bar{t}} f^i_{\alpha\gamma}\overline{f^j_\beta}\,\overline{f^t_\delta}\\
&\quad + f^i_{\alpha\gamma} h_{i\bar{j}}\overline{f^j_{\beta\delta}} + \sum_{q=1}^{n} f^i_\alpha h_{i\bar{j},q} f^q_\gamma \overline{f^j_{\beta\delta}}\Big]\\
&= - \sum_{\alpha,\beta=1}^{\ell} \frac{1}{|\lambda_\alpha|^2|\lambda_\beta|^2}(f^\alpha_{\beta\gamma}\overline{\lambda_\alpha} + \lambda_\beta\lambda_\gamma\overline{\lambda_\alpha} h_{\beta\overline{\alpha},\gamma})(h_{\alpha\overline{\beta},\overline{\delta}}\lambda_\alpha\overline{\lambda_\beta\lambda_\delta} + \lambda_\alpha\overline{f^\alpha_{\beta\delta}})\\
&\quad + \sum_{\alpha=1}^{\ell}\sum_{i=1}^{n}\frac{1}{|\lambda_\alpha|^2}(h_{\alpha\overline{\alpha},\overline{\delta}\gamma}\lambda_\gamma|\lambda_\alpha|^2\overline{\lambda_\delta} + h_{i\overline{\alpha},\overline{\delta}} f^i_{\alpha\gamma}\overline{\lambda_\alpha\lambda_\delta}\\
&\quad + f^i_{\alpha\gamma}\overline{f^i_{\alpha\delta}} + \lambda_\alpha h_{\alpha\bar{j},\gamma}\lambda_\gamma\overline{f^i_{\alpha\delta}})\\
&= \sum_{\alpha=1}^{\ell}\sum_{i=\ell+1}^{n}\frac{1}{|\lambda_\alpha|^2}(f^i_{\alpha\gamma} + h_{\alpha\bar{i},\gamma}\lambda_\alpha\lambda_\gamma)(\overline{f^i_{\alpha\delta}} + h_{i\overline{\alpha},\overline{\delta}}\overline{\lambda_\alpha\lambda_\delta}) - \sum_{\alpha=1}^{\ell} R^N_{\gamma\overline{\delta}\alpha\overline{\alpha}}\lambda_\gamma\overline{\lambda_\delta}.
\end{aligned}
$$

Putting all of the above together, we can get the formula (1.1). □

Proof of Theorem 1.2 Here we will use the method in the proof of Lemma 4.1 in [13], which is a slightly simpler proof. Let $V_\ell = \sum_{\alpha=1}^{\ell}\sum_{i=1}^{n} f^i_\alpha dz^\alpha \otimes e_i \in \Gamma(M,E)$, $E = T'^*M \otimes f^*(T'N)$,

$e_i = f^*\frac{\partial}{\partial \omega^i}$. Since f is a holomorphic map, V_ℓ is a holomorphic section of E. Clearly, $U_\ell = |V_\ell|^2 = \sum\limits_{\alpha,\beta=1}^{\ell} g^{\alpha\overline{\beta}}A_{\alpha\overline{\beta}}$. Thus, by Bochner's formula, we have

$$\sqrt{-1}\partial\overline{\partial}|V_\ell|^2 = \langle \nabla' V_\ell, \nabla' V_\ell\rangle - \langle \Theta^E V_\ell, V_\ell\rangle\ ,$$

where Θ^E is the curvature of the vector bundle E with respect to the induced metric, since

$$\Theta^E = \Theta^{T'^*M} \otimes Id_{f^*(T'N)} + Id_{T'^*M} \otimes f^*(\Theta^{T'N})\ .$$

More precisely, we can assume that

$$\Theta^{T'^*M} = -\sum_{\alpha,\beta,\delta,\eta=1}^{m} (R^M)^{\delta}_{\alpha\overline{\beta}\eta} dz^\alpha \wedge d\overline{z}^\beta \otimes \frac{\partial}{\partial z_\delta} \otimes dz_\eta,$$

$$f^*(\Theta^{T'N}) = \sum_{i,j,k,l=1}^{n}\sum_{\alpha,\beta=1}^{m} (R^N)^{l}_{i\overline{j}k} f^i_\alpha \overline{f^j_\beta} dz^\alpha \wedge d\overline{z}^\beta \otimes e^*_k \otimes e_l,$$

where $(R^M)^{\delta}_{\alpha\overline{\beta}\eta} = \sum\limits_{\xi=1}^{m} g^{\delta\overline{\xi}} R^M_{\alpha\overline{\beta}\eta\overline{\xi}}$ and $(R^N)^{l}_{i\overline{j}k} = \sum\limits_{s=1}^{n} h^{l\overline{s}} R^N_{i\overline{j}k\overline{s}}$.

Hence, at the point p assumed in the theorem condition, we have

$$\langle \Theta^E V_\ell, V_\ell\rangle = \Big(-\sum_{\delta=1}^{\ell} R^M(\alpha,\overline{\beta},\delta,\overline{\delta})|\lambda_\delta|^2 + \sum_{\gamma=1}^{\ell} R^N(\alpha,\overline{\beta},\gamma,\overline{\gamma})|\lambda_\gamma|^2\lambda_\alpha\overline{\lambda_\beta}\Big) dz^\alpha \wedge d\overline{z}^\beta.$$

Putting all the above together, we can get the formula (1.2). □

4 Applications

Since, in general, $\| \wedge^\ell \partial f\|_0^2$ and σ_ℓ are not smooth, we will consider that $W_\ell(x)$ serves as a smooth barrier for $\| \wedge^\ell \partial f\|_0(x)$ and U_ℓ serves as a smooth barrier for σ_ℓ. As stated in the previous section, we have Hermitian form $f^*h = A_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta$ with $A_{\alpha\overline{\beta}} = \sum\limits_{i,j=1}^{n} f^i_\alpha \overline{f^j_\beta} h_{i\overline{j}}$. We assume that $g_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}}$ at $p \in M$, $h_{i\overline{j}} = \delta_{i\overline{j}}$ at $f(p)$. After a constant unitary change of coordinates z and ω, at p, we have $df(\frac{\partial}{\partial z^\alpha}) = \sum\limits_{i=1}^{n} \lambda_\alpha \delta_{i\alpha} \frac{\partial}{\partial \omega^i}$ with $|\lambda_1| \geq \cdots \geq |\lambda_\alpha| \geq \cdots |\lambda_m|$. For the local Hermitian metrics $A = (A_{\alpha\overline{\beta}})$ and $G = (g_{\alpha\overline{\beta}})$, we denote that A_ℓ and G_ℓ be the upper-left $\ell \times \ell$ blocks. It is easy to see that $|\lambda_1|^2 \geq \cdots \geq |\lambda_m|^2$ are the eigenvalues of A (with respect to g). We start with some necessary algebraic results (see [6]).

Proposition 4.1 ([6], Proposition 2.1) For any $1 \leq \ell \leq m$, the following holds:

$$\sigma_\ell = \sum_{\alpha=1}^{\ell} |\lambda_\alpha|^2 \geq \sum_{\alpha,\beta=1}^{\ell} g^{\alpha\overline{\beta}} A_{\alpha\overline{\beta}} = U_\ell\ . \tag{4.1}$$

Proposition 4.2 ([6], Proposition 2.2) For any $1 \leq \ell \leq m$, the following holds:

$$\| \wedge^\ell \partial f\|_0^2 = \Pi_{\alpha=1}^{\ell} |\lambda_\alpha|^2 \geq \frac{\det(A_\ell)}{\det(G_\ell)} = W_\ell\ . \tag{4.2}$$

Proof of Theorem 1.4 To prove part (a), let $D = \frac{(f^*h)^m}{g^m} = W_m$. Since M is compact, D attains its maximum at some point p. We assume that at p, D is not equal to zero. Then in a neighborhood of p, $D \neq 0$. The maximum principle then implies that at p, with respect to the coordinates specified in Theorem 1.1,

$$\begin{aligned} 0 \geq \Delta \log D &= S^M - \sum_{\delta=1}^{m} \mathrm{Ric}_m^{(1)}(\delta, \overline{\delta})|\lambda_\delta|^2 + \sum_{\alpha=1}^{\ell}\sum_{i=\ell+1}^{n} \frac{1}{|\lambda_\alpha|^2}|f^i_{\alpha\delta} + h_{\alpha\bar{i},\delta}\lambda_\alpha\lambda_\delta|^2 \\ &\geq -K + \kappa\Big(\sum_{\delta=1}^{m}|\lambda_\delta|^2\Big) \\ &\geq -K + mD^{\frac{1}{m}}\kappa. \end{aligned}$$

where $\mathrm{Ric}_m^{(1)}$ denotes the m-first Ricci curvature of (N, h). The above inequality implies the result.

For (b), clearly $\| \wedge^\ell \partial f\|_0^2$ attains a maximum somewhere at p in M. We assume that the coordinates at p and $f(p)$ satisfy the conditions of Theorem 1.1. Since (M, g) is Kähler, we can also assume that at p, $g_{\alpha\overline{\beta},\delta} = g_{\alpha\overline{\beta},\overline{\delta}} = 0$, $\forall\, 1 \leq \alpha, \beta, \delta \leq m$. Then we have $\| \wedge^\ell \partial f\|_0^2(p) = W_\ell(p)$ and $W_\ell(x) \leq \| \wedge^\ell \partial f\|_0^2(x) \leq \| \wedge^\ell \partial f\|_0^2(p) = W_\ell(p)$ for x in the small neighborhood of p. Hence, W_ℓ also attains its local maximum at p. Now, at p, we get

$$\begin{aligned} 0 &\geq \sum_{\gamma=1}^{\ell} \frac{\partial^2}{\partial z^\gamma \partial \overline{z}^\gamma}(\log(W_\ell)) \geq S_\ell^M - \sum_{\gamma=1}^{\ell} \mathrm{Ric}_\ell^{(1)}(\gamma, \overline{\gamma})|\lambda_\gamma|^2 \\ &\geq -K + \ell(W_\ell(p))^{\frac{1}{\ell}}\kappa = -K + \ell(\| \wedge^\ell \partial f\|_0^2(p))^{\frac{1}{\ell}}\kappa, \end{aligned}$$

where $\mathrm{Ric}_\ell^{(1)}$ denotes the ℓ-first Ricci curvature of (N, h).

The proof of part (c) is similar. We assume that σ_ℓ attains a maximum at p. We also assume that the coordinates at p and $f(p)$ satisfy the conditions of Theorem 1.2, so $U_\ell(x) \leq \sigma_\ell(x) \leq \sigma_\ell(p) = U_\ell(p)$ for x in the small neighborhood of p. Thus, at p,

$$\begin{aligned} 0 &\geq \sum_{\alpha=1}^{\ell} \frac{1}{2}(\nabla_\alpha \nabla_{\overline{\alpha}} + \nabla_{\overline{\alpha}}\nabla_\alpha)U_\ell \\ &\geq \sum_{\delta=1}^{\ell} \mathrm{Ric}_\ell^{(2)}(\delta, \overline{\delta})|\lambda_\delta|^2 - \sum_{\alpha,\gamma=1}^{\ell} R^N(\alpha, \overline{\alpha}, \gamma, \overline{\gamma})|\lambda_\alpha|^2|\lambda_\gamma|^2 \\ &\geq -K\sum_{\delta=1}^{\ell}|\lambda_\delta|^2 + \kappa\sum_{\alpha=1}^{\ell}|\lambda_\alpha|^4 \\ &\geq -KU_\ell(p) + \frac{\kappa}{\ell}U_\ell^2(p), \end{aligned}$$

where $\mathrm{Ric}_\ell^{(2)}$ denotes the ℓ-second Ricci curvature of (M, g). $\square$

Combining Theorems 1.1, 1.2 and 1.4, we can now easily prove Theorem 1.5.

Proof of Theorem 1.5 If f is not degenerate, then $D = \frac{(f^*h)^m}{g^m} = W_m$ has a nonzero maximum somewhere at p. By using the coordinates around p and $f(p)$, specified as in Theorem 1.4, at p, we have that

$$0 \geq \Delta \log D \geq S^M - \sum_{\delta=1}^{m} \mathrm{Ric}_m^{(1)}(\delta, \overline{\delta})|\lambda_\delta|^2.$$

This leads to a contradiction under the assumption that either $S^M \geq 0$ and manifold (N, h) has $\mathrm{Ric}_m^{(1)} < 0$, or that $S^M > 0$ and $\mathrm{Ric}_m^{(1)} \leq 0$. For the second part of (a), we introduce the operator

$$\Psi = \sum_{\gamma=1}^{m} \frac{1}{2|\lambda_\gamma|^2}(\nabla_\gamma \nabla_{\overline{\gamma}} + \nabla_{\overline{\gamma}} \nabla_\gamma).$$

Since, at p, $D \neq 0$, the above operator is well defined in a small neighborhood of p. Then, at p,

$$0 \geq \Psi(\log D) \geq \sum_{\gamma=1}^{m} \frac{1}{|\lambda_\gamma|^2} \mathrm{Ric}^M(\gamma, \overline{\gamma}) - S_m^N.$$

The above also induces a contradiction under either $\mathrm{Ric}^M \geq 0$ and $S_m^N < 0$, or $\mathrm{Ric}^M > 0$ and $S_m^N \leq 0$.

The proof of (b) is similar to that of (a). It is worth noting that in the second part of (b), we need to introduce the operator

$$\Psi_\ell = \sum_{\gamma=1}^{\ell} \frac{1}{2|\lambda_\gamma|^2}(\nabla_\gamma \nabla_{\overline{\gamma}} + \nabla_{\overline{\gamma}} \nabla_\gamma).$$

For (c), if f is not constant, σ_ℓ will attain a maximum somewhere, say at p and $\sigma_\ell(p) > 0$. By using the coordinates around p and $f(p)$, specified as in Theorem 1.4, at p, we have that

$$\begin{aligned} 0 &\geq \sum_{\alpha=1}^{\ell} \frac{1}{2}(\nabla_\alpha \nabla_{\overline{\alpha}} + \nabla_{\overline{\alpha}} \nabla_\alpha) U_\ell \\ &\geq \sum_{\delta=1}^{\ell} \mathrm{Ric}_\ell^{(2)}(\delta, \overline{\delta})|\lambda_\delta|^2 - \sum_{\alpha,\gamma=1}^{\ell} R^N(\alpha, \overline{\alpha}, \gamma, \overline{\gamma})|\lambda_\alpha|^2|\lambda_\gamma|^2 \ , \end{aligned}$$

if $\mathrm{Ric}_\ell^{(2)} > 0$, the first term is positive, and the second one is nonnegative, since $\widetilde{B}^N \leq 0$. Hence, we have a contradiction. The same holds if $\mathrm{Ric}_\ell^{(2)} \geq 0$ and $\widetilde{B}^N < 0$. □

5 An Integral Inequality for Non-degenerate Holomorphic Maps

In this section we prove Theorem 1.7. The proof does not use any maximum principle argument, since the curvatures and target spaces may not be signed in a pointwise sense. This method was essentially derived by Zhang in [15].

Proof of Theorem 1.7 Because f is a non-degenerate holomorphic map, we assume that $p \in M$ with $D(p) = \frac{(f^*h)^m}{g^m}(p) > 0$. We also assume that the coordinates at p and $f(p)$ satisfy the conditions of Theorem 1.1. Let ϵ be an arbitrary positive constant. By the formula (1.1), at p, we have

$$\begin{aligned} \Delta \log(D+\epsilon) &= \frac{\Delta D}{D+\epsilon} - \frac{|\partial D|^2}{(D+\epsilon)^2} \\ &= \frac{D}{D+\epsilon}(\frac{\Delta D}{D} - \frac{|\partial D|^2}{D^2}) + \frac{\epsilon|\partial D|^2}{D(D+\epsilon)^2} \\ &= \frac{D}{D+\epsilon}\Delta \log D + \frac{\epsilon|\partial D|^2}{D(D+\epsilon)^2} \\ &\geq \frac{D}{D+\epsilon}\{S^M - \mathrm{tr}_g(f^*(\mathrm{Ric}_m^{(1)}(h)))\}. \end{aligned} \tag{5.1}$$

Here we note that the above inequality is independent of the choice of coordinates.

Set $V = \{x \in M \mid D = 0 \text{ at } x\}$, which is a proper subvariety (may be empty) of M. Then, the above inequality (5.1) holds on $M \setminus V$, and by continuity we know that it holds on the whole of M.

Next, we fix a $\psi \in C^\infty(M, \mathbb{R})$ such that $e^\psi g^m$ is a Gauduchon metric on M. Intergrating the above inequality with respect to $e^{(m-1)\psi} g^m$ over M gives

$$\int_M \frac{D}{D+\epsilon}\{S^M - \mathrm{tr}_g(f^*(\mathrm{Ric}_m^{(1)}(h)))\}e^{(m-1)\psi}g^m \le \int_M \Delta \log(D+\epsilon)e^{(m-1)\psi}g^m = 0 \ .$$

Here we have used that

$$\int_M \Delta \log(D+\epsilon)e^{(m-1)\psi}g^m = \int_M (\Delta_{e^\psi g} \log(D+\epsilon))(e^\psi g)^m = 0 \ .$$

Now, we can easily use the same arguments as to those in Theorem 1.1 in [15] to complete the proof. □

Acknowledgements The author is grateful to Professor Fangyang Zheng for constant encouragement and support. He wishes to express his gratitude to Professor Lei Ni for many useful discussions on [5, 6].

References

[1] Gauduchon P. Le théorème de l'excentricité nulle. C R Acad Sci Paris Sér A-B, 1977, **285**: A387–A390

[2] Kobayashi S. Hyperbolic Complex Spaces. New York: Springer, 1998

[3] Lee M C, Streets J. Complex manifolds with negative curvature operator. arXiv:1903.12645

[4] Lu Y C. Holomorphic mappings of complex manifolds. J Differential Geometry, 1968, **2**: 299–312

[5] Ni L. Liouville theorems and a Schwarz Lemma for holomorphic mappings between Kähler manifolds. Comm Pure Appl Math, 2021, **74**: 1100–1126

[6] Ni L. General Schwarz Lemmata and their applications. International J Math, 2019, **30**(13): 1940007

[7] Ni L, Zheng F Y. Positivity and Kodaira embedding theorem. ArXiv:1804.096096

[8] Royden H L. The Ahlfors-Schwarz lemma in several complex variables. Commentarii Mathematici Helvetici, 1980, **55**(1): 547–558

[9] Tossati V. A general Schwarz lemma for almost-Hermitian manifolds. Comm Anal Geom, 2007, **15**(5): 1063–1086

[10] Wu D, Yau S T. Negative holomorphic curvature and positive canonical bundle. Invent Math, 2016, **204**(2): 595–604

[11] Yang X K. RC-positivity, vanishing theorems and rigidity of holomorphic maps. arXiv:1807.02601

[12] Yang X K. RC-positivity and the generalized energy density I: Rigidity. arXiv:1810.03276

[13] Yang X K, Zheng F Y. On real bisectional curvature for Hermitian manifolds. Trans Amer Math Soc, 2019, **371**(4): 2703–2718

[14] Yau S T. A general Schwarz lemma for Kähler manifolds. Amer J Math, 1978, **100**(1): 197–203

[15] Zhang Y S. Integral inequalities for holomorphic maps and applications. Tran Amer Math Soc, 2021, **374**(4): 2341–2358

[16] Zhang Y S. A note on conical Kähler-Ricci flow on minimal elliptic Kähler surface. Acta Math Sci, 2018, **38B**(1): 169–176