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THE PRODUCT OPERATOR BETWEEN BLOCH-TYPE SPACES OF SLICE REGULAR FUNCTIONS*

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Abstract There is little work concerning the properties of quaternionic operators acting on slice regular function spaces defined on quaternions. In this paper, we present an equivalent characterization for the boundedness of the product operator $C_\varphi D^m$ acting on Bloch-type spaces of slice regular functions. After that, an equivalent estimation for its essential norm is established, which can imply several existing results on holomorphic spaces.

Key words essential norm; differentiation; composition operator; Bloch-type space; slice regular functions

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1 Introduction

Gentili and Struppa first introduced slice hyperholomorphic functions in 2006 (see, e.g.,[12]). Since then, many mathematicians have been involved in creating a theory regarding the functions of quaternionic variables, and this has several applications, for example in Schur analysis and operator theory. Associated with slice hyperholomorphic functions, there is a very rich literature on Schur analysis; see the book [1] and the references therein. Slice hyperholomorphic functions can also be referred to as slice regular, as they are defined on quaternions and are quaternionic-valued. With the development of the theory of slice regular functions, there have appeared various slice regular function spaces, such as Fock space [2, 11, 26], Hardy and Bergman spaces [7, 23], and Bloch, Besov and Dirichlet spaces [25] and so on. These quaternionic function spaces play important roles in quaternionic operator theory, which is different from the complex operator theory. As far as we are concerned, one of the main difference is the definition of the spectrum of a linear operator, which is called the S-spectrum in quaternionic operator theory, and is widely used in fractional powers and fractional diffusion processes (see the excellent books [5, 6]).

For a long time, describing the behavior of linear operators acting on various complex holomorphic function spaces has been a very fundamental topic. The linear operators include the (weighted differentiation) composition operators ([4, 17, 19, 20, 24, 27, 28, 30, 31]), the

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integral-type operator ([3]) and so on. To the best of our knowledge, there is very little work on the properties of quaternionic operators on slice regular function spaces. Considering that the Bloch-type space is a convenient setting for many problems in functional analysis and the product operator $C_\varphi D^m$ is a very general operator, we fix our attention in this paper on some classical and challenging problems to characterize the boundedness and compactness of this product operator acting between the Bloch-type spaces of slice regular functions.

We now recall some preliminaries regarding slice regular functions. For more information regarding the ensuing facts, we refer readers to [9, 10]. Let the symbol $\mathbb{H}$ denote the noncommutative, associative, real algebra of quaternions $q = x_0 + x_1e_1 + x_2e_2 + x_3e_3 = \mathrm{Re}q + \mathrm{Im}q$, with $\mathrm{Re}q = x_0$ and $\mathrm{Im}q = x_1e_1 + x_2e_2 + x_3e_3$, where x_j are real numbers for $j = 0, 1, 2, 3$, and the imaginary units e_1, e_2, e_3 are subject to the rule $e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1$. The set $\{e_0 = 1, e_1, e_2, e_3\}$ is the usual basis of the quaternions. An element $q \in \mathbb{H}$ can also be written as a linear combination of two complex numbers, that is, $q = (x_0 + x_1e_1) + (x_2 + x_3e_1)e_2$. Moreover, we can consider the space $\mathbb{R}^3$ embedded in $\mathbb{H}$ as follows:

$$(x_1, x_2, x_3) \longrightarrow x_1e_1 + x_2e_2 + x_3e_3.$$

We say the conjugate of $q \in \mathbb{H}$ is $\overline{q} = \mathrm{Re}q - \mathrm{Im}q$ and its modulus is $|q|^2 = q\overline{q} = |\mathrm{Re}q|^2 + |\mathrm{Im}q|^2$. Every $q \in \mathbb{H}$ can be expressed as $q = x + yI$, where $x, y \in \mathbb{R}$ and $I = \mathrm{Im}q/|\mathrm{Im}q|$ if $\mathrm{Im}q \neq 0$, otherwise we take I arbitrarily such that $I^2 = -1$. Let the symbol $\mathbb{S}$ denote the two-dimensional unit sphere of purely imaginary quaternions, meaning that

$$\mathbb{S} = \{q = x_1e_1 + x_2e_2 + x_3e_3 : \ x_1^2 + x_2^2 + x_3^2 = 1\},$$

hence $I \in \mathbb{S}$. It is obvious that $q^2 = -1$ for all $q \in \mathbb{S}$.

In the sequel, we take $i \in \mathbb{S}$ and let $\mathbb{C}(i)$ denote the space generated by $\{1, i\}$, which can be identified as the usual complex plane. It is easy to check that

$$\mathbb{H} = \bigcup_{i\in\mathbb{S}} \mathbb{C}(i).$$

The set $\mathbb{B} = \{q \in \mathbb{H} : \ |q| < 1\}$ is a unit ball in $\mathbb{H}$, and so $\mathbb{B}_i = \mathbb{B} \cap \mathbb{C}(i)$ is identified as the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}(i)$ for $i \in \mathbb{S}$.

We denote the space of holomorphic functions on $\mathbb{D}$ by $H(\mathbb{D})$. For $0 < \alpha < \infty$, an $f \in H(\mathbb{D})$ is said to be in the complex Bloch-type space or α-Bloch space $\mathcal{B}^\alpha_{\mathbb{D}}$ if

$$\|f\|_{\alpha,\mathbb{D}} = \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

Then $\mathcal{B}^\alpha_{\mathbb{D}}$ is a Banach space under the norm

$$\|f\|_{\mathcal{B}^\alpha_{\mathbb{D}}} = |f(0)| + \|f\|_{\alpha,\mathbb{D}}.$$

That the complex Bloch-type space is important in operator theory is due to its invariance with respect to Möbius transformation. We refer the readers to the book [31] by Zhu, which is an excellent source concerning the development of theory on complex holomorphic function spaces.

As regards the theory of slice regular functions, it has been developed systematically in recent decades and is widely applied in quaternionic quantum mechanics; see e.g., [10]. Here we first present the definition of slice regular functions and cite some basic properties; see, e.g., [8, 9, 13].

Definition 1.1 (Slice regular functions) Let Ω be a domain in $\mathbb{H}$. A real differentiable quaternionic-valued function $f : \Omega \to \mathbb{H}$ is called slice regular if, for any $i \in \mathbb{S}$, its restriction f_i on $\Omega_i = \Omega \cap \mathbb{C}(i)$ satisfies

$$\overline{\partial_i} f(x+yi) := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f_i(x+yi) = 0$$

for all $x + yi \in \Omega_i$. We denote by $\mathcal{R}(\Omega)$ the set of all slice regular functions on Ω.

Proposition 1.2 (Splitting lemma) If $f \in \mathcal{R}(\Omega)$, then for any $i \in \mathbb{S}$ and every $j \in \mathbb{S}$ orthogonal to i, there are two holomorphic functions $F, G :\ \Omega \cap \mathbb{C}(i) \to \mathbb{C}(i)$ such that, for any $z = x + iy$, it holds that

$$f_i = f\,|_{\Omega_i}\,(z) = F(z) + G(z)j.$$

Slice regular functions possess good properties on specific open sets that we will call axially symmetric slice domains. On these domains, slice regular functions satisfy the representation formula, which allows us to reconstruct the values of the function once we know its values on some complex plane $\mathbb{C}(i)$.

Definition 1.3 Let $\Omega \subset \mathbb{H}$ be a domain. Then

(1) Ω is called a slice domain (or s-domain for short) if it intersects the real axis and if, for any $i \in \mathbb{S}$, Ω_i is a domain in $\mathbb{C}(i)$.

(2) Ω is called an axially symmetric domain if, for any $x + yi \in \Omega$ with $x, y \in \mathbb{R}$ and $i \in \mathbb{S}$, the entire two-sphere $x + y\mathbb{S}$ is contained in Ω.

Proposition 1.4 (Representation Formula) Let f be a slice regular function on an axially symmetric s-domain $\Omega \subset \mathbb{H}$. Choose any $j \in \mathbb{S}$. Then the following equality holds for all $q = x + yi \in \Omega$:

$$f(x+yi) = \frac{1}{2}\left((1+ij)f(x-yj) + (1-ij)f(x+yj)\right).$$

Letting $i, j \in \mathbb{S}$ be mutually orthogonal vectors and $\Omega \subset \mathbb{H}$ an axially symmetric s-domain, the splitting lemma and the representation formula entail the ensuing definitions, which relate the slice regular function space $\mathcal{R}(\Omega)$ on Ω with the space of pairs of holomorphic functions on Ω_i, denoted by $H(\Omega_i)$. Afterwards, define

$$Q_i :\ \mathcal{R}(\Omega) \to H(\Omega_i) + H(\Omega_i)j$$

by $Q_i[f] = f_i = f|_{\Omega_i}$ and

$$P_i :\ H(\Omega_i) + H(\Omega_i)j \to \mathcal{R}(\Omega)$$

by

$$P_i[f](q) = P_i[f](x + yI_q) = \frac{1}{2}[(1 + I_q i)f(x - yi) + (1 - I_q i)f(x + yi)]$$

for $f \in H(\Omega_i) + H(\Omega_i)j$.

Based on the above mappings, we first recall a definition for the slice regular α-Bloch space on the unit ball $\mathbb{B}$ from [15, Definition 2.1]. Then we present its relation with a complex Bloch-type space.

Definition 1.5 For $0 < \alpha < \infty$, the slice regular α-Bloch space associated with the unit ball $\mathbb{B}$ is the quaternionic right linear space of slice regular function f on $\mathbb{B}$ such that

$$\|f\|_\alpha = \sup_{p\in\mathbb{B}}(1-|p|^2)^\alpha \left|\frac{\partial f}{\partial x_0}(p)\right| < \infty,$$

where Re $p = x_0$; that is to say,

$$\mathcal{B}^\alpha := \mathcal{B}^\alpha(\mathbb{B}) = \{f \in \mathcal{R}(\mathbb{B}) : \ \ \|f\|_\alpha < \infty\}.$$

It is easy to check that $\mathcal{B}^\alpha$ is a Banach space endowed with the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha.$$

The space $\mathcal{B}_i^\alpha$ has a close relation to the slice regular α-Bloch space, which is the quaternionic right linear space of slice regular functions on the unit ball $\mathbb{B}$ such that

$$\|f\|_{\alpha,i} = \sup_{z\in\mathbb{B}_i}(1-|z|^2)^\alpha |Q_i[f]'(z)| < \infty,$$

where

$$Q_i[f]'(z) = \frac{\partial Q_i[f]}{\partial x_0}(z) = \frac{\partial f_i}{\partial x_0}(z)$$

is a holomorphic map of complex variable $z = x_0 + iy$ and $i \in \mathbb{S}$. Hence,

$$\mathcal{B}_i^\alpha := \mathcal{B}_i^\alpha(\mathbb{B}) = \{f \in \mathcal{R}(\mathbb{B}) : \|f\|_{\mathcal{B}_i^\alpha} = |f(0)| + \|f\|_{\alpha,i} < \infty\}$$

is a Banach space under the norm $\|f\|_{\mathcal{B}_i^\alpha}$. The following remark further reveals the relationship between $\mathcal{B}_i^\alpha$ and $\mathcal{B}_{\mathbb{D}}^\alpha$:

Remark 1.6 Let $i \in \mathbb{S}$ and $f \in \mathcal{B}_i^\alpha$. Then, for any $j \in \mathbb{S}$ with $j \perp i$, there exist holomorphic functions f_1 and $f_2 : \mathbb{B}_i \to \mathbb{C}(i)$ such that

$$Q_i[f](z) = f_i(z) = f_1(z) + f_2(z)j.$$

In addition,

$$|Q_i[f]'(z)|^2 = |f_i'(z)|^2 = |f_1'(z)|^2 + |f_2'(z)|^2,$$

which implies the conclusion that $f \in \mathcal{B}_i^\alpha$ if and only if both f_1 and f_2 belong to the complex α-Bloch space $\mathcal{B}_{\mathbb{D}}^\alpha$. It turns out that

$$\begin{aligned}\|f\|_{\alpha,i}^2 &= \sup_{z\in\mathbb{B}_i}(1-|z|^2)^{2\alpha}|f_i'(z)|^2\\ &= \sup_{z\in\mathbb{B}_i}(1-|z|^2)^{2\alpha}[|f_1'(z)|^2 + |f_2'(z)|^2]\\ &= \|f_1\|_{\alpha,\mathbb{D}}^2 + \|f_2\|_{\alpha,\mathbb{D}}^2. \end{aligned} \tag{1.1}$$

The next proposition shows that the spaces $\mathcal{B}^\alpha$ and $\mathcal{B}_i^\alpha$ contain the same elements.

Proposition 1.7 ([25, Proposition 2.6]) Let $i \in \mathbb{S}$. Then $f \in \mathcal{B}_i^\alpha$ if and only if $f \in \mathcal{B}^\alpha$. More precisely, one has

$$\|f\|_{\mathcal{B}_i^\alpha} \le \|f\|_{\mathcal{B}^\alpha} \le 2\|f\|_{\mathcal{B}_i^\alpha}. \tag{1.2}$$

In addition, it also holds that

$$\|f\|_{\alpha,i} \le \|f\|_\alpha \le 2\|f\|_{\alpha,i}.$$

Furthermore, the function $f \in \mathcal{B}_i^\alpha$ if and only if $f \in \mathcal{B}_j^\alpha$ and the norms $\|\cdot\|_{\mathcal{B}_i^\alpha}$ and $\|\cdot\|_{\mathcal{B}_j^\alpha}$ are equivalent for $i, j \in \mathbb{S}$.

In the sequel, we define the composition operator on slice regular α-Bloch spaces.

Definition 1.8 Let $\varphi: \mathbb{B} \to \mathbb{B}$ be a slice regular map such that $\varphi(\mathbb{B}_i) \subset \mathbb{B}_i$ for some $i \in \mathbb{S}$. Then the $\mathbb{C}(i)$-composition operator $(C_\varphi)_i : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ of the unit ball $\mathbb{B}$, with the domain consisting of all $h \in \mathcal{B}^\alpha$ such that $C_\varphi h$ belongs to $\mathcal{B}^\beta$, is defined by

$$(C_\varphi f)_i = f_i \circ \varphi_i, \quad f \in \mathcal{B}^\alpha.$$

Using the Representation Formula, we can obtain all values of $C_\varphi f$ on $\mathbb{H}$.

Readers can also refer to [22] for the definition of the slice regular composition operator C_φ. It is well-known that the study of composition operators is a fairly active field. For general references on the theory of composition operators on the holomorphic functions of complex variables, see the excellent books [4], by Cowen and MacCluer, and [24], by Shapiro. As regards the slice regular composition operators, Ren and Wang ([22]) studied their properties acting on the quaternionic Hardy spaces. As a generalization, the basic properties of the slice regular weighted composition operator were systematically characterized in the recent papers [16, 18]. As far as we are concerned, there has been no investigation on the product operator of differentiation and composition operators acting on slice regular function spaces, so we concentrate on this characterization with regard to Bloch-type spaces.

A natural notion of differentiation can be given for slice regular functions (see [12, 13]), and this is called the slice (or Cullen) derivative of f.

Definition 1.9 Let Ω be a slice domain in $\mathbb{H}$, and let $f : \Omega \to \mathbb{H}$ be a slice regular function. The slice derivative of f at $q = x + yi \in \Omega_i$ is defined by

$$\partial_i f(x + yi) := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f_i(x + yi).$$

We notice that the operators ∂_i and that $\overline{\partial}_i$ can commute, and that $\partial_i f = \frac{\partial f}{\partial x}$ for regular function f. Therefore, the slice derivative of a regular function is still regular, so we can iterate the differentiation to obtain the m-th slice derivative

$$\partial_i^m f = \frac{\partial^m f}{\partial x^m}, \quad m \in \mathbb{N}.$$

In the ensuing sections, we will directly denote the m-th slice derivative $\partial_i^m f$ by $f^{(m)}$ for $i \in \mathbb{S}$ and $m \in \mathbb{N}$, and denote

$$Df := \partial_i f = \frac{\partial f}{\partial x}, \quad f \in \mathcal{R}(\mathbb{B}).$$

Generally, for a nonnegative integer $m \in \mathbb{N}$, we denote

$$D^m f := \partial_i^m f = \frac{\partial^m f}{\partial x^m}, \quad f \in \mathcal{R}(\mathbb{B}).$$

Combining the differentiation operator with a map φ satisfying $\varphi(\mathbb{B}_i) \subset \mathbb{B}_i$ for some $i \in \mathbb{S}$, we define the product operator of differentiation and composition operators as

$$(C_\varphi D^m f)_i = [f^{(m)}]_i \circ \varphi_i, \ f \in \mathcal{R}(\mathbb{B}).$$

Using the Representation Formula, we can extend all values of the operator $C_\varphi D^m$ on $\mathcal{R}(\mathbb{B})$.

As is well known, the composition operator is a typical bounded operator on the complex classical Bloch space $\mathcal{B}_{\mathbb{D}}^\alpha$ with $\alpha = 1$, while the differentiation operators are typically unbounded on various complex Banach spaces of holomorphic functions. Recently, a lot of work has appeared on new characterizations in terms of φ^m for composition and differentiation operators between complex holomorphic function spaces. For example, Zhao [30] obtained the new characterization for the compactness of composition operator C_φ from $\mathcal{B}^p_{\mathbb{D}}$ to $\mathcal{B}^q_{\mathbb{D}}$ as $\left(\frac{e}{2p}\right)^p \limsup\limits_{m\to\infty} m^{p-1}\|\varphi^m\|_{q,\mathbb{D}} = 0$, where φ^m means the m-th power of φ. For more similar characterizations regarding the boundedness and compactness of some classical linear operators, we refer readers to [3, 17, 20, 27–29] and the references therein.

Regarding the complex-valued product operator $C_\varphi D^m$, we have deduced an equivalent description for its boundedness and estimated its essential norm in terms of monomial z^n in the complex Bloch-type spaces, which have concise representations, as follows:

Theorem A ([19, Theorem 1]) Let $0 < \alpha, \beta < \infty$, m be a nonnegative integer and let φ be a holomorphic self-map of the unit disk $\mathbb{D}$. Then $C_\varphi D^m : \mathcal{B}^\alpha_{\mathbb{D}} \to \mathcal{B}^\beta_{\mathbb{D}}$ is bounded if and only if

$$\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi D^m(z^n)\|_{\beta,\mathbb{D}} < \infty,$$

where $z \in \mathbb{D},\ n \in \mathbb{N}$.

Theorem B ([19, Theorem 2]) Let $0 < \alpha, \beta < \infty$, m be a nonnegative integer and let φ be a holomorphic self-map of the unit disk $\mathbb{D}$. Suppose that $C_\varphi D^m : \mathcal{B}^\alpha_{\mathbb{D}} \to \mathcal{B}^\beta_{\mathbb{D}}$ is bounded, then the estimate for the essential norm of $C_\varphi D^m : \mathcal{B}^\alpha_{\mathbb{D}} \to \mathcal{B}^\beta_{\mathbb{D}}$ is

$$\|C_\varphi D^m\|_e \asymp \left(\frac{e}{\alpha+m}\right)^{\alpha+m} \limsup_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m(z^n)\|_{\beta,\mathbb{D}}.$$

Building on the above foundations, we continue to provide the corresponding characterizations for the boundedness and essential norm estimation of $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ of slice regular functions. Throughout the remainder of this paper, $\mathbb{N}$ will denote the set of all nonnegative integers and C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notations $A \asymp B,\ A \preceq B,\ A \succeq B$ mean that there may be different positive constants C such that $B/C \le A \le CB,\ A \le CB,\ CB \le A$.

2 The Characterizations for $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ of Slice Regular Functions

In this section, we first present a characterization for the boundedness of $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ acting between the slice regular Bloch-type spaces containing Theorem A as a particular case.

Theorem 2.1 Let $0 < \alpha, \beta < \infty$, m be a nonnegative integer and let $\varphi :\ \mathbb{B} \to \mathbb{B}$ be a slice regular map such that $\varphi(\mathbb{B}_i) \subset \mathbb{B}_i$ for some $i \in \mathbb{S}$. Then the product operator $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded if and only if

$$\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi D^m I_n(p)\|_\beta < \infty, \tag{2.1}$$

where $I_n(p) = p^n,\ p \in \mathbb{H},\ n \in \mathbb{N}$.

Proof **Necessity** Assume that the operator $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Combining this with Proposition 1.7 ensures that the operator $C_\varphi D^m : \mathcal{B}^\alpha_i \to \mathcal{B}^\beta_i$ is bounded. Suppose that $p = x_0 + Iy \in \mathbb{B}$ with some $I \in \mathbb{S}$, and denote $z = x_0 + iy$ and $\overline{z} = x_0 - iy$. It then holds that $|z| = |\overline{z}| = |p|$. Since

$$\|C_\varphi D^m I_n(p)\|_\beta = \sup_{p\in\mathbb{B}} (1-|p|^2)^\beta \left|\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(p)\right|, \tag{2.2}$$

we will estimate the right part by the Representation Formula, due to the fact that $\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}$ is a slice regular function. First, it is true that

$$\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(p) = \frac{1}{2}(1-Ii)\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(z) + \frac{1}{2}(1+Ii)\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(\overline{z}).$$

Then, the above calculations yield that

$$(1-|p|^2)^\beta \left|\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(p)\right| \le (1-|z|^2)^\beta \left|\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(z)\right| + (1-|\overline{z}|^2)^\beta \left|\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(\overline{z})\right|,$$

which further implies that

$$\begin{aligned}&\sup_{p\in\mathbb{B}}(1-|p|^2)^\beta \left|\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(p)\right| \\ &\le \sup_{i\in\mathbb{S}}\sup_{z\in\mathbb{B}_i}(1-|z|^2)^\beta \left|\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(z)\right| + \sup_{i\in\mathbb{S}}\sup_{\overline{z}\in\mathbb{B}_i}(1-|\overline{z}|^2)^\beta \left|\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(\overline{z})\right| \\ &\le 2\sup_{i\in\mathbb{S}}\sup_{z\in\mathbb{B}_i}(1-|z|^2)^\beta \left|\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(z)\right|. \end{aligned} \tag{2.3}$$

Since $I_n(p)=p^n$ defined on $\mathbb{B}_i$ coincides with the usual complex monomial z^n, it is clear that $I_n(p)\in\mathcal{B}_i^\alpha$. Furthermore, by the boundness of $C_\varphi D^m:\mathcal{B}_i^\alpha\to\mathcal{B}_i^\beta$, we conclude that

$$\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi D^m I_n(p)\|_{\beta,i} < \infty. \tag{2.4}$$

Actually, this is due to the fact that $I_n(p)$ is bounded in $\mathcal{B}_i^\alpha$ and that

$$\|I_n\|_{\alpha,i} = \|I_n\|_{\mathcal{B}_i^\alpha} = \sup_{z\in\mathbb{B}_i}(1-|z|^2)^\alpha|nz^{n-1}| = \|z^n\|_{\mathcal{B}_{\mathbb{D}}^\alpha} = \|z^n\|_{\alpha,\mathbb{D}}.$$

By the direct arguments in $\mathcal{B}_{\mathbb{D}}^\alpha$, we obtain that

$$\lim_{n\to\infty} n^{\alpha-1}\|I_n\|_{\mathcal{B}_i^\alpha} = \lim_{n\to\infty} n^{\alpha-1}\|I_n\|_{\alpha,i} = \lim_{n\to\infty} n^{\alpha-1}\|z^n\|_{\alpha,\mathbb{D}} = \left(\frac{\alpha+m}{e}\right)^{\alpha+m}. \tag{2.5}$$

The above inequality ensures that there is a constant $C>0$, independent of n, such that

$$\|I_n\|_{\mathcal{B}_i^\alpha} \le Cn^{1-\alpha}.$$

It turns out that

$$\begin{aligned}\frac{1}{C}n^{\alpha-1}\|C_\varphi D^m I_n(p)\|_{\beta,i} &\le \frac{\|C_\varphi D^m I_n(p)\|_{\beta,i}}{\|I_n\|_{\mathcal{B}_i^\alpha}} = \left\|C_\varphi D^m \frac{I_n(p)}{\|I_n\|_{\mathcal{B}_i^\alpha}}\right\|_{\beta,i} \\ &\le \|C_\varphi D^m\|_{\mathcal{B}_i^\alpha\to\mathcal{B}_i^\beta} < \infty.\end{aligned}$$

Taking $\sup\limits_{n\in\mathbb{N}}$ in the above inequality, we can obtain the formula (2.4).

Letting $j\in\mathbb{S}$ with $j\perp i$, we can write $(I_n)_i(z) = I_{n,1}(z) + I_{n,2}(z)j$ with two complex holomorphic functions $I_{n,1}\in\mathcal{B}_{\mathbb{D}}^\alpha$ and $I_{n,2}\in\mathcal{B}_{\mathbb{D}}^\alpha$, by Remark 1.6. Indeed, for $I_n(p)=p^n$, we can directly write $I_{n,1}(z)=z^n$ and $I_{n,2}(z)=0$. More generally, we can deduce that

$$\begin{aligned}&\sup_{i\in\mathbb{S}}\sup_{z\in\mathbb{B}_i}(1-|z|^2)^\beta \left|\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(z)\right| \\ &\le \sup_{i\in\mathbb{S}}\sup_{z\in\mathbb{B}_i}(1-|z|^2)^\beta \left|\frac{\partial(C_\varphi D^m I_{n,1})}{\partial x_0}(z)\right| + \sup_{i\in\mathbb{S}}\sup_{z\in\mathbb{B}_i}(1-|z|^2)^\beta \left|\frac{\partial(C_\varphi D^m I_{n,2})}{\partial x_0}(z)\right| \\ &= \|C_\varphi D^m I_{n,1}(z)\|_{\beta,\mathbb{D}} + \|C_\varphi D^m I_{n,2}(z)\|_{\beta,\mathbb{D}} \\ &= \|C_\varphi D^m I_{n,1}(z)\|_{\beta,\mathbb{D}} = \|C_\varphi D^m I_n(p)\|_{\beta,i}.\end{aligned} \tag{2.6}$$

Here we have used (1.1) with $I_{n,2}(z)=0$. Putting the display (2.6) into (2.3) yields that

$$\sup_{p\in\mathbb{B}}(1-|p|^2)^\beta\left|\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(p)\right|\le 2\|C_\varphi D^m I_n(p)\|_{\beta,i},$$

which, together with the formulas (2.4) and (2.2), implies that

$$\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi D^m I_n(p)\|_\beta\le 2\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi D^m I_n(p)\|_{\beta,i}<\infty.$$

Sufficiency Supposing the formula (2.1) holds, we get that

$$\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi D^m I_n(p)\|_{\beta,i}<\infty. \tag{2.7}$$

Letting $j\in\mathbb{S}$ with $j\perp i$, it holds that

$$(I_n)_i(z)=I_{n,1}(z)+I_{n,2}(z)j=z^n.$$

Therefore, by $\varphi(\mathbb{B}_i)\subset\mathbb{B}_i$, we deduce that

$$\begin{aligned}&\sup_{n\in\mathbb{N}} n^{\alpha-1}\sup_{z\in\mathbb{B}_i}(1-|z|^2)^\beta\left|\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(z)\right|\\ =&\sup_{n\in\mathbb{N}} n^{\alpha-1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|\frac{\partial(C_\varphi D^m I_{n,1})}{\partial x_0}(z)\right|\\ =&\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi D^m I_{n,1}(z)\|_{\beta,\mathbb{D}}.\end{aligned}$$

The above arguments, together with (2.7), ensure that

$$\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi D^m I_n(z)\|_{\beta,\mathbb{D}}<\infty.$$

Observing Theorem A, the above formula holds if and only if the operator $C_\varphi D^m:\mathcal{B}^\alpha_{\mathbb{D}}\to\mathcal{B}^\beta_{\mathbb{D}}$ is bounded. By Remark 1.6, the boundedness of $C_\varphi D^m:\mathcal{B}^\alpha_{\mathbb{D}}\to\mathcal{B}^\beta_{\mathbb{D}}$ can entail the boundedness of the operator $C_\varphi D^m:\mathcal{B}^\alpha_i\to\mathcal{B}^\beta_i$. This means that $\|C_\varphi D^m f\|_{\beta,i}<\infty$ for any $f\in\mathcal{B}^\alpha_i$. Employing Proposition 1.7, it turns out that $\|C_\varphi D^m f\|_\beta<\infty$ for all $f\in\mathcal{B}^\alpha$, which implies the boundedness of the operator $C_\varphi D^m:\mathcal{B}^\alpha\to\mathcal{B}^\beta$, ending the proof. □

As stated in [25, Definition 2.9], the space $\mathcal{H}^\infty(\mathbb{B})$ is the collection of all quaternionic right linear bounded slice regular functions on $\mathbb{B}$; that is,

$$\mathcal{H}^\infty(\mathbb{B})=\{f\in\mathcal{R}(\mathbb{B}):\ \|f\|_\infty:=\sup_{q\in\mathbb{B}}|f(q)|<\infty\}.$$

We can verify that $\mathcal{H}^\infty(\mathbb{B})$ is a Banach space under the norm $\|f\|_\infty$. Immediately, we go on showing that $C_\varphi D^m:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is compact if $\|\varphi\|_\infty<1$. The following proposition can be deduced by a similar way as [14, Proposition 3.3], which plays a critical role in proving the compactness of operators:

Proposition 2.2 Let $0<\alpha,\beta<\infty$, m be a nonnegative integer and let $\varphi:\ \mathbb{B}\to\mathbb{B}$ be a slice regular map such that $\varphi(\mathbb{B}_i)\subset\mathbb{B}_i$ for some $i\in\mathbb{S}$. Then the product operator $C_\varphi D^m:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is compact if and only if, for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{B}^\alpha$ with $f_k\to 0$ as $k\to\infty$ on compact sets, $\|C_\varphi D^m f_k\|_{\mathcal{B}^\beta}\to 0$ as $k\to\infty$.

Theorem 2.3 Let $0<\alpha,\beta<\infty$, m be a nonnegative integer and let $\varphi:\ \mathbb{B}\to\mathbb{B}$ be a slice regular map such that $\varphi(\mathbb{B}_i)\subset\mathbb{B}_i$ for some $i\in\mathbb{S}$. If $\|\varphi\|_\infty<1$, then the product operator $C_\varphi D^m:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is compact.

Proof Let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}^\alpha$ satisfying $f_k \to 0$ as $k\to\infty$ on compact sets of $\mathbb{B}$. Then it also holds that $f_k \to 0$ as $k\to\infty$ on compact sets of $\mathbb{B}_i$ for $i\in\mathbb{S}$. Let $j\in\mathbb{S}$ be such that $j\perp i$, and let $f_{k,1},\ f_{k,2}:\mathbb{B}_i\to\mathbb{C}(i)$ be holomorphic functions satisfying $(f_k)_i(z)=f_{k,1}(z)+f_{k,2}(z)j$ for some $z=x_0+iy\in\mathbb{B}_i$. By Remark 1.6, it follows that the two functions $f_{k,1}(z)$ and $f_{k,2}(z)$ lie in the complex Bloch-type spaces $\mathcal{B}^\alpha_{\mathbb{D}}$ on $\mathbb{B}_i$, where $\mathbb{B}_i$ is identified with $\mathbb{D}\subset\mathbb{C}(i)$. In addition, it is obvious that $f_{k,l}\to 0$ as $k\to\infty$ on compact sets of $\mathbb{D}$ and $l=1,2$. Hence we obtain that

$$\begin{aligned}
&\sup_{i\in\mathbb{S}}\sup_{z\in\mathbb{B}_i}(1-|z|^2)^\beta\left|\frac{\partial(C_\varphi D^m f_k)}{\partial x_0}(z)\right|\\
\le{}& \sup_{i\in\mathbb{S}}\sup_{z\in\mathbb{B}_i}(1-|z|^2)^\beta\left|\frac{\partial(C_\varphi D^m f_{k,1})}{\partial x_0}(z)\right| + \sup_{i\in\mathbb{S}}\sup_{z\in\mathbb{B}_i}(1-|z|^2)^\beta\left|\frac{\partial(C_\varphi D^m f_{k,2})}{\partial x_0}(z)\right|\\
={}& \|C_\varphi D^m f_{k,1}\|_{\beta,\mathbb{D}}+\|C_\varphi D^m f_{k,2}\|_{\beta,\mathbb{D}}\\
\to{}& 0,\quad \text{as } k\to\infty,
\end{aligned}\tag{2.8}$$

where the last line is due to the corresponding result in complex Bloch-type spaces (see, e.g., [19, page 356]) under the case $\|\varphi\|_\infty<1$. Based on the fact that

$$\begin{aligned}
&(1-|p|^2)^\beta\left|\frac{\partial(C_\varphi D^m f_k)}{\partial x_0}(p)\right|\\
\le{}&(1-|z|^2)^\beta\left|\frac{\partial(C_\varphi D^m f_k)}{\partial x_0}(z)\right|+(1-|\overline{z}|^2)^\beta\left|\frac{\partial(C_\varphi D^m f_k)}{\partial x_0}(\overline{z})\right|,
\end{aligned}$$

we use (2.8) to deduce that

$$\begin{aligned}
\|C_\varphi D^m f_k(p)\|_\beta &= \sup_{p\in\mathbb{B}}(1-|p|^2)^\beta\left|\frac{\partial(C_\varphi D^m f_k)}{\partial x_0}(p)\right|\\
&\le 2\sup_{i\in\mathbb{S}}\sup_{z\in\mathbb{B}_i}(1-|z|^2)^\beta\left|\frac{\partial(C_\varphi D^m f_k)}{\partial x_0}(z)\right|\\
&\to 0,\quad \text{as } k\to\infty.
\end{aligned}$$

Employing Proposition 2.2, the compactness of $C_\varphi D^m:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ follows. □

Definition 2.4 The essential norm of a bounded linear operator T between two normed linear spaces X and Y is its distance from the compact operator K; that is,

$$\|T\|_e=\inf\{\|T-K\|_{X\to Y},\ K:X\to Y \text{ is compact}\},$$

where $\|.\|_{X\to Y}$ denotes the operator norm (see, e.g., [21]).

It is obvious that T is compact if and only if $\|T\|_e=0$. Thus Theorem 2.3 can yield $\|C_\varphi D^m\|_e=0$ for the case $\|\varphi\|_\infty<1$. Next, we continue to estimate the essential norm of $C_\varphi D^m:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ under the case $\|\varphi\|_\infty=1$, which contains Theorem B as a special case.

Theorem 2.5 Let $0<\alpha,\beta<\infty$, m be a nonnegative integer and let $\varphi:\ \mathbb{B}\to\mathbb{B}$ be a slice regular map such that $\varphi(\mathbb{B}_i)\subset\mathbb{B}_i$ for some $i\in\mathbb{S}$ and $\|\varphi\|_\infty=1$. Suppose that $C_\varphi D^m:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is bounded. Then the estimation for the essential norm of $C_\varphi D^m:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is

$$\|C_\varphi D^m\|_e\asymp\limsup_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m I_n\|_\beta.\tag{2.9}$$

Proof **The lower estimation** Let $I_n(p)=p^n$. Then $I_n(z)=z^n$ is a sequence on $\mathbb{B}_i$ (identified as the unit disk $\mathbb{D}$) associated with $n\in\mathbb{N}$. Since I_n converges to zero uniformly on compact subsets of $\mathbb{B}_i$, Proposition 2.2 ensures that

$$\|KI_n\|_{\mathcal{B}_i^\beta} \to 0, \quad \text{as } n \to \infty$$

for every compact operator $K : \mathcal{B}_i^\alpha \to \mathcal{B}_i^\beta$. Therefore, it follows that

$$\begin{aligned}
\|C_\varphi D^m - K\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} &\geq \left\|(C_\varphi D^m - K)\frac{I_n}{\|I_n\|_{\mathcal{B}^\alpha}}\right\|_{\mathcal{B}^\beta} \\
&\geq \left\|(C_\varphi D^m - K)\frac{I_n}{\|I_n\|_{\mathcal{B}^\alpha}}\right\|_{\mathcal{B}_i^\beta} \\
&\geq \frac{1}{2\|I_n\|_{\mathcal{B}_i^\alpha}} \|(C_\varphi D^m - K)I_n\|_{\mathcal{B}_i^\beta} \\
&\geq \frac{1}{2\|I_n\|_{\mathcal{B}_i^\alpha}} \left(\|C_\varphi D^m I_n\|_{\mathcal{B}_i^\beta} - \|KI_n\|_{\mathcal{B}_i^\beta}\right).
\end{aligned}$$

Here the second and third inequalities are both due to (1.2) in Proposition 1.7. Combining the formula (2.5) with the norm relation in Proposition 1.7, it follows that

$$\begin{aligned}
\|C_\varphi D^m\|_e &= \inf_K \|C_\varphi D^m - K\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \\
&\succeq \inf_K \limsup_{n\to\infty} \frac{1}{2\|I_n\|_{\mathcal{B}_i^\alpha}} \left(\|C_\varphi D^m I_n\|_{\mathcal{B}_i^\beta} - \|KI_n\|_{\mathcal{B}_i^\beta}\right) \\
&\asymp \limsup_{n\to\infty} \frac{1}{2} n^{\alpha-1} \left(\frac{e}{\alpha+m}\right)^{\alpha+m} \|C_\varphi D^m I_n\|_{\mathcal{B}_i^\beta} \\
&\succeq \limsup_{n\to\infty} n^{\alpha-1} \|C_\varphi D^m I_n\|_{\mathcal{B}^\beta} \\
&\asymp \limsup_{n\to\infty} n^{\alpha-1} \|C_\varphi D^m I_n\|_\beta.
\end{aligned}$$

The lower estimation follows from the above arguments.

The upper estimation Let L_n be the sequence of operators given in [19, Lemma 3, 4, 5]; that is, L_n is compact as an operator from $\mathcal{B}_{\mathbb{D}}^\alpha$ to $\mathcal{B}_{\mathbb{D}}^\alpha$. Using Remark 1.6, for every $f \in \mathcal{B}_i^\alpha$, let $j \in \mathbb{S}$ be such that $j \perp i$, and let $f_1, f_2 : \mathbb{B}_i \to \mathbb{C}(i)$ be holomorphic functions satisfying $f(z) = f_1(z) + f_2(z)j$ for some $z = x_0 + \mathrm{i}y \in \mathbb{B}_i$. It follows that $f_k \in \mathcal{B}_{\mathbb{D}}^\alpha$, $k = 1, 2$, and so

$$\|L_n f\|_{\alpha,i}^2 = \|L_n f_1\|_{\alpha,\mathbb{D}}^2 + \|L_n f_2\|_{\alpha,\mathbb{D}}^2.$$

This verifies that $L_n : \mathcal{B}_i^\alpha \to \mathcal{B}_i^\alpha$ is also compact. Furthermore, $C_\varphi D^m L_n : \mathcal{B}_i^\alpha \to \mathcal{B}_i^\beta$ and $C_\varphi D^m L_n : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ are a sequence of compact operators. Thus

$$\begin{aligned}
\|C_\varphi D^m\|_e &\leq \limsup_{n\to\infty} \|C_\varphi D^m - C_\varphi D^m L_n\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \\
&= \limsup_{n\to\infty} \|C_\varphi D^m (I - L_n)\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta} \\
&= \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}^\alpha}\leq 1} \|C_\varphi D^m (I - L_n) f\|_{\mathcal{B}^\beta} \\
&\leq 2 \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}_i^\alpha}\leq 1} \|C_\varphi D^m (I - L_n) f\|_{\mathcal{B}_i^\beta} \\
&\leq 2 \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}_i^\alpha}\leq 1} |C_\varphi D^m (I - L_n) f(0)| \qquad (2.10) \\
&\quad +2 \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}_i^\alpha}\leq 1} \|C_\varphi D^m (I - L_n) f\|_{\beta,i}. \qquad (2.11)
\end{aligned}$$

The formula (1.1) implies that

$$\begin{aligned}\|f\|_{\mathcal{B}_i^\alpha} &= |f(0)| + \|f\|_{\alpha,i}\\ &= \sqrt{|f_1(0)|^2+|f_2(0)|^2} + \sqrt{\|f_1\|^2_{\alpha,\mathbb{D}}+\|f_2\|^2_{\alpha,\mathbb{D}}}\\ &\geq \max\{\|f_1\|_{\mathcal{B}_{\mathbb{D}}^\alpha}, \|f_2\|_{\mathcal{B}_{\mathbb{D}}^\alpha}\},\end{aligned}$$

so if $\|f\|_{\mathcal{B}_i^\alpha} \leq 1$, we get that $\|f_k\|_{\mathcal{B}_{\mathbb{D}}^\alpha} \leq 1$ for $k = 1, 2$, but the converse is not true.

On the one hand, since

$$|C_\varphi D^m(I-L_n)f(0)| \leq |C_\varphi D^m(I-L_n)f_1(0)| + |C_\varphi D^m(I-L_n)f_2(0)|,$$

and $\limsup\limits_{n\to\infty} \sup\limits_{\|f_k\|_{\mathcal{B}_{\mathbb{D}}^\alpha}\leq 1} |C_\varphi D^m(I-L_n)f_k(0)| = 0$ for $k = 1, 2$, the term (2.10) is zero due to the fact that

$$\begin{aligned}&\limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}_i^\alpha}\leq 1} |C_\varphi D^m(I-L_n)f(0)|\\ &\leq \limsup_{n\to\infty} \sup_{\|f_1\|_{\mathcal{B}_{\mathbb{D}}^\alpha}\leq 1} |C_\varphi D^m(I-L_n)f_1(0)| + \limsup_{n\to\infty} \sup_{\|f_2\|_{\mathcal{B}_{\mathbb{D}}^\alpha}\leq 1} |C_\varphi D^m(I-L_n)f_2(0)|.\end{aligned}$$

On the other hand, we turn to estimate the term (2.11). By (1.1), it follows that

$$\|C_\varphi D^m(I-L_n)f\|^2_{\beta,i} = \|C_\varphi D^m(I-L_n)f_1\|^2_{\beta,\mathbb{D}} + \|C_\varphi D^m(I-L_n)f_2\|^2_{\beta,\mathbb{D}},$$

which leads to

$$\|C_\varphi D^m(I-L_n)f\|_{\beta,i} \leq \|C_\varphi D^m(I-L_n)f_1\|_{\beta,\mathbb{D}} + \|C_\varphi D^m(I-L_n)f_2\|_{\beta,\mathbb{D}}.$$

This further implies that

$$\begin{aligned}&\limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{B}_i^\alpha}\leq 1} \|C_\varphi D^m(I-L_n)f\|_{\beta,i}\\ &\leq \limsup_{n\to\infty} \sup_{\|f_1\|_{\mathcal{B}_{\mathbb{D}}^\alpha}\leq 1} \|C_\varphi D^m(I-L_n)f_1\|_{\beta,\mathbb{D}} + \limsup_{n\to\infty} \sup_{\|f_2\|_{\mathcal{B}_{\mathbb{D}}^\alpha}\leq 1} \|C_\varphi D^m(I-L_n)f_2\|_{\beta,\mathbb{D}}\\ &\preceq \left(\frac{e}{\alpha+m}\right)^{\alpha+m} \limsup_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m I_n\|_{\beta,\mathbb{D}}\\ &\leq \left(\frac{e}{\alpha+m}\right)^{\alpha+m} \limsup_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m I_n\|_{\beta},\end{aligned}$$

where we use the complex upper estimation (see [19, page 357-359])

$$\limsup_{n\to\infty} \sup_{\|h\|_{\mathcal{B}_{\mathbb{D}}^\alpha}\leq 1} \|C_\varphi D^m(I-L_n)h\|_{\beta,\mathbb{D}} \preceq \left(\frac{e}{\alpha+m}\right)^{\alpha+m} \limsup_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m I_n\|_{\beta,\mathbb{D}}$$

and $\|C_\varphi D^m I_n\|_{\beta,\mathbb{D}} = \|C_\varphi D^m I_n\|_{\beta,i} \leq \|C_\varphi D^m I_n\|_\beta$. Combining the estimations for the parts (2.10) and (2.11), we deduce that

$$\|C_\varphi D^m\|_e \preceq \limsup_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m I_n\|_\beta,$$

completing the proof of the upper estimation. □

Remark 2.6 Let $0 < \alpha, \beta < \infty$, m be a nonnegative integer and let $\varphi:\ \mathbb{B} \to \mathbb{B}$ be a slice regular map such that $\varphi(\mathbb{B}_i) \subset \mathbb{B}_i$ for some $i \in \mathbb{S}$. Then (2.9) also holds for $\|\varphi\|_\infty < 1$.

Proof Based on Theorem 2.3, we only need to show that the right part of (2.9) equals 0. Take $0 < r < 1$ such that $\|\varphi\|_\infty = r < 1$. Letting $i, j \in \mathbb{S}$ with $i \perp j$, there exists a monomial $z^n : \mathbb{B}_i \to \mathbb{B}_i$ such that $(I_n)_i = z^n$. Furthermore, we obtain that

$$\begin{aligned}
\limsup_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m I_n(p)\|_\beta &\le 2\limsup_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m I_n(p)\|_{\beta,i}\\
&= 2\limsup_{n\to\infty} n^{\alpha-1}\sup_{z\in\mathbb{B}_i}(1-|z|^2)^\beta\left|\frac{\partial(C_\varphi D^m I_n)}{\partial x_0}(z)\right|\\
&= 2\limsup_{n\to\infty} n^{\alpha-1}\sup_{z\in\mathbb{B}_i}(1-|z|^2)^\beta\left|\frac{\partial(C_\varphi D^m z^n)}{\partial x_0}(z)\right|\\
&= 2\limsup_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m z^n\|_{\beta,\mathbb{D}} = 0,
\end{aligned}$$

implying that the formula (2.9) is true. □

The last corollary is a consequence of Theorem 2.5 and Remark 2.6.

Corollary 2.7 Let $0 < \alpha, \beta < \infty$, m be a nonnegative integer and let $\varphi: \mathbb{B} \to \mathbb{B}$ be a slice regular map such that $\varphi(\mathbb{B}_i) \subset \mathbb{B}_i$ for some $i \in \mathbb{S}$. Then the product operator $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if

$$\limsup_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m I_n(p)\|_\beta = 0.$$

To conclude, we pose a question for exploring in the near future:

Question How does one present the corresponding characterizations for a general slice regular map $\varphi: \mathbb{B} \to \mathbb{B}$ without the assumption $\varphi(\mathbb{B}_i) \subset \mathbb{B}_i$?

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