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A STABILITY PROBLEM FOR THE 3D MAGNETOHYDRODYNAMIC EQUATIONS NEAR EQUILIBRIUM*

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Abstract This paper is concerned with a stability problem on perturbations near a physically important steady state solution of the 3D MHD system. We obtain three major results. The first assesses the existence of global solutions with small initial data. Second, we derive the temporal decay estimate of the solution in the L^2-norm, where to prove the result, we need to overcome the difficulty caused by the presence of linear terms from perturbation. Finally, the decay rate in L^2 space for higher order derivatives of the solution is established.

Key words Background magnetic field; MHD equations with perturbation; stability; decay rate

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1 Introduction

The 3D incompressible magnetohydrodynamic equations can be written as

$$\begin{cases}\partial_t U + U\cdot\nabla U = -\nabla P + \mu\Delta U + B\cdot\nabla B,\\ \partial_t B + U\cdot\nabla B = \nu\Delta B + B\cdot\nabla U,\\ \nabla\cdot U = 0, \nabla\cdot B = 0,\\ U(x,0) = U_0(x), B(x,0) = B_0(x),\end{cases} \tag{1.1}$$

where $x\in\mathbb{R}^3$ and $t>0$. $U=U(x,t)$ represents the fluid velocity, $B=B(x,t)$ the magnetic field, and $P=P(x,t)$ the pressure. For simplicity, we set the viscosity coefficient μ and the magnetic field coefficient ν to 1.

The MHD equations (1.1) involve the coupling of the incompressible Navier-Stokes equation and Maxwell's equation. They play an important role in many fields, such as geophysics, astrophysics, cosmology and engineering (see [3, 5, 19]). Because of the physical and engineering applications of MHD equations, the mathematical study of them is very important; as a result, the MHD equations have been extensively studied. For instance, G. Duvaut and J. L. Lions [8] obtained the local existence and uniqueness of a solution in the Sobolev space $H^s(\mathbb{R}^d), s>d$,

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and proved the global existence of the solution for small initial data. M. Sermange and R. Teman [25] further studied the properties of the solutions. Miao, Yuan and Zhang [18] studied the well-posedness of solutions of MHD equations in $\mathrm{BMO}^{-1}(\mathbb{R}^3)$ and $bmo^{-1}(\mathbb{R}^3)$, and proved that the solution of the Cauchy problem of MHD with small initial value is globally unique in $\mathrm{BMO}^{-1}(\mathbb{R}^3)$ and locally unique in $bmo^{-1}(\mathbb{R}^3)$. Cao, Wu and Yuan [4] studied the 2D incompressible MHD with partial dissipation for data in $H^s(\mathbb{R}^2), s > 2$. Recently, the global regularity issue concerning equations (1.1) has attracted much interest, and considerable results have been obtained (see [9, 11, 29]).

For the 3D equations around the equilibrium state (U^0, B^0), it is clear that $U^0(x,t) = (0,0,0)$ and $B^0 = (1,0,0)$ are the special solutions of (1.1). The perturbation (u,b) around this equilibrium with $u \triangleq U - U^0, b \triangleq B - B^0$ obeys

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \Delta u + b \cdot \nabla b + \partial_1 b, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u + \Delta b + \partial_1 u, \\ \nabla \cdot u = 0, \nabla \cdot b = 0, \\ u(x,0) = u_0(x), b(x,0) = b_0(x), \end{cases} \tag{1.2}$$

where $x \in \mathbb{R}^3$, $t > 0$.

The stability problem of equations (1.2) was first raised by H.Alfvén in [2], and now it has aroused people's attention again. The stability problem on MHD equations can be extremely challenging. Recently, in the 2D case, Ji, Lin, Wu and Yan [10] studied stability for the 2D MHD equations with partial dissipation in $H^s(\mathbb{R}^2), s \geq 0$. Ren et al. [20] proved the global existence and decay of smooth solutions for the 2-D MHD equations for general perturbations. For the 3D case, Abidi and Zhang [1] proved the global solution near the equilibrium. Deng and Zhang [6] obtained the large time behavior of solutions near the equilibrium. Wu and Zhu [28] proved global solutions of the 3D incompressible MHD equations with mixed partial dissipation and magnetic diffusion near an equilibrium.

In this paper, inspired by references [23, 30], we study the global solutions and decay estimates to equations (1.2) in $H^s(\mathbb{R}^3)$. For the global solution of equations (1.2), we refer to [16, 27]. For the decay estimation of equations (1.2), we refer to [7, 15, 21, 24]. However, the classical method does not apply here, due to the appearance of linear terms $\partial_1 u$ and $\partial_1 b$ from the perturbation in equations (1.2). We introduce a diagonalization method to eliminate the linear terms, then we prove the temporal decay estimates by the classical Fourier splitting method. Our main results are stated as follows:

First, to prove the global well-posedness of small initial data in Theorem 1.2, we need the local well-posedness result of the strong solution in Proposition 1.1, which can be obtained by a standard procedure with Friedrichs' method. The key estimate is that

$$\begin{aligned} \int_{\mathbb{R}^3} \Lambda^s (u \cdot \nabla u) \cdot \Lambda^s u \mathrm{d}x &= \int_{\mathbb{R}^3} [\Lambda^s, u] \cdot \nabla u \cdot \Lambda^s u \mathrm{d}x \\ &\leq C \|\Lambda^s u\|_{L^2} \|\Lambda^s u\|_{L^6} \|\nabla u\|_{L^3} \\ &\leq C \|\Lambda^s u\|_{L^2} \|\Lambda^{s+1} u\|_{L^2} \|u\|_{H^s} \\ &\leq \delta \|\Lambda^{s+1} u\|_{L^2}^2 + C(\delta) \|\Lambda^s u\|_{L^2}^2 \|u\|_{H^s}^2 \end{aligned}$$

for $s \geq \frac{3}{2}$, where δ is small constant number.

Proposition 1.1 (Local well-posedness of a strong solution) Assume $(u_0, b_0) \in H^s(\mathbb{R}^3)$ with $\mathrm{div}u_0 = \mathrm{div}b_0 = 0$, $s \geq \frac{3}{2}$. Then, there exists a small time $T = T(\|u_0\|_{H^s}, \|b_0\|_{H^s}) > 0$ such that (1.2) has a unique strong solution (u, b) satisfying $(u, b) \in C([0, T]; H^s(\mathbb{R}^3))$.

Theorem 1.2 Letting $s \geq \frac{3}{2}$, assume that $(u_0, b_0) \in H^s(\mathbb{R}^3)$ with $\mathrm{div}u_0 = \mathrm{div}b_0 = 0$, and that the initial data satisfies

$$\|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|b_0\|_{\dot{H}^{\frac{1}{2}}} \leq \epsilon$$

for a small constant $\epsilon > 0$. Then the perturbation MHD equations (1.2) have a unique global solution (u, b) satisfying

$$\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 + \int_0^t \|\nabla u(\tau)\|_{H^s}^2 + \|\nabla b(\tau)\|_{H^s}^2 \mathrm{d}\tau \leq \|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2$$

for all $t > 0$.

Theorem 1.3 If $(u_0, b_0) \in L^1(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$ with $\mathrm{div}u_0 = \mathrm{div}b_0 = 0$, $s \geq \frac{3}{2}$, then the small global solution (u, b) to equations (1.2) constructed in Theorem 1.2 has the following optimal decay rate:

$$\|u(t)\|_{L^2} + \|b(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}. \tag{1.3}$$

The decay rate of the higher order derivative of the solution is also obtained.

Theorem 1.4 Under the assumption of Theorem 1.3, for any integer $m \geq 0$, the small global solution (u, b) satisfies

$$\|\Lambda^m u(t)\|_{L^2} + \|\Lambda^m b(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{m}{2}}, \tag{1.4}$$

where C is a constant which depends on m and the initial data. $\Lambda = \sqrt{-\Delta}$ is defined in the end of this section.

Remark 1.5 The decay rates (1.3) and (1.4) are optimal in the sense that they coincide with the ones of the heat equation.

Remark 1.6 For the real number $s > 0$, we can also obtain the time decay rate of the L^2-norm for the s-order derivative of the solution by the interpolation relation

$$\|\Lambda^s f\|_{L^2} \leq C\|\Lambda^m f\|_{L^2}^{m+1-s}\|\Lambda^{m+1} f\|_{L^2}^{s-m}$$

with $m < s < m+1$, which is

$$\|\Lambda^s u(t)\|_{L^2} + \|\Lambda^s b(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{s}{2}}. \tag{1.5}$$

The rest of this paper is divided into four parts. In the second section, we shall collect some lemmas and prove Lemma 2.3 by a diagonalization process. In the third section, we give the proof of Theorem 1.1 by a priori estimates. In the fourth section, we give the proof of Theorem 1.2 by a classical Fourier splitting method, which was obtained first by Schonbek [22]. In the fifth section, we will show the proof of Theorem 1.3 by making use of an induction argument. Noting here that C denotes a positive constant that may change from line to line, $\mathcal{F}f$ denotes the Fourier transform of a function f and the fractional Laplacian operator $\Lambda = \sqrt{-\Delta}$ is defined through a Fourier transform, namely $\widehat{\Lambda f}(\xi) = |\xi|\hat{f}(\xi)$.

2 Preliminaries

The primary purpose of this section is to give three Lemmas; the first one is the product type estimate, the second and the third are mainly used for the decay estimate of a solution. The detailed processes are as follows:

Lemma 2.1 (Product estimate [13, 14, 17]) Let $1 < p < \infty$, and $s > 0$. Then there exists a constant C such that

$$\|\Lambda^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}) \tag{2.1}$$

for $f \in L^{p_1} \cap \dot{W}^{s,p_3}$ and $g \in \dot{W}^{s,p_2} \cap L^{p_4}$, with $p_2, p_3 \in (1, +\infty)$ and $p_1, p_4 \in [1, +\infty]$ satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

We will use the following L^2 estimate of the Fourier transform of the initial datum in a ball (the proof of Lemma 2.2 is based on the Hausdorff-Young theorem; for more details please refer to [12, 22, 30]):

Lemma 2.2 Let $u_0 \in L^p(\mathbb{R}^3)$, $1 \le p < 2$. Then

$$\int_{S(t)} |\mathcal{F}u_0|^2 \mathrm{d}\xi \le C(t+1)^{-\frac{3}{2}(\frac{2}{p}-1)}, \tag{2.2}$$

where $S(t) = \{\xi \in \mathbb{R}^3 :| \xi |\le g(t)\}$ is a ball with

$$g(t) = \Big(\frac{\gamma}{t+1}\Big)^{\frac{1}{2}}.$$

Here, $\gamma > 0$ is a constant which will be determined later, and C is a constant which depends upon γ and the L^p norm of u_0.

In order to prove Theorem 1.2, we need to calculate the estimates of $|\widehat{u}(\xi)|$ and $|\widehat{b}(\xi)|$, which will play a key role in this paper. For more details, readers can refer to [7, 26].

Lemma 2.3 Letting $(u, b) \in C([0, T]; H^s(\mathbb{R}^3))$ be a global solution to the Cauchy problem (1.2) with initial data $(u_0, b_0) \in L^1(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$, there exists a constant $C > 0$ depending only on $\|u_0\|_{L^2}$ and $\|b_0\|_{L^2}$ such that

$$|\widehat{u}(\xi,t)| \le C\left(\left(|\widehat{u_0}(\xi)| + |\widehat{b_0}(\xi)|\right) + \frac{1}{|\xi|}\right),$$

$$|\widehat{b}(\xi,t)| \le C\left(\left(|\widehat{u_0}(\xi)| + |\widehat{b_0}(\xi)|\right) + \frac{1}{|\xi|}\right).$$

Proof We rewrite (1.2) in the following form:

$$\begin{cases} \partial_t u - \Delta u - \partial_1 b = G, \\ \partial_t b - \Delta b - \partial_1 u = F. \end{cases} \tag{2.3}$$

Here $G := -u \cdot \nabla u + b \cdot \nabla b - \nabla p$, $F := -u \cdot \nabla b + b \cdot \nabla u$. We take the Fourier transform for (2.3) as

$$\partial_t \begin{pmatrix} \widehat{u}_i \\ \widehat{b}_i \end{pmatrix}(\xi) = A \begin{pmatrix} \widehat{u}_i \\ \widehat{b}_i \end{pmatrix}(\xi) + \begin{pmatrix} \widehat{G}_i \\ \widehat{F}_i \end{pmatrix}(\xi), \quad i = 1, 2, 3, \tag{2.4}$$

where

$$A=\begin{pmatrix}-|\xi|^2 & \mathrm{i}\xi_1\\ \mathrm{i}\xi_1 & -|\xi|^2\end{pmatrix}.$$

Then the eigenvalues of matrix A can be calculated as follows:

$$\lambda_1=-|\xi|^2+\mathrm{i}|\xi_1|,\quad \lambda_2=-|\xi|^2-\mathrm{i}|\xi_1|.$$

The associated eigenvectors are

$$f=\begin{pmatrix}\mathrm{i}\xi_1\\ \mathrm{i}|\xi_1|\end{pmatrix},\quad g=\begin{pmatrix}\mathrm{i}\xi_1\\ -\mathrm{i}|\xi_1|\end{pmatrix}.$$

The matrix C of the eigenvectors and its inverse are given by

$$C\triangleq\begin{pmatrix}f & g\end{pmatrix}=\begin{pmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}=\begin{pmatrix}\mathrm{i}\xi_1 & \mathrm{i}\xi_1\\ \mathrm{i}|\xi_1| & -\mathrm{i}|\xi_1|\end{pmatrix}$$

and

$$C^{-1}\triangleq\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}=\begin{pmatrix}\dfrac{1}{2\mathrm{i}\xi_1} & \dfrac{1}{2\mathrm{i}\mid\xi_1\mid}\\ \dfrac{1}{2\mathrm{i}\xi_1} & -\dfrac{1}{2\mathrm{i}\mid\xi_1\mid}\end{pmatrix}.$$

If we define

$$\begin{pmatrix}\widehat{U_i}\\ \widehat{B_i}\end{pmatrix}\triangleq C^{-1}\begin{pmatrix}\widehat{u_i}\\ \widehat{b_i}\end{pmatrix}, \tag{2.5}$$

then $\widehat{U_i}$ and $\widehat{B_i}$ satisfy

$$\partial_t\begin{pmatrix}\widehat{U_i}\\ \widehat{B_i}\end{pmatrix}(\xi)=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\widehat{U_i}\\ \widehat{B_i}\end{pmatrix}(\xi)+C^{-1}\begin{pmatrix}\widehat{G_i}\\ \widehat{F_i}\end{pmatrix}(\xi),\quad i=1,2,3.$$

Integrating in time, by Duhamel's formula, we have

$$\widehat{U_i}(\xi,t)=e^{\lambda_1 t}\widehat{U_i}(\xi,0)+\int_0^t e^{\lambda_1(t-\tau)}\big(c_{11}\widehat{G_i}(\xi,\tau)+c_{12}\widehat{F}_i(\xi,\tau)\big)\mathrm{d}\tau, \tag{2.6}$$

$$\widehat{B_i}(\xi,t)=e^{\lambda_2 t}\widehat{B_i}(\xi,0)+\int_0^t e^{\lambda_2(t-\tau)}\big(c_{21}\widehat{G_i}(\xi,\tau)+c_{22}\widehat{F}_i(\xi,\tau)\big)\mathrm{d}\tau \tag{2.7}$$

for $i=1,2,3$, and, according to (2.5), we have

$$\begin{pmatrix}\widehat{U_i}\\ \widehat{B_i}\end{pmatrix}(\xi,0)=\begin{pmatrix}c_{11}\ c_{12}\\ c_{21}\ c_{22}\end{pmatrix}\begin{pmatrix}\widehat{u_{0i}}\\ \widehat{b_{0i}}\end{pmatrix}(\xi)=\begin{pmatrix}c_{11}\widehat{u_{0i}}+c_{12}\widehat{b_{0i}}\\ c_{21}\widehat{u_{0i}}+c_{22}\widehat{b_{0i}}\end{pmatrix}(\xi),$$

and

$$\begin{aligned}\begin{pmatrix}\widehat{u_i}\\ \widehat{b_i}\end{pmatrix}&=\begin{pmatrix}C_{11}\ C_{12}\\ C_{21}\ C_{22}\end{pmatrix}\begin{pmatrix}\widehat{U_i}\\ \widehat{B_i}\end{pmatrix}\\ &=\begin{pmatrix}C_{11}\widehat{U_i}+C_{12}\widehat{B_i}\\ C_{21}\widehat{U_i}+C_{22}\widehat{B_i}\end{pmatrix}.\end{aligned} \tag{2.8}$$

We can get from (2.6)–(2.8) that

$$\begin{aligned}\widehat{u_i}(\xi,t)=&\widehat{M_{11}}(\xi,t)\widehat{u_{0i}}(\xi)+\widehat{M_{12}}(\xi,t)\widehat{b_{0i}}(\xi)\\&+\int_0^t\widehat{M_{11}}(\xi,t-\tau)\widehat{G_i}\mathrm{d}\tau+\int_0^t\widehat{M_{12}}(\xi,t-\tau)\widehat{F_i}\mathrm{d}\tau,\end{aligned}\tag{2.9}$$

$$\begin{aligned}\widehat{b_i}(\xi,t)=&\widehat{M_{21}}(\xi,t)\widehat{u_{0i}}(\xi)+\widehat{M_{22}}(\xi,t)\widehat{b_{0i}}(\xi)\\&+\int_0^t\widehat{M_{21}}(\xi,t-\tau)\widehat{G_i}\mathrm{d}\tau+\int_0^t\widehat{M_{22}}(\xi,t-\tau)\widehat{F_i}\mathrm{d}\tau,\end{aligned}\tag{2.10}$$

where

$$M=\begin{pmatrix}\widehat{M_{11}}(\xi,t)&\widehat{M_{12}}(\xi,t)\\\widehat{M_{21}}(\xi,t)&\widehat{M_{22}}(\xi,t)\end{pmatrix}=\begin{pmatrix}\dfrac{e^{\lambda_1t}+e^{\lambda_2t}}{2}&\mathrm{sgn}(\xi_1)\dfrac{e^{\lambda_1t}-e^{\lambda_2t}}{2}\\\mathrm{sgn}(\xi_1)\dfrac{e^{\lambda_1t}-e^{\lambda_2t}}{2}&\dfrac{e^{\lambda_1t}+e^{\lambda_2t}}{2}\end{pmatrix},$$

and

$$\mathrm{sgn}(\xi_1)=\begin{cases}1&\xi_1>0\\-1&\xi_1<0\end{cases}.$$

It is clear that

$$|\widehat{M_{ij}}(\xi,t)|\le Ce^{-|\xi|^2t},i,j=1,2.\tag{2.11}$$

Taking the divergence of $(1.2)_1$, and using the divergence free condition of u and b, we have

$$-\Delta p=\nabla\cdot(-u\cdot\nabla u+b\cdot\nabla b)=-\nabla\cdot\mathrm{div}(u\otimes u)+-\nabla\cdot\mathrm{div}(b\otimes b).$$

Since the Fourier transform is a bounded map from L^1 to L^∞, this leads to

$$\begin{aligned}|\widehat{\nabla p}(\xi,t)|\le&|\xi||\widehat{p}(\xi,t)|\le|\xi|(|\widehat{u\otimes u}|+|\widehat{b\otimes b}|)\\\le&|\xi|(\|u(t)u(t)\|_{L^1}+\|b(t)b(t)\|_{L^1})\\\le&C|\xi|(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2).\end{aligned}\tag{2.12}$$

Similarly, by the divergence free condition of u and b, we can obtain that

$$|\widehat{b\cdot\nabla b}|\le|\xi||\widehat{b\otimes b}|\le|\xi|\|b\|_{L^2}^2,\tag{2.13}$$

$$|\widehat{u\cdot\nabla u}|\le|\xi||\widehat{u\otimes u}|\le|\xi|\|u\|_{L^2}^2,\tag{2.14}$$

$$|\widehat{b\cdot\nabla u}|\le|\xi||\widehat{b\otimes u}|\le|\xi|\|b\otimes u\|_{L^1}\le C|\xi|(\|u\|_{L^2}^2+\|b\|_{L^2}^2),\tag{2.15}$$

and

$$|\widehat{u\cdot\nabla b}|\le C|\xi|(\|u\|_{L^2}^2+\|b\|_{L^2}^2).\tag{2.16}$$

Combining the estimates (2.12)–(2.16), by the expression formula of G and F, we get

$$|\widehat{G}(\xi,t)|\le C|\xi|(\|u\|_{L^2}^2+\|b\|_{L^2}^2),\tag{2.17}$$

and

$$|\widehat{F}(\xi,t)|\le C|\xi|(\|u\|_{L^2}^2+\|b\|_{L^2}^2).\tag{2.18}$$

Inserting (2.17), (2.18) and (2.11) into (2.9), we deduce that

$$|\widehat{u}|(t,x)\le e^{-|\xi|^2t}(|\widehat{u_0}|+|\widehat{b_0}|)+C\int_0^te^{-|\xi|^2(t-\tau)}|\xi|(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)\mathrm{d}\tau$$

$$\leq C(|\widehat{u_0}| + |\widehat{b_0}|) + \sup_{0\leq\tau\leq t}(\|u(\tau)\|_{L^2}^2 + \|b(\tau)\|_{L^2}^2)\int_0^t e^{-|\xi|^2(t-\tau)}|\xi|\mathrm{d}\tau$$
$$\leq C(|\widehat{u_0}| + |\widehat{b_0}| + \frac{1}{|\xi|}). \tag{2.19}$$

Using an argument similar to (2.10), we have

$$|\widehat{b}|(t,x) \leq C(|\widehat{u_0}| + |\widehat{b_0}| + \frac{1}{|\xi|}).$$

The proof of Lemma 2.3 is thus completed. □

3 Proof of Theorem 1.2

This section aims to prove Theorem 1.2. We will show that for any given $T > 0$, $\|(u,b)\|_{H^s}$ is uniformly bounded over $(0,T)$. Therefore, in order to achieve this goal, we first give the basic L^2-estimate, then establish the H^s-estimate. For further details, readers may refer to Yuan and Liu [30] and Schonbek, Schonbek and Süli [24]. The proof of Theorem 1.1 is split into two subsections: the L^2 estimate and the H^s estimate.

3.1 L^2 estimate

Taking the L^2 inner products of the equations $(1.2)_{1,2}$ with u and b, and adding the results and integrating by parts, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 = 0, \tag{3.1}$$

where we have used the fact that

$$\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot u\mathrm{d}x = 0,\ \int_{\mathbb{R}^3}(u\cdot\nabla)b\cdot b\mathrm{d}x = 0,$$
$$\int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot u\mathrm{d}x + \int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot b\mathrm{d}x = 0 \quad\text{and}\quad \int_{\mathbb{R}^3}\partial_1 b\cdot u + \partial_1 u\cdot b\mathrm{d}x = 0.$$

3.2 H^s estimate

Applying Λ^s to the equations of $(1.2)_{1,2}$, dotting the resulting equations with $\Lambda^s u$ and $\Lambda^s b$, and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2) + \|\Lambda^{s+1}u\|_{L^2}^2 + \|\Lambda^{s+1}b\|_{L^2}^2\\
\leq\ & -\int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\mathrm{d}x + \int_{\mathbb{R}^3}\Lambda^s(b\cdot\nabla b)\cdot\Lambda^s u\mathrm{d}x + \int_{\mathbb{R}^3}\Lambda^s\partial_1 b\cdot\Lambda^s u\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla b)\cdot\Lambda^s b\mathrm{d}x + \int_{\mathbb{R}^3}\Lambda^s(b\cdot\nabla u)\cdot\Lambda^s b\mathrm{d}x + \int_{\mathbb{R}^3}\Lambda^s\partial_1 u\cdot\Lambda^s b\mathrm{d}x\\
:=\ & \sum_{i=1}^6 I_i.
\end{aligned}\tag{3.2}$$

Now, we estimate I_1–I_6. For the term I_1, by the divergence free condition of u, integration by parts, Lemma 2.1 and $\|f\|_{L^3} \leq C\|f\|_{\dot{H}^{\frac{1}{2}}}$, we obtain

$$\begin{aligned}
I_1 &= -\int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\mathrm{d}x\\
&= \int_{\mathbb{R}^3}\Lambda^s(u\otimes u)\cdot\Lambda^{s+1}u\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\le (\|\Lambda^s u\|_{L^6}\|u\|_{L^3} + \|u\|_{L^3}\|\Lambda^s u\|_{L^6})\,\|\Lambda^{s+1}u\|_{L^2}\\
&\le C\|u\|_{L^3}\|\Lambda^s u\|_{L^6}\|\Lambda^{s+1}u\|_{L^2}\\
&\le C\|u\|_{\dot H^{\frac12}}\|\Lambda^{s+1}u\|_{L^2}^2. \qquad (3.3)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &= -\int_{\mathbb{R}^3}\Lambda^s(b\cdot\nabla b)\cdot\Lambda^s u\mathrm{d}x\\
&\le (\|b\|_{L^3}\|\Lambda^s b\|_{L^6} + \|\Lambda^s b\|_{L^6}\|b\|_{L^3})\|\Lambda^{s+1}u\|_{L^2}\\
&\le C\|b\|_{\dot H^{\frac12}}\|\Lambda^{s+1}b\|_{L^2}\|\Lambda^{s+1}u\|_{L^2}\\
&\le C\|b\|_{\dot H^{\frac12}}(\|\Lambda^{s+1}b\|_{L^2}^2 + \|\Lambda^{s+1}u\|_{L^2}^2). \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
|I_4+I_5| &= \left|-\int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla b)\cdot\Lambda^s b\mathrm{d}x + \int_{\mathbb{R}^3}\Lambda^s(b\cdot\nabla u)\cdot\Lambda^s b\mathrm{d}x\right|\\
&= \left|\int_{\mathbb{R}^3}\Lambda^s(u\otimes b)\cdot\Lambda^{s+1}b\mathrm{d}x - \int_{\mathbb{R}^3}\Lambda^s(b\otimes u)\cdot\Lambda^{s+1}b\mathrm{d}x\right|\\
&\le C(\|u\|_{L^3}\|\Lambda^s b\|_{L^6} + \|\Lambda^s u\|_{L^6}\|b\|_{L^3})\|\Lambda^{s+1}b\|_{L^2}\\
&\le C\|u\|_{L^3}\|\Lambda^{s+1}b\|_{L^2}^2 + C\|b\|_{L^3}\|\Lambda^{s+1}u\|_{L^2}\|\Lambda^{s+1}b\|_{L^2}\\
&\le C(\|u\|_{\dot H^{\frac12}} + \|b\|_{\dot H^{\frac12}})(\|\Lambda^{s+1}b\|_{L^2}^2 + \|\Lambda^{s+1}u\|_{L^2}^2). \qquad (3.5)
\end{aligned}
$$

For the terms I_3 and I_6, integrating by parts, we get

$$
I_3 + I_6 = \int_{\mathbb{R}^3}\Lambda^s\partial_1 b\cdot\Lambda^s u\mathrm{d}x + \int_{\mathbb{R}^3}\Lambda^s\partial_1 u\cdot\Lambda^s b\mathrm{d}x = 0. \qquad (3.6)
$$

Inserting estimates (3.3)–(3.5) into (3.2), we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2) + (2 - C(\|u\|_{\dot H^{\frac12}} + \|b\|_{\dot H^{\frac12}}))(\|\Lambda^{s+1}u\|_{L^2}^2 + \|\Lambda^{s+1}b\|_{L^2}^2) \le 0. \qquad (3.7)
$$

By Proposition 1.1 , we know that if $\|u_0\|_{\dot H^{\frac12}} + \|b_0\|_{\dot H^{\frac12}} < \frac{2}{C} < \epsilon$, then for a small time interval $(0,t_1]$, $\|u\|_{\dot H^{\frac12}} + \|b\|_{\dot H^{\frac12}} < \frac{1}{C}$, so one has

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2) + \|\Lambda^{s+1}u\|_{L^2}^2 + \|\Lambda^{s+1}b\|_{L^2}^2 \le 0. \qquad (3.8)
$$

Combining this with estimate (3.1), we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^s}^2 + \|b\|_{H^s}^2) + \|\nabla u\|_{H^s}^2 + \|\nabla b\|_{H^s}^2 \le 0 \qquad (3.9)
$$

or

$$
\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 + \int_0^{t_1}\|\nabla u(t)\|_{H^s}^2 + \|\nabla b(t)\|_{H^s}^2\mathrm{d}t \le \|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2 \qquad (3.10)
$$

for $0 < t \le t_1$. In estimate (3.8), choosing $s = \frac12$ and integrating the resulting estimate, it follows that

$$
\|u(t_1)\|_{\dot H^{\frac12}}^2 + \|b(t_1)\|_{\dot H^{\frac12}}^2 \le \|u_0\|_{\dot H^{\frac12}}^2 + \|b_0\|_{\dot H^{\frac12}}^2. \qquad (3.11)
$$

Thus, by a continuous extending argument, we get

$$
\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 + \int_0^t\|\nabla u(\tau)\|_{H^s}^2 + \|\nabla b(\tau)\|_{H^s}^2\mathrm{d}\tau \le \|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2 \qquad (3.12)
$$

for all $0 < t < \infty$. This completes the proof of Theorem 1.1. □

4 Proof of Theorem 1.3

In this section, we prove the decay rate of the global solution in $L^2(\mathbb{R}^3)$ by the classic Fourier splitting method.

Proof Applying Plancherel's theorem to (3.1), and by splitting the phase space $\mathbb{R}^3$ into two time-dependent parts, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\widehat{u}(\xi)|^2+|\widehat{b}(\xi)|^2\mathrm{d}\xi\\
&=-\int_{\mathbb{R}^3}|\xi|^2(|\widehat{u}(\xi)|^2+|\widehat{b}(\xi)|^2)\mathrm{d}\xi\\
&=-\int_{S(t)}|\xi|^2(|\widehat{u}(\xi)|^2+|\widehat{b}(\xi)|^2)\mathrm{d}\xi-\int_{S(t)^c}|\xi|^2(|\widehat{u}(\xi)|^2+|\widehat{b}(\xi)|^2)\mathrm{d}\xi\\
&\le-\int_{S(t)}|\xi|^2(|\widehat{u}(\xi)|^2+|\widehat{b}(\xi)|^2)\mathrm{d}\xi-\int_{S(t)^c}g(t)^2(|\widehat{u}(\xi)|^2+|\widehat{b}(\xi)|^2)\mathrm{d}\xi\\
&\le-\int_{\mathbb{R}^3}g(t)^2(|\widehat{u}(\xi)|^2+|\widehat{b}(\xi)|^2)\mathrm{d}\xi+\int_{S(t)}g(t)^2(|\widehat{u}(\xi)|^2+|\widehat{b}(\xi)|^2)\mathrm{d}\xi,
\end{aligned}\tag{4.1}$$

where $S(t)$ and $g(t)$ are defined in Lemma 2.2, and $\gamma>0$ is a constant which will be determined later. Thus, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2\right)+g(t)^2\left(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2\right)\\
&\le g(t)^2\int_{S(t)}|\widehat{u}(\xi,t)|^2+|\widehat{b}(\xi,t)|^2\mathrm{d}\xi.
\end{aligned}\tag{4.2}$$

Multiplying (4.2) by the integrating factor $(t+1)^\gamma$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}((1+t)^\gamma(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2))\le\gamma(1+t)^{\gamma-1}\int_{|\xi|\le g(t)}|\widehat{u}(\xi,t)|^2+|\widehat{b}(\xi,t)|^2\mathrm{d}\xi.\tag{4.3}$$

By Lemma 2.3, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}((1+t)^\gamma(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2))&\le\gamma(1+t)^{\gamma-1}\int_{|\xi|\le g(t)}\left(|\widehat{u_0}(\xi)|^2+|\widehat{b_0}(\xi)|^2+\frac{1}{|\xi|^2}\right)\mathrm{d}\xi\\
&\le C(t+1)^{\gamma-1-\frac{3}{2}}+C(t+1)^{\gamma-1-\frac{1}{2}}.
\end{aligned}\tag{4.4}$$

Integrating (4.4) in time from 0 to t leads to the result

$$\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2\le C(t+1)^{-\frac{3}{2}}+C(t+1)^{-\frac{1}{2}}+C(1+t)^{-\gamma}.\tag{4.5}$$

By choosing $\gamma>\frac{1}{2}$, we have

$$\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2\le C(t+1)^{-\frac{1}{2}}.\tag{4.6}$$

Inserting estimate (4.6) into (2.19), we can obtain that in the ball $S(t)$,

$$\begin{aligned}
|\widehat{u}(t,\xi)|&\le e^{-|\xi|^2t}(|\widehat{u_0}(\xi)|+|\widehat{b_0}(\xi)|)+C\int_0^t e^{-|\xi|^2(t-\tau)}|\xi|(\tau+1)^{-\frac{1}{2}}\mathrm{d}\tau\\
&\le C(|\widehat{u_0}(\xi)|+|\widehat{b_0}(\xi)|)+C|\xi|((t+1)^{\frac{1}{2}}-1)\\
&\le C(|\widehat{u_0}(\xi)|+|\widehat{b_0}(\xi)|+1).
\end{aligned}\tag{4.7}$$

Similarly, we obtain

$$|\widehat{b}(t,\xi)|\le C(|\widehat{u_0}(\xi)|+|\widehat{b_0}(\xi)|+1).\tag{4.8}$$

Inserting (4.7) and (4.8) into (4.3), and by Lemma 2.2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((1+t)^{\gamma}(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)\right)\leq C\gamma(1+t)^{\gamma-1}(t+1)^{-\frac{3}{2}}. \tag{4.9}$$

Integrating (4.9) in time and choosing $\gamma>\frac{3}{2}$ leads to

$$\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2\leq C(1+t)^{-\frac{3}{2}}.$$

This completes the proof of Theorem 1.2. □

5 Proof of Theorem 1.4

In this section, we will prove the higher order derivative's decay estimate of the small global solution to the equations of (1.2) in $L^2(\mathbb{R}^3)$ space.

Proof We use the Fourier splitting method again. Let

$$S(t)=\{\xi\in\mathbb{R}^3\big||\xi|\leq f(t)\},\ f(t)=(\frac{\gamma}{1+t})^{\frac{1}{2}}. \tag{5.1}$$

Then Plancherel's theorem and (5.1) imply that

$$\begin{aligned}
&\|\Lambda^{m+1}u(t)\|_{L^2}^2+\|\Lambda^{m+1}b(t)\|_{L^2}^2\\
=&\int_{\mathbb{R}^3}|\xi|^2(|\mathcal{F}\Lambda^m u(\xi,t)|^2+|\mathcal{F}\Lambda^m b(\xi,t)|^2)\mathrm{d}\xi\\
\geq&\int_{|\xi|\geq f(t)}|\xi|^2(|\mathcal{F}\Lambda^m u(\xi,t)|^2+|\mathcal{F}\Lambda^m b(\xi,t)|^2)\mathrm{d}\xi\\
\geq& f(t)^2\Big(\|\Lambda^m u\|_{L^2}^2+\|\Lambda^m b\|_{L^2}^2\Big)-f(t)^2\int_{|\xi|\leq f(t)}|\xi|^2\Big(|\mathcal{F}\Lambda^{m-1}u(\xi,t)|^2+|\mathcal{F}\Lambda^{m-1}b(\xi,t)|^2\Big)\mathrm{d}\xi\\
\geq& f(t)^2(\|\Lambda^m u\|_{L^2}^2+\|\Lambda^m b\|_{L^2}^2)-f(t)^4\int_{|\xi|\leq f(t)}|\mathcal{F}\Lambda^{m-1}u(\xi,t)|^2+|\mathcal{F}\Lambda^{m-1}b(\xi,t)|^2\mathrm{d}\xi\\
\geq&(\frac{\gamma}{1+t})(\|\Lambda^m u\|_{L^2}^2+\|\Lambda^m b\|_{L^2}^2)-(\frac{\gamma}{1+t})^2(\|\Lambda^{m-1}u\|_{L^2}^2+\|\Lambda^{m-1}b\|_{L^2}^2),
\end{aligned} \tag{5.2}$$

where $m\geq 1$ is an integer. Inserting estimate (5.2) into (3.8), it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^m u\|_{L^2}^2+\|\Lambda^m b\|_{L^2}^2)+(\frac{\gamma}{1+t})(\|\Lambda^m u\|_{L^2}^2+\|\Lambda^m b\|_{L^2}^2)\\
\leq&(\frac{\gamma}{1+t})^2(\|\Lambda^{m-1}u\|_{L^2}^2+\|\Lambda^{m-1}b\|_{L^2}^2).
\end{aligned} \tag{5.3}$$

By induction, when $m=1$, multiplying both sides of inequality (5.3) by the integrating factor $(t+1)^{\gamma}$ yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}((t+1)^{\gamma}(\|\Lambda u\|_{L^2}^2+\|\Lambda b\|_{L^2}^2))&\leq C(1+t)^{\gamma-2}(\|u\|_{L^2}^2+\|b\|_{L^2}^2)\\
&\leq C(1+t)^{\gamma-2-\frac{3}{2}}.
\end{aligned} \tag{5.4}$$

Integrating inequality (5.4) from 0 to t, we have

$$(t+1)^{\gamma}(\|\Lambda u\|_{L^2}^2+\|\Lambda b\|_{L^2}^2)\leq C(\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2)+C(1+t)^{\gamma-1-\frac{3}{2}}. \tag{5.5}$$

Thus, choosing $\gamma>\frac{5}{2}$, we can obtain

$$\|\Lambda u\|_{L^2}^2+\|\Lambda b\|_{L^2}^2\leq C(1+t)^{-\frac{3}{2}-1}. \tag{5.6}$$

Assuming that for $m > 1$,

$$\|\Lambda^{m-1}u\|_{L^2}^2 + \|\Lambda^{m-1}b\|_{L^2}^2 \le C(1+t)^{-\frac{3}{2}-(m-1)}. \tag{5.7}$$

Therefore, inserting the above estimate into (5.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m b\|_{L^2}^2) + \frac{\gamma}{1+t}(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m b\|_{L^2}^2) \le C(\frac{\gamma}{1+t})^2(1+t)^{-\frac{3}{2}-(m-1)}. \tag{5.8}$$

Multiplying both sides of (5.8) by $(1+t)^\gamma$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[(1+t)^\gamma(\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m b\|_{L^2}^2)] \le (1+t)^{\gamma-\frac{3}{2}-(m-1)-2}. \tag{5.9}$$

Integrating (5.9) on $[0,t]$, we get

$$(1+t)^\gamma\left(\|\Lambda^m u(t)\|_{L^2}^2 + \|\Lambda^m b(t)\|_{L^2}^2\right) \le \|\Lambda^m u_0\|_{L^2}^2 + \|\Lambda^m b_0\|_{L^2}^2 + (1+t)^{\gamma-\frac{3}{2}-(m-1)-1}. \tag{5.10}$$

Similarly, by choosing $\gamma > \frac{3}{2} + m$, we obtain

$$\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m b\|_{L^2}^2 \le (1+t)^{-\frac{3}{2}-m}. \tag{5.11}$$

Therefore, the proof of Theorem 1.3 is completed. □

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