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ON SCHWARZ-PICK TYPE INEQUALITY FOR MAPPINGS SATISFYING POISSON DIFFERENTIAL INEQUALITY*

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Abstract Let f be a twice continuously differentiable self-mapping of a unit disk satisfying Poisson differential inequality $|\Delta f(z)| \le B \cdot |Df(z)|^2$ for some $B > 0$ and $f(0) = 0$. In this note, we show that f does not always satisfy the Schwarz-Pick type inequality

$$\frac{1-|z|^2}{1-|f(z)|^2} \le C(B),$$

where $C(B)$ is a constant depending only on B. Moreover, a more general Schwarz-Pick type inequality for mapping that satisfies general Poisson differential inequality is established under certain conditions.

Key words Schwarz-Pick inequality; Poisson differential inequality; hyperbolically Lipschitz continuity

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1 Introduction and Main Results

Let $\mathbb{D}$ be an open unit disk in the complex plane $\mathbb{C}$ and denote $\mathbb{T} = \partial\mathbb{D}$. For a domain $\Omega \subset \mathbb{C}$, let $\mathcal{C}^n(\Omega)$ be the set of all complex-valued n-times of continuously differentiable functions from Ω into $\mathbb{C}$. In particular, let $\mathcal{C}(\Omega) := \mathcal{C}^0(\Omega)$ be the set of all continuous functions in Ω.

A real-valued function u, defined in an open subset Ω of the complex plane $\mathbb{C}$, is real harmonic if it is twice continuously differentiable in Ω and satisfies Laplace's equation

$$\Delta u(z) = \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = 0, \quad z \in \Omega.$$

A complex-valued function $\omega = u + \mathrm{i}v$ is harmonic if both u and v are real harmonic. We refer the readers to [7] for more properties of harmonic mappings in the plane.

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Let P be the Poisson kernel, that is, the function

$$P(z,e^{\mathrm{i}\theta})=\frac{1-|z|^2}{|z-e^{\mathrm{i}\theta}|^2},\quad z\in\mathbb{D},\theta\in\mathbb{R},$$

and let G be the Green function of the unit disk, that is, the function

$$G(z,w)=\frac{1}{2\pi}\log\left|\frac{1-z\overline{w}}{z-w}\right|,\quad z,w\in\mathbb{D},z\neq w.$$

Let $f:\mathbb{T}\to\mathbb{C}$ be a bounded integrable function on the unit circle $\mathbb{T}$ and let $g:\mathbb{D}\to\mathbb{C}$ be continuous. The solution of Poisson's equation $\Delta\omega=g$ in $\mathbb{D}$ satisfying the boundary condition $\omega|_{\mathbb{T}}=f\in L^1(\mathbb{T})$ is given by

$$\omega(z)=P[f](z)-G[g](z),\quad z\in\mathbb{D}, \tag{1.1}$$

where

$$P[f](z)=\frac{1}{2\pi}\int_0^{2\pi}P(z,e^{\mathrm{i}\varphi})f(e^{\mathrm{i}\varphi})\mathrm{d}\varphi, G[g](z)=\int_{\mathbb{D}}G(z,w)g(w)\mathrm{d}m(w), \tag{1.2}$$

where $\mathrm{d}m$ denotes the Lebesgue measure in the plane. It was proved in [5] that for any $g\in\mathcal{C}(\mathbb{D})$, the function $G:=G[g]$ satisfies the inequality

$$|G(z)|\leq\frac{||g||_\infty}{4}\cdot(1-|z|^2), \tag{1.3}$$

where $||g||_\infty:=\sup\limits_{z\in\mathbb{D}}|g(z)|$. Suppose that $\Omega\subset\mathbb{C}$ is a hyperbolic type domain with a hyperbolic metric $\lambda_\Omega(z)|\mathrm{d}z|$. The hyperbolic distance between two points $z_1,z_2\in\Omega$ is defined by

$$d_{h_\Omega}(z_1,z_2):=\inf_\gamma\Big\{\int_\gamma\lambda_\Omega(z)|\mathrm{d}z|\Big\},$$

where the infimum is taken over all rectifiable curves γ in Ω connected z_1 and z_2. It is well known that when $D=\mathbb{D}$,

$$\lambda_{\mathbb{D}}(z)=\frac{2}{1-|z|^2}\ \text{ and }\ d_{h_{\mathbb{D}}}(z_1,z_2)=\log\frac{|1-z_1\overline{z_2}|+|z_1-z_2|}{|1-z_1\overline{z_2}|-|z_1-z_2|}.$$

A function f from a hyperbolic type domain Ω into a hyperbolic type domain Ω' is said to be hyperbolically Lipschitz continuous if there exists a constant $L>0$ such that the inequality

$$d_{h_{\Omega'}}(f(z_1),f(z_2))\leq L\cdot d_{h_\Omega}(z_1,z_2)$$

holds for any $z_1,z_2\in\Omega$.

Suppose that Ω and Ω' are two simply connected domains of hyperbolic type in $\mathbb{C}$ with hyperbolic metrics $\lambda_\Omega(z)|\mathrm{d}z|$ and $\lambda_{\Omega'}(w)|\mathrm{d}w|$, respectively. The classical Schwarz-Pick lemma states that if f is holomorphic from Ω into Ω', then

$$\frac{\lambda_{\Omega'}\circ f(z)}{\lambda_\Omega(z)}\cdot|f'(z)|\leq 1, z\in\Omega. \tag{1.4}$$

Moreover, equality occurs in (1.4) when f is conformal from Ω onto Ω'. In particular, if $\Omega=\Omega'=\mathbb{D}$, then inequality (1.4) becomes

$$|f'(z)|\cdot\frac{1-|z|^2}{1-|f(z)|^2}\leq 1.$$

The Schwarz-Pick lemma (1.4) has lots of generalizations. For example, Ahlfors [1] extended it to holomorphic mappings from a unit disk into a Riemann surface equipped with a Riemann metric whose Gaussian curvature is less than or equal to -1. In [19], Yau generalized it to holomorphic mappings between a complete Kähler manifold with Ricci curvature bounded from below by a constant and a Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant. Osserman, in [16], obtained a general finite shrinking lemma from a geodesic disk of radius ρ_1 with respect to a metric $d\hat{s}^2$ which is circularly symmetric into a geodesic disk on a surface. We refer readers to the multipoint version [3] and the higher-order derivatives version [6, 14] of the Schwarz-Pick lemma and references therein.

In this paper, we are particularly interested in the Schwarz-Pick type inequality for mappings satisfying Poisson differential inequality (1.9). Let f be a quasiconformal self-mapping of a unit disk satisfying the Poisson differential equation

$$|\Delta f(z)| \le B \cdot |Df(z)|^2, \tag{1.5}$$

where $|Df(z)| = |f_z(z)| + |f_{\overline{z}}(z)|$. In [12], Kalaj obtained the following result:

Lemma 1.1 ([12, Lemma 2.3]) Suppose that f is a K-quasiconformal self-mapping of $\mathbb{D}$ satisfying Poisson differential equation (1.5) and $f(0) = 0$. Then there exists $C(B, K)$ such that

$$\frac{1-|z|^2}{1-|f(z)|^2} \le C(B, K). \tag{1.6}$$

This lemma can be viewed as a kind of Schwarz-Pick type inequality for the K-quasiconformal self-mapping of $\mathbb{D}$ satisfying Poisson differential inequality (1.5). Now, let $\mathcal{F}(\mathbb{D}, B) = \{f : \mathbb{D} \to \mathbb{D} : f(0) = 0, |\Delta f| \le B \cdot |Df|^2, f \in \mathcal{C}^2\}$. We can derive, by using the Schwarz lemma for harmonic mapping [9], the Schwarz-Pick type inequality

$$\frac{1-|z|^2}{1-|f(z)|^2} \le \frac{\pi}{2} \tag{1.7}$$

for the class $\mathcal{F}(\mathbb{D}, 0)$. In [12], Kalaj asked if the quasiconformality assumption is important for Lemma 1.1. In other words, does (1.7) hold for some constant $C = C(B)$ instead of $\pi/2$ for the class $\mathcal{F}(\mathbb{D}, B)$? We summarize this as follows:

Question 1.2 Is the Schwarz-Pick type inequality

$$\frac{1-|z|^2}{1-|f(z)|^2} \le C(B) \tag{1.8}$$

always valid for mappings in the class $\mathcal{F}(\mathbb{D}, B)$?

The first aim of this paper is to give a negative answer to the question. We have

Theorem 1.3 The mappings in the class $\mathcal{F}(\mathbb{D}, B)$ do not always enjoy Schwarz-Pick type inequality (1.8).

Although the answer to Question 1.2 is negative, one can establish a Schwarz-Pick type inequality (1.8) for twice continuously differentiable self-mappings of a unit disk satisfying Poisson differential inequality (1.8) under certain conditions. For example, in [15], a Schwarz-Pick type inequality (1.8) for (K, K')–quasiconformal self-mappings of a unit disk satisfying the Poisson differential inequality (1.8) was obtained. However, those mappings discussed in Lemma 1.1 and [15, Lemma 2.1] require quasiconformality. Here, we establish one kind of inequality (1.8) for those mappings satisfying (1.5) under certain conditions (with no requirement of quasiconformality). Actually, our result also works with twice continuously differentiable self-mappings of a unit disk satisfying the following more general Poisson differential inequality:

$$|\Delta f(z)| \leq a \cdot |Df(z)|^2 + b. \tag{1.9}$$

It is noted that both harmonic mappings and holomorphic mappings are contained in the class of functions satisfying inequality (1.9). We refer the reader to the research works on those mappings for two dimensions [4, 8] and higher dimensions [11]. Next, we will state our result on Schwarz-Pick type inequality (1.8) for twice continuously differentiable self-mappings of a unit disk satisfying the general Poisson differential inequality (1.9) under certain conditions. Throughout the rest of this paper, $L(a,b,K)$ always refers to the constant in Lemma 2.5. According to [8], one can give the explicit expression of $L(a,b,K)$ when the values of a, b are small enough.

Theorem 1.4 For a given $q \in \{1\} \cup [2, +\infty)$, let $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be continuous in $\overline{\mathbb{D}}$, $f|_{\mathbb{D}} \in \mathcal{C}^2$, $f|_{\mathbb{T}} \in \mathcal{C}^2$, $f(0) = 0$ and satisfy the following Poisson differential inequality:

$$|\Delta f(z)| \leq a \cdot |Df(z)|^2 + b$$

and $\left|\frac{\partial^2 f(e^{i\varphi})}{\partial \varphi^2}\right| \leq K$, where $0 < a < \frac{1}{2}, 0 < b, K < \infty$. Supposing, for $q \geq 2$, that

$$\frac{(2^{q-1}q+1) \cdot \max\{a,b\} \cdot \left(L^2(a,b,K)+1\right)}{4} < \frac{2}{\pi},$$

we get

$$\frac{1-|z|^q}{1-|f(z)|^q} \leq \begin{cases} \dfrac{1}{\frac{2}{\pi} - \frac{3\max\{a,b\}\cdot(L^2(a,b,K)+1)}{4}}, & \text{when } q = 1, \\[2ex] \dfrac{1}{\frac{2}{\pi} - \frac{(2^{q-1}q+1)\cdot\max\{a,b\}\cdot(L^2(a,b,K)+1)}{4}}, & \text{when } q \geq 2. \end{cases}$$

Under which conditions, the subject of a harmonic self-mapping of the unit disk that has Lipschitz continuity with respect to a given metric has attracted the attention of many researchers; see the papers [17, 20] and the references cited therein. A direct and interesting corollary of Theorem 1.4 is the property of hyperbolically Lipschitz continuity for those mappings mentioned in Theorem 1.4.

Corollary 1.5 Let $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be continuous in $\overline{\mathbb{D}}$, $f|_{\mathbb{D}} \in \mathcal{C}^2$, $f|_{\mathbb{T}} \in \mathcal{C}^2$, $f(0) = 0$ and satisfy the following Poisson differential inequality:

$$|\Delta f(z)| \leq a \cdot |Df(z)|^2 + b$$

and $\left|\frac{\partial^2 f(e^{i\varphi})}{\partial \varphi^2}\right| \leq K$, where $0 < a < \frac{1}{2}, 0 < b, K < \infty$. If

$$\frac{5 \cdot \max\{a,b\} \cdot \left(L^2(a,b,K)+1\right)}{4} < \frac{2}{\pi},$$

then f is a hyperbolically Lipschitz continuity; that is, the inequality

$$d_{h_{\mathbb{D}}}(f(z_1), f(z_2)) \leq \frac{L(a,b,K)}{\frac{2}{\pi} - \frac{5\cdot\max\{a,b\}\cdot(L^2(a,b,K)+1)}{4}} \cdot d_{h_{\mathbb{D}}}(z_1, z_2)$$

holds for any $z_1, z_2 \in \mathbb{D}$.

The organization of the rest of this paper is as follows: in Section 2 we make some preparations which will be used in the proof of Theorems 1.2 and 1.3. The proof of Theorem 1.2 will be presented in Section 3. The proof of Theorem 1.3 is given in Section 4. The last section is devoted to the proof of Corollary 1.4.

2 Some Preparations

In [13], Kalaj and Zhu obtained a Schwarz-Pick type inequality for the harmonic self-mapping f of $\mathbb{D}$ with $f(0) = 0$. It is read as follows:

Lemma 2.1 ([13, Proposition 3.6]) If f is a harmonic self-mapping of $\mathbb{D}$ with $f(0) = 0$, then the inequality

$$\frac{1-|z|^q}{1-|f(z)|^q} \leq \frac{\pi}{2} \tag{2.1}$$

holds for every $z \in \mathbb{D}$ and $q > 0$. In particular, we have, for $q = 2$, that

$$\frac{1-|z|^2}{1-|f(z)|^2} \leq \frac{\pi}{2}, z \in \mathbb{D}. \tag{2.2}$$

Next, we will establish a Schwarz-Pick type inequality (2.1) of the harmonic self-mapping of a unit disk with the removed of the assumption that $f(0) = 0$. The result is as follows:

Lemma 2.2 Let f be a harmonic self-mapping of $\mathbb{D}$ satisfying $|f(0)| < 2/\pi$. Then for any $q \geq 1$, the inequality

$$\frac{1-|z|^q}{1-|f(z)|^q} \leq \frac{1}{\frac{2}{\pi} - |f(0)|} \tag{2.3}$$

holds for every $z \in \mathbb{D}$.

Proof Apparently, inequality (2.3) is true when $z = 0$. Hence, we assume that $z \neq 0$. By a generalized Schwarz type inequality for harmonic self-mapping [10, 18] of the unit disk, we get

$$\begin{aligned} \frac{1-|f(z)|^q}{1-|z|^q} &= \frac{1-\left|f(z) - \frac{1-|z|^2}{1+|z|^2}\cdot f(0) + \frac{1-|z|^2}{1+|z|^2}\cdot f(0)\right|^q}{1-|z|^q} \\ &\geq \frac{1-\left(\left|f(z) - \frac{1-|z|^2}{1+|z|^2}\cdot f(0)\right| + \frac{1-|z|^2}{1+|z|^2}\cdot |f(0)|\right)^q}{1-|z|^q} \\ &\geq \frac{1-\left(\frac{4}{\pi}\cdot \arctan|z| + \frac{1-|z|^2}{1+|z|^2}\cdot |f(0)|\right)^q}{1-|z|^q}. \end{aligned} \tag{2.4}$$

Next, we will use L'Hospital's rules for monotonicity [2] to decide the monotonicity of the function

$$\phi(x) = \frac{1-\left(\frac{4}{\pi}\cdot \arctan x + \frac{1-x^2}{1+x^2}\cdot m\right)^q}{1-x^q},$$

where $x \in (0,1), m \in [0, 2/\pi)$. To do this, we introduce the function

$$\varphi(x) = \left(\frac{4}{\pi}\cdot\frac{\arctan x}{x} + \frac{1-x^2}{x(1+x^2)}\cdot m\right)^{q-1}\cdot\left[\frac{-4x}{(1+x^2)^2}\cdot m + \frac{4}{\pi}\cdot\frac{1}{1+x^2}\right],$$

where $x \in (0,1), m \in [0, 2/\pi)$. Then we have

$$\varphi'(x) = \left[\frac{4(3x^2-1)}{(1+x^2)^3}\cdot m - \frac{8}{\pi}\cdot\frac{x}{(1+x^2)^2}\right]\cdot\left(\frac{4}{\pi}\cdot\frac{\arctan x}{x} + \frac{1-x^2}{x(1+x^2)}\cdot m\right)^{q-1}$$

$$+ (q-1)\left[\frac{-4x}{(1+x^2)^2}\cdot m+\frac{4}{\pi}\cdot\frac{1}{1+x^2}\right]\cdot\left(\frac{4}{\pi}\cdot\frac{\arctan x}{x}+\frac{1-x^2}{x(1+x^2)}\cdot m\right)^{q-2}$$
$$\times\left[\frac{x^4-4x^2-1}{x^2(1+x^2)}\cdot m+\frac{4}{\pi}\cdot\frac{x-(1+x^2)\arctan x}{x^2(1+x^2)}\right]$$
$$= A^{q-2}\cdot\left\{\left[\frac{4(3x^2-1)}{x(1+x^2)^3}\cdot m-\frac{8}{\pi}\cdot\frac{1}{(1+x^2)^2}\right]\cdot\left(\frac{4}{\pi}\cdot\arctan x+\frac{1-x^2}{1+x^2}\cdot m\right)+(q-1)\right.$$
$$\left.\times\left[\frac{-4x}{(1+x^2)^2}\cdot m+\frac{4}{\pi}\cdot\frac{1}{1+x^2}\right]\cdot\left[\frac{x^4-4x^2-1}{x^2(1+x^2)}\cdot m+\frac{4}{\pi}\cdot\frac{x-(1+x^2)\arctan x}{x^2(1+x^2)}\right]\right\},$$

where $A=\frac{4}{\pi}\cdot\frac{\arctan x}{x}+\frac{1-x^2}{x(1+x^2)}\cdot m$. Obviously, we have

$$\frac{4}{\pi}\cdot\arctan x+\frac{1-x^2}{1+x^2}\cdot m\geq 0$$

and

$$\frac{-4x}{(1+x^2)^2}\cdot m+\frac{4}{\pi}\cdot\frac{1}{1+x^2}\geq\frac{-2m(1+x^2)}{(1+x^2)^2}+\frac{4}{\pi}\cdot\frac{1}{1+x^2}>0,$$

when $x\in(0,1), m\in[0,2/\pi)$. In addition, the elementary inequality $\frac{x}{1+x^2}\leq\arctan x\leq\frac{x}{1-x^2}$ implies that inequality

$$\frac{x^4-4x^2-1}{x^2(1+x^2)}\cdot m+\frac{4}{\pi}\cdot\frac{x-(1+x^2)\arctan x}{x^2(1+x^2)}<0$$

holds for any $x\in(0,1), m\in[0,2/\pi)$. Finally, we verify that

$$j(x):=\frac{3x^2-1}{x(1+x^2)}\cdot m-\frac{2}{\pi}<0 \tag{2.5}$$

for any $x\in(0,1), m\in[0,2/\pi)$. Since

$$j'(x)=\frac{-3x^4+6x^2+1}{x^2(1+x^2)^2}\cdot m>0,$$

j is monotonically increasing on $(0,1)$, which implies that $j(x)<j(1)=m-\frac{2}{\pi}<0$. Hence, we have $\varphi'(x)<0$ for any $x\in(0,1), m\in[0,2/\pi)$. Therefore, by L'Hospital's rules for monotonicity [2], we see that $\phi(x)$ is monotonically decreasing on $(0,1)$. Using L'Hospital's law, we get $\phi(x)\geq\lim\limits_{x\to1^-}\phi(x)=\frac{2}{\pi}-m$. This completes the proof. □

Lemma 2.3 For any $0\leq y<1, 0\leq\varepsilon<1, q>1$, we have

$$(y+\varepsilon)^q\leq y^q+2^{q-1}q\cdot\varepsilon. \tag{2.6}$$

Proof Let $k(y)=(y+\varepsilon)^q-y^q, y\in[0,1)$. Then we have $k'(y)=q[(y+\varepsilon)^{q-1}-y^{q-1}]\geq 0$. Hence, k is monotonic increasing when $0\leq y<1, 0\leq\varepsilon<1, q>1$, which implies that

$$k(y)=(y+\varepsilon)^q-y^q\leq k(1)=(1+\varepsilon)^q-1. \tag{2.7}$$

Let $\lambda(\varepsilon)=(1+\varepsilon)^q-2^{q-1}q\varepsilon-1$. Then, $\lambda'(\varepsilon)=q[(1+\varepsilon)^{q-1}-2^{q-1}]\leq 0$. Therefore, $\lambda(\varepsilon)\leq\lambda(0)=0$; namely, for any $0\leq\varepsilon<1$, it holds that $(1+\varepsilon)^q-1\leq 2^{q-1}q\varepsilon$. Combining this inequality with (2.7), we get the desired inequality. □

Lemma 2.4 For any $t\in[0,1)$, we have

$$\frac{4}{\pi}\cdot\arctan t\leq\frac{2}{\pi}(t-1)+1. \tag{2.8}$$

Proof Considering the function $\phi(t) = \frac{4}{\pi} \cdot \arctan t, t \in [0,1)$ we get

$$\begin{cases} \phi'(t) = \dfrac{4}{\pi} \cdot \dfrac{1}{1+t^2} > 0, \\ \phi''(t) = \dfrac{4}{\pi} \cdot \dfrac{-2t}{(1+t^2)^2} < 0. \end{cases} \tag{2.9}$$

This shows that ϕ is monotonically increasing and concave in $[0,1)$, which implies that the tangent of ϕ at the point $(1,1)$ lies above the image of ϕ. Hence, the desired inequality (2.8) follows. □

The following result, obtained by Heinz-Bernstein [4, 8], is crucial for us to get a Schwarz-Pick type inequality for mappings satisfying the Poisson differential inequality (1.9) under certain conditions:

Lemma 2.5 ([4, 8]) Let $u : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be continuous in $\overline{\mathbb{D}}$, $u|_{\mathbb{D}} \in \mathcal{C}^2, u|_{\mathbb{T}} \in \mathcal{C}^2$ and satisfy the inequality

$$|\Delta u| \leq a|Du|^2 + b,$$

and $\left|\frac{\partial^2 u(e^{\mathrm{i}\varphi})}{\partial \varphi^2}\right| \leq K$, where $0 < a < \frac{1}{2}, 0 < b, K < \infty$. Then $|Du| \leq L(a,b,K)$ holds on $\mathbb{D}$, where $L(a,b,K)$ is a positive constant that depends only on a, b, K.

3 Proof of Theorem 1.2

To finish the proof, we find a self-mapping of the unit disk that satisfies the Poisson differential inequality (1.5) and $f(0)=0$, but that has no inequality (1.8); that is, it does not satisfy $\frac{1-|z|^2}{1-|f(z)|^2} \leq M$ for any $z \in \mathbb{D}$ and $M \geq 0$. To do this, we take the function $f(z) = \frac{3z - z^2\overline{z}}{2}, z \in \mathbb{D}$. A simple calculation shows that $f_z(z) = \frac{3-2|z|^2}{2}, f_{\overline{z}}(z) = -\frac{z^2}{2}, \Delta f(z) = -4z, f(e^{\mathrm{i}\theta}) = e^{\mathrm{i}\theta}$, $|f(z)| < 1$ and $f(0) = 0$. Hence, f is a self-mapping of the unit disk satisfying the Poisson differential inequality

$$|\Delta f(z)| \leq a \cdot |Df(z)|^2$$

for any $a \geq 4$. Next, by letting $t = |z|^2 \in [0,1)$, we get that

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|f(z)|^2} = \lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|z|^2\left(\frac{3-|z|^2}{2}\right)^2} = \lim_{t\to 1^-} \frac{4}{t^2-5t+4} = +\infty. \tag{3.1}$$

This finishes the proof.

4 Proof of Theorem 1.3

By Lemma 2.5, there is constant $L(a,b,K) > 0$ such that $|Df| \leq L(a,b,K)$ holds for any $z \in \mathbb{D}$. Now, let

$$\Delta f(z) = h(z), \quad z \in \mathbb{D}, \tag{4.1}$$

where $h(z) = l(z)(|Df(z)|^2 + 1)$ and $||l||_\infty \leq \max\{a,b\}$; one can simply define $l(z) := \Delta f(z) \cdot \left(|Df(z)|^2+1\right)^{-1}, z \in \mathbb{D}$. By assumption, we have $h \in \mathcal{C}(\mathbb{D})$. Hence, we get that $||h||_\infty \leq \max\{a,b\} \cdot \left(L^2(a,b,K)+1\right)$, by Lemma 2.5. Now, using formula (1.1), we have

$$f(z) = P[k](z) - G[h](z), \tag{4.2}$$

where $f|_{\mathbb{T}} := k$. For the case of $q = 1$, by using formula (4.2) and Lemma 2.4, we get

$$\begin{aligned}
\frac{1-|f(z)|}{1-|z|} &= \frac{1-\left|P[k](z)-\frac{1-|z|^2}{1+|z|^2}\cdot P[k](0)+\frac{1-|z|^2}{1+|z|^2}\cdot P[k](0)-G[h](z)\right|}{1-|z|}\\
&\geq \frac{1-\left(\left|P[k](z)-\frac{1-|z|^2}{1+|z|^2}\cdot P[k](0)\right|+\frac{1-|z|^2}{1+|z|^2}\cdot|P[k](0)|+|G[h](z)|\right)}{1-|z|}\\
&\geq \frac{1-\left(\frac{4}{\pi}\cdot\arctan|z|+\frac{1-|z|^2}{1+|z|^2}\cdot|P[k](0)|+\frac{||h||_\infty(1-|z|^2)}{4}\right)}{1-|z|}\\
&\geq \frac{1-\left(\frac{2}{\pi}\cdot(|z|-1)+1+\frac{1-|z|^2}{1+|z|^2}\cdot|P[k](0)|+\frac{||h||_\infty(1-|z|^2)}{4}\right)}{1-|z|}\\
&= \frac{2}{\pi}-\frac{1+|z|}{1+|z|^2}\cdot|P[k](0)|-\frac{||h||_\infty(1+|z|)}{4}\\
&\geq \frac{2}{\pi}-\frac{1+|z|}{4(1+|z|^2)}\cdot||h||_\infty-\frac{||h||_\infty(1+|z|)}{4}\geq\frac{2}{\pi}-\frac{3||h||_\infty}{4}\\
&\geq \frac{2}{\pi}-\frac{3\max\{a,b\}\cdot\left(L^2(a,b,K)+1\right)}{4}>0. \qquad (4.3)
\end{aligned}$$

For the case of $q \geq 2$, since $f(0) = 0$ and

$$\frac{\max\{a,b\}\cdot\left(L^2(a,b,K)+1\right)}{4}<\frac{(2^{q-1}q+1)\cdot\max\{a,b\}\cdot\left(L^2(a,b,K)+1\right)}{4}<\frac{2}{\pi},$$

we get (by using estimate (1.3)) that

$$|P[k](0)|=|G[h](0)|\leq\frac{||h||_\infty}{4}\leq\frac{\max\{a,b\}\cdot\left(L^2(a,b,K)+1\right)}{4}<2/\pi. \qquad (4.4)$$

Hence, $P[k]$ is a harmonic self-mapping of $\mathbb{D}$ satisfying $|P[k](0)| < 2/\pi$. By virtue of Lemma 2.2, we get that

$$\frac{1-|P[k](z)|^q}{1-|z|^q}\geq\frac{2}{\pi}-|P[k](0)|\geq\frac{2}{\pi}-\frac{\max\{a,b\}\cdot\left(L^2(a,b,K)+1\right)}{4}. \qquad (4.5)$$

Now, using (4.5) and Lemma 2.3, we get that

$$\begin{aligned}
\frac{1-|f(z)|^q}{1-|z|^q} &\geq \frac{1-(|P[k](z)|+|G[h](z)|)^q}{1-|z|^q}\\
&\geq \frac{1-\left(|P[k](z)|+\frac{||h||_\infty(1-|z|^2)}{4}\right)^q}{1-|z|^q}\geq\frac{1-|P[k](z)|^q-2^{q-1}q\frac{||h||_\infty(1-|z|^2)}{4}}{1-|z|^q}\\
&= \frac{1-|P[k](z)|^q}{1-|z|^q}-\frac{2^{q-1}q\cdot||h||_\infty}{4}\cdot\frac{1-|z|^2}{1-|z|^q}\geq\frac{2}{\pi}-|P[k](0)|-\frac{2^{q-1}q\cdot||h||_\infty}{4}\\
&\geq \frac{2}{\pi}-\frac{(2^{q-1}q+1)||h||_\infty}{4}\geq\frac{2}{\pi}-\frac{(2^{q-1}q+1)\cdot\max\{a,b\}\cdot\left(L^2(a,b,K)+1\right)}{4}\\
&> 0. \qquad (4.6)
\end{aligned}$$

Hence, the proof of Theorem 1.3 is complete.

5 Proof of Corollary 1.4

From above proof of Theorem 1.3, we see that if $\frac{5\cdot\max\{a,b\}\cdot\left(L^2(a,b,K)+1\right)}{4} < \frac{2}{\pi}$, then $|Df| \leq L(a,b,K)$ and

$$\frac{1-|z|^2}{1-|f(z)|^2} \leq \frac{1}{\frac{2}{\pi} - \frac{5\cdot\max\{a,b\}\cdot(L^2(a,b,K)+1)}{4}}.$$

These inequalities imply that

$$\frac{|Df|(1-|z|^2)}{1-|f(z)|^2} \leq \frac{L(a,b,K)}{\frac{2}{\pi} - \frac{5\cdot\max\{a,b\}\cdot(L^2(a,b,K)+1)}{4}}.$$

This completes the proof.

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