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UNIQUENESS OF THE INVERSE TRANSMISSION SCATTERING WITH A CONDUCTIVE BOUNDARY CONDITION*

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Abstract This paper considers the inverse acoustic wave scattering by a bounded penetrable obstacle with a conductive boundary condition. We will show that the penetrable scatterer can be uniquely determined by its far-field pattern of the scattered field for all incident plane waves at a fixed wave number. In the first part of this paper, adequate preparations for the main uniqueness result are made. We establish the mixed reciprocity relation between the far-field pattern corresponding to point sources and the scattered field corresponding to plane waves. Then the well-posedness of a modified interior transmission problem is deeply investigated by the variational method. Finally, the a priori estimates of solutions to the general transmission problem with boundary data in $L^p(\partial\Omega)$ $(1<p<2)$ are proven by the boundary integral equation method. In the second part of this paper, we give a novel proof on the uniqueness of the inverse conductive scattering problem.

Key words Acoustic wave; uniqueness; mixed reciprocity relation; modified interior transmission problem; a priori estimates

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1 Introduction

The inverse scattering problem we are concerned with here is determining the shape of an obstacle by measurements of the far-field patterns of acoustic waves. We are interested in the scattering of a penetrable obstacle covered by a thin layer of high conductivity; that is, the so-called conductive boundary condition ([1, 2]), which is a generalization of the classical transmission problem.

Let Ω denote a penetrable bounded open domain in $\mathbb{R}^3$ with $\mathbb{R}^3\backslash\overline{\Omega}$ connected. Let $n(x)$ be the refractive index, let $k>0$ be the wave number, and set a jump parameter $\lambda\in\mathbb{C}\backslash\{0\}$ and a complex-valued function μ on the smooth boundary $\partial\Omega$. Then the conductive scattering

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problem we consider is modeled as follows:

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in} \quad \mathbb{R}^3\backslash\overline{\Omega}, \\ \Delta u + k^2 n u = 0, & \text{in} \quad \Omega, \\ u_+ - u_- = 0, & \text{on} \quad \partial\Omega, \\ \dfrac{\partial u_+}{\partial \nu} - \lambda \dfrac{\partial u_-}{\partial \nu} + \mu u_+ = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $u = u^i + u^s$ is the total field, which is a superposition of the incident wave $u^i = u^i(x, d) := e^{\mathrm{i}kx\cdot d}$ (note that the incident wave will be a plane wave or a point source in our later proofs and that d denotes the incident direction) and the scattered wave u^s, and ν is the unit outward normal to the boundary $\partial\Omega$. Here, $u_\pm$ and $\frac{\partial u_\pm}{\partial \nu}$ denote the limit of u and $\frac{\partial u}{\partial \nu}$ from the exterior $(+)$ and interior $(-)$, respectively. Furthermore, the scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\Big(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\Big) = 0, \quad r = |x|, \tag{1.2}$$

and the convergence holds uniformly with respect to $\hat{x} = x/|x| \in \mathbb{S}$, where $\mathbb{S}$ denotes the unit sphere in $\mathbb{R}^3$.

Referring to Section 2 of paper [4], we make following assumptions on n, λ and μ to guarantee the well-posedness of the direct problem (1.1):

Assumption 1.1 (1) The refractive function $n \in L^\infty(\Omega)$ satisfies $\mathrm{Re}(n) > 0$ and $\mathrm{Im}(n) \geq 0$ almost everywhere (a.e.) in Ω.

(2) λ is a non-zero complex constant, such that there exists $\hat{c} > 0$, such that $\mathrm{Re}(\lambda) \geq \hat{c}$ and $\mathrm{Im}(\lambda) \leq 0$, $\mathrm{Im}(\lambda n) \geq 0$ a.e. in Ω.

(3) $\mu \in L^\infty(\partial\Omega)$ with $\mathrm{Re}(\mu) \leq 0$ and $\mathrm{Im}(\mu) \geq 0$ a.e. on $\partial\Omega$.

It is well known that the radiating solution u^s has the asymptotic expansion

$$u^s(x) = \frac{e^{\mathrm{i}kr}}{r}\Big\{u^\infty(\hat{x}) + O\Big(\frac{1}{r}\Big)\Big\}, \quad \text{as} \quad r = |x| \longrightarrow \infty \tag{1.3}$$

uniformly for all directions $\hat{x} = x/|x| \in \mathbb{S}$. Here, u^∞ is called the far-field pattern of u^s, which is an analytic function defined on $\mathbb{S}$.

The problem of uniqueness in the inverse obstacle scattering theory is of central importance both for the theoretical study and the implementation of numerical algorithms in acoustic, electromagnetic, fluid-solid interaction and elastic waves, etc.. The first uniqueness result was shown by Schiffer [20] in acoustic waves with a Dirichlet boundary condition whose argument cannot be transferred to other boundary conditions. In 1990, Isakov [17] gave a uniqueness proof for transmission problems ($u^i = u^e$, $\partial u^e/\partial\nu = \mu\partial u^i/\partial\nu$, $\mu \neq 1$) based on the variational method by constructing a sequence of singular solutions. In 1993, Kirsch and Kress [18] simplified Isakov's proof and also transferred it to the Neumann boundary condition by proving a continuous dependence result in a weighted Banach space of continuous functions. In the same year, Ramm [31] used a new method to prove the uniqueness of the impenetrable obstacle with a Dirichlet or Neumann boundary condition.

In 1994, Hettlich [16] achieved the uniqueness theorem for the general conductive boundary condition ($u_+ - u_- = 0$, $\partial u_+/\partial\nu - \mu\partial u_-/\partial\nu = \lambda u$, the interior wave number is a constant) based on the idea of Isakov. Furthermore, the uniqueness of coefficients μ, λ and the constant interior wave number were also proven. Later, in 1996, Gerlach and Kress [12] simplified and shortened the analysis of Hettlich. In order to present a refinement in the case when the boundary of the scatterer is allowed to have irregularities, in 1998, Mitrea [24] studied its uniqueness dependent upon boundary integral techniques and the Calderón-Zygmund theory. In 2004, Valdivia [33] worked on the uniqueness again based on the original idea of Isakov. Since then, there have been many other uniqueness problems, such as impenetrable scatterers with an unknown type of boundary condition ([10, 19]), local uniqueness ([13, 32]), penetrable orthotropic [11] or anisotropic inhomogeneous obstacles ([14, 25]), a piecewise homogeneous medium ([21–23]), etc..

In this paper, we again consider the uniqueness of the inverse transmission scattering with a conductive boundary condition by an inhomogeneous medium. The idea is inspired by [29] (an inhomogeneous acoustic cavity), [30] (fluid-solid interaction with embedded obstacles) and [34] (penetrable obstacles with embedded objects in acoustic and electromagnetic scattering). Hence, before showing the main uniqueness proof, we discuss some important preparations, which are also of interest in their own right.

Firstly, we establish a mixed reciprocity relation between the far-field pattern corresponding to point sources and the scattered field corresponding to plane waves of this general transmission problem. The mixed reciprocity relation was shown in [26] (Theorem 1) for sound-soft and sound-hard obstacles, in [9] (Theorem 3.24) for generalized impedance objects, in [28] (Theorem 2.2.4) for inhomogeneous media, and in [23] for a piecewise homogeneous medium, etc.. In the derivation of (2.1) and the above references, we can conclude that the relation for $y \in \mathbb{R}^3 \backslash \overline{\Omega}$ is valid for all possible boundary conditions of penetrable or impenetrable scatterers. Furthermore, for $y \in \Omega$, relation (2.1) has a close connection with the jump parameter λ (Lemma 3.2 in [23]), but that disregards the complex-valued function μ.

Secondly, we study the well-posedness of a modified interior transmission problem by the variational method. Though the interior transmission problems have been deeply investigated in the book [6], there are few results about the conductive boundary ([3, 15] for $\lambda = 1$). The well-posedness is achieved under some limitations on λ and μ, and the discreteness of interior transmission eigenvalues is a by-product. In the future, we want to conduct further research on the interior transmission eigenvalues problem.

Thirdly, we prove the a priori estimates of solutions to the general transmission problem with boundary data in $L^p(\partial\Omega)$ $(1 < p < 2)$ by the boundary integral equation method. In a manner different from Section 2.2 in [34] (Theorem 2.5), where the authors make a small modification to the boundary conditions (2.14), we revise the representations of potential functions (refer to Section 3.8 in the book [8]) and keep the conductive boundary conditions unchanged. Those two different modifications are made in order to obtain compactness of the matrix A in Theorem 2.12 (or the matrix L in [34] Theorem 2.5). The key point is that the operators S_k, K_k, $K_k^{'}$ and the difference $T_{k_1} - T_k$ are compact in $L^p(\partial\Omega)$ for $1 < p < 2$ (Lemma 9 in [26] and Lemma 1 in [27]).

Finally, the novel and simple method for proving the uniqueness of the conductive boundary by its far field pattern is easy to implement for our inverse transmission problem.

The remainder section of the paper is organized as follows: Section 2 is devoted to making preparations; we show a mixed reciprocity relation, investigate the well-posedness of a modified interior transmission problem, and construct the a priori estimates of solutions to the general transmission problem with boundary data in $L^p(\partial\Omega)$ $(1 < p < 2)$ by the boundary integral equation method. In Section 3, the main result on the uniqueness of the inverse penetrable transmission problem is proven in detail.

2 Preparations

In this section, we make some necessary preparations before showing the uniqueness of the inverse problem (1.1).

2.1 Mixed reciprocity relation

In this subsection, we establish the mixed reciprocity relation, which has been proven both for impenetrable and penetrable scatterers. In the sequel, we view $\Phi_k(x,y) = \frac{e^{\mathrm{i}k|x-y|}}{4\pi|x-y|}$ as an incident point source wave, and correspondingly, denote by $u^s(x;y)$, $u(x;y)$ and $u^\infty(\hat{x};y)$ the scattered, total wave and the far-field pattern associated with $\Phi_k(x,y)$ to problem (1.1). Similarly, $u^s(x,d)$, $u(x,d)$ and $u^\infty(\hat{x},d)$ denote the scattered, total wave and the far-field pattern associated with $u^i(x,d)$ to problem (1.1), respectively.

Theorem 2.1 (Mixed reciprocity relation) For the scattering of plane waves $u^i(\cdot,d)$ and the far-field pattern of point sources $\Phi_k(\cdot,y)$, we have

$$4\pi u^\infty(-d;y) = \begin{cases} u^s(y,d), & d\in\mathbb{S},\ y\in\mathbb{R}^3\backslash\overline{\Omega}, \\ \lambda u^s(y,d) + (\lambda-1)u^i(y,d), & d\in\mathbb{S},\ y\in\Omega. \end{cases} \tag{2.1}$$

Proof Firstly, we consider the case $y\in\mathbb{R}^3\backslash\overline{\Omega}$. Since $u^s(x;y)$ and $u^s(x,d)$ both fulfil the Sommerfeld radiation condition (1.2), it holds that

$$\int_{\partial\Omega}\Big[u^s_+(z;y)\frac{\partial u^s_+(z,d)}{\partial\nu(z)} - \frac{\partial u^s_+(z;y)}{\partial\nu(z)}u^s_+(z,d)\Big]\mathrm{d}s(z) = 0. \tag{2.2}$$

From Green's representation theorem [5], for the radiating solution $u^s(x;y)$ to the Helmholtz equation, one can derive the following integral representation for $x\in\mathbb{R}^3\backslash\overline{\Omega}$:

$$u^s(x;y) = \int_{\partial\Omega}\Big\{\frac{\partial\Phi_k(x,z)}{\partial\nu(z)}u^s_+(z;y) - \Phi_k(x,z)\frac{\partial u^s_+(z;y)}{\partial\nu(z)}\Big\}\mathrm{d}s(z).$$

Letting $|x|\longrightarrow\infty$, we have

$$u^\infty(\hat{x};y) = \frac{1}{4\pi}\int_{\partial\Omega}\Big\{\frac{\partial e^{-\mathrm{i}k\hat{x}\cdot z}}{\partial\nu(z)}u^s_+(z;y) - e^{-\mathrm{i}k\hat{x}\cdot z}\frac{\partial u^s_+(z;y)}{\partial\nu(z)}\Big\}\mathrm{d}s(z).$$

Hence,

$$\begin{aligned} 4\pi u^\infty(-d;y) &= \int_{\partial\Omega}\Big\{\frac{\partial e^{\mathrm{i}kd\cdot z}}{\partial\nu(z)}u^s_+(z;y) - e^{\mathrm{i}kd\cdot z}\frac{\partial u^s_+(z;y)}{\partial\nu(z)}\Big\}\mathrm{d}s(z) \\ &= \int_{\partial\Omega}\Big\{\frac{\partial u^i(z,d)}{\partial\nu(z)}u^s_+(z;y) - u^i(z,d)\frac{\partial u^s_+(z;y)}{\partial\nu(z)}\Big\}\mathrm{d}s(z) \\ &= \int_{\partial\Omega}\Big\{\frac{\partial u_+(z,d)}{\partial\nu(z)}u^s_+(z;y) - u_+(z,d)\frac{\partial u^s_+(z;y)}{\partial\nu(z)}\Big\}\mathrm{d}s(z), \end{aligned} \tag{2.3}$$

where the last equality is obtained by adding formula (2.2).

Using Green's representation theorem again, for $y\in\mathbb{R}^3\backslash\overline{\Omega}$,

$$u^s(y,d) = \int_{\partial\Omega}\Big\{\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u^s_+(z,d) - \Phi_k(z,y)\frac{\partial u^s_+(z,d)}{\partial\nu(z)}\Big\}\mathrm{d}s(z).$$

Applying Green's second integral theorem to $u^i(\cdot,d)$ and $\Phi_k(\cdot,y)$ in Ω yields

$$\int_{\partial\Omega}\Big[\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u^i(z,d)-\Phi_k(z,y)\frac{\partial u^i(z,d)}{\partial\nu(z)}\Big]\mathrm{d}s(z)=0.$$

Adding up the previous two equalities, we arrive at

$$u^s(y,d)=\int_{\partial\Omega}\Big\{\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u_+(z,d)-\Phi_k(z,y)\frac{\partial u_+(z,d)}{\partial\nu(z)}\Big\}\mathrm{d}s(z). \tag{2.4}$$

Combining (2.3) with (2.4), we find that

$$\begin{aligned}4\pi u^\infty(-d;y)-u^s(y,d)=&\int_{\partial\Omega}\Big\{\frac{\partial u_+(z,d)}{\partial\nu(z)}u_+^s(z;y)-u_+(z,d)\frac{\partial u_+^s(z;y)}{\partial\nu(z)}\Big\}\mathrm{d}s(z)\\&+\int_{\partial\Omega}\Big\{\Phi_k(z,y)\frac{\partial u_+(z,d)}{\partial\nu(z)}-\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u_+(z,d)\Big\}\mathrm{d}s(z)\\=&\int_{\partial\Omega}\Big\{\frac{\partial u_+(z,d)}{\partial\nu(z)}u_+(z;y)-u_+(z,d)\frac{\partial u_+(z;y)}{\partial\nu(z)}\Big\}\mathrm{d}s(z).\end{aligned}$$

Use the conductive boundary condition and Green's formula in Ω, we conclude that

$$\begin{aligned}4\pi u^\infty(-d;y)-u^s(y,d)=&\int_{\partial\Omega}\Big\{\Big[\lambda\frac{\partial u_-(z,d)}{\partial\nu(z)}-\mu u_-(z,d)\Big]u_-(z;y)\\&-u_-(z,d)\Big[\lambda\frac{\partial u_-(z;y)}{\partial\nu(z)}-\mu u_-(z;y)\Big]\Big\}\mathrm{d}s(z)\\=&\lambda\int_{\partial\Omega}\Big\{\frac{\partial u_-(z,d)}{\partial\nu(z)}u_-(z;y)-u_-(z,d)\frac{\partial u_-(z;y)}{\partial\nu(z)}\Big\}\mathrm{d}s(z)\\=&\lambda\int_{\Omega}\big[u(z;y)\Delta u(z,d)-u(z,d)\Delta u(z;y)\big]dz=0.\end{aligned}$$

This implies that $4\pi u^\infty(-d;y)=u^s(y,d)$ for all $d\in\mathbb{S}$, $y\in\mathbb{R}^3\backslash\overline{\Omega}$.

Secondly, we consider the case $y\in\Omega$. Recalling equality (2.3), which holds also for $y\in\Omega$, using the conductive boundary condition, we obtain

$$\begin{aligned}4\pi u^\infty(-d;y)=&\int_{\partial\Omega}\Big\{\frac{\partial u_+(z,d)}{\partial\nu(z)}u_+^s(z;y)-u_+(z,d)\frac{\partial u_+^s(z;y)}{\partial\nu(z)}\Big\}\mathrm{d}s(z)\\=&\int_{\partial\Omega}\Big\{\mu\Phi_k(z,y)u_-(z,d)-(\lambda-1)\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u_-(z,d)\Big\}\mathrm{d}s(z)\\&+\lambda\int_{\partial\Omega}\Big\{\frac{\partial u_-(z,d)}{\partial\nu(z)}u_-^s(z;y)-u_-(z,d)\frac{\partial u_-^s(z;y)}{\partial\nu(z)}\Big\}\mathrm{d}s(z)\\=&\int_{\partial\Omega}\Big\{\mu\Phi_k(z,y)u_-(z,d)-(\lambda-1)\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u_-(z,d)\Big\}\mathrm{d}s(z)\\&+\lambda\int_{\Omega}k^2(n-1)\Phi_k(z,y)u(z,d)\mathrm{d}z.\end{aligned}\tag{2.5}$$

The last equality is obtained completely similar to the proof of (3.13) to (3.14) in Lemma 3.2 ([23]).

On the other hand, with the help of Green's representation formula, we know that

$$\begin{aligned}u(y,d)=&\int_{\partial\Omega}\Big\{\Phi_k(z,y)\frac{\partial u_-(z,d)}{\partial\nu(z)}-\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u_-(z,d)\Big\}\mathrm{d}s(z)\\&+\int_{\Omega}k^2(n-1)\Phi_k(z,y)u(z,d)\mathrm{d}z.\end{aligned}\tag{2.6}$$

Combining (2.5) with (2.6) and using the conductive boundary condition, we have

$$\begin{aligned}
&4\pi u^{\infty}(-d;y)-\lambda u(y,d)\\
=&\int_{\partial\Omega}\Big\{\mu\Phi_k(z,y)u_-(z,d)-(\lambda-1)\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u_-(z,d)\Big\}\mathrm{d}s(z)\\
&+\int_{\partial\Omega}\Big\{\lambda\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u_-(z,d)-\lambda\Phi_k(z,y)\frac{\partial u_-(z,d)}{\partial\nu(z)}\Big\}\mathrm{d}s(z)\\
=&\int_{\partial\Omega}\Big\{\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u_-(z,d)+\big[\mu u_-(z,d)-\lambda\frac{\partial u_-(z,d)}{\partial\nu(z)}\big]\Phi_k(z,y)\Big\}\mathrm{d}s(z)\\
=&\int_{\partial\Omega}\Big\{\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u_+(z,d)-\frac{\partial u_+(z,d)}{\partial\nu(z)}\Phi_k(z,y)\Big\}\mathrm{d}s(z).
\end{aligned}$$

For $y\in\Omega$, both $\Phi_k(\cdot,y)$ and $u^s(\cdot,d)$ satisfy the Helmholtz equation in $\mathbb{R}^3\backslash\overline{\Omega}$ and the Sommerfeld radiation condition (1.2), hence

$$\int_{\partial\Omega}\Big\{\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u^s_+(z,d)-\frac{\partial u^s_+(z,d)}{\partial\nu(z)}\Phi_k(z,y)\Big\}\mathrm{d}s(z)=0.$$

Consequently,

$$\begin{aligned}
4\pi u^{\infty}(-d;y)-\lambda u(y,d)&=\int_{\partial\Omega}\Big\{\frac{\partial\Phi_k(z,y)}{\partial\nu(z)}u^i(z,d)-\frac{\partial u^i(z,d)}{\partial\nu(z)}\Phi_k(z,y)\Big\}\mathrm{d}s(z)\\
&=-u^i(y,d).
\end{aligned}$$

This implies that $4\pi u^{\infty}(-d;y)=\lambda u^s(y,d)+(\lambda-1)u^i(y,d)$ for all $d\in\mathbb{S}$, $y\in\Omega$. The proof is complete. □

Remark 2.2 If there is a buried object inside Ω, Theorem 2.1 also holds. Lemma 3.2 in [23] is a special case when n is a constant and $\mu=0$ a.e. on $\partial\Omega$.

Remark 2.3 Theorem 2.1 also holds in two dimensional space with some modifications of the coefficient.

2.2 Modified interior transmission problem

Given $\ell_1,\ell_2\in L^2(\Omega)$, $f_1\in H^{1/2}(\partial\Omega)$, $f_2\in H^{-1/2}(\partial\Omega)$, we consider the following modified interior transmission problem:

$$\begin{cases}
\Delta v-v=\ell_1, & \text{in}\quad\Omega,\\
\Delta w-w=\ell_2, & \text{in}\quad\Omega,\\
v-w=f_1, & \text{on}\quad\partial\Omega,\\
\dfrac{\partial v}{\partial\nu}-\lambda\dfrac{\partial w}{\partial\nu}+\mu v=f_2, & \text{on}\quad\partial\Omega.
\end{cases}\tag{2.7}$$

In order to reformulate (2.7) as an equivalent variational problem, we define the Hilbert space

$$X:=\Big\{\boldsymbol{\psi}\in[L^2(\Omega)]^2:\nabla\cdot\boldsymbol{\psi}\in L^2(\Omega)\ \text{and}\ \nabla\times\boldsymbol{\psi}=0\Big\},$$

equipped with the norm $\|\boldsymbol{\psi}\|_X^2=\|\boldsymbol{\psi}\|^2_{[L^2(\Omega)]^2}+\|\nabla\cdot\boldsymbol{\psi}\|^2_{L^2(\Omega)}$.

Now, we multiply the second equation in (2.7) by a test function $\varphi\in H^1(\Omega)$ to get

$$\int_{\Omega}\ell_2\overline{\varphi}\mathrm{d}x=\int_{\Omega}(\Delta w-w)\overline{\varphi}\mathrm{d}x=\int_{\partial\Omega}\frac{\partial w}{\partial\nu}\overline{\varphi}\mathrm{d}s-\int_{\Omega}(\nabla w\cdot\nabla\overline{\varphi}+w\overline{\varphi})\mathrm{d}x.$$

Using the conductive boundary condition in (2.7), we see that

$$\lambda\int_{\partial\Omega}\frac{\partial w}{\partial\nu}\overline{\varphi}\mathrm{d}s=\int_{\partial\Omega}\Big(\frac{\partial v}{\partial\nu}+\mu v-f_2\Big)\overline{\varphi}\mathrm{d}s$$
$$=\int_{\partial\Omega}(\mathbf{v}\cdot\nu)\overline{\varphi}\mathrm{d}s+\int_{\partial\Omega}\mu\big(w+f_1\big)\overline{\varphi}\mathrm{d}s-\int_{\partial\Omega}f_2\overline{\varphi}\mathrm{d}s,$$

where $\mathbf{v}=\nabla v$, and then $\mathbf{v}\in X$. After arranging, we obtain that

$$\lambda\int_{\Omega}(\nabla w\cdot\nabla\overline{\varphi}+w\overline{\varphi})\mathrm{d}x-\int_{\partial\Omega}(\mathbf{v}\cdot\nu)\overline{\varphi}\mathrm{d}s-\int_{\partial\Omega}\mu w\overline{\varphi}\mathrm{d}s$$
$$=\int_{\partial\Omega}\mu f_1\overline{\varphi}\mathrm{d}s-\int_{\partial\Omega}f_2\overline{\varphi}\mathrm{d}s-\lambda\int_{\Omega}\ell_2\overline{\varphi}\mathrm{d}x.$$

We multiply the first equation in (2.7) by a test function $\boldsymbol{\psi}\in X$. Similarly, integrating in Ω and using the boundary condition, we obtain

$$\int_{\Omega}(\nabla\cdot\mathbf{v})(\nabla\cdot\overline{\boldsymbol{\psi}})\mathrm{d}x=\int_{\Omega}(\nabla\cdot(\nabla v))(\nabla\cdot\overline{\boldsymbol{\psi}})\mathrm{d}x=\int_{\Omega}(v+\ell_1)(\nabla\cdot\overline{\boldsymbol{\psi}})\mathrm{d}x$$
$$=\int_{\partial\Omega}v(\overline{\boldsymbol{\psi}}\cdot\nu)\mathrm{d}s-\int_{\Omega}(\nabla v)\cdot\overline{\boldsymbol{\psi}}\mathrm{d}x+\int_{\Omega}\ell_1(\nabla\cdot\overline{\boldsymbol{\psi}})\mathrm{d}x$$
$$=\int_{\partial\Omega}(w+f_1)(\overline{\boldsymbol{\psi}}\cdot\nu)\mathrm{d}s-\int_{\Omega}\mathbf{v}\cdot\overline{\boldsymbol{\psi}}\mathrm{d}x+\int_{\Omega}\ell_1(\nabla\cdot\overline{\boldsymbol{\psi}})\mathrm{d}x;$$

that is

$$\int_{\Omega}\Big[(\nabla\cdot\mathbf{v})(\nabla\cdot\overline{\boldsymbol{\psi}})+\mathbf{v}\cdot\overline{\boldsymbol{\psi}}\Big]\mathrm{d}x-\int_{\partial\Omega}w(\overline{\boldsymbol{\psi}}\cdot\nu)\mathrm{d}s=\int_{\Omega}\ell_1(\nabla\cdot\overline{\boldsymbol{\psi}})\mathrm{d}x+\int_{\partial\Omega}f_1(\overline{\boldsymbol{\psi}}\cdot\nu)\mathrm{d}s.$$

Based on the above calculations, we introduce the sesquilinear form $\mathcal{A}_1(\mathbf{U},\mathbf{V})$, defined on $\{H^1(\Omega)\times X\}^2$ by

$$\mathcal{A}_1(\mathbf{U},\mathbf{V})=\lambda\int_{\Omega}(\nabla w\cdot\nabla\overline{\varphi}+w\overline{\varphi})\mathrm{d}x-\int_{\partial\Omega}(\mathbf{v}\cdot\nu)\overline{\varphi}\mathrm{d}s-\int_{\partial\Omega}\mu w\overline{\varphi}\mathrm{d}s$$
$$+\int_{\Omega}\Big[(\nabla\cdot\mathbf{v})(\nabla\cdot\overline{\boldsymbol{\psi}})+\mathbf{v}\cdot\overline{\boldsymbol{\psi}}\Big]\mathrm{d}x-\int_{\partial\Omega}w(\overline{\boldsymbol{\psi}}\cdot\nu)\mathrm{d}s,$$

where $\mathbf{U}:=(w,\mathbf{v})$ and $\mathbf{V}:=(\varphi,\boldsymbol{\psi})$ are in $H^1(\Omega)\times X$. We denote by $L_1:H^1(\Omega)\times X\longrightarrow\mathbb{C}$ the bounded antilinear functional given by

$$L_1(\mathbf{V})=\int_{\partial\Omega}(\mu f_1-f_2)\overline{\varphi}\mathrm{d}s-\lambda\int_{\Omega}\ell_2\overline{\varphi}\mathrm{d}x+\int_{\Omega}\ell_1(\nabla\cdot\overline{\boldsymbol{\psi}})\mathrm{d}x+\int_{\partial\Omega}f_1(\overline{\boldsymbol{\psi}}\cdot\nu)\mathrm{d}s.$$

Therefore, the variational formulation of problem (2.7) is to find $\mathbf{U}=(w,\mathbf{v})\in H^1(\Omega)\times X$ such that

$$\mathcal{A}_1(\mathbf{U},\mathbf{V})=L_1(\mathbf{V}),\quad\forall\ \mathbf{V}\in H^1(\Omega)\times X.\tag{2.8}$$

Changing the roles of w and v, we can obtain another different variational formulation of problem (2.7); namely, we multiply the first equation in (2.7) by a test function $\varphi\in H^1(\Omega)$ and the second equation by a test function $\boldsymbol{\psi}\in X$, integrate in Ω, and use the boundary condition to obtain

$$\int_{\Omega}(\nabla v\cdot\nabla\overline{\varphi}+v\overline{\varphi})\mathrm{d}x-\lambda\int_{\partial\Omega}(\mathbf{w}\cdot\nu)\overline{\varphi}\mathrm{d}s+\int_{\partial\Omega}\mu v\overline{\varphi}\mathrm{d}s=\int_{\partial\Omega}f_2\overline{\varphi}\mathrm{d}s-\int_{\Omega}\ell_1\overline{\varphi}\mathrm{d}x$$

and

$$\int_{\Omega}\Big[(\nabla\cdot\mathbf{w})(\nabla\cdot\overline{\boldsymbol{\psi}})+\mathbf{w}\cdot\overline{\boldsymbol{\psi}}\Big]\mathrm{d}x-\int_{\partial\Omega}v(\overline{\boldsymbol{\psi}}\cdot\nu)\mathrm{d}s=\int_{\Omega}\ell_2(\nabla\cdot\overline{\boldsymbol{\psi}})\mathrm{d}x-\int_{\partial\Omega}f_1(\overline{\boldsymbol{\psi}}\cdot\nu)\mathrm{d}s,$$

where $\mathbf{w} = \nabla w$, and then $\mathbf{w} \in X$.

We introduce the sesquilinear form $\mathcal{A}_2(\mathbb{U}, \mathbb{V})$ defined on $\{H^1(\Omega) \times X\}^2$ and the bounded antilinear functional $L_2 : H^1(\Omega) \times X \longrightarrow \mathbb{C}$ given by

$$\begin{aligned}\mathcal{A}_2(\mathbb{U}, \mathbb{V}) &= \int_\Omega (\nabla v \cdot \nabla \overline{\varphi} + v\overline{\varphi})\mathrm{d}x - \lambda \int_{\partial\Omega} (\mathbf{w} \cdot \nu)\overline{\varphi}\mathrm{d}s + \int_{\partial\Omega} \mu v\overline{\varphi}\mathrm{d}s \\ &\quad + \lambda \int_\Omega \Big[(\nabla \cdot \mathbf{w})(\nabla \cdot \overline{\boldsymbol{\psi}}) + \mathbf{w} \cdot \overline{\boldsymbol{\psi}}\Big]\mathrm{d}x - \lambda \int_{\partial\Omega} v(\overline{\boldsymbol{\psi}} \cdot \nu)\mathrm{d}s, \\ L_2(\mathbb{V}) &= \int_{\partial\Omega} f_2\overline{\varphi}\mathrm{d}s - \int_\Omega \ell_1\overline{\varphi}\mathrm{d}x + \lambda \int_\Omega \ell_2(\nabla \cdot \overline{\boldsymbol{\psi}})\mathrm{d}x - \lambda \int_{\partial\Omega} f_1(\overline{\boldsymbol{\psi}} \cdot \nu)\mathrm{d}s,\end{aligned}$$

where $\mathbb{U} := (v, \mathbf{w})$ and $\mathbb{V} := (\varphi, \boldsymbol{\psi})$ are in $H^1(\Omega) \times X$. Then, the variational formulation of problem (2.7) is to find $\mathbb{U} = (v, \mathbf{w}) \in H^1(\Omega) \times X$ such that

$$\mathcal{A}_2(\mathbb{U}, \mathbb{V}) = L_2(\mathbb{V}), \quad \forall\, \mathbb{V} \in H^1(\Omega) \times X. \tag{2.9}$$

The following Theorem states the equivalence between problems (2.7) and (2.8) or (2.9) (the detailed proof is the same as that of Theorem 3.3 in the paper [7] and Theorem 6.5 in the book [5], so for brevity we omit it here):

Theorem 2.4 Problem (2.7) has a unique solution $(w, v) \in H^1(\Omega) \times H^1(\Omega)$ if and only if problem (2.8) has a unique solution $\mathbf{U} = (w, \mathbf{v}) \in H^1(\Omega) \times X$ or problem (2.9) has a unique solution $\mathbb{U} = (v, \mathbf{w}) \in H^1(\Omega) \times X$.

Now, we investigate the modified interior transmission problem in the variational formulations (2.8) and (2.9).

Theorem 2.5 (1) If $\mathrm{Re}(\lambda) \geq \hat{c} > 1$ and $\mathrm{Re}(\mu) \leq 0$, then the variational problem (2.8) has a unique solution $\mathbf{U} = (w, \mathbf{v}) \in H^1(\Omega) \times X$ that satisfies

$$\|w\|_{H^1(\Omega)} + \|\mathbf{v}\|_X \leq c_1\big(\|\ell_1\|_{L^2(\Omega)} + \|\ell_2\|_{L^2(\Omega)} + \|f_1\|_{H^{1/2}(\partial\Omega)} + \|f_2\|_{H^{-1/2}(\partial\Omega)}\big).$$

(2) If $0 < \hat{c} \leq \mathrm{Re}(\lambda) < 1$ and $\mathrm{Re}(\mu) \equiv 0$, then the variational problem (2.9) has a unique solution $\mathbb{U} = (v, \mathbf{w}) \in H^1(\Omega) \times X$ that satisfies

$$\|v\|_{H^1(\Omega)} + \|\mathbf{w}\|_X \leq c_2\big(\|\ell_1\|_{L^2(\Omega)} + \|\ell_2\|_{L^2(\Omega)} + \|f_1\|_{H^{1/2}(\partial\Omega)} + \|f_2\|_{H^{-1/2}(\partial\Omega)}\big),$$

where $c_j > 0$ $(j = 1, 2)$ is independent of ℓ_1, ℓ_2, f_1 and f_2.

Proof The trace theorems and Schwarz's inequality ensure the continuity of the antilinear functional L_j $(j = 1, 2)$ on $H^1(\Omega) \times X$ and the existence of a constant c_j which is independent of ℓ_1, ℓ_2, f_1 and f_2 such that

$$\|L_j\| \leq c_j\big(\|\ell_1\|_{L^2(\Omega)} + \|\ell_2\|_{L^2(\Omega)} + \|f_1\|_{H^{1/2}(\partial\Omega)} + \|f_2\|_{H^{-1/2}(\partial\Omega)}\big).$$

For the first part, if $\mathbf{U} = (w, \mathbf{v}) \in H^1(\Omega) \times X$, the assumptions that $\mathrm{Re}(\lambda) \geq \hat{c} > 1$ and $\mathrm{Re}(\mu) \leq 0$ imply that

$$\begin{aligned}|\mathcal{A}_1(\mathbf{U}, \mathbf{U})| \geq |\mathrm{Re}(\mathcal{A}_1(\mathbf{U}, \mathbf{U}))| &= \Big|\mathrm{Re}(\lambda) \int_\Omega (|\nabla w|^2 + |w|^2)\mathrm{d}x - \int_{\partial\Omega} \mathrm{Re}(\mu)|w|^2\mathrm{d}s \\ &\quad + \int_\Omega (|\nabla \cdot \mathbf{v}|^2 + |\mathbf{v}|^2)\mathrm{d}x - \int_{\partial\Omega} (\mathbf{v} \cdot \nu)\overline{w}\mathrm{d}s - \int_{\partial\Omega} w(\overline{\mathbf{v}} \cdot \nu)\mathrm{d}s\Big| \\ &\geq \mathrm{Re}(\lambda)\|w\|^2_{H^1(\Omega)} + \|\mathbf{v}\|^2_X - 2\Big|\int_{\partial\Omega} w(\overline{\mathbf{v}} \cdot \nu)\mathrm{d}s\Big| \\ &\geq (\hat{c} - \varepsilon_1^{-1})\|w\|^2_{H^1(\Omega)} + (1 - \varepsilon_1)\|\mathbf{v}\|^2_X, \quad \hat{c}^{-1} < \varepsilon_1 < 1.\end{aligned}$$

For the second part, if $\mathbb{U}=(v,\mathbf{w})\in H^1(\Omega)\times X$, the assumptions that $0<\hat{c}\le \mathrm{Re}(\lambda)<1$ and $\mathrm{Re}(\mu)\equiv 0$ imply that

$$\begin{aligned}|\mathcal{A}_2(\mathbb{U},\mathbb{U})| \ge |\mathrm{Re}(\mathcal{A}_2(\mathbb{U},\mathbb{U}))| = \Big| &\int_\Omega (|\nabla v|^2+|v|^2)\mathrm{d}x + \int_{\partial\Omega}\mathrm{Re}(\mu)|v|^2\mathrm{d}s \\ &+\mathrm{Re}(\lambda)\int_\Omega(|\nabla\cdot\mathbf{w}|^2+|\mathbf{w}|^2)\mathrm{d}x - 2\mathrm{Re}(\lambda)\mathrm{Re}\Big(\int_{\partial\Omega}(\mathbf{w}\cdot\nu)\overline{v}\mathrm{d}s\Big)\Big| \\ &\ge \|v\|^2_{H^1(\Omega)}+\mathrm{Re}(\lambda)\|\mathbf{w}\|^2_X - 2\mathrm{Re}(\lambda)\Big|\int_{\partial\Omega}(\mathbf{w}\cdot\nu)\overline{v}\mathrm{d}s\Big| \\ &\ge (1-\mathrm{Re}(\lambda)\varepsilon_2^{-1})\|v\|^2_{H^1(\Omega)}+\mathrm{Re}(\lambda)(1-\varepsilon_2)\|\mathbf{w}\|^2_X, \quad \mathrm{Re}(\lambda)<\varepsilon_2<1.\end{aligned}$$

Hence $\mathcal{A}_j$ $(j=1,2)$ is coercive. The continuity of $\mathcal{A}_j$ follows easily from Schwarz's inequality and the classical trace theorems. Then Theorem 2.5 is a direct consequence of the Lax-Milgram Lemma applied to (2.8) and (2.9). □

Combining the above two Theorems 2.4 and 2.5, we obtain the well-posedness of the modified interior transmission problem (2.7).

Theorem 2.6 Assume that $\mathrm{Re}(\lambda)\ge\hat{c}>1$, $\mathrm{Re}(\mu)\le 0$ or $0<\hat{c}\le\mathrm{Re}(\lambda)<1$, $\mathrm{Re}(\mu)\equiv 0$. Then the modified interior transmission problem (2.7) has a unique solution $(w,v)\in H^1(\Omega)\times H^1(\Omega)$ that satisfies

$$\|w\|_{H^1(\Omega)}+\|v\|_{H^1(\Omega)}\le c\big(\|\ell_1\|_{L^2(\Omega)}+\|\ell_2\|_{L^2(\Omega)}+\|f_1\|_{H^{1/2}(\partial\Omega)}+\|f_2\|_{H^{-1/2}(\partial\Omega)}\big),$$

and $c>0$ is independent of ℓ_1, ℓ_2, f_1 and f_2.

Using the analytic Fredholm theory (see Section 8.5 in the book [9]), we get a by-product regarding the discreteness of the following interior transmission eigenvalues problem:

$$\begin{cases}\Delta v+k^2v=0, & \text{in}\quad \Omega,\\ \Delta w+k^2nw=0, & \text{in}\quad \Omega,\\ v-w=0, & \text{on}\quad \partial\Omega,\\ \dfrac{\partial v}{\partial\nu}-\lambda\dfrac{\partial w}{\partial\nu}+\mu v=0, & \text{on}\quad\partial\Omega.\end{cases}\tag{2.10}$$

Definition 2.7 Values of k for which the above interior transmission problem (2.10) has a nontrivial solution pair $(v,w)\in H^1(\Omega)\times H^1(\Omega)$ are called transmission eigenvalues.

Before we establish the discreteness result, we first study the case when there are no real transmission eigenvalues.

Lemma 2.8 Assume that n, λ and μ satisfy Assumption 1.1. If either $\mathrm{Im}(\lambda)<0$ or $\mathrm{Im}(n)>0$ almost everywhere in Ω, then there are no real transmission eigenvalues of the problem (2.10).

Proof Let v and w be a solution pair of the interior transmission problem (2.10). Applying Green's identity to v and w, we have

$$\begin{aligned}\int_\Omega(|\nabla v|^2-k^2|v|^2)\mathrm{d}x &= \int_{\partial\Omega}\overline{v}\frac{\partial v}{\partial\nu}\mathrm{d}s=\lambda\int_{\partial\Omega}\overline{w}\frac{\partial w}{\partial\nu}\mathrm{d}s-\int_{\partial\Omega}\mu|v|^2\mathrm{d}s\\ &=\lambda\int_\Omega(|\nabla w|^2-k^2n|w|^2)\mathrm{d}x-\int_{\partial\Omega}\mu|v|^2\mathrm{d}s.\end{aligned}$$

Since $\mathrm{Im}(\mu) \geq 0$, $\mathrm{Im}(\lambda) \leq 0$, $\mathrm{Im}(\lambda n) \geq 0$, we have that

$$\mathrm{Im}(\lambda) \int_\Omega |\nabla w|^2 \mathrm{d}x = 0, \quad \mathrm{Im}\Big(\int_\Omega \lambda n |w|^2 \mathrm{d}x \Big) = 0, \quad \mathrm{Im}\Big(\int_{\partial\Omega} \mu |v|^2 \mathrm{d}s \Big) = 0.$$

If $\mathrm{Im}(\lambda) < 0$ a.e. in Ω, then $\nabla w = 0$ in Ω, from the equation $w = 0$. From the boundary condition in (2.10) and the integral representation formula, v also vanishes in Ω.

If $\mathrm{Im}(\lambda) = 0$ and $\mathrm{Im}(n) > 0$ a.e. in Ω, then $\lambda \geq \hat{c} > 0$ and $\mathrm{Im}(\lambda n) = \lambda \mathrm{Im}(n) > 0$. Hence, $w = 0$ and $v = 0$ in Ω. This completes the proof. □

Remark 2.9 From the proof of Lemma 2.8, we conclude that k may be an interior transmission eigenvalue of (2.10) if $\mathrm{Im}(\lambda) = 0$ and $\mathrm{Im}(n) = 0$. In this case, if $\mathrm{Im}(\mu) > 0$ almost everywhere on $\partial\Omega$, we further obtain that $v = 0$ on $\partial\Omega$, whence the eigenvalues of (2.10) form a subset of the classical Dirichlet eigenvalues of $-\Delta$ in Ω.

Theorem 2.10 Assume that n, λ and μ satisfy Assumption 1.1 and that $\mathrm{Im}(\lambda) = 0$ and $\mathrm{Im}(n) = 0$. If either $\lambda \geq \hat{c} > 1$, $\mathrm{Re}(\mu) \leq 0$ or $0 < \hat{c} \leq \lambda < 1$, $\mathrm{Re}(\mu) \equiv 0$, then the transmission eigenvalues of (2.10) form a discrete (possibly empty) set with $+\infty$ as the only possible accumulation point.

Proof Let us set

$$\mathcal{H}(\Omega) = \Big\{ (v, w) \in H^1(\Omega) \times H^1(\Omega) : \Delta v \in L^2(\Omega) \text{ and } \Delta w \in L^2(\Omega) \Big\},$$

and consider the operator $\mathcal{F}_{k,n}$ from $\mathcal{H}(\Omega)$ into $L^2(\Omega) \times L^2(\Omega) \times H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ defined by

$$\mathcal{F}_{k,n}(v, w) = \Big(\Delta v + k^2 v, \Delta w + k^2 n w, (v - w)|_{\partial\Omega}, \Big(\frac{\partial v}{\partial \nu} - \lambda \frac{\partial w}{\partial \nu} + \mu v \Big)\Big|_{\partial\Omega} \Big).$$

Then the family of operators $\mathcal{F}_{k,n}$ depends analytically on k. Based on Theorem 2.6 above, we conclude that $\mathcal{F}_{i,1}$ is invertible and has a bounded inverse operator $\mathcal{F}_{i,1}^{-1}$ if either $\lambda \geq \hat{c} > 1$, $\mathrm{Re}(\mu) \leq 0$ or $0 < \hat{c} \leq \lambda < 1$, $\mathrm{Re}(\mu) \equiv 0$. Then,

$$\mathcal{F}_{k,n} = \mathcal{F}_{i,1}(I - \mathcal{F}_{i,1}^{-1}(\mathcal{F}_{i,1} - \mathcal{F}_{k,n})).$$

Since $(\mathcal{F}_{i,1} - \mathcal{F}_{k,n})(w, v) = (-(1+k^2)v, -(1+k^2 n)w, 0, 0)$ is compact based on the compact embedding of $H^1(\Omega)$ to $L^2(\Omega)$, we conclude that the transmission eigenvalues form a discrete (possibly empty) set with $+\infty$ as the only possible accumulation point by the analytic Fredholm theory (Section 8.5 of the book [9]). The proof is complete. □

Remark 2.11 For the case $\lambda = 1$, the discreteness and existence of the transmission eigenvalues have been proven clearly in [3].

2.3 A priori estimates for the transmission problem with L^p data

By employing the boundary integral equation method ([8, 27, 34]), we establish the a priori estimates of the solution to the following general transmission problem (2.11) with boundary data in $L^p(\partial\Omega)$ $(1 < p < 2)$; that is, $h_1 \in L^p(\partial\Omega)$, $h_2 \in L^p(\partial\Omega)$ (these a priori estimates are needed later in the uniqueness proof of the inverse problem and are also interesting in their own right):

$$\begin{cases} \Delta w_1 + k^2 w_1 = 0, & \text{in} \quad \mathbb{R}^3\backslash\overline{\Omega}, \\ \Delta w_2 + k^2 n w_2 = 0, & \text{in} \quad \Omega, \\ w_1 - w_2 = h_1, & \text{on} \quad \partial\Omega, \\ \dfrac{\partial w_1}{\partial \nu} - \lambda \dfrac{\partial w_2}{\partial \nu} + \mu w_1 = h_2, & \text{on} \quad \partial\Omega, \\ \lim\limits_{r\to\infty} r(\dfrac{\partial w_1}{\partial r} - \mathrm{i}k w_1) = 0. \end{cases} \tag{2.11}$$

We introduce the single- and double-layer boundary integral operators

$$(S_k\psi)(x) = \int_{\partial\Omega} \Phi_k(x,y)\psi(y)\mathrm{d}s(y), \ \ x \in \partial\Omega,$$
$$(K_k\psi)(x) = \int_{\partial\Omega} \frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\psi(y)\mathrm{d}s(y), \ \ x \in \partial\Omega,$$

and their normal derivative operators

$$(K_k^{'}\psi)(x) = \int_{\partial\Omega} \frac{\partial \Phi_k(x,y)}{\partial \nu(x)}\psi(y)\mathrm{d}s(y), \ \ x \in \partial\Omega,$$
$$(T_k\psi)(x) = \frac{\partial}{\partial \nu(x)}\int_{\partial\Omega} \frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\psi(y)\mathrm{d}s(y), \ \ x \in \partial\Omega,$$

where $\Phi_k(x,y) = \frac{e^{\mathrm{i}k|x-y|}}{4\pi|x-y|}$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$. Referring to Lemma 9 in [26] and Lemma 1 in [27], we see that the operators S_k, K_k and $K_k^{'}$ are both bounded and compact in $L^p(\partial\Omega)$ $(1 < p < 2)$.

Theorem 2.12 Assuming that n, λ and μ satisfy Assumption 1.1. For h_1, $h_2 \in L^p(\partial\Omega)$ with $4/3 \le p < 2$, the transmission problem (2.11) has a unique solution pair $(w_1, w_2) \in L^2(B_R\backslash\overline{\Omega}) \times L^2(\Omega)$ satisfying that

$$\|w_1\|_{L^2(B_R\backslash\overline{\Omega})} + \|w_2\|_{L^2(\Omega)} \le C\Big(\|h_1\|_{L^p(\partial\Omega)} + \|h_2\|_{L^p(\partial\Omega)}\Big), \tag{2.12}$$

where B_R denotes a large ball centered at the origin with radius R such that $\overline{\Omega} \subset B_R$ and C is a positive constant depending on R.

Proof In order to apply the boundary integral equation method, we divide our proof into two steps (refer to Theorem 2.5 in [34]).

Step One Assume that $k^2 n(x) \equiv k_1^2 > 0$ is a constant. We seek a solution pair $(\widehat{w}_1, \widehat{w}_2)$ of problem (2.11) in the following form:

$$\widehat{w}_1(x) = \int_{\partial\Omega} \Big\{\lambda \Phi_k(x,y)\varphi(y) + \frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\psi(y)\Big\}\mathrm{d}s(y), \quad x \in \mathbb{R}^3\backslash\overline{\Omega},$$
$$\widehat{w}_2(x) = \int_{\partial\Omega} \Big\{\Phi_{k_1}(x,y)\varphi(y) + \frac{1}{\lambda}\frac{\partial \Phi_{k_1}(x,y)}{\partial \nu(y)}\psi(y)\Big\}\mathrm{d}s(y), \quad x \in \Omega,$$

where $\Phi_k(x,y) = \frac{e^{\mathrm{i}k|x-y|}}{4\pi|x-y|}$, $\Phi_{k_1}(x,y) = \frac{e^{\mathrm{i}k_1|x-y|}}{4\pi|x-y|}$ and $\mathrm{Re}(\lambda) \ge \hat{c} > 0$, that is, $\lambda \ne 0$. Then by the jump relations of the single- and double-layer potentials, we have

$$\widehat{w}_1|_{\partial\Omega} = \lambda S_k\varphi + K_k\psi + \frac{1}{2}\psi, \qquad \frac{\partial \widehat{w}_1}{\partial \nu}\Big|_{\partial\Omega} = \lambda K_k^{'}\varphi - \frac{\lambda}{2}\varphi + T_k\psi,$$
$$\widehat{w}_2|_{\partial\Omega} = S_{k_1}\varphi + \frac{1}{\lambda}K_{k_1}\psi - \frac{1}{2\lambda}\psi, \qquad \frac{\partial \widehat{w}_2}{\partial \nu}\Big|_{\partial\Omega} = K_{k_1}^{'}\varphi + \frac{1}{2}\varphi + \frac{1}{\lambda}T_{k_1}\psi.$$

Note that the boundary integral operators S_{k_1}, K_{k_1}, K'_{k_1} and T_{k_1} are defined as S_k, K_k, K'_k and T_k with wave number k in the fundamental solution replaced by k_1. Recalling the conductive boundary condition, we obtain that

$$h_1 = (\widehat{w}_1 - \widehat{w}_2)|_{\partial\Omega} = (\lambda S_k - S_{k_1})\varphi + \Big(K_k - \frac{1}{\lambda}K_{k_1} + \frac{\lambda+1}{2\lambda}I\Big)\psi,$$

$$(\lambda+1)\widehat{w}_1|_{\partial\Omega} = \big(\widehat{w}_1 + \lambda\widehat{w}_2 + \lambda(\widehat{w}_1 - \widehat{w}_2)\big)\big|_{\partial\Omega} = \lambda(S_k + S_{k_1})\varphi + (K_k + K_{k_1})\psi + \lambda h_1,$$

$$\begin{aligned} h_2 =& \Big(\frac{\partial \widehat{w}_1}{\partial\nu} - \lambda\frac{\partial \widehat{w}_2}{\partial\nu} + \mu\widehat{w}_1\Big)\Big|_{\partial\Omega} \\ =& \Big(\lambda(K'_k - K'_{k_1}) + \frac{\lambda\mu}{\lambda+1}(S_k + S_{k_1}) - \lambda I\Big)\varphi \\ &+ \Big(T_k - T_{k_1} + \frac{\mu}{\lambda+1}(K_k + K_{k_1})\Big)\psi + \frac{\lambda\mu}{\lambda+1}h_1. \end{aligned}$$

Here, I denotes the identity operator. Then the transmission problem (2.11) can be reduced to a system of the integral equations

$$A\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}\dfrac{\mu}{\lambda+1}h_1 - \dfrac{1}{\lambda}h_2 \\ \dfrac{2\lambda}{\lambda+1}h_1\end{pmatrix}, \tag{2.13}$$

where the integral matrix operator A is given by

$$A = \begin{pmatrix} K'_{k_1} - K'_k - \dfrac{\mu}{\lambda+1}(S_k + S_{k_1}) & \dfrac{1}{\lambda}(T_{k_1} - T_k) - \dfrac{\mu}{\lambda(\lambda+1)}(K_k + K_{k_1}) \\ \dfrac{2\lambda}{\lambda+1}(\lambda S_k - S_{k_1}) & \dfrac{2}{\lambda+1}(\lambda K_k - K_{k_1}) \end{pmatrix}.$$

Since all elements of A are compact operators in the corresponding Banach spaces, it is easy to see that $A + \mathbb{I}$ ($\mathbb{I}$ denotes the identity matrix) is a Fredholm operator with index zero. Together with the uniqueness of the direct transmission problem (2.11), there exists a unique solution $(\varphi,\psi) \in L^p(\partial\Omega)\times L^p(\partial\Omega)$ of system (2.13) satisfying the estimate

$$\|\varphi\|_{L^p(\partial\Omega)} + \|\psi\|_{L^p(\partial\Omega)} \le C\Big(\|h_1\|_{L^p(\partial\Omega)} + \|h_2\|_{L^p(\partial\Omega)}\Big).$$

Referring to inequalities (2.22) and (2.23) in the paper [34] (Theorem 2.5), that is,

$$\Big\|\int_{\partial\Omega}\Phi_{k_1}(\cdot,y)\varphi(y)\mathrm{d}s(y)\Big\|_{L^2(\Omega)} \le |\partial\Omega|^{1/q}\sup_{y\in\partial\Omega}\|\Phi_{k_1}(\cdot,y)\|_{L^2(\Omega)}\|\varphi\|_{L^p(\partial\Omega)},$$

where $1/p + 1/q = 1$ and

$$\Big\|\int_{\partial\Omega}\frac{\partial\Phi_{k_1}(\cdot,y)}{\partial\nu(y)}\psi(y)\mathrm{d}s(y)\Big\|_{L^2(\Omega)} \le C\|\psi\|_{L^p(\partial\Omega)}, \quad 2 < q \le 4,$$

we achieve estimate (2.12).

Step Two For the general case $n(x) \in L^\infty(\Omega)$, we consider the following problem:

$$\begin{cases} \Delta\widetilde{w}_1 + k^2\widetilde{w}_1 = 0, & \text{in} \quad \mathbb{R}^3\backslash\overline{\Omega}, \\ \Delta\widetilde{w}_2 + k^2 n\widetilde{w}_2 = \ell, & \text{in} \quad \Omega, \\ \widetilde{w}_1 - \widetilde{w}_2 = 0, & \text{on} \quad \partial\Omega, \\ \dfrac{\partial\widetilde{w}_1}{\partial\nu} - \lambda\dfrac{\partial\widetilde{w}_2}{\partial\nu} + \mu\widetilde{w}_1 = 0, & \text{on} \quad \partial\Omega, \\ \lim\limits_{r\to\infty} r\Big(\dfrac{\partial\widetilde{w}_1}{\partial r} - \mathrm{i}k\widetilde{w}_1\Big) = 0, & \end{cases} \tag{2.14}$$

where $l := (k_1^2 - k^2 n(x))\widehat{w}_2 \in L^2(\Omega)$ and $(\widehat{w}_1, \widehat{w}_2)$ denotes the solution of problem (2.11) with $k^2 n(x) \equiv k_1^2$. By Step One, we have that

$$\|\widehat{w}_1\|_{L^2(B_R\backslash\overline{\Omega})} + \|\widehat{w}_2\|_{L^2(\Omega)} \le C\big(\|h_1\|_{L^p(\partial\Omega)} + \|h_2\|_{L^p(\partial\Omega)}\big).$$

Note that problem (2.14) is a special case of the problem (2.1)–(2.4) in paper [4] (that is, $f_2 = 0$ in (2.3)), which has been deeply investigated by the variational method. Therefore, for every $\ell \in L^2(\Omega)$, problem (2.14) has a unique solution $(\widetilde{w}_1, \widetilde{w}_2) \in H^1(B_R\backslash\overline{\Omega}) \times H^1(\Omega)$ satisfying the estimate $\|\widetilde{w}_1\|_{H^1(B_R\backslash\overline{\Omega})} + \|\widetilde{w}_2\|_{H^1(\Omega)} \le C\|\ell\|_{L^2(\Omega)}$.

Define $w_1 := \widetilde{w}_1 + \widehat{w}_1$ and $w_2 := \widetilde{w}_2 + \widehat{w}_2$. Then $(w_1, w_2) \in L^2(B_R\backslash\overline{\Omega}) \times L^2(\Omega)$ is a unique solution of problem (2.11) satisfying estimate (2.12). Thus, the proof is complete. □

3 Uniqueness of the Inverse Transmission Problem

In this section, we consider the uniqueness of the inverse transmission problem (1.1). Under some restrictions on λ and μ, we use a simple and novel method to show that the penetrable obstacle can be uniquely determined by its far-field pattern associated with plane waves.

Theorem 3.1 Assume that (n, λ, μ) and $(\widetilde{n}, \widetilde{\lambda}, \widetilde{\mu})$ satisfy Assumption 1.1 and either $\hat{c} > 1$, $\mathrm{Re}(\mu) \le 0$, $\mathrm{Re}(\widetilde{\mu}) \le 0$ or $0 < \hat{c} \le \mathrm{Re}(\lambda) < 1$, $0 < \hat{c} \le \mathrm{Re}(\widetilde{\lambda}) < 1$, $\mathrm{Re}(\mu) \equiv \mathrm{Re}(\widetilde{\mu}) \equiv 0$. Let $u^\infty(\hat{x}, d)$ and $\widetilde{u}^\infty(\hat{x}, d)$ be the far-field patterns of the scattering solutions $u^s(x, d)$ and $\widetilde{u}^s(x, d)$ to the transmission problem (1.1), with respect to the penetrable scatterers $(\Omega, n, \lambda, \mu)$ and $(\widetilde{\Omega}, \widetilde{n}, \widetilde{\lambda}, \widetilde{\mu})$, respectively. If these satisfy

$$u^\infty(\hat{x}, d) = \widetilde{u}^\infty(\hat{x}, d), \quad \text{for all } \hat{x}, d \in \mathbb{S}, \tag{3.1}$$

then $\Omega = \widetilde{\Omega}$.

Proof Let G be the unbounded connected domain of $\mathbb{R}^3\backslash\overline{(\Omega \cup \widetilde{\Omega})}$. By Rellich's lemma, the assumption $u^\infty(\hat{x}, d) = \widetilde{u}^\infty(\hat{x}, d)$ for all $\hat{x}$, $d \in \mathbb{S}$ implies that $u^s(x, d) = \widetilde{u}^s(x, d)$, $x \in G$. According to Theorem 2.1, for the far-field pattern of incident point source $\Phi_k(x, y)$, we obtain that $u^\infty(-d; x) = \widetilde{u}^\infty(-d; x)$, $d \in \mathbb{S}$, $x \in G$. Thus, Rellich's lemma again gives that $u^s(y; x) = \widetilde{u}^s(y; x)$, $x, y \in G$.

Assume that $\Omega \ne \widetilde{\Omega}$. Then, without loss of generality, we may assume that there exists a point $x^* \in \partial G$ such that $x^* \in \partial\Omega$ and $x^* \notin \partial\widetilde{\Omega}$. We can choose $\delta > 0$ such that $x_j = x^* + \frac{\delta}{j}\nu(x^*)$, $j = 1, 2, \cdots$, is contained in G, where $\nu(x^*)$ is the unit outward normal vector to $\partial\Omega$ at x^*. We let δ be sufficiently small such that $x_j \in O_\delta(x^*)$ for all $j \in \mathbb{N}$, where $O_\delta(x^*)$ is a small ball centered at x^* with radius $\delta > 0$ such that $O_\delta(x^*) \cap \widetilde{\Omega} = \emptyset$ (see Figure 1). Then, we have that $u^s(x; x_j) = \widetilde{u}^s(x; x_j)$, for $x \in G$, $j \in \mathbb{N}$.

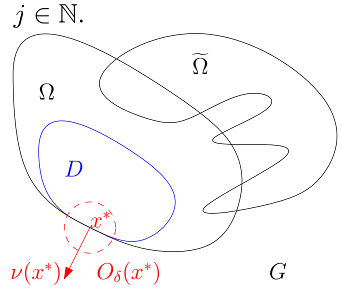

Figure 1 Possible choice of x^*

Since $x^* \in \partial\Omega$, $x^* \notin \partial\widetilde{\Omega}$ and $\partial\Omega$ is smooth enough, there exists a small smooth domain D such that $O_\delta(x^*) \cap \Omega \subset D \subset \Omega\backslash\overline{\widetilde{\Omega}}$. Denoting $\widetilde{u}^j(x) := \widetilde{u}(x;x_j)$, $u^j(x) := u(x;x_j)$ for simplicity, we can then verify that $(\widetilde{u}^j, u^j)$ solves the modified interior transmission problem

$$\begin{cases} \Delta\widetilde{u}^j - \widetilde{u}^j = \ell_1^j, & \text{in} \quad D, \\ \Delta u^j - u^j = \ell_2^j, & \text{in} \quad D, \\ \widetilde{u}_-^j - u_-^j = f_1^j, & \text{on} \quad \partial D, \\ \dfrac{\partial\widetilde{u}_-^j}{\partial\nu} - \lambda\dfrac{\partial u_-^j}{\partial\nu} + \mu\widetilde{u}_-^j = f_2^j, & \text{on} \quad \partial D, \end{cases} \tag{3.2}$$

where $\ell_1^j := -(k^2+1)\widetilde{u}^j|_D$, $\ell_2^j := -(k^2n+1)u^j|_D$, $f_1^j := (\widetilde{u}_-^j - u_-^j)|_{\partial D}$, $f_2^j := (\frac{\partial\widetilde{u}_-^j}{\partial\nu} - \lambda\frac{\partial u_-^j}{\partial\nu} + \mu\widetilde{u}_-^j)|_{\partial D}$. It has been proven that problem (3.2) is well-posed if $\mathrm{Re}(\lambda) \geq \hat{c} > 1$, $\mathrm{Re}(\mu) \leq 0$ or $0 < \hat{c} \leq \mathrm{Re}(\lambda) < 1$, $\mathrm{Re}(\mu) \equiv 0$ (see Theorem 2.6), and its solution $(\widetilde{u}^j, u^j)$ satisfies the a priori estimates

$$\|\widetilde{u}^j\|_{H^1(D)} + \|u^j\|_{H^1(D)} \leq C_1\Big(\|\ell_1^j\|_{L^2(D)} + \|\ell_2^j\|_{L^2(D)} + \|f_1^j\|_{H^{1/2}(\partial D)} + \|f_2^j\|_{H^{-1/2}(\partial D)}\Big) \tag{3.3}$$

for some positive constant C_1 independent of ℓ_1^j, ℓ_2^j, f_1^j and f_2^j.

Next, we claim that

$$\|\ell_1^j\|_{L^2(D)} + \|\ell_2^j\|_{L^2(D)} + \|f_1^j\|_{H^{1/2}(\partial D)} + \|f_2^j\|_{H^{-1/2}(\partial D)} \leq C_2 \tag{3.4}$$

uniformly for all $j \in \mathbb{N}$, where $C_2 > 0$ is independent of j.

In fact, let $y_j = x^* - \frac{\delta}{j}\nu(x^*)$ for sufficiently small $\delta > 0$ such that $y_j \in \Omega$. Define $U^j(x) := u^s(x;x_j) - \Phi_k(x,y_j)$ in $\mathbb{R}^3\backslash\overline{\Omega}$ and $V^j(x) := u(x;x_j)$ in Ω. Then (U^j, V^j) solves the transmission problem:

$$\begin{cases} \Delta U^j + k^2U^j = 0, & \text{in} \quad \mathbb{R}^3\backslash\overline{\Omega}, \\ \Delta V^j + k^2nV^j = 0, & \text{in} \quad \Omega, \\ U_+^j - V_-^j = h_1^j, & \text{on} \quad \partial\Omega, \\ \dfrac{\partial U_+^j}{\partial\nu} - \lambda\dfrac{\partial V_-^j}{\partial\nu} + \mu U_+^j = h_2^j, & \text{on} \quad \partial\Omega, \\ \lim\limits_{r\to\infty} r(\dfrac{\partial U^j}{\partial r} - \mathrm{i}kU^j) = 0, \end{cases} \tag{3.5}$$

where

$$h_1^j := -[\Phi_k(x,y_j) + \Phi_k(x,x_j)]|_{\partial\Omega},$$
$$h_2^j := -\Big[\frac{\partial\Phi_k(x,y_j)}{\partial\nu(x)} + \frac{\partial\Phi_k(x,x_j)}{\partial\nu(x)} + \mu(\Phi_k(x,y_j) + \Phi_k(x,x_j))\Big]\Big|_{\partial\Omega}.$$

Recalling the definition of the fundamental solution $\Phi_k(x,y)$, we conclude that $h_1^j \in L^p(\partial\Omega)$ and $h_2^j \in L^p(\partial\Omega)$ for $1 < p < 2$; that is, $\|h_1^j\|_{L^p(\partial\Omega)} \leq C_3$ and $\|h_2^j\|_{L^p(\partial\Omega)} \leq C_4$ for any $j \in \mathbb{N}$, where C_3, C_4 are positive constants independent of j.

Using Theorem 2.12, we conclude that, for $4/3 \leq p < 2$,

$$\|U^j\|_{L^2(B_R\backslash\overline{\Omega})} + \|V^j\|_{L^2(\Omega)} \leq C_5(\|h_1^j\|_{L^p(\partial\Omega)} + \|h_2^j\|_{L^p(\partial\Omega)}) \leq C_6.$$

Since $\ell_2^j := -(k^2n+1)u^j|_D = -(k^2n+1)V^j|_D$ and $D \subset \Omega$, we obtain that ℓ_2^j is uniformly bounded in $L^2(D)$.

Due to the fact that the distance between x_j and $\widetilde{\Omega}$ is strictly positive, the well-posedness of the transmission problem (1.1) implies that $\widetilde{u}^j$ is uniformly bounded in $L^2(D)$. Hence, $\ell_1^j := -(k^2+1)\widetilde{u}^j|_D$ is uniformly bounded in $L^2(D)$.

Next, we prove that $f_1^j := (\widetilde{u}_-^j - u_-^j)|_{\partial D}$ and $f_2^j := (\frac{\partial \widetilde{u}_-^j}{\partial \nu} - \lambda\frac{\partial u_-^j}{\partial \nu} + \mu\widetilde{u}_-^j)|_{\partial D}$ are uniformly bounded in $H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)$, respectively. According to the above proof, for $x \in \partial G \cap \partial D$, we have

$$u_-^j = u_+^j = \Phi_k(x,x_j) + u_+^s(x;x_j) = \Phi_k(x,x_j) + \widetilde{u}_+^s(x;x_j) = \widetilde{u}_+^j = \widetilde{u}_-^j,$$

$$\begin{aligned}\lambda\frac{\partial u_-^j}{\partial \nu} &= \frac{\partial u_+^j}{\partial \nu} + \mu u_+^j = \frac{\partial[u_+^s(x;x_j)+\Phi_k(x,x_j)]}{\partial\nu(x)} + \mu[u_+^s(x;x_j)+\Phi_k(x,x_j)] \\ &= \frac{\partial[\widetilde{u}_+^s(x;x_j)+\Phi_k(x,x_j)]}{\partial\nu(x)} + \mu[\widetilde{u}_+^s(x;x_j)+\Phi_k(x,x_j)] \\ &= \frac{\partial \widetilde{u}_+^j}{\partial \nu} + \mu\widetilde{u}_+^j = \frac{\partial \widetilde{u}_-^j}{\partial \nu} + \mu\widetilde{u}_-^j;\end{aligned}$$

that is, $f_1^j = 0$ and $f_2^j = 0$ on $\partial G \cap \partial D$. Thus, it is sufficient to show that f_1^j and f_2^j are uniformly bounded in $H^{1/2}(\partial D\backslash\Gamma)$ and $H^{-1/2}(\partial D\backslash\Gamma)$, where $\Gamma = \partial G \cap \partial D$.

Now, considering the special small domain $D^* = D\backslash\overline{O_\delta(x^*)}$, we find that $\|u^j\|_{H^1(D^*)}$ and $\|\widetilde{u}^j\|_{H^1(D^*)}$ are uniformly bounded for all $j \in \mathbb{N}$. By the trace theorem, we obtain the uniformly bounded properties of f_1^j and f_2^j.

Therefore, the claim is complete.

Combining the above with (3.3) and (3.4), we have

$$\|\Phi_k(x,x_j) + \widetilde{u}^s(x;x_j)\|_{H^1(D)} = \|\widetilde{u}^j\|_{H^1(D)} \le C, \quad \text{for all } j \in \mathbb{N}.$$

However, there is a contradiction; since $\|\widetilde{u}^s(x;x_j)\|_{H^1(D)}$ is uniformly bounded, $\|\Phi_k(x,x_j)\|_{H^1(D)}$ is unbounded as $j \longrightarrow \infty$. Hence, $\Omega = \widetilde{\Omega}$. This completes the proof. □

Remark 3.2 If there are impenetrable buried objects inside Ω, the penetrable obstacle can also be uniquely determined by our method, with small modifications in subsections 2.1 (Remark 2.2) and 2.3 (Theorem 2.5 in [34]). Furthermore, the buried object will be determined by the mixed reciprocity relation (2.1), after discovering the penetrable surface (Theorem 3.7 in [23]).

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