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THE TWO-LEVEL STABILIZED FINITE ELEMENT METHOD BASED ON MULTISCALE ENRICHMENT FOR THE STOKES EIGENVALUE PROBLEM*

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Abstract In this paper, we first propose a new stabilized finite element method for the Stokes eigenvalue problem. This new method is based on multiscale enrichment, and is derived from the Stokes eigenvalue problem itself. The convergence of this new stabilized method is proved and the optimal priori error estimates for the eigenfunctions and eigenvalues are also obtained. Moreover, we combine this new stabilized finite element method with the two-level method to give a new two-level stabilized finite element method for the Stokes eigenvalue problem. Furthermore, we have proved a priori error estimates for this new two-level stabilized method. Finally, numerical examples confirm our theoretical analysis and validate the high effectiveness of the new methods.

Key words Two-level; multiscale finite element method; P_1/P_1 elements; the Stokes eigenvalue problem

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1 Introduction

The eigenvalue problems [1] are extensively used in many areas such as structural mechanics and fluid mechanics. Therefore, providing more efficient computational methods for eigenvalue

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problems is a significant element of computing science, and many works [2–17] had addressed this.

In this paper, we turn our attention to the stabilized finite element method for the Stokes eigenvalue problem. It is well known that the simplest confirming elements of this problem, like the lowest equal-order elements, may work well and be of practical importance in scientific computation, because they provide a simple and practical uniform data structure and enough accuracy. However the core problem is that the lowest equal-order finite elements do not satisfy the inf-sup condition and lead to nonphysical pressure oscillations.

In order to overcome these difficulties, stabilized techniques [18–30] have been used to ameliorate the compatibility condition. In particular, Araya, Barrenechea and Valentin in [24] proposed a new stabilized finite element method based on multiscale enrichment for the source Stokes problem. Based on their work, we derive a stabilized finite element method based on multiscale enrichment with the lowest equal order elements for the Stokes eigenvalue problem, prove the convergence of this new method, and obtain the optimal a priori error estimates for the eigenfunctions and eigenvalues. Furthermore, we combine this new stabilized method with two-level method to give a new two-level stabilized finite element method for the Stokes eigenvalue problem. Finally, numerical examples confirm our theory analysis for this new stabilized finite element method and two-level stabilized finite element method and validate the high effectiveness of these methods for the Stokes eigenvalue problem.

The rest of this paper is arranged as follows: in the second section we introduce some basic Sobolev spaces and the Stokes eigenvalue problem. In the third section, a new stabilized finite element approximation based on multiscale enrichment for the Stokes eigenvalue problem is proposed. In the fourth section, we establish convergence analysis of this new method for the Stokes eigenvalues and eigenfunctions and obtain the optimal a priori error estimates. In the fifth section, a new two-level stabilized finite element method for the Stokes eigenvalue problem is given. In the last section, we provide some numerical examples that confirm our theoretical analysis.

2 Preliminaries

2.1 Statement of the problem

Let Ω be a bounded and convex domain in $\Re^2$ with a Lipschitz-continuous boundary $\partial\Omega$. In this paper we consider the following classic Stokes eigenvalue problem with Dirichlet boundary conditions:

$$\begin{cases} -\nu\Delta\mathbf{u}+\nabla p=\lambda\mathbf{u} & \text{in } \ \Omega, \\ \nabla\cdot\mathbf{u}=0 & \text{in } \ \Omega, \\ \mathbf{u}=\mathbf{0} & \text{on } \ \partial\Omega, \end{cases} \tag{2.1}$$

where $\nu>0$ is the viscosity, $\mathbf{u}=(u_1(x_1,x_2),u_2(x_1,x_2))$ the velocity, $p=p(x_1,x_2)$ the pressure and $\lambda\in\Re$ the eigenvalue.

Set

$$X\triangleq(H_0^1(\Omega))^2,\quad Y\triangleq(L^2(\Omega))^2,\quad M\triangleq L_0^2(\Omega)=\{q\in L^2(\Omega):\int_\Omega q\mathrm{d}x=0\}.$$

These are Sobolev spaces. Furthermore, $(L^2(\Omega))^m$ for $m = 1, 2, 4$ are endowed with the L^2-scalar product $(\cdot, \cdot)$ and L^2-norm $\|\cdot\|_0$, and X is equipped with the usual scalar product and norm, which are defined as follows:

$$((\mathbf{u}, \mathbf{v})) \triangleq (\nabla\mathbf{u}, \nabla\mathbf{v})_\Omega, \quad |\mathbf{u}|_1 \triangleq ((\mathbf{u}, \mathbf{u}))^{\frac{1}{2}} = (\nabla\mathbf{u}, \nabla\mathbf{u})_\Omega^{\frac{1}{2}}.$$

Meanwhile, we use the standard Sobolev spaces $W^{m,p}(\Omega)$ with the norm $\|\cdot\|_{m,p}$ and the seminorm $|\cdot|_{m,p}$, where $m, p \geq 0$. In particular, we will denote $H^m(\Omega)$ for $W^{m,2}(\Omega)$, $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$ and $|\cdot|_m$ for $|\cdot|_{m,2}$.

We use the following inequalities [31]:

$$\|\mathbf{u}\|_0 \leq \gamma_0|\mathbf{u}|_1 \quad \forall\ \mathbf{u} \in X, \tag{2.2}$$

$$|\mathbf{u}|_1 \leq \gamma_0\|A\mathbf{u}\|_0 \quad \forall\ \mathbf{u} \in D(A), \tag{2.3}$$

where γ_0 is a positive constant depending only on Ω and A is a Laplace operator that is defined as

$$A\mathbf{u} = -\Delta\mathbf{u} \quad \forall\mathbf{u} \in D(A) := (H^2(\Omega))^2 \cap X. \tag{2.4}$$

The weak form of problem (2.1) is to find $(\mathbf{u}, p, \lambda) \in X \times M \times \Re$ with $\|\mathbf{u}\|_0 = 1$ such that

$$\mathcal{C}((\mathbf{u}, p); (\mathbf{v}, q)) = \lambda(\mathbf{u}, \mathbf{v})_\Omega, \tag{2.5}$$

where

$$\mathcal{C}((\mathbf{u}, p); (\mathbf{v}, q)) \triangleq a(\mathbf{u}, \mathbf{v}) - d(p, \mathbf{v}) + d(q, \mathbf{u}) \quad \forall\ (\mathbf{u}, p)\ ,\ (\mathbf{v}, q)\ \in X \times M,$$

with $a(\mathbf{u}, \mathbf{v}) \triangleq \nu(\nabla\mathbf{u}, \nabla\mathbf{v})_\Omega,\ d(p, \mathbf{v}) \triangleq (p, \nabla\cdot\mathbf{v})_\Omega$.

Moreover, the bilinear form $\mathcal{C}((\mathbf{u}, p); (\mathbf{v}, q))$ satisfies the following property and inf-sup condition [32]:

$$|\mathcal{C}((\mathbf{u}, p); (\mathbf{v}, q))| \leq C(\nu\|\nabla\mathbf{u}\|_0 + \|p\|_0)(\|\nabla\mathbf{v}\|_0 + \|q\|_0) \quad \forall(\mathbf{u}, p), (\mathbf{v}, q) \in X \times M,$$

$$\beta(\nu\|\nabla\mathbf{u}\|_0 + \|p\|_0) \leq \sup_{(\mathbf{v},q)\in X\times M} \frac{|\mathcal{C}((\mathbf{u}, p); (\mathbf{v}, q))|}{\|\nabla\mathbf{v}\|_0 + \|q\|_0} \qquad \forall(\mathbf{u}, p) \in X \times M,$$

where C and β are the positive constants depending only on Ω.

Recalling the spectral theory [1], we know that the Stokes eigenvalue problem (2.5) has a positive eigenvalue sequence λ_j which is assumed to be increasingly ordered as follows:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \leq \lim_{j\to+\infty} \lambda_j = +\infty.$$

We also know the corresponding eigenfunctions

$$(\mathbf{u}_1, p_1), (\mathbf{u}_2, p_2), \cdots, (\mathbf{u}_j, p_j), \cdots,$$

with $(\mathbf{u}_i, \mathbf{u}_j) = \delta_{ij}$. In this paper, for simplicity, we only consider simple eigenvalues.

Furthermore, C will denote a positive constant depending only on the data $(\Omega, \nu, \mathbf{f})$ and not on the mesh parameter h.

3 New Stabilized Finite Element Method

In this section, we introduce the finite element approximation for problem (2.1). Let τ_h be a triangulation of Ω formed by closed triangle elements K with boundary ∂K. $\{p_j, j = 1, 2, \cdots, N\}$ is the set of all vertices of τ_h. ε_h denotes the set of all edges of τ_h and $\varepsilon_h^{\text{int}}$ denotes the set of the interior edges of τ_h that are no part of $\partial\Omega$. Furthermore, h_K denotes the diameter of the element K, and $h = \max\limits_{K\in\tau_h} h_K$ is the mesh parameter. Assume that the partition τ_h is regular and quasi-uniform; that is to say, there exists $C > 0$ such that

$$\frac{h_K}{\varrho_K} \le C, \quad \forall\ K \in \tau_h, \tag{3.1}$$

where ϱ_K denotes the diameter of the largest ball contained in K. X_h and M_h denote the standard finite element spaces of the approximation for velocity and pressure on τ_h, respectively. These spaces are defined as

$$X_h \triangleq \{\mathbf{v} \in X \cap (C^0(\overline{\Omega}))^2 : \mathbf{v}|_K \in P_1(K)^2,\ \forall\ K \in \tau_h\},$$
$$M_h \triangleq \{q \in M : q|_K \in P_1(K),\ \forall\ K \in \tau_h\},$$

where $P_1(K)$ denotes the space of linear functions in domain K.

3.1 Some useful lemmas

Lemma 3.1 ([31]) We have the trace inequalities

$$\|\mathbf{v}\|^2_{(L^2(E))^2} \le C(h_K^{-1}\|\mathbf{v}\|^2_{(L^2(K))^2} + h_K|\mathbf{v}|^2_{(H^1(K))^2}) \quad \forall\ \mathbf{v} \in (H^1(K))^2, \tag{3.2}$$
$$\|\partial_{\mathbf{n}}\mathbf{v}\|^2_{(L^2(E))^2} \le C(h_K^{-1}|\mathbf{v}|^2_{(H^1(K))^2} + h_K|\mathbf{v}|^2_{(H^2(K))^2}) \quad \forall\ \mathbf{v} \in (H^2(K))^2, \tag{3.3}$$

where $E \in \partial K$, and C depends on the minimum angle of $K \in \tau_h$.

Lemma 3.2 ([36]) Under the regular and quasi-uniform assumption of τ_h, the following properties hold:

$$|\mathbf{v}_h|_1 \le Ch^{-1}\|\mathbf{v}_h\|_0 \quad \forall \mathbf{v}_h \in X_h, \tag{3.4}$$
$$\|\mathbf{v}_h\|_{\infty,K} \le Ch^{-1}\|\mathbf{v}_h\|_{0,K} \quad \forall \mathbf{v}_h \in X_h, \tag{3.5}$$
$$|p_h|_1 \le Ch^{-1}\|p_h\|_0 \quad \forall p_h \in M_h. \tag{3.6}$$

3.2 Stabilized finite element method for the stokes eigenvalue problem

In this section, we introduce a new stabilized finite element method based on multiscale enrichment for the Stokes eigenvalue problem. The specific process is as follows:

Let E_h be a finite dimensional space called a multiscale space, such that

$$E_h \subset (H^1(\tau_h))^2,\ E_h \cap X_h = \{\mathbf{0}\},$$

where $(H^1(\tau_h))^2 \triangleq \{\mathbf{v} \in Y : \mathbf{v}|_K \in (H^1(K))^2\}$.

Then, according to the above notations we consider the following Petrov-Galerkin variational formulation: find $\mathbf{u}_h + \mathbf{u}_e \in X_h \oplus E_h$ and $q_h \in M_h$ such that

$$\nu(\nabla(\mathbf{u}_h + \mathbf{u}_e), \nabla\mathbf{v}_h)_\Omega - (p_h, \nabla\cdot\mathbf{v}_h)_\Omega + (q_h, \nabla\cdot(\mathbf{u}_h + \mathbf{u}_e))_\Omega = \lambda_h(\mathbf{u}_h, \mathbf{v}_h)_\Omega \tag{3.7}$$

for all $\mathbf{v}_h \in X_h \oplus E_h^0$ and $q_h \in M_h$, where $E_h^0 := \{\mathbf{v} \in (H^1(\tau_h))^2 : \mathbf{v}|_K \in (H_0^1(K))^2\}$.

The formulation (3.7) can be rewritten as

$$\nu(\nabla(\mathbf{u}_h + \mathbf{u}_e), \nabla\mathbf{v}_h)_\Omega - (p_h, \nabla\cdot\mathbf{v}_h)_\Omega + (q_h, \nabla\cdot(\mathbf{u}_h + \mathbf{u}_e))_\Omega = \lambda_h(\mathbf{u}_h, \mathbf{v}_h)_\Omega, \tag{3.8}$$
$$\nu(\nabla(\mathbf{u}_h + \mathbf{u}_e), \nabla\mathbf{v}_b)_K - (p_h, \nabla\cdot\mathbf{v}_b)_K = \lambda_h(\mathbf{u}_h, \mathbf{v}_b)_K \tag{3.9}$$

for all $\mathbf{v}_h \in X_h, \mathbf{v}_b \in E_h^0, q_h \in M_h$ and $K \in \tau_h$. The formulation (3.9) is equivalent to

$$\begin{cases} -\nu\Delta\mathbf{u}_e = \nu\Delta\mathbf{u}_h + \lambda_h\mathbf{u}_h - \nabla p_h & \text{in } \Omega, \\ \mathbf{u}_e = \mathbf{g}_e & \text{on } E \subset \partial K, K \in \tau_h, \end{cases} \tag{3.10}$$

where $\mathbf{g}_e = \mathbf{0}$ if $E \in \partial\Omega$, and on the interval edges, $\mathbf{g}_e$ is the solution of

$$\begin{cases} -\nu \partial_{ss} \mathbf{g}_e = \dfrac{1}{h_E} [[\nu \partial_{\mathbf{n}} \mathbf{u}_h + p_h \mathbf{I} \cdot \mathbf{n}]]_E \quad \text{in } E, \\ \mathbf{g}_e = \mathbf{0} \quad \text{at the vertices}, \end{cases} \tag{3.11}$$

where $h_E = |E| = \text{meas}(E)$ denotes the length of $E \subset \partial K$, $\mathbf{n}$ is the normal outward vector on ∂K, ∂_s is the tangential derivative, $\partial\mathbf{n}$ is the normal derivative, and $\mathbf{I}$ is the $\Re_{2\times 2}$ identity matrix. Furthermore, we use $[[\mathbf{v}]]_E$ to denote the jump of function $\mathbf{v} \in (H^1(\tau_h))^2$ across the edge E; that is,

$$[[\mathbf{v}]]_E := (\mathbf{v}|_{K_1})|_E \cdot \mathbf{n}_1 + (\mathbf{v}|_{K_2})|_E \cdot \mathbf{n}_2, \quad \forall\, K_1,\ K_2 \in \tau_h,$$

and $K_1 \cap K_2 = E \in \varepsilon^{\text{int}}$. Here, $\mathbf{n}_1$ and $\mathbf{n}_2$ are the outer normals about K_1 and K_2, respectively (see Figure 1). In particular, if $E \in \partial\Omega$, then we define $[[\mathbf{v}]]_E = \mathbf{v} \cdot \mathbf{n}$.

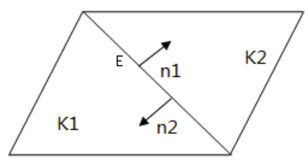

Figure 1 Two neighboring elements $K_1, K_2 \in \tau_h$ sharing the edge $E \in \varepsilon^{\text{int}}$

On each $K \in \tau_h$, we can set $\mathbf{u}_e = \mathbf{u}_e^K + \mathbf{u}_e^{\partial K}$ and get the following auxiliary problems:

$$\begin{cases} -\nu \Delta \mathbf{u}_e^K = \nu \Delta \mathbf{u}_h + \lambda_h \mathbf{u}_h - \nabla p_h & \text{in } K \in \tau_h, \\ \mathbf{u}_e^K = \mathbf{0} & \text{on } \partial K \in \tau_h, \end{cases} \tag{3.12}$$

and

$$\begin{cases} -\nu \Delta \mathbf{u}_e^{\partial K} = \mathbf{0} & \text{in } K \in \tau_h, \\ \mathbf{u}_e^{\partial K} = \mathbf{g}_e & \text{on } \partial K \in \tau_h, \end{cases} \tag{3.13}$$

where $\mathbf{g}_e$ is the solution of (3.11). Such problems as (3.12) and (3.13) are well-posed, and (3.9) is satisfied.

Now we define two operators, $\mathcal{M}_K : (L^2(K))^2 \to (H_0^1(K))^2$ and $\mathcal{B}_K : (L^2(\partial K))^2 \to (H^1(K))^2$, such that

$$\mathbf{u}_e^K := \frac{1}{\nu} \mathcal{M}_K (\nu \Delta \mathbf{u}_h + \lambda_h \mathbf{u}_h - \nabla p_h) \qquad \forall K \in \tau_h \tag{3.14}$$

and

$$\mathbf{u}_e^{\partial K} := \frac{1}{\nu} \mathcal{B}_K ([[\nu \partial_{\mathbf{n}} \mathbf{u}_h + p_h \mathbf{I} \cdot \mathbf{n}]]) \qquad \forall K \in \tau_h. \tag{3.15}$$

Now, the enriched part $\mathbf{u}_e$ is identified via (3.14) and (3.15). Therefore, we can perform the static condensation to derive a new multiscale finite element method for the Stokes eigenvalue problem (2.1). Firstly, integrating by parts, we have the following equalities on each $K \in \tau_h$:

$$\nu (\nabla \mathbf{u}_e, \nabla \mathbf{v}_h)_K = -\nu (\mathbf{u}_e, \Delta \mathbf{v}_h)_K + (\mathbf{u}_e, \nu \partial_{\mathbf{n}} \mathbf{v}_h)_{\partial K}, \tag{3.16}$$

$$(q_h, \nabla \cdot \mathbf{u}_e)_K = -(\mathbf{u}_e, \nabla q_h)_K + (\mathbf{u}_e, q_h \mathbf{I} \cdot \mathbf{n})_{\partial K}. \tag{3.17}$$

We make use of (3.16) and (3.17), and after a static condensation process, which is similar to the method presented in [24], we can rewrite the formulation (3.8) in the following form:

$$\mathcal{C}_h((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))=\lambda_h(\mathbf{u}_h,\mathbf{v}_h)_\Omega+\sum_{K\in\tau_h}\frac{\tau_1}{\nu}(\lambda_h\mathbf{u}_h,\nabla q_h)_K, \tag{3.18}$$

where $\tau_1=\frac{h_K^2}{\alpha_1},\tau_2=\frac{h_E}{\alpha_2},\alpha_1,\alpha_2>0$, and $\mathcal{C}_h(\cdot;\cdot)$ is defined as

$$\begin{aligned}\mathcal{C}_h((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))=&\,\nu(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h)_\Omega-(p_h,\nabla\cdot\mathbf{v}_h)_\Omega+(q_h,\nabla\cdot\mathbf{u}_h)_\Omega\\&+\sum_{K\in\tau_h}\frac{\tau_1}{\nu}(\nabla p_h,\nabla q_h)_K+\sum_{E\in\varepsilon_h^{\rm int}}\frac{\tau_2}{\nu}([[\nu\partial_{\mathbf{n}}\mathbf{u}_h]]_E,[[\nu\partial_{\mathbf{n}}\mathbf{v}_h]]_E)_E.\end{aligned} \tag{3.19}$$

Problem (3.18) can be simplified to a generalized eigenvalue problem which attains a finite number of eigenpairs $(\lambda_{h,j},(\mathbf{u}_{h,j},p_{h,j})),1\le j\le N$, with positive eigenvalues. We assume the eigenvalues to be increasingly ordered as follows:

$$0<\lambda_{h,1}\le\lambda_{h,2}\le\cdots\le\lambda_{h,j}\le\cdots\le\lambda_{h,N},$$

and $(\mathbf{u}_{h,i},\mathbf{u}_{h,j})=\delta_{i,j},1\le i,j\le N$.

In order to simplify the notations, in the ensuing paragraphs the subindex j in $\lambda_j,\lambda_{h,j},\mathbf{u}_j,\mathbf{u}_{h,j}$, p_j and $p_{h,j}$ will be dropped.

Our first goal is to prove that the approximation solutions of the discrete eigenvalue problem (3.18) converge to the solutions of the spectral problem (2.5). We first present the a priori error estimates for the following classic Stokes problem: $\forall\mathbf{f}\in Y$, find $(\mathbf{u},p)\in X\times M$ such that

$$\mathcal{C}((\mathbf{u},p);(\mathbf{v},q))=(\mathbf{f},\mathbf{v})_\Omega\quad\forall(\mathbf{v},q)\in X\times M. \tag{3.20}$$

Since the inf-sup condition holds and Ω is convex, problem (3.20) has a unique solution and for a given $\mathbf{f}\in Y$ it holds that [31, 32]

$$\nu|\mathbf{u}|_2+|p|_1\le C\|\mathbf{f}\|_0. \tag{3.21}$$

The stabilized finite element method for the classic Stokes problem is arrived at by finding $(\mathbf{u}_h,p_h)\in X_h\times M_h$ such that

$$\mathcal{C}_h((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))=(\mathbf{f},\mathbf{v}_h)+\sum_{K\in\tau_h}\frac{\tau_1}{\nu}(\mathbf{f},\nabla q_h)_K,\quad\forall(\mathbf{v}_h,q_h)\in X_h\times M_h. \tag{3.22}$$

Here we introduce the following mesh-dependent norms:

$$|||\mathbf{v}|||_h^2\triangleq\nu|\mathbf{v}|_1^2+\sum_{E\in\varepsilon_h}\frac{\tau_2}{\nu}\|[[\nu\partial_{\mathbf{n}}\mathbf{v}]]_E\|_{0,E}^2,\ \|q\|_h^2\triangleq\sum_{K\in\tau_h}\frac{\tau_1}{\nu}|q|_{1,K}^2.$$

Lemma 3.3 ([37]) It holds that

$$\sup_{0\ne(\mathbf{v}_h,q_h)\in X_h\times M_h}\frac{\mathcal{C}_h((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))}{|\mathbf{v}_h|_1+\|q_h\|_0}\ge\beta_1(|\mathbf{u}_h|_1+\|p_h\|_0)\ \ \forall\ (\mathbf{u}_h,p_h)\in X_h\times M_h,$$

where $\beta_1>0$ depending on ν, but not on h.

Theorem 3.4 ([24]) Assume that $(\mathbf{u},p)\in D(A)\times(M\cap H^1(\Omega))$ is the solution of (3.20) and $(\mathbf{u}_h,p_h)$ is the solution of (3.22). Then it holds that

$$\|\mathbf{u}-\mathbf{u}_h\|_0+h(\nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0+\|p-p_h\|_0)\le Ch^2(\nu|\mathbf{u}|_2+|p|_1), \tag{3.23}$$

$$|||\mathbf{u}-\mathbf{u}_h|||_h+\|p-p_h\|_h\le Ch(\nu|\mathbf{u}|_2+|p|_1). \tag{3.24}$$

4 Error Analysis

4.1 Spectral approximation

In this section, we use the classic spectral approximation theory [1, 2] to obtain the convergence of the eigenvalues and eigenfunctions with optimal order. Let $N = (X, M)$. Then (3.20) is expressed equivalently as follows: $\forall \mathbf{f} \in Y$, so there exists an unique solution $(\mathbf{u}, p) \in N$ such that

$$\mathcal{C}((\mathbf{u}, p); (\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v})_\Omega \quad \forall (\mathbf{v}, q) \in N. \tag{4.1}$$

Then we define the operators $T : N \to N$ and $R : Y \to Y$ as follows:

$$T\mathbf{F} = (\mathbf{u}, p), \qquad \mathbf{F} = (\mathbf{f}, \sigma),$$
$$R\mathbf{f} = \mathbf{u}.$$

From the definitions of the operators T and R, we know that T and R are continuous operators. In fact, making use of (3.21) and the Poincaré inequality, we get

$$\|T\mathbf{F}\|_N = \|(\mathbf{u}, p)\|_N = \|\nabla \mathbf{u}\|_0 + \|p\|_0 \le C\|\mathbf{f}\|_0 \le C\|\mathbf{F}\|_N,$$
$$\|R\mathbf{f}\|_0 = \|\mathbf{u}\|_0 \le C\|\mathbf{f}\|_0.$$

On the other hand, there exits a unique solution $(\mathbf{u}_h, p_h) \in (X_h, M_h) \subseteq N$ such that

$$\mathcal{C}_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) = (\mathbf{f}, \mathbf{v}_h)_\Omega + \sum_{K \in \tau_h} \frac{\tau_1}{\nu}(\mathbf{f}, \nabla q_h)_K \quad \forall (\mathbf{v}_h, q_h) \in N. \tag{4.2}$$

Then we define the operators $T_h : N \to N$ and $R_h : Y \to Y$ as follows:

$$T_h\mathbf{F} = (\mathbf{u}_h, p_h), \qquad \mathbf{F} = (\mathbf{f}, \sigma),$$
$$R_h\mathbf{f} = \mathbf{u}_h.$$

T_h and R_h are also continuous operators. In fact, making use of (3.21), Theorem 3.4 and the Poincaré inequality, we get

$$\begin{aligned}\|T_h\mathbf{F}\|_N = \|(\mathbf{u}_h, p_h)\|_N &= |||\mathbf{u}_h|||_h + \|p_h\|_h \\ &\le C|||\mathbf{u} - \mathbf{u}_h|||_h + \|p - p_h\|_h + |||\mathbf{u}|||_h + \|p\|_h \\ &\le C\|\mathbf{f}\|_0 \le C\|\mathbf{F}\|_N,\end{aligned}$$
$$\|R_h\mathbf{f}\|_0 = \|\mathbf{u}_h\|_0 \le \|\mathbf{u} - \mathbf{u}_h\|_0 + \|\mathbf{u}\|_0 \le C\|\mathbf{f}\|_0.$$

Letting $\lambda \ne 0$, we notice that $\mathbf{f}$ is an eigenfunction of R of eigenvalue λ if and only if $(\mathbf{f}, p, \frac{1}{\lambda})$ is the solution of (2.5) for some $p \in M$, and $F = (\mathbf{f}, p)$ is an eigenfunction of T of eigenvalue λ if and only if $(\mathbf{f}, p, \frac{1}{\lambda})$ is the solution of (2.5). Meanwhile, letting $\lambda_h \ne 0$, $\mathbf{f}$ is an eigenfunction of R_h of eigenvalue λ_h if and only if $(\mathbf{f}, p_h, \frac{1}{\lambda_h})$ is the solution of (3.18) for some $p_h \in M_h$, and $F = (\mathbf{f}, p_h)$ is an eigenfunction of T_h of eigenvalue λ_h if and only if $(\mathbf{f}, p_h, \frac{1}{\lambda_h})$ is the solution of (3.18).

4.2 Error analysis

In order to get the error estimate, we first need to introduce the Galerkin projection $(\mathbf{R}_h(\mathbf{u}, p), Q_h(\mathbf{u}, p)) : (X, M) \to (X_h, M_h)$ defined by

$$\mathcal{C}_h((\mathbf{R}_h, Q_h); (\mathbf{v}_h, q_h)) = \mathcal{C}((\mathbf{u}, p); (\mathbf{v}_h, q_h)) \quad \forall (\mathbf{v}_h, q_h) \in X_h \times M_h; \tag{4.3}$$

that is to say,

$$\begin{aligned}&\nu(\nabla\mathbf{R}_h,\nabla\mathbf{v}_h)_\Omega-(Q_h,\nabla\cdot\mathbf{v}_h)_\Omega+(q_h,\nabla\cdot\mathbf{R}_h)_\Omega\\&+\sum_{K\in\tau_h}\frac{h_K^2}{8\nu}(\nabla Q_h,\nabla q_h)_K+\sum_{E\in\varepsilon_h^{\rm int}}\frac{h_E}{12\nu}([[\nu\partial_{\mathbf{n}}\mathbf{R}_h]],[[\nu\partial_{\mathbf{n}}\mathbf{v}_h]])_E\\=\;&\nu(\nabla\mathbf{u},\nabla\mathbf{v}_h)_\Omega-(p,\nabla\cdot\mathbf{v}_h)_\Omega+(q_h,\nabla\cdot\mathbf{u})_\Omega.\end{aligned}\tag{4.4}$$

We remark that if $\mathbf{u}\in D(A)$ and $p\in H^1(\Omega)\cap M$, we know that

$$\mathcal{C}_h((\mathbf{R}_h,Q_h);(\mathbf{v}_h,q_h))=\mathcal{C}_h((\mathbf{u},p);(\mathbf{v}_h,q_h))-\sum_{K\in\tau_h}\frac{h_K^2}{8\nu}(\nabla p,\nabla q_h)_K\tag{4.5}$$

for all $(\mathbf{v}_h,q_h)\in X_h\times M_h$, because $[[\nu\partial_{\mathbf{n}}\mathbf{u}]]_E=\mathbf{0}$.

Lemma 4.1 ([37]) Let $(\mathbf{u},p)\in X\times M$ and (4.3) hold. Then the Galerkin projection $(\mathbf{R}_h,Q_h)$ satisfies

$$|\mathbf{u}-\mathbf{R}_h|_1+\|p-Q_h\|_0\le C(|\mathbf{u}|_1+\|p\|_0).\tag{4.6}$$

Furthermore, if $\forall(\mathbf{u},p)\in D(A)\times(H^1(\Omega)\cap M)$, then it holds that

$$\|\mathbf{u}-\mathbf{R}_h\|_0+h(\nu|\mathbf{u}-\mathbf{R}_h|_1+\|p-Q_h\|_0)\le Ch^2(\nu|\mathbf{u}|_2+|p|_1)\tag{4.7}$$

and

$$|||\mathbf{u}-\mathbf{R}_h|||_h+\|p-Q_h\|_h\le Ch(\nu|\mathbf{u}|_2+|p|_1).\tag{4.8}$$

Theorem 4.2 Assume that $(\mathbf{u},p,\lambda)\in X\times M\times\Re$ with $\|\mathbf{u}\|_0=1$ is the solution of (2.5) and that $(\mathbf{u},p)\in D(A)\times(M\cap H^1(\Omega))$. Then there exists a discrete solution $(\mathbf{u}_h,p_h,\lambda_h)\in X_h\times M_h\times\Re$ of (3.18) with $\|\mathbf{u}_h\|_0=1$ such that

$$\begin{aligned}&|\lambda-\lambda_h|\le Ch^2,\\&\nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0+\|p-p_h\|_0\le Ch,\\&|||\mathbf{u}-\mathbf{u}_h|||_h+||p-p_h||_h\le Ch,\\&\|\mathbf{u}-\mathbf{u}_h\|_0\le Ch^2.\end{aligned}$$

Proof We will use the spectral approximation theory [1] to prove that the operator T_h converges to T and that R_h converges to R as h goes to zero.

Using Theorem 3.4 and (3.21), we can get that for $\forall\mathbf{f}\in Y$ and $\mathbf{F}\in N$,

$$\begin{aligned}&\|R\mathbf{f}-R_h\mathbf{f}\|_0=\|\mathbf{u}-\mathbf{u}_h\|_0\le Ch^2\|\mathbf{f}\|_0,\\&\|T\mathbf{F}-T_h\mathbf{F}\|_N=|||\mathbf{u}-\mathbf{u}_h|||_h+||p-p_h||_h\le Ch\|\mathbf{f}\|_0\le Ch\|\mathbf{F}\|_N,\end{aligned}$$

so $R_h\to R$ and $T_h\to T$ in the norm when h goes to zero.

For any Hilbert space χ, the space of compact operators in χ is close in $\mathcal{B}(\chi)$, where $\mathcal{B}(\chi)=\{L:\chi\to\chi, L \text{ is linear and continuous}\}$.

Letting $(\mathbf{u},p,\lambda),\|\mathbf{u}\|_0=1,\lambda\ne0$ be the solution of (2.5), by Remark 7.3 and 7.4 in [1], for small h we obtain that there exists $(\mathbf{u}_h,p_h,\lambda_h),\|\mathbf{u}_h\|_0=1,\lambda_h\ne0$ such that

$$\begin{aligned}&|\lambda-\lambda_h|\le C\|R-R_h\|\le Ch^2,\\&|||\mathbf{u}-\mathbf{u}_h|||_h+||p-p_h||_h\le C\|T-T_h\|\le Ch,\\&\|\mathbf{u}-\mathbf{u}_h\|_0\le C\|R-R_h\|\le Ch^2.\end{aligned}$$

Now, in what follows, our main work is to estimate the term $\|p-p_h\|_0$. Making use of (4.3), Lemma 3.3 and the Poincaré inequality, we have

$$\begin{aligned}
&\|Q_h-p_h\|_0\\
&\le \frac{1}{\beta_1}\sup_{0\neq(\mathbf{v}_h,q_h)\in X_h\times M_h}\frac{\mathcal{C}_h((\mathbf{R}_h-\mathbf{u}_h,Q_h-p_h);(\mathbf{v}_h,q_h))}{|\mathbf{v}_h|_1+\|q_h\|_0}\\
&=\frac{1}{\beta_1}\sup_{0\neq(\mathbf{v}_h,q_h)\in X_h\times M_h}\frac{\mathcal{C}_h((\mathbf{R}_h,Q_h);(\mathbf{v}_h,q_h))-\lambda_h(\mathbf{u}_h,\mathbf{v}_h)_\Omega-\sum\limits_{K\in\tau_h}\frac{h_K^2}{8\nu}(\lambda_h\mathbf{u}_h,\nabla q_h)_K}{|\mathbf{v}_h|_1+\|q_h\|_0}\\
&=\frac{1}{\beta_1}\sup_{0\neq(\mathbf{v}_h,q_h)\in X_h\times M_h}\left\{\frac{\mathcal{C}((\mathbf{R}_h,Q_h);(\mathbf{v}_h,q_h))-\mathcal{C}_h((\mathbf{u},p);(\mathbf{v}_h,q_h))}{|\mathbf{v}_h|_1+\|q_h\|_0}\right.\\
&\quad\left.+\frac{-\sum\limits_{K\in\tau_h}\frac{h_K^2}{8\nu}(\lambda_h\mathbf{u}_h,\nabla q_h)_K+\lambda(\mathbf{u},\mathbf{v}_h)_\Omega-\lambda_h(\mathbf{u}_h,\mathbf{v}_h)_\Omega}{|\mathbf{v}_h|_1+\|q_h\|_0}\right\}\\
&=\frac{1}{\beta_1}\sup_{0\neq(\mathbf{v}_h,q_h)\in X_h\times M_h}\frac{\lambda(\mathbf{u},\mathbf{v}_h)_\Omega-\lambda_h(\mathbf{u}_h,\mathbf{v}_h)_\Omega-\sum\limits_{K\in\tau_h}\frac{h_K^2}{8\nu}(\lambda_h\mathbf{u}_h,\nabla q_h)_K}{|\mathbf{v}_h|_1+\|q_h\|_0}\\
&=\mathrm{I}+\mathrm{II}.
\end{aligned}\tag{4.9}$$

Now we estimate the terms I and II as follows:

$$\begin{aligned}
\mathrm{I}&=\frac{1}{\beta_1}\sup_{0\neq(\mathbf{v}_h,q_h)\in X_h\times M_h}\frac{\lambda(\mathbf{u},\mathbf{v}_h)_\Omega-\lambda_h(\mathbf{u}_h,\mathbf{v}_h)_\Omega}{|\mathbf{v}_h|_1+\|q_h\|_0}\\
&=\frac{1}{\beta_1}\sup_{0\neq(\mathbf{v}_h,q_h)\in X_h\times M_h}\frac{\lambda(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h)_\Omega+(\lambda-\lambda_h)(\mathbf{u}_h,\mathbf{v}_h)_\Omega}{|\mathbf{v}_h|_1+\|q_h\|_0}\\
&\le\frac{1}{\beta_1}\frac{\lambda\|\mathbf{u}-\mathbf{u}_h\|_0\gamma_0|\mathbf{v}_h|_1+|\lambda-\lambda_h|\cdot\|\mathbf{u}_h\|_0\gamma_0|\mathbf{v}_h|_1}{|\mathbf{v}_h|_1+\|q_h\|_0}\\
&\le Ch^2,
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
\mathrm{II}&=\frac{1}{\beta_1}\sup_{0\neq(\mathbf{v}_h,q_h)\in X_h\times M_h}\frac{-\sum\limits_{K\in\tau_h}\frac{h_K^2}{8\nu}(\lambda_h\mathbf{u}_h,\nabla q_h)_K}{|\mathbf{v}_h|_1+\|q_h\|_0}\\
&=\frac{1}{\beta_1}\sup_{0\neq(\mathbf{v}_h,q_h)\in X_h\times M_h}\frac{(\lambda-\lambda_h)\sum\limits_{K\in\tau_h}\frac{h_K^2}{8\nu}(\mathbf{u}_h,\nabla q_h)_K-\lambda\sum\limits_{K\in\tau_h}\frac{h_K^2}{8\nu}(\mathbf{u}_h,\nabla q_h)_K}{|\mathbf{v}_h|_1+\|q_h\|_0}\\
&=\frac{1}{\beta_1}\sup_{0\neq(\mathbf{v}_h,q_h)\in X_h\times M_h}\left\{\frac{(\lambda-\lambda_h)\sum\limits_{K\in\tau_h}\frac{h_K^2}{8\nu}(\mathbf{u}_h-\mathbf{u},\nabla q_h)_K+(\lambda-\lambda_h)\sum\limits_{K\in\tau_h}\frac{h_K^2}{8\nu}(\mathbf{u},\nabla q_h)_K}{|\mathbf{v}_h|_1+\|q_h\|_0}\right.\\
&\quad\left.+\frac{-\lambda\sum\limits_{K\in\tau_h}\frac{h_K^2}{8\nu}(\mathbf{u}_h-\mathbf{u},\nabla q_h)_K-\lambda\sum\limits_{K\in\tau_h}\frac{h_K^2}{8\nu}(\mathbf{u},\nabla q_h)_K}{|\mathbf{v}_h|_1+\|q_h\|_0}\right\}\\
&\le Ch.
\end{aligned}\tag{4.11}$$

Then, gathering (4.9), (4.10) and (4.11), it holds that

$$\|Q_h-p_h\|_0\le Ch.\tag{4.12}$$

Furthermore, we can apply the triangular inequality and (4.7) to yield that the following inequality holds:

$$\|p-p_h\|_0 \le \|p-Q_h\|_0 + \|Q_h-p_h\|_0 \le Ch. \tag{4.13}$$

This finishes the proof of Theorem 4.2. □

5 Two-level Stabilized Finite Element Method

In this section, we combine the two-level method with our new stabilized finite element method given in (3.18) to present the two-level stabilized finite element method for the Stokes eigenvalue problem and to derive the optimal error estimates. This two-level stabilized finite element involves solving a Stokes eigenvalue problem on a coarse mesh with mesh size H and a Stokes problem on a fine mesh with mesh size $h=\mathcal{O}(H^2)$ which can still maintain the optimal accuracy. The method provides an approximation solution with the convergence rate of the same order as the one-level stabilized finite element solution, which solves a Stokes eigenvalue problem on a fine mesh with mesh size h. From now on, H and $h \ll H$ will be two real positive parameters tending to 0. The two-level stabilized finite element method is as follows:

Algorithm Two-level stabilized finite element approximation

Step I Solve the Stokes eigenvalue problem on a coarse mesh by finding $(\mathbf{u}_H, p_H, \lambda_H) \in X_H \times M_H \times \Re$ with $\|\mathbf{u}_H\|_0 = 1$ such that

$$\mathcal{C}_H((\mathbf{u}_H,p_H);(\mathbf{v}_H,q_H)) = \lambda_H(\mathbf{u}_H,\mathbf{v}_H) + \lambda_H \sum_{K\in\tau_H} \frac{\tau_1}{\nu}(\mathbf{u}_H,\nabla q_H)_K \quad \forall(\mathbf{v}_H,q_H)\in(X_H,M_H). \tag{5.1}$$

Step II Solve the Stokes problem on a fine mesh by finding $(\mathbf{u}^h,p^h)\in(X_h,M_h)$ such that

$$\mathcal{C}_h((\mathbf{u}^h,p^h);(\mathbf{v}_h,q_h)) = \lambda_H(\mathbf{u}_H,\mathbf{v}_h) + \lambda_H \sum_{K\in\tau_h} \frac{\tau_1}{\nu}(\mathbf{u}_H,\nabla q_h)_K \quad \forall(\mathbf{v}_h,q_h)\in(X_h,M_h). \tag{5.2}$$

Step III Set

$$\lambda^h = \frac{\mathcal{C}_h((\mathbf{u}^h,p^h);(\mathbf{u}^h,p^h)) - \lambda_H \sum\limits_{K\in\tau_h} \frac{\tau_1}{\nu}(\mathbf{u}_H,\nabla p^h)_K}{(\mathbf{u}^h,\mathbf{u}^h)}, \tag{5.3}$$

where $\mathbf{u}^h \in X_h\backslash\{0\}$.

In order to do the error estimates, we first need the following lemma:

Lemma 5.1 ([15]) Let $(\mathbf{u},p,\lambda)$ be an eigenvalue pair of (2.5). For any $\mathbf{w}\in X\backslash\{0\}$ and $s\in M$, it holds that

$$\frac{\mathcal{C}((\mathbf{w},s),(\mathbf{w},s))}{(\mathbf{w},\mathbf{w})} - \lambda = \frac{\mathcal{C}((\mathbf{w}-\mathbf{u},s-p),(\mathbf{w}-\mathbf{u},s-p))}{(\mathbf{w},\mathbf{w})} - \lambda\frac{(\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u})}{(\mathbf{w},\mathbf{w})}.$$

Theorem 5.2 Let $(\mathbf{u},p,\lambda)$ be an eigenvalue pair of (2.5) and $(\mathbf{u},p)\in D(A)\times(M\cap H^1(\Omega))$. The $(\mathbf{u}^h,p^h)$ is the solution of (5.2). Then $(\mathbf{u}^h,p^h,\lambda^h)$ satisfies the error estimates

$$\nu\|\nabla(\mathbf{u}-\mathbf{u}^h)\|_0 + \|p-p^h\|_0 \le C(h+H^2)(\nu|\mathbf{u}|_2+|p|_1) \tag{5.4}$$

and

$$|\lambda-\lambda^h| \le C(h+H^2)^2(\nu|\mathbf{u}|_2+|p|_1)^2. \tag{5.5}$$

Proof Subtracting (5.2) from (2.5), we get the following error equation:

$$\begin{aligned}&\nu(\nabla(\mathbf{u}-\mathbf{u}^h),\nabla\mathbf{v}_h)_\Omega-(p-p^h,\nabla\cdot\mathbf{v}_h)_\Omega+(q_h,\nabla\cdot(\mathbf{u}-\mathbf{u}^h))_\Omega\\&-\sum_{K\in\tau_h}\frac{\tau_1}{\nu}(\nabla p^h,\nabla q_h)_K-\sum_{E\in\varepsilon_h^{\rm int}}\frac{\tau_2}{\nu}([[\nu\partial_{\mathbf{n}}\mathbf{u}^h]]_E,[[\nu\partial_{\mathbf{n}}\mathbf{v}_h]]_E)_E\\=&\lambda(\mathbf{u}-\mathbf{u}_H,\mathbf{v}_h)_\Omega+(\lambda-\lambda_H)(\mathbf{u}_H,\mathbf{v}_h)_\Omega-\lambda_H\sum_{K\in\tau_h}\frac{\tau_1}{\nu}(\mathbf{u}_H,\nabla q_h)_K.\end{aligned}\tag{5.6}$$

Using the Galerkin projection, the above formulation (5.6) can be rewritten as

$$\begin{aligned}&\mathcal{C}_h((\mathbf{R}_h-\mathbf{u}^h,Q_h-p^h);(\mathbf{v}_h,q_h))\\=&\lambda(\mathbf{u}-\mathbf{u}_H,\mathbf{v}_h)_\Omega+(\lambda-\lambda_H)(\mathbf{u}_H,\mathbf{v}_h)_\Omega-\lambda_H\sum_{K\in\tau_h}\frac{\tau_1}{\nu}(\mathbf{u}_H,\nabla q_h)_K.\end{aligned}\tag{5.7}$$

Setting $\mathbf{v}_h=\mathbf{R}_h-\mathbf{u}^h, q_h=Q_h-p^h$ in (5.7), we get

$$\begin{aligned}|||\mathbf{R}_h-\mathbf{u}^h|||_h^2+\|Q_h-p^h\|_h^2=&\lambda(\mathbf{u}-\mathbf{u}_H,\mathbf{R}_h-\mathbf{u}^h)_\Omega+(\lambda-\lambda_H)(\mathbf{u}_H,\mathbf{R}_h-\mathbf{u}^h)_\Omega\\&-\lambda_H\sum_{K\in\tau_h}\frac{\tau_1}{\nu}(\mathbf{u}_H,\nabla(Q_h-p^h))_K.\end{aligned}\tag{5.8}$$

Now applying Young's inequality and Poincaré's inequality, we estimate the terms on the right hand side of (5.8):

$$\begin{aligned}&\lambda(\mathbf{u}-\mathbf{u}_H,\mathbf{R}_h-\mathbf{u}^h)_\Omega+(\lambda-\lambda_H)(\mathbf{u}_H,\mathbf{R}_h-\mathbf{u}^h)_\Omega-\lambda_H\sum_{K\in\tau_h}\frac{\tau_1}{\nu}(\mathbf{u}_H,\nabla(Q_h-p^h))_K\\\le&\lambda\|\mathbf{u}-\mathbf{u}_H\|_0\gamma_0|\mathbf{R}_h-\mathbf{u}^h|_1+|\lambda-\lambda_H|\|\mathbf{u}_H\|_0\gamma_0|\mathbf{R}_h-\mathbf{u}^h|_1\\&+\lambda_H\Big(\sum_{K\in\tau_h}\frac{\tau_1}{\nu}\|\mathbf{u}_H\|_{0,K}^2\Big)^{\frac12}\|Q_h-p^h\|_h\\\le&C\nu^{-1}\|\mathbf{u}-\mathbf{u}_H\|_0^2+\frac{\nu}{2}\|\nabla(\mathbf{R}_h-\mathbf{u}^h)\|_0^2+C|\lambda-\lambda_H|^2\|\mathbf{u}_H\|_0^2+C\lambda_H^2h^2\|\mathbf{u}_H\|_0^2\\&+\frac12\|Q_h-p^h\|_h^2.\end{aligned}\tag{5.9}$$

Using Theorem 4.2, and combining (5.8) with (5.9), we get

$$\begin{aligned}&\nu\|\nabla(\mathbf{R}_h-\mathbf{u}^h)\|_0^2+\|Q_h-p^h\|_h^2\\\le&CH^4(\nu|\mathbf{u}|_2+|p|_1)^2+CH^4(\nu|\mathbf{u}|_2+|p|_1)^4\|\mathbf{u}_H\|_0^2+C|\lambda_H|^2h^2\|\mathbf{u}_H\|_0^2\\\le&C(h+H^2)^2(\nu|\mathbf{u}|_2+|p|_1)^2.\end{aligned}\tag{5.10}$$

The above inequality implies that

$$\nu\|\nabla(\mathbf{R}_h-\mathbf{u}^h)\|_0+\|Q_h-p^h\|_h\le C(h+H^2)(\nu|\mathbf{u}|_2+|p|_1).\tag{5.11}$$

We now use the triangular inequality, Lemma 4.1 and (5.11) to get

$$\nu\|\nabla(\mathbf{u}-\mathbf{u}^h)\|_0\le\|\nabla(\mathbf{u}-\mathbf{R}^h)\|_0+\|\nabla(\mathbf{R}_h-\mathbf{u}^h)\|_0\le C(h+H^2)(\nu|\mathbf{u}|_2+|p|_1)\tag{5.12}$$

and

$$\|p-p^h\|_h\le\|p-Q_h\|_h+\|Q_h-p^h\|_h\le C(h+H^2)(\nu|\mathbf{u}|_2+|p|_1).\tag{5.13}$$

In order to prove $\|p-p^h\|_0$, we need to apply Lemma 3.3, Theorem 4.2 and (5.7) to obtain

$$\beta_1\|Q_h-p^h\|_0\le\frac{\mathcal{C}_h((\mathbf{R}_h-\mathbf{u}^h,Q_h-p^h);(\mathbf{v}_h,q_h))}{|\mathbf{v}_h|_1+\|q_h\|_0}$$

$$\leq \frac{\lambda(\mathbf{u}-\mathbf{u}_H,\mathbf{v}_h)_\Omega+(\lambda-\lambda_H)(\mathbf{u}_H,\mathbf{v}_h)_\Omega-\lambda_H\sum\limits_{K\in\tau_h}\frac{\tau_1}{\nu}(\mathbf{u}_H,\nabla q_h)_K}{|\mathbf{v}_h|_1+\|q_h\|_0}$$
$$\leq C(h+H^2)(\nu|\mathbf{u}|_2+|p|_1). \tag{5.14}$$

Again using the triangular inequality, Lemma 4.1 and (5.14), we get that

$$\|p-p^h\|_0\leq\|p-Q_h\|_0+\|Q_h-p^h\|_0\leq C(h+H^2)(\nu|\mathbf{u}|_2+|p|_1). \tag{5.15}$$

Gathering together (5.12) and (5.15), the formulation (5.4) of Theorem 5.2 holds.

In what follows, we prove (5.5) of Theorem 5.2 to complete the proof. Here we use Lemma 5.1 and get

$$\begin{aligned}\lambda^h-\lambda&=\frac{\mathcal{C}_h((\mathbf{u}^h,p^h);(\mathbf{u}^h,p^h))-\lambda_H\sum\limits_{K\in\tau_h}\frac{\tau_1}{\nu}(\mathbf{u}_H,\nabla p^h)_K}{(\mathbf{u}^h,\mathbf{u}^h)}-\lambda\\&=\frac{\mathcal{C}((\mathbf{u}^h-\mathbf{u},p^h-p);(\mathbf{u}^h-\mathbf{u},p^h-p))-\lambda_H\sum\limits_{K\in\tau_h}\frac{\tau_1}{\nu}(\mathbf{u}_H,\nabla p^h)_K}{(\mathbf{u}^h,\mathbf{u}^h)}\\&\quad+\frac{\sum\limits_{K\in\tau_h}\frac{\tau_1}{\nu}\|\nabla p^h\|_{0,K}^2+\sum\limits_{E\in\varepsilon_h^{\rm int}}\frac{\tau_2}{\nu}\|[[\nu\partial_{\mathbf{n}}\mathbf{u}^h]]_E\|_{0,E}^2}{(\mathbf{u}^h,\mathbf{u}^h)}-\lambda\frac{(\mathbf{u}^h-\mathbf{u},\mathbf{u}^h-\mathbf{u})}{(\mathbf{u}^h,\mathbf{u}^h)}.\end{aligned} \tag{5.16}$$

Now, taking the norm and using Young's inequality, Lemma 3.1, Lemma 3.2, (5.4) and (5.13), we obtain that

$$\begin{aligned}|\lambda^h-\lambda|&\leq C(\|\nabla(\mathbf{u}^h-\mathbf{u})\|_0^2+\|p^h-p\|_0^2)+C\|p^h-p\|_h^2+C\sum_{K\in\tau_h}\frac{\tau_1}{\nu}\|\nabla p\|_{0,K}^2\\&\quad+C\lambda_H^2h^2\|\mathbf{u}_H\|_0^2+C(\|\nabla(\mathbf{u}^h-\mathbf{u})\|_0^2+h^2|\mathbf{u}|_2^2)+\lambda\|\mathbf{u}^h-\mathbf{u}\|_0^2\\&\leq C(h+H^2)^2(\nu|\mathbf{u}|_2+|p|_1)^2.\end{aligned} \tag{5.17}$$

Thus the formulation (5.5) holds. This completes the proof of Theorem 5.2. □

6 Numerical Example

In this section, we have two aims: one is to present a numerical example to confirm the theoretical results of Theorem 4.2 for this new stabilized finite element method; the other is to use the same example to check the theoretical results for our two-level stabilized finite element method for this Stokes eigenvalue problem. The algorithms are implemented by the finite element software Freefem++ [38].

In the example that follows, the velocity and pressure are approximated by the lowest equal-order finite element pairs by using uniform triangulation. The algorithm for the Stokes eigenvalue problem with the viscosity $\nu=1.0$ is carried out in the domain $\Omega=\{(x,y)|0<x,y<1\}$. For simplicity, we here only consider the first eigenvalue of the Stokes eigenvalue problem. Since the exact solution for this problem is unknown, we solve the problem by using the Galerkin finite element method with a P_2/P_1 element on a very fine mesh (6724 grid points) to take its approximation solution as the exact solution for the purpose of comparison. Here we take $\lambda=52.3447$ as the first exact eigenvalue, and we select $\alpha_1=8,\alpha_2=72$, because they can deal with the considered problem well.

(1) Results for the new stabilized finite element method.

The result of the new multiscale finite element method for the Stokes eigenvalue problem is presented in Table 1. From Table 1, we see that the new stabilized finite element method keeps the convergence rate just as the theoretical analysis in Theorem 4.2; that is to say that our new stabilized finite element method is efficient.

Table 1 Result of the new multiscale finite element method for the Stokes eigenvalue problem

$1/h$	CPU-time	λ_h	$\frac{\lvert\lambda-\lambda_h\rvert}{\lvert\lambda\rvert}$	λ_h-rate
9	0.14	55.7956	0.065926	
18	0.53	53.2441	0.017182	1.93998
27	1.232	52.7478	0.007702	1.979
36	2.382	52.572	0.004342	1.99224
45	3.806	52.4902	0.002780	1.99751
54	5.959	52.4458	0.001931	1.99998
63	8.658	52.4189	0.001418	2.00127

Furthermore, in order to show the stability and efficiency of the new stabilized finite element method for the Stokes eigenvalue problem, we present the velocity streamline and the pressure contour with $h = \frac{1}{63}$ in Figure 3. For comparison, we also present the results of the standard Galerkin finite element method with a P_2/P_1 element on a very fine mesh (6724 grid points) in Figure 2. From these two figures, we can see that the new stabilized finite element method for the Stokes eigenvalue problem has good stability and computational efficiency.

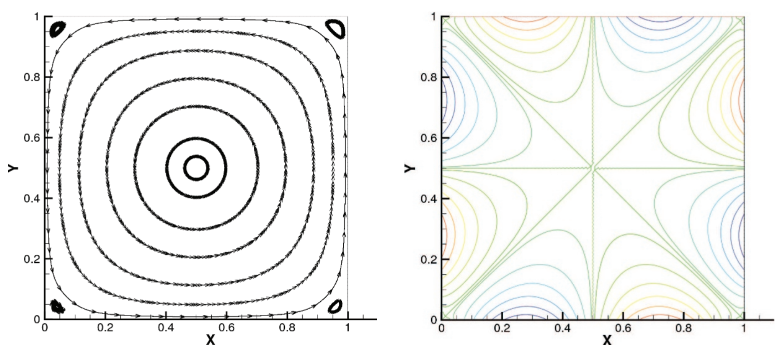

Figure 2 The velocity streamlines (left) and the pressure contour map (right) for the standard Galerkin method with P_2/P_1 element

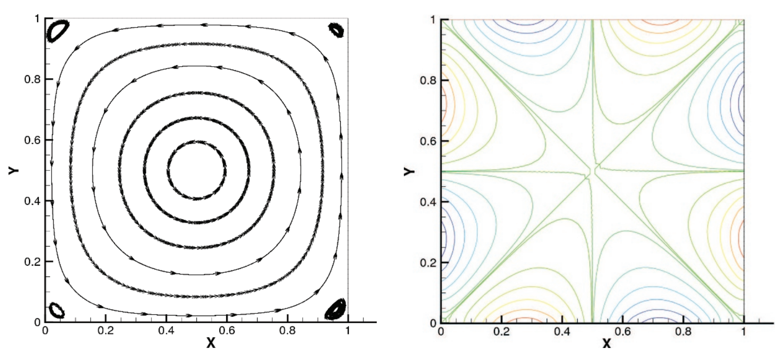

Figure 3 The velocity streamlines (left) and the pressure contour map (right) for the stabilized finite element method with P_1/P_1 element

(2) Results for the new two-level stabilized finite element method.

Now our goal is to validate the superiority of this new two-level stabilized finite element method while comparing it with the one-level method. Here we compare the solution of the one-level method solved at a fixed mesh size h with the results obtained by using the two-level method to solve the Stokes eigenvalue problem at the fine mesh size h and coarse mesh size H, which is chosen such that $h = H^2$. The result is shown in Table 2. From Table 2, we see that the two-level stabilized finite element method can save computation time compared to the one-level method. Moreover, this becomes more and more significant as h decreases. The other interesting discovery is that the value of λ_h becomes too small to converge to the exact solution.

Table 2 Comparing the one-level method with the two-level method

	$1/H$	$1/h$	CPU	λ_h	$\frac{\vert\lambda-\lambda_h\vert}{\lambda}$	λ_h-rate
one-level		16	0.43	53.4781	0.02165223	
two-level	4	16	0.415	54.5241	0.0416365	
one-level		25	1.077	52.8144	0.00897387	1.9736
two-level	5	25	0.993	53.138	0.0151551	2.26455
one-level		36	2.832	52.572	0.00434181	1.99106
two-level	6	36	2.064	52.7215	0.00719769	2.04193
one-level		49	4.767	52.4674	0.002345	1.99806
two-level	7	49	3.9	52.5502	0.00392511	1.9668
one-level		64	9.116	52.4166	0.00137424	2.00097
two-level	8	64	6.848	52.4672	0.00234077	1.93556
one-level		81	16.635	52.3896	0.0008575	2.00224
two-level	9	81	11.423	52.4226	0.0014877	1.92403
one-level		100	29.249	52.3741	0.000562	2.00286
two-level	10	100	19.288	52.3966	0.00099209	1.92284
one-level		121	50.983	52.3648	0.000384	2.00325
two-level	11	121	28.142	52.3807	0.00068714	1.92673

7 Conclusion

Firstly, a new stabilized finite element method for the Stokes eigenvalue problem has been proposed in this paper. The convergence of this new stabilized finite element method for the Stokes eigenvalue problem has been proved and the optimal a priori error estimates for the eigenfunctions and eigenvalues have also been obtained. Secondly, combining the above new stabilized finite element method with the two-level method, a new two-level stabilized finite element method for the Stokes eigenvalue problem have been given. The a priori error estimates for the new two-level stabilized method have been gained. Finally, numerical examples confirming our theoretical analysis and validating the high effectiveness of new methods have been given.

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