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ON THE EXISTENCE WITH EXPONENTIAL DECAY AND THE BLOW-UP OF SOLUTIONS FOR COUPLED SYSTEMS OF SEMI-LINEAR CORNER-DEGENERATE PARABOLIC EQUATIONS WITH SINGULAR POTENTIALS*

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Abstract In this article, we study the initial boundary value problem of coupled semi-linear degenerate parabolic equations with a singular potential term on manifolds with corner singularities. Firstly, we introduce the corner type weighted p-Sobolev spaces and the weighted corner type Sobolev inequality, the Poincaré inequality, and the Hardy inequality. Then, by using the potential well method and the inequality mentioned above, we obtain an existence theorem of global solutions with exponential decay and show the blow-up in finite time of solutions for both cases with low initial energy and critical initial energy. Significantly, the relation between the above two phenomena is derived as a sharp condition. Moreover, we show that the global existence also holds for the case of a potential well family.

Key words coupled parabolic equations; totally characteristic degeneracy; singular potentials; asymptotic stability; blow-up

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1 Introduction and Main Results

Let $\mathbb{M} \subset [0,1) \times X \times [0,1)$ be a corner type domain with finite corner measure $|\mathbb{M}| = \int_{\mathbb{M}} \frac{\mathrm{d}r}{r}\mathrm{d}x\frac{\mathrm{d}w}{rw}$, which is a local model of stretched corner-manifolds (i.e., the manifolds with corner singularities) with dimension $N = n + 2 \geq 3$. Here, let X is a closed compact sub-manifold of dimension n emdedded in the unit sphere of $\mathbb{R}^{n+1}$. Let $\mathbb{M}_0$ denote the interior of $\mathbb{M}$ and $\partial\mathbb{M} = \{0\} \times X \times \{0\}$ denote the boundary of $\mathbb{M}$. The corner-Laplacian is defined as

$$\Delta_{\mathbb{M}} = (r\partial_r)^2 + (\partial_{x_1})^2 + \cdots + (\partial_{x_n})^2 + (rw\partial_w)^2,$$

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which is a degenerate elliptic operator on the boundary $\partial\mathbb{M}$. The present paper is concerned with the initial boundary value problem of a class of coupled semi-linear corner-degenerate parabolic equations with singular potential term of the form

$$\begin{cases} \partial_t u - \Delta_{\mathbb{M}} u - \mu V_1 u = F_u(u,v), \text{ in } \mathbb{M}_0 \times (0,T), \\ \partial_t v - \Delta_{\mathbb{M}} v - \mu V_2 v = F_v(u,v), \text{ in } \mathbb{M}_0 \times (0,T), \\ u(z,0) = u_0, v(z,0) = v_0, \qquad \text{in } \mathbb{M}_0, \\ u = 0, v = 0, \qquad \text{on } \partial\mathbb{M} \times (0,T), \end{cases} \tag{1.1}$$

where $z := (r,x,w) \in \mathbb{M}, 0 < T \le +\infty$ is the maximal existence time. The singular potential term $V_i (i = 1,2)$ is unbounded on $\partial\mathbb{M}$ and satisfies the corner Hardy inequality

$$\|V^{\frac{1}{2}}\varphi\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} \le C\|\nabla_{\mathbb{M}}\varphi\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} \tag{1.2}$$

for $\varphi \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Chen et al. [13] obtained two kinds of singular potential functions $V = \frac{r^{-2}\mathrm{e}^{-\frac{1}{r^2}}}{\mathrm{e}^{-\frac{1}{r^2}}+x_1^2+\cdots+x_n^2+w^2}$ and $V = \frac{1}{r^2+x_1^2+\cdots+x_n^2+w^2}$, which satisfy the corner type Hardy inequality (1.2). Define

$$C^* = \sup\left\{\frac{\|V^{\frac{1}{2}}\varphi\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}}{\|\nabla_{\mathbb{M}}\varphi\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}}; \varphi \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}), \varphi \ne 0\right\}, \tag{1.3}$$

and then let

$$0 < \mu < \frac{1}{C^{*2}}.$$

The function $F : \mathbb{R}^2 \to \mathbb{R}$ is a C^1-function given by

$$F(u,v) = \alpha|u+v|^{p+1} + 2\beta|uv|^{\frac{p+1}{2}}, \tag{1.4}$$

where $1 < p < \frac{N+2}{N-2}, \alpha > 1$ and $\beta > 0$. In addition,

$$\begin{aligned} f_1(u,v) &:= \frac{\partial F}{\partial u}(u,v) = (p+1)[\alpha|u+v|^{p-1}(u+v) + \beta|u|^{\frac{p-3}{2}}|v|^{\frac{p+1}{2}}u], \\ f_2(u,v) &:= \frac{\partial F}{\partial v}(u,v) = (p+1)[\alpha|u+v|^{p-1}(u+v) + \beta|v|^{\frac{p-3}{2}}|u|^{\frac{p+1}{2}}v], \\ uf_1(u,v) &+ vf_2(u,v) = (p+1)F(u,v) \quad \text{for all } (u,v) \in \mathbb{R}^2. \end{aligned} \tag{1.5}$$

Chen et al. [4] studied the initial-boundary problem of a single semi-linear parabolic equation on a stretched cone. The corresponding cone is Laplacian $\Delta_{\mathbb{B}} = (x_1\partial_{x_1})^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$, which is degenerate at $x_1 = 0$. This kind of operator is a simple example of conical dfferential operators. Alimohammady and Kalleji [2] studied a similar problem for a class of single semi-linear parabolic equations with a positive potential function on stretched cone. The authors of this paper [3] studied the initial-boundary problem of a single semi-linear parabolic equation with a singular potential function for the edge Laplacian $\Delta_{\mathbb{E}} = (w\partial_w)^2 + \partial_{x_1}^2 + \cdots + \partial_{x_n}^2 + (w\partial_{y_1})^2 + \cdots + (w\partial_{y_q})^2$, with edge singularity at $w = 0$. A powerful technique for treating the above problems is the so-called potential well method, which was developed by Sattinger [21] in the context of hyperbolic equations. At the same time, the pseudo-differential operators with conical singularities and edge singularities have been widely studied with various motivations by Egorov and Schulze [14], Schulze [23], Schrohe and Seiler [22], Melrose and Mendoza [18] and Mazzeo [17], Chen et al. [5–10].

Motivated by the above work, in this article we generalize the above results for scale parabolic equations to a coupled system of nonlinear parabolic equations. We further study a class of coupled systems of semi-linear parabolic equations with singular potentials on a manifold with corner singularities. Here the so-called corner Laplacian $\Delta_{\mathbb{M}} = (r\partial_r)^2 + (\partial_{x_1})^2 + \cdots + (\partial_{x_n})^2 + (rw\partial_w)^2$ is degenerate at both $r = 0$ and $w = 0$, and it is named after the local structure of a manifold with corner singularities. Recently, Chen et al. [11] established the so-called corner type Sobolev inequality and Poincar*é* inequality in the weighted Sobolev spaces. Such kinds of inequalities will be of fundamental importance in proving the existence of weak solutions for nonlinear problems with corner degeneracy. Melrose and Piazza studied the structure of manifolds with corners in [19]. Schulze discussed the calculus of corner degenerate pseudo-dfferential operators in [24]. Chen et al. studied multiple solutioms and multiple sign changing solutions for semi-linear corner degenerate elliptic equations with singular potential in [13] and [12], respectively.

First, we introduce the following definition of the weak solution:

Definition 1.1 Function $(u, v) = (u(z, t), v(z, t))$ is called a weak solution of problem (1.1) on $\mathbb{M} \times [0, T)$, with $0 < T \leq +\infty$ being the maximal existence time, if $u, v \in L^\infty(0, T; \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))$ with $u_t, v_t \in L^2(0, T; \mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M}))$ satisfies problem (1.1) in the distribution sense, i.e.,

$$\begin{aligned}
&\int_{\mathbb{M}} ru_t \cdot \varphi \mathrm{d}\sigma + \int_{\mathbb{M}} r\nabla_{\mathbb{M}}u \cdot \nabla_{\mathbb{M}}\varphi \mathrm{d}\sigma - \int_{\mathbb{M}} r\mu V_1 u \cdot \varphi \mathrm{d}\sigma = \int_{\mathbb{M}} rf_1(u, v) \cdot \varphi \mathrm{d}\sigma, \\
&\int_{\mathbb{M}} rv_t \cdot \varphi \mathrm{d}\sigma + \int_{\mathbb{M}} r\nabla_{\mathbb{M}}v \cdot \nabla_{\mathbb{M}}\varphi \mathrm{d}\sigma - \int_{\mathbb{M}} r\mu V_2 v \cdot \varphi \mathrm{d}\sigma = \int_{\mathbb{M}} rf_2(u, v) \cdot \varphi \mathrm{d}\sigma
\end{aligned} \tag{1.6}$$

for any $\varphi \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$, $t \in (0, T)$ with $u(z, 0) = u_0(z)$, $v(z, 0) = v_0(z)$ for $z \in \mathbb{M}$.

From the variational point of view, there are two natural functionals on $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$ associated with problem (1.1): the energy functional and the Nehari functional. These are defined respectively, by

$$E(u, v) = \frac{1}{2}\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}}u|^2 + |\nabla_{\mathbb{M}}v|^2)\mathrm{d}\sigma - \frac{1}{2}\int_{\mathbb{M}} r\mu(V_1|u|^2 + V_2|v|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} rF(u, v)\mathrm{d}\sigma, \tag{1.7}$$

$$K(u, v) = \int_{\mathbb{M}} r(|\nabla_{\mathbb{M}}u|^2 + |\nabla_{\mathbb{M}}v|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} r\mu(V_1|u|^2 + V_2|v|^2)\mathrm{d}\sigma - (p + 1)\int_{\mathbb{M}} rF(u, v)\mathrm{d}\sigma. \tag{1.8}$$

Remark 1.2 The weak solution in the above definition satisfies the conservation of energy

$$\int_0^t \|(\partial_\tau u, \partial_\tau v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} \mathrm{d}\tau + E(u, v) = E(u_0, v_0), \quad 0 \leq t < T. \tag{1.9}$$

We are now in a position to state our main results. Our main results are concerned with the global existence with exponential decay and the finite time blow-up of a solution for problem (1.1). Let

$$d := \inf\Big\{\sup_{\lambda \geq 0} E(\lambda u, \lambda v) | (u, v) \in (\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2 \setminus \{(0, 0)\}\Big\}. \tag{1.10}$$

Theorem 1.3 Let $u_0, v_0 \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Assume that $E(u_0, v_0) \leq d$ and $K(u_0, v_0) \geq 0$. Then problem (1.1) admits a global weak solution $u, v \in L^\infty(0, \infty; \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))$ with

$u_t, v_t \in L^2(0,\infty;\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M}))$. Moreover, there exist constants $\lambda > 0$ and $C > 0$, such that

$$\|(u(t),v(t))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} \le C\mathrm{e}^{-\lambda t} \quad \text{for} \quad 0 \le t < \infty.$$

Theorem 1.4 Let $u_0, v_0 \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Assume that $E(u_0,v_0) \le d$ and $K(u_0,v_0) \le 0$. Then the weak solution of problem (1.1) blows up in finite time, i.e., the maximal existence time T is finite and

$$\lim_{t\to T^-} \|(u(t),v(t))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} = +\infty.$$

This paper is organized as follows: in Section 2 we give some preliminaries, such as the definition of a corner type weighted p-Sobolev space, the properties of corner type weighted p-Sobolev space and some useful inequalities, such as the Sobolev inequality, the Poincaré inequality and the Hardy inequality (more details can be seen in [11, 13]). In Section 3, we introduce a family of potential wells relative to problem (1.1) and prove a series of corresponding properties. Then, we discuss the invariance of some sets under the solution flow of (1.1) and the vacuum isolating behavior of solutions. Finally, we give the proof of Theorem 1.3 and Theorem 1.4 in Section 4. Moreover, we show that the global existence also holds for the case of a potential well family.

2 Corner Type Weighted p-Sobolev Spaces

Let $X \subset S^n$ be a bounded open set in the unit sphere of $\mathbb{R}^{n+1}_{\tilde{x}}$. Then the finite corner is defined as

$$M = (E \times [0,1))/(E \times \{0\}),$$

where the base E is a finite cone defined as $E = ([0,1) \times X)/(\{0\} \times X)$. Thus, the finite stretched corner is

$$\mathbb{M} \subset \mathbb{E} \times [0,1) = [0,1) \times X \times [0,1), \tag{2.1}$$

with the smooth boundary $\partial\mathbb{M} = \{0\} \times X \times \{0\}$. Here we denote $\mathbb{M}_0$ as the interior of $\mathbb{M}$. In this paper, we shall use the coordinates $(r,x,w) \in \mathbb{M}$.

The typical degenerate differential operator A on the stretched cone $\mathbb{E}$ is as follows:

$$A = r^{-\mu} \sum_{j\le\mu} a_j(r)(r\partial_r)^j = r^{-\mu} A_{\mathbb{E}},$$

with coefficients $a_j(r) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(X))$. Here $A_{\mathbb{E}}$ is degenerate cone operator. Denote $\mathrm{Diff}^{\mu}_{\deg}(\mathbb{E})$ for the set of cone differential operators as A. The typical differential operator B on the stretched corner $\mathbb{M}$ is then of the form

$$B = w^{-\nu} \sum_{j\le(\nu-l)} b_{jl}(w)(w\partial_w)^j,$$

where we have the coefficients $b_l(w) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\nu-l}_{\deg}(E))$, i.e.,

$$b_l(w) = r^{-(\nu-l)} \sum_{j\le(\nu-l)} a_{jl}(r,w)(r\partial_r)^j,$$

with $a_{jl}(r,w) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-l-j}(X))$. This implies that

$$B = (rw)^{-\nu} \sum_{j+l\le\nu} \tilde{a}_{jl}(r,w)(r\partial_r)^j(rw\partial_w)^l = (rw)^{-\nu} B_{\mathbb{M}},$$

where $\tilde{a}_{jl}(r,w) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-l-j}(X))$ and $B_{\mathbb{M}}$ is called a degenerate corner operator. In fact, we have following Riemannian metric on the corner M:

$$dw^2 + w^2(dr^2 + r^2 g_X),$$

where g_X is a Riemannian metric on X. Then the corresponding gradient operator with corner degeneracy is

$$\nabla_{\mathbb{M}} = (r\partial_r, \partial_{x_1}, \cdots, \partial_{x_n}, rw\partial_w).$$

Now we define the weighted $L_p^{\gamma_1,\gamma_2}$ space on $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$ as follows:

Definition 2.1 Let $(r,x,w) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$, with weight data $\gamma_i \in \mathbb{R}, i = 1,2$ and $1 \le p < +\infty$. Then $\mathcal{L}_p^{\gamma_1,\gamma_2}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+, \frac{\mathrm{d}r}{r}\mathrm{d}x\frac{\mathrm{d}w}{rw})$ denotes the space of all $u(r,x,w) \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$ such that

$$\|u\|_{\mathcal{L}_p^{\gamma_1,\gamma_2}} = \left(\int_{\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+} |r^{\frac{N}{p}-\gamma_1} w^{\frac{N}{p}-\gamma_2} u(r,x,w)|^p \frac{\mathrm{d}r}{r}\mathrm{d}x\frac{\mathrm{d}w}{rw}\right)^{\frac{1}{p}} < +\infty.$$

By the above weighted $L_p^{\gamma_1,\gamma_2}$ space, we can define the following weighted p-Sobolev spaces on $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$, with a natural scale for all $1 \le p < +\infty$:

Definition 2.2 Let $m \in \mathbb{N}, \gamma_i \in \mathbb{R}, i = 1,2$, and set $N = n+2$, with the weighted Sobolev space

$$\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+) = \Big\{u \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+) | (r\partial_r)^l \partial_x^\alpha (rw\partial_w)^k u(r,x,w) \in \mathcal{L}_p^{\gamma_1,\gamma_2}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+, \frac{\mathrm{d}r}{r}\mathrm{d}x\frac{\mathrm{d}w}{rw})\Big\}$$

for $k, l \in \mathbb{R}$ and the multi-index $\alpha \in \mathbb{R}^n$, with $k + |\alpha| + l \le m$. Moreover, the closure of C_0^∞ functions in $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$ is denoted by $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$.

Similarly, we can define the following weighted p-Sobolev spaces on an open stretched corner $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$:

$$\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+) = \Big\{u \in \mathcal{D}'(\mathbb{R}_+ \times X \times \mathbb{R}_+) | (r\partial_r)^l \partial_x^\alpha (rw\partial_w)^k u(r,x,w) \in \mathcal{L}_p^{\gamma_1,\gamma_2}(\mathbb{R}_+ \times X \times \mathbb{R}_+, \frac{\mathrm{d}r}{r}\mathrm{d}x\frac{\mathrm{d}w}{rw})\Big\},$$

for $k, l \in \mathbb{R}$ and the multi-index $\alpha \in \mathbb{R}^n$, with $k + |\alpha| + l \le m$, which is a Banach space with the norm

$$\|u\|_{\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}} = \sum_{l+|\alpha|+k\le m} \left\{\int_{\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+} |r^{\frac{N}{p}-\gamma_1} w^{\frac{N}{p}-\gamma_2} (r\partial_r)^l \partial_x^\alpha (rw\partial_w)^k u(r,x,w)| \frac{\mathrm{d}r}{r}\mathrm{d}x\frac{\mathrm{d}w}{rw}\right\}^{\frac{1}{p}}.$$

Moreover, the subspace $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+)$ denoting the closure of C_0^∞ functions in $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+)$ is denoted by $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+)$.

In a fashion similar to the definition in [11], we can introduce the following weighted p-Sobolev space on the finite stretched corner $\mathbb{M}$ defined in (2.1):

Definition 2.3 Letting $m \in \mathbb{N}, i = 1,2, 1 \le p < \infty, \gamma_i \in \mathbb{R}$, $W_{\mathrm{loc}}^{m,p}(\mathbb{M}_0)$ is the classical local Sobolev space. Then $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M})$ denotes the subspace of all $u \in W_{\mathrm{loc}}^{m,p}(\mathbb{M}_0)$, such that

$$\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M}) = \{u \in W_{\mathrm{loc}}^{m,p}(\mathbb{M}_0) | (\omega\sigma)u \in \mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)\}$$

for any cut-off functions $\omega = \omega(r,x)$ and $\sigma = \sigma(x,w)$, supported by collar neighborhoods of $(0,1)\times\partial\mathbb{M}$ and $\partial\mathbb{M}\times(0,1)$, respectively. Moreover, the complement of C_0^∞ functions in $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M})$ is $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{M})$.

It can be deduced from Definition 2.3 that $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M})$ is a Banach space with $1\le p<\infty$, and a Hilbert space with $p=2$. Also we have that $r^{\gamma_1'}w^{\gamma_2'}\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M}) = \mathcal{H}_p^{m,(\gamma_1+\gamma_1',\gamma_2+\gamma_2')}(\mathbb{M})$. Next, the following proposition gives us the embedding property for the weighted Sobolev space $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M})$:

Proposition 2.4 The embedding $\mathcal{H}_p^{m',(\gamma_1',\gamma_2')}(\mathbb{M}) \hookrightarrow \mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M})$ is continuous for $m'\ge m$, $\gamma_1'\ge\gamma_2'$.

Proof See [11], Proposition 2.4. □

Proposition 2.5 (corner Sobolev inequality) Assume $1\le p<N$, $\frac{1}{p^*}=\frac{1}{p}-\frac{1}{N}$, $N=1+n+1$ and $\gamma_1,\gamma_2\in\mathbb{R}$. For $u(r,x,w)\in C_0^\infty(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)$, the following estimate holds:

$$\begin{aligned}\|u\|_{L_{p^*}^{\gamma_1^*,\gamma_2^*}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)} \le{}& \alpha(c_3+c_4)\|r\partial_r u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)}\\ &+\alpha(c_1+c_2+c_3+c_4)\sum_{i=1}^n\|\partial_{x_i}u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)}\\ &+\alpha(c_2+c_4)\|rw\partial_w u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)}\\ &+(c_1+c_2)\|u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)}\\ &+(c_1+c_3)\|u\|_{L_p^{\gamma_1-1,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)},\end{aligned}\tag{2.2}$$

where $\gamma_1^*=\gamma_1-1$, $\gamma_2^*=\gamma_2-1$, and $\alpha=\frac{(N-1)p}{N-p}$ with constants $c_1=\frac{1}{N}|\frac{(N-1)(N-\gamma_1 p)}{N-p}|^{\frac{1}{N}}\times|\frac{(N-1)(N-\gamma_2 p)}{N-p}|^{\frac{1}{N}}$, $c_2=\frac{1}{N}|\frac{(N-1)(N-\gamma_1 p)}{N-p}|^{\frac{1}{N}}$, $c_3=\frac{1}{N}|\frac{(N-1)(N-\gamma_2 p)}{N-p}|^{\frac{1}{N}}$, and $c_4=\frac{1}{N}$.

Proof See [11], Proposition 3.1. □

In the case of $\gamma_1=\gamma_2=\frac{N}{p}$, we have the constant in (2.2): $c_1=c_2=c_3=0$. Then the Hölder inequality implies that, for $u\in\mathcal{H}_p^{1,(\gamma_1,\gamma_2)}(\mathbb{M})$,

$$\|u\|_{\mathcal{L}_{p^*}^{\gamma_1-1,\gamma_2-1}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)}\le c\|\nabla_{\mathbb{M}}u\|_{\mathcal{L}_p^{\gamma_1,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)},\tag{2.3}$$

where $\nabla_{\mathbb{M}}=(r\partial_r,\partial_{x_1},\cdots,\partial_{x_n},rw\partial_w)$ is the corner type gradient operator on $\mathbb{M}=\mathbb{R}_+\times X\times\mathbb{R}_+$, and the constant $c=\frac{(N-1)p}{(N-p)N}$ is the best constant (as we had in standard Sobolev spaces).

Proposition 2.6 The embedding

$$\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})\hookrightarrow\mathcal{H}_{l,0}^{0,(\frac{N-1}{l},\frac{N}{l})}(\mathbb{M})$$

is compact for $2<l<2^*$.

Proof See [11], Proposition 3.3. □

If $u\in\mathcal{L}_p^{1,(\frac{N-1}{p},\frac{N}{p})}(\mathbb{M})$, $v\in\mathcal{L}_{p'}^{1,(\frac{N-1}{p'},\frac{N}{p'})}(\mathbb{M})$, with $p,p'\in(1,\infty)$ and $\frac{1}{p}+\frac{1}{p'}=1$, then we have the following corner type Hölder inequality:

$$\int_{\mathbb{M}}r|uv|\mathrm{d}\sigma\le\left(\int_{\mathbb{M}}r|u|^p\mathrm{d}\sigma\right)^{\frac{1}{p}}\left(\int_{\mathbb{M}}r|v|^{p'}\mathrm{d}\sigma\right)^{\frac{1}{p'}}.\tag{2.4}$$

Proposition 2.7 (corner Poincaré inequality) For $u(r,x,w)\in\mathcal{H}_{p,0}^{1,(\gamma_1,\gamma_2)}(\mathbb{M}), 1\le p<\infty$, it holds that

$$\|u\|_{\mathcal{L}_p^{\gamma_1-1,\gamma_2}(\mathbb{M})}\le d_{\mathbb{M}}\|\nabla_{\mathbb{M}}u\|_{\mathcal{L}_p^{\gamma_1-1,\gamma_2}(\mathbb{M})},\tag{2.5}$$

where $d_{\mathbb{M}}$ is the diameter of $\mathbb{M}$.

Proof See [11], Proposition 3.2. $\square$

Proposition 2.8 Let $(r,x,w)\in\mathbb{M}=[0,1)\times X\times[0,1)$ and $1\le p<\infty$. Then

(i) let

$$V_1(r,x,w)=\frac{r^{-2}\mathrm{e}^{-\frac{1}{r^2}}}{\mathrm{e}^{-\frac{1}{r^2}}+x_1^2+\cdots+x_n^2+w^2},$$

so for $u\in\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$, we have

$$(\frac{N-2}{2})^2\int_{\mathbb{M}}rV_1u^2\mathrm{d}\sigma\le\int_{\mathbb{M}}r|\nabla_{\mathbb{M}}u|^2\mathrm{d}\sigma, \tag{2.6}$$

where $\mathrm{d}\sigma=\frac{\mathrm{d}r}{r}\mathrm{d}x\frac{\mathrm{d}w}{rw}$;

(ii) let

$$V_2(r,x,w)=\frac{1}{r^2+x_1^2+\cdots+x_n^2+w^2},$$

so for $u\in\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$, we have

$$(\frac{N-4}{2})^2\int_{\mathbb{M}}rV_2u^2\mathrm{d}\sigma\le\int_{\mathbb{M}}r|\nabla_{\mathbb{M}}u|^2\mathrm{d}\sigma. \tag{2.7}$$

Proof See [13], Proposition 3.1. $\square$

Using the corner Hardy inequality (1.2), the operator $-\Delta_{\mathbb{M}}u-\mu Vu$, for $0<\mu<\frac{1}{C^{*2}}$, is a positive operator defined on the Hilbert space $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Thus, in a fashion similar to the result of Proposition 2.5 in [11], we have the following lemma:

Proposition 2.9 Let $0<\mu<\frac{1}{C^{*2}}$. Then the Dirichlet problem

$$\begin{cases}-\Delta_{\mathbb{M}}\psi-\mu V\psi=\lambda\psi, & \text{in } \mathbb{M}_0,\\ \psi=0, & \text{on } \partial\mathbb{M}\end{cases}$$

has a discrete set of positive eigenvalues $\{\lambda_k\}_{k\ge1}$ which can be ordered, after counting (finite) multiplicity, as $0<\lambda_1\le\lambda_2\le\lambda_3\le\cdots\le\lambda_k\le\cdots$, and $\lambda_k\to\infty$ as $k\to+\infty$. Also, the corresponding eigenfunctions $\{\psi_k\}_{k\ge1}$ constitute an orthonormal basis of the Hilbert space $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$.

3 A Family of Potential Wells and Vacuum Isolating of Solutions

In this section, we shall introduce a family of potential wells, the exterior of the corresponding potential well sets, and give a series of properties of these. Then, the invariant sets and the vacuum isolating of solutions for problem (1.1) are discussed. First, let the definitions of functionals $E(u,v)$ and $K(u,v)$ be defined by (1.7) and (1.8). Next, we give some properties of the above functionals as follows:

Lemma 3.1 Let $(u,v)\in(\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2\setminus\{(0,0)\}$. Then we have

(i) $\lim\limits_{\lambda\to0}E(\lambda u,\lambda v)=0$, and $\lim\limits_{\lambda\to+\infty}E(\lambda u,\lambda v)=-\infty$;

(ii) on the interval $0<\lambda<\infty$, there exists a unique $\lambda^*=\lambda^*(u,v)>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}E(\lambda u,\lambda v)|_{\lambda=\lambda^*}=0;$$

(iii) $E(\lambda u, \lambda v)$ is strictly increasing on $0 \le \lambda < \lambda^*$, strictly decreasing on $\lambda > \lambda^*$, and takes the maximum at $\lambda = \lambda^*$;

(iv) $K(\lambda u, \lambda v) > 0$ for $0 < \lambda < \lambda^*$, $K(\lambda u, \lambda v) < 0$ for $\lambda > \lambda^*$ and $K(\lambda^* u, \lambda^* v) = 0$, which means that $\lambda^* = 1$.

Proof (i) Let $(u, v) \in (\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2 \setminus \{(0,0)\}$. From the definition of $E(u, v)$, we have

$$\begin{aligned}E(\lambda u, \lambda v) = &\frac{\lambda^2}{2}\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} u|^2 + |\nabla_{\mathbb{M}} v|^2)\mathrm{d}\sigma - \frac{\lambda^2}{2}\int_{\mathbb{M}} r\mu(V_1|u|^2 + V_2|v|^2)\mathrm{d}\sigma\\ &- \lambda^{p+1}\int_{\mathbb{M}} rF(u, v)\mathrm{d}\sigma,\end{aligned}$$

which gives

$$\lim_{\lambda\to 0} E(\lambda u, \lambda v) = 0,$$

and

$$\lim_{\lambda\to +\infty} E(\lambda u, \lambda v) = -\infty.$$

(ii) An easy calculation shows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}E(\lambda u, \lambda v) = \lambda\Big[&\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} u|^2 + |\nabla_{\mathbb{M}} v|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} r\mu(V_1|u|^2 + V_2|v|^2)\mathrm{d}\sigma\\ &- (p+1)\lambda^{p-1}\int_{\mathbb{M}} rF(u, v)\mathrm{d}\sigma\Big],\end{aligned} \tag{3.1}$$

which leads to the conclusion.

(iii) By a direct calculation, (3.1) gives that for $(u, v) \neq (0, 0)$, there exists a unqiue

$$\lambda^* = \left[\frac{\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} u|^2 + |\nabla_{\mathbb{M}} v|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} r\mu(V_1|u|^2 + V_2|v|^2)\mathrm{d}\sigma}{(p+1)\int_{\mathbb{M}} rF(u, v)\mathrm{d}\sigma}\right]^{\frac{1}{p-1}} > 0$$

such that $\frac{\mathrm{d}}{\mathrm{d}\lambda}E(\lambda^* u, \lambda^* v) = 0$. Moreover,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\lambda}E(\lambda u, \lambda v) > 0 \quad \text{for } 0 < \lambda < \lambda^*,\\ &\frac{\mathrm{d}}{\mathrm{d}\lambda}E(\lambda u, \lambda v) < 0 \quad \text{for } \lambda^* < \lambda < \infty.\end{aligned}$$

Hence, the conclusion of (iii) holds.

(iv) The conclusion follows from

$$\begin{aligned}K(\lambda u, \lambda v) &= \lambda^2\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} u|^2 + |\nabla_{\mathbb{M}} v|^2)\mathrm{d}\sigma - \lambda^2\int_{\mathbb{M}} r\mu(V_1|u|^2 + V_2|v|^2)\mathrm{d}\sigma\\ &\quad - (p+1)\lambda^{p+1}\int_{\mathbb{M}} rF(u, v)\mathrm{d}\sigma\\ &= \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}E(\lambda u, \lambda v).\end{aligned}$$

□

Define the Nehari manifold by

$$\mathcal{N} = \{(u, v) \in (\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2 | K(u, v) = 0, (u, v) \neq (0, 0)\}.$$

Then the definition of d (1.10) and Lemma 3.1 implies that

$$d = \inf_{(u,v)\in\mathcal{N}} E(u, v) > 0. \tag{3.2}$$

Thus, the potential well associated with problem (1.1) is the set

$$W=\{(u,v)\in(\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2|E(u,v)<d,K(u,v)>0\}\cup\{(0,0)\}.$$

Here, d is the depth of the potential well, which is defined by (1.10) and satisfies (3.2).

The exterior of the potential well is the set

$$Z=\{(u,v)\in(\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2|E(u,v)<d,K(u,v)<0\}.$$

For $\delta>0$, we further define

$$K_\delta(u,v)=\delta\int_{\mathbb{M}}r(|\nabla_{\mathbb{M}}u|^2+|\nabla_{\mathbb{M}}v|^2)\mathrm{d}\sigma-\delta\int_{\mathbb{M}}r(V_1|u|^2+V_2|v|^2)\mathrm{d}\sigma-(p+1)\int_{\mathbb{M}}rF(u,v)\mathrm{d}\sigma \quad (3.3)$$

and

$$r(\delta)=\left[\frac{\delta(1-\mu C^{*2})}{(p+1)(2^p\alpha+\beta)C_*^{p+1}}\right]^{\frac{1}{p-1}},$$

where

$$C_*=\sup\left\{\frac{\|\varphi\|_{\mathcal{L}_{p+1}^{\frac{N-1}{p+1},\frac{N}{p+1}}(\mathbb{M})}}{\|\nabla_{\mathbb{M}}\varphi\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}};\varphi\in\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}),\varphi\neq0\right\}, \quad (3.4)$$

and the constant C_* can be obtained from Proposition 2.7 and Proposition 2.6.

The following lemmas are given to exhibit the relation between $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}$ and $K_\delta(u,v)$:

Lemma 3.2 Assume that $u,v\in\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. If $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<r(\delta)$, then $K_\delta(u,v)>0$. In particular, if $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<r(1)$, then $K(u,v)>0$.

Proof From $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<r(\delta)$, we have

$$\begin{aligned}(p+1)\int_{\mathbb{M}}rF(u,v)\mathrm{d}\sigma&\le(p+1)(2^p\alpha+\beta)\int_{\mathbb{M}}r(|u|^{p+1}+|v|^{p+1})\mathrm{d}\sigma\\&\le(p+1)(2^p\alpha+\beta)(\|u\|^2_{\mathcal{L}_{p+1}^{\frac{N-1}{p+1},\frac{N}{p+1}}(\mathbb{M})}+\|v\|^2_{\mathcal{L}_{p+1}^{\frac{N-1}{p+1},\frac{N}{p+1}}(\mathbb{M})})^{\frac{p+1}{2}}\\&\le(p+1)(2^p\alpha+\beta)C_*^{p+1}(\|\nabla_{\mathbb{M}}u\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}+\|\nabla_{\mathbb{M}}v\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})})^{\frac{p+1}{2}}\\&\le(p+1)(2^p\alpha+\beta)C_*^{p+1}r(\delta)^{p-1}\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\\&=\delta(1-\mu C^{*2})\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})},\end{aligned}$$

$$\delta\int_{\mathbb{M}}r\mu(V_1|u|^2+V_2|v|^2)\mathrm{d}\sigma\le\delta\mu C^{*2}\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})},$$

and so by the definition of $K_\delta(u,v)$ by (3.3), the lemma is proved. □

Lemma 3.3 Assume that $u,v\in\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. If $K_\delta(u,v)<0$, then

$$\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}>r(\delta).$$

In particular, if $K(u,v)<0$, then $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}>r(1)$.

Proof From $K_\delta(u,v)<0$, we have

$$\begin{aligned}
&\delta(1-\mu C^{*2})\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\\
\le{}&\delta\int_{\mathbb{M}}r(|\nabla_{\mathbb{M}}u|^2+|\nabla_{\mathbb{M}}v|^2)\mathrm{d}\sigma-\delta\int_{\mathbb{M}}r(V_1|u|^2+V_2|v|^2)\mathrm{d}\sigma\\
<{}&(p+1)\int_{\mathbb{M}}rF(u,v)\mathrm{d}\sigma\\
\le{}&(p+1)(2^p\alpha+\beta)\int_{\mathbb{M}}r(|u|^{p+1}+|v|^{p+1})\mathrm{d}\sigma\\
\le{}&(p+1)(2^p\alpha+\beta)C_*^{p+1}\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^{p+1}_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})},
\end{aligned}$$

which leads to

$$\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}>\left[\frac{\delta(1-\mu C^{*2})}{(p+1)(2^p\alpha+\beta)C_*^{p+1}}\right]^{\frac{1}{p-1}}=r(\delta).$$

□

Lemma 3.4 Assume that $u,v\in\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$ and $(u,v)\neq(0,0)$. If $K_\delta(u,v)=0$, then

$$\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\ge r(\delta).$$

In particular, if $K(u,v)=0$, then $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\ge r(1)$.

Proof If $K_\delta(u,v)=0$ and $(u,v)\neq(0,0)$, then from

$$\begin{aligned}
&\delta(1-\mu C^{*2})\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\\
\le{}&\delta\int_{\mathbb{M}}r(|\nabla_{\mathbb{M}}u|^2+|\nabla_{\mathbb{M}}v|^2)\mathrm{d}\sigma-\delta\int_{\mathbb{M}}r(V_1|u|^2+V_2|v|^2)\mathrm{d}\sigma\\
={}&(p+1)\int_{\mathbb{M}}rF(u,v)\mathrm{d}\sigma\\
\le{}&(p+1)(2^p\alpha+\beta)\int_{\mathbb{M}}r(|u|^{p+1}+|v|^{p+1})\mathrm{d}\sigma\\
\le{}&(p+1)(2^p\alpha+\beta)C_*^{p+1}\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^{p+1}_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})},
\end{aligned}$$

we get

$$\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\ge\left[\frac{\delta(1-\mu C^{*2})}{(p+1)(2^p\alpha+\beta)C_*^{p+1}}\right]^{\frac{1}{p-1}}=r(\delta).$$

□

For $\delta>0$, we define the depth of a family of potential wells as

$$d(\delta)=\inf_{(u,v)\in\mathcal{N}_\delta}E(u,v),\tag{3.5}$$

where the Nehari manifold is

$$\mathcal{N}_\delta=\{(u,v)\in(\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2|K_\delta(u,v)=0,(u,v)\neq(0,0)\}.\tag{3.6}$$

Then, the depth $d(\delta)$ and its expression can be estimated as follows:

Lemma 3.5 Let $0<\delta<\frac{p+1}{2}$. Then

$$d(\delta)\ge a(\delta)r^2(\delta),$$

where $a(\delta) = (\frac{1}{2} - \frac{\delta}{p+1})(1 - \mu C^{*2})$. In particular,

$$d \geq \frac{p-1}{2(p+1)}(1 - \mu C^{*2})\Big[\frac{1 - \mu C^{*2}}{(p+1)(2^p\alpha + \beta)C_*^{p+1}}\Big]^{\frac{2}{p-1}} > 0.$$

Moreover, we have

$$d(\delta) = \frac{2(p+1)}{p-1}(\frac{1}{2} - \frac{\delta}{p+1})\delta^{\frac{2}{p-1}} d. \tag{3.7}$$

Proof From the definition of $\mathcal{N}_\delta$ by (3.6), we have $\|(\nabla_{\mathbb{M}} u, \nabla_{\mathbb{M}} v)\|_{\mathcal{L}_2^{\frac{N-1}{2}, \frac{N}{2}}(\mathbb{M})} \geq r(\delta)$ for any $(u, v) \in \mathcal{N}_\delta$, by Lemma 3.4. Thus, from the definition of $E(u, v)$ by (1.7) and the definition of $K_\delta(u, v)$ by (3.3), we deduce that

$$\begin{aligned} E(u,v) &= (\frac{1}{2} - \frac{\delta}{p+1})\Big[\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} u|^2 + |\nabla_{\mathbb{M}} v|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} r(V_1|u|^2 + V_2|v|^2)\mathrm{d}\sigma\Big] + \frac{1}{p+1}K_\delta(u,v) \\ &\geq (\frac{1}{2} - \frac{\delta}{p+1})(1 - \mu C^{*2})\|(\nabla_{\mathbb{M}} u, \nabla_{\mathbb{M}} v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2}, \frac{N}{2}}(\mathbb{M})} \\ &\geq a(\delta) r^2(\delta) \end{aligned} \tag{3.8}$$

for any $(u, v) \in \mathcal{N}_\delta$, which infers that

$$d(\delta) \geq a(\delta) r^2(\delta),$$

from the definition of $d(\delta)$ when $0 < \delta < \frac{p+1}{2}$.

The first part of this lemma is proved. Now let us prove eq. (3.7).

(1) If $\delta > 0$, $(\bar{u}, \bar{v}) \in \mathcal{N}_\delta$ is a minimizer of $d(\delta) = \inf\limits_{(u,v)\in\mathcal{N}_\delta} E(u, v)$, i.e., $E(\bar{u}, \bar{v}) = d(\delta)$. In this case we define $\lambda = \lambda(\delta)$ by $(\lambda\bar{u}, \lambda\bar{v}) \in \mathcal{N}$. Then, there exists a unique λ which satisfies

$$\lambda = \Big[\frac{\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} \bar{u}|^2 + |\nabla_{\mathbb{M}} \bar{v}|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} r(V_1|\bar{u}|^2 + V_2|\bar{v}|^2)\mathrm{d}\sigma}{(p+1)\int_{\mathbb{M}} rF(\bar{u}, \bar{v})\mathrm{d}\sigma}\Big]^{\frac{1}{p-1}} = \delta^{-\frac{1}{p-1}}$$

for each $\delta > 0$. Thus, from the definition of $d(\delta)$ by (3.5) for $\delta = 1$ and $(\lambda\bar{u}, \lambda\bar{v}) \in \mathcal{N}$, we can obtain that

$$\begin{aligned} d &\leq E(\lambda\bar{u}, \lambda\bar{v}) \\ &= \frac{p-1}{2(p+1)}\lambda^2\Big[\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} \bar{u}|^2 + |\nabla_{\mathbb{M}} \bar{v}|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} r(V_1|\bar{u}|^2 + V_2|\bar{v}|^2)\mathrm{d}\sigma\Big] + \frac{1}{p+1}K(\lambda\bar{u}, \lambda\bar{v}) \\ &= \frac{p-1}{2(p+1)}\delta^{-\frac{2}{p-1}}\Big[\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} \bar{u}|^2 + |\nabla_{\mathbb{M}} \bar{v}|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} r(V_1|\bar{u}|^2 + V_2|\bar{v}|^2)\mathrm{d}\sigma\Big]. \end{aligned}$$

Notice that

$$d(\delta) = E(\bar{u}, \bar{v}) = (\frac{1}{2} - \frac{\delta}{p+1})\Big[\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} \bar{u}|^2 + |\nabla_{\mathbb{M}} \bar{v}|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} r(V_1|\bar{u}|^2 + V_2|\bar{v}|^2)\mathrm{d}\sigma\Big],$$

so we get

$$d \leq \delta^{-\frac{2}{p-1}}\frac{p-1}{2(p+1)}(\frac{1}{2} - \frac{\delta}{p+1})^{-1} d(\delta),$$

which implies that

$$d(\delta) \geq \frac{2(p+1)}{p-1}(\frac{1}{2} - \frac{\delta}{p+1})\delta^{\frac{2}{p-1}} d \tag{3.9}$$

for $0 < \delta < \frac{p+1}{2}$.

(2) If $\delta > 0$, $(\tilde{u},\tilde{v}) \in \mathcal{N}$ is a minimizer of $d = \inf\limits_{(u,v)\in\mathcal{N}} E(u,v)$, i.e., $E(\tilde{u},\tilde{v}) = d$. In this case, we define $\lambda = \lambda(\delta)$ by $(\tilde{u},\tilde{v}) \in \mathcal{N}_\delta$. Then, there exists a unique λ which satisfies

$$\lambda = \left[\frac{\delta\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}}\tilde{u}|^2 + |\nabla_{\mathbb{M}}\tilde{v}|^2)\mathrm{d}\sigma - \delta\int_{\mathbb{M}} r(V_1|\tilde{u}|^2 + V_2|\tilde{v}|^2)\mathrm{d}\sigma}{(p+1)\int_{\mathbb{M}} rF(\tilde{u},\tilde{v})\mathrm{d}\sigma}\right]^{\frac{1}{p-1}} = \delta^{\frac{1}{p-1}}$$

for each $\delta > 0$. Thus, from the definition of $d(\delta)$ by (3.5) and $(\lambda\tilde{u},\lambda\tilde{v}) \in \mathcal{N}_\delta$, we can obtain that

$$\begin{aligned} d(\delta) &\le E(\lambda\tilde{u},\lambda\tilde{v}) \\ &= (\frac{1}{2} - \frac{\delta}{p+1})\lambda^2\left[\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}}\tilde{u}|^2 + |\nabla_{\mathbb{M}}\tilde{v}|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} r(V_1|\tilde{u}|^2 + V_2|\tilde{v}|^2)\mathrm{d}\sigma\right] \\ &\quad + \frac{1}{p+1}K_\delta(\lambda\tilde{u},\lambda\tilde{v}) \\ &= (\frac{1}{2} - \frac{\delta}{p+1})\delta^{\frac{2}{p-1}}\left[\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}}\tilde{u}|^2 + |\nabla_{\mathbb{M}}\tilde{v}|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} r(V_1|\tilde{u}|^2 + V_2|\tilde{v}|^2)\mathrm{d}\sigma\right]. \end{aligned}$$

Notice that

$$d = E(\tilde{u},\tilde{v}) = \frac{p-1}{2(p+1)}\left[\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}}\tilde{u}|^2 + |\nabla_{\mathbb{M}}\tilde{v}|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} r(V_1|\tilde{u}|^2 + V_2|\tilde{v}|^2)\mathrm{d}\sigma\right],$$

so we get

$$d(\delta) \le \frac{2(p+1)}{p-1}(\frac{1}{2} - \frac{\delta}{p+1})\delta^{\frac{2}{p-1}}d \tag{3.10}$$

for $0 < \delta < \frac{p+1}{2}$. From (3.9) and (3.10), we obtain the conclusion, (3.7). □

Additionally, we show how $d(\delta)$ behaves with respect to δ in the following lemma:

Lemma 3.6 $d(\delta)$ satisfies the following properties :

(i) $\lim\limits_{\delta\to 0} d(\delta) = 0$, $d(\frac{p+1}{2}) = 0$ and $d(\delta)$ is continuous and $d(\delta) > 0$ for $0 < \delta \le \frac{p+1}{2}$;

(ii) $d(\delta)$ is increasing on $0 < \delta \le 1$, decreasing on $1 \le \delta \le \frac{p+1}{2}$, and takes the maximum $d = d(1)$ at $\delta = 1$.

Proof From Lemma 3.5, we immediately obtain the result of (i) and we also have that

$$d'(\delta) = \frac{2(p+1)}{(p-1)^2}\delta^{\frac{2}{p-1}-1}(1-\delta),$$

which implies the conclusion of (ii). □

Lemma 3.7 Assume that $0 < E(u,v) < d$ for some $u,v \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$, and $\delta_1 < \delta_2$ are the two roots of the equation $E(u,v) = d(\delta)$. Then the sign of $K_\delta(u,v)$ does not change for $\delta \in (\delta_1,\delta_2)$.

Proof $E(u,v) > 0$ implies $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} \ne 0$. If the sign of $K_\delta(u,v)$ is changeable for $\delta \in (\delta_1,\delta_2)$, we can choose $\bar{\delta} \in (\delta_1,\delta_2)$ satisfying $K_{\bar{\delta}}(u,v) = 0$. Thus, by the definition of $d(\delta)$, we have $E(u,v) \ge d(\bar{\delta})$. However, from Lemma 3.6, $E(u,v) = d(\delta_1) = d(\delta_2) < d(\bar{\delta})$, which is a contraction. □

After the definition of the depth of the family of potential wells $d(\delta)$, the following lemmas are given to exhibit the relation between $K_\delta(u,v)$ and $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}$ when $E(u,v) \le d(\delta)$:

Lemma 3.8 Let $u,v \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$ and $0 < \delta < \frac{p+1}{2}$. Assume that $E(u,v) \le d(\delta)$.

(i) If $K_\delta(u,v)>0$, then $0<\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<\frac{d(\delta)}{a(\delta)}$. In particular, if $K(u,v)>0$, then

$$0<\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<\frac{2(p+1)}{p-1}d(1-\mu C^{*2})^{-1}.$$

(ii) If $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}>\frac{d(\delta)}{a(\delta)}$, then $K_\delta(u,v)<0$. In particular, if

$$\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}>\frac{2(p+1)}{p-1}d(1-\mu C^{*2})^{-1},$$

then $K(u,v)<0$.

(iii) If $K_\delta(u,v)=0$, then $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\le\frac{d(\delta)}{a(\delta)}$. In particular, if $K(u,v)=0$, then

$$\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\le\frac{2(p+1)}{p-1}d(1-\mu C^{*2})^{-1}.$$

Proof (i) If $E(u,v)\le d(\delta)$ and $K_\delta(u,v)>0$, we can get

$$\begin{aligned}
d(\delta)&\ge E(u,v)\\
&=(\frac{1}{2}-\frac{\delta}{p+1})\left[\int_{\mathbb{M}}r(|\nabla_{\mathbb{M}}u|^2+|\nabla_{\mathbb{M}}v|^2)\mathrm{d}\sigma-\int_{\mathbb{M}}r(V_1|u|^2+V_2|v|^2)\mathrm{d}\sigma\right]+\frac{1}{p+1}K_\delta(u,v)\\
&\ge(\frac{1}{2}-\frac{\delta}{p+1})(1-\mu C^{*2})\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}+\frac{1}{p+1}K_\delta(u,v)\\
&>a(\delta)\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}.
\end{aligned}$$

(ii) If $E(u,v)\le d(\delta)$ and $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}>\frac{d(\delta)}{a(\delta)}$, then

$$\begin{aligned}
K_\delta(u,v)&=(p+1)E(u,v)\\
&\quad-(\frac{p+1}{2}-\delta)\left[\int_{\mathbb{M}}r(|\nabla_{\mathbb{M}}u|^2+|\nabla_{\mathbb{M}}v|^2)\mathrm{d}\sigma-\int_{\mathbb{M}}r(V_1|u|^2+V_2|v|^2)\mathrm{d}\sigma\right]\\
&\le(p+1)d(\delta)-(\frac{p+1}{2}-\delta)(1-C^{*2})\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\\
&<(p+1)d(\delta)-(\frac{p+1}{2}-\delta)(1-C^{*2})\frac{d(\delta)}{a(\delta)}=0.
\end{aligned}$$

(iii) If $E(u,v)\le d(\delta)$ and $K_\delta(u,v)=0$, then

$$\begin{aligned}
d(\delta)\ge E(u,v)&\ge(\frac{1}{2}-\frac{\delta}{p+1})(1-\mu C^{*2})\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}+\frac{1}{p+1}K_\delta(u,v)\\
&=a(\delta)\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}.
\end{aligned}$$

□

Remark 3.9 The results of Lemmas 3.2–3.4 and Lemma 3.8 show that the space $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$ is divided into two parts – $K_\delta(u,v)>0$ and $K_\delta(u,v)<0$ – by surface $K_\delta(u,v)=0((u,v)\ne(0,0))$. The inside part of $K_\delta(u,v)=0$ is $K_\delta(u,v)>0$ and the outside part of $K_\delta(u,v)=0$ is $K_\delta(u,v)<0$. Sphere $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}=r(\delta)$ lies inside of $K_\delta(u,v)>0$ and sphere $\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}=\frac{d(\delta)}{a(\delta)}$ lies inside of $K_\delta(u,v)<0$.

Now we are in a position to introduce a family of potential wells. For $0<\delta<\frac{p+1}{2}$, define

$$W_\delta=\Big\{(u,v)\in(\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2|K_\delta(u,v)>0,E(u,v)<d(\delta)\}\cup\{(0,0)\Big\},$$
$$\bar{W}_\delta=W_\delta\cap\partial W_\delta=\Big\{(u,v)\in(\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2|K_\delta(u,v)\geq 0,E(u,v)\leq d(\delta)\Big\}.$$

In addition, we define the exterior of the corresponding potential well sets by

$$Z_\delta=\Big\{(u,v)\in(\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2|K_\delta(u,v)<0,E(u,v)<d(\delta)\Big\}$$

for $0<\delta<\frac{p+1}{2}$.

Lemma 3.10 Let $0<\delta<\frac{p+1}{2}$. Then

(i) $B_{r_1(\delta)}\subset W_\delta\subset B_{r_2(\delta)}$, where

$$B_{r_1(\delta)}=\Big\{(u,v)\in(\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2|\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<\min\{r^2(\delta),2d(\delta)\}\Big\},$$
$$B_{r_2(\delta)}=\Big\{(u,v)\in(\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2|\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<\frac{d(\delta)}{a(\delta)}\Big\};$$

(ii) $Z_\delta\subset B_\delta^c$, where

$$B_\delta=\Big\{(u,v)\in(\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2|\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<r(\delta)\Big\}.$$

Proof (i) If $(u,v)\in B_{r_1(\delta)}$, then

$$\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<r(\delta)$$

and

$$\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<2d(\delta).$$

Then, from Lemma 3.2 and $E(u,v)\leq\frac{1}{2}\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}$, we have $K_\delta(u,v)>0$ and $E(u,v)<d(\delta)$. Hence, we can obtain that $(u,v)\in W_\delta$.

If $(u,v)\in W_\delta$, from Lemma 3.8 we can obtain that

$$\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<\frac{d(\delta)}{a(\delta)},$$

from which we can obtain that $(u,v)\in B_{r_2(\delta)}$.

(ii) If $(u,v)\in Z_\delta$, from Lemma 3.3 we can obtain that

$$\|(\nabla_{\mathbb{M}}u,\nabla_{\mathbb{M}}v)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}>r(\delta),$$

from which we can obtain that $(u,v)\in B_\delta^c$. □

From the definition of W_δ and Z_δ and Lemma 3.6 we can obtain

Lemma 3.11 The potential well sets and its outsiders have the following properties:

(i) If $0<\delta'<\delta''\leq 1$, then $W_{\delta'}\subset W_{\delta''}$.

(ii) If $1\leq\delta'<\delta''<\frac{p+1}{2}$, then $Z_{\delta'}\subset Z_{\delta''}$.

Next, by using the above potential wells, we establish the invariant sets and the vacuum isolating of solutions for problem (1.1).

Proposition 3.12 Assume that $u_0, v_0 \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Let $0<\delta<\frac{p+1}{2}$. Suppose that $0<e<d$, $\delta_1<\delta_2$ are the two roots of equation $d(\delta)=e$. Then all solutions of problem (1.1) with $0<E(u_0,v_0)\le e, K(u_0,v_0)>0$ belong to W_δ for $\delta_1<\delta<\delta_2$.

Proof Let $(u(t),v(t))$ be any solution of problem (1.1) with $E(u_0,v_0)=e, K(u_0,v_0)>0$, and let T is the maximal existence time of $(u(t),v(t))$. From Lemma 3.6, it follows that

$$0<E(u_0,v_0)\le e=d(\delta_1)=d(\delta_2)<d(\delta)<d$$

for $\delta_1<\delta<\delta_2$. From Lemma 3.7, it follows that $K_\delta(u_0,v_0)>0$ for $\delta_1<\delta<\delta_2$. Hence, we can obtain that $(u_0,v_0)\in W_\delta$ for $\delta_1<\delta<\delta_2$.

Our goal is to prove that $(u(t),v(t))\in W_\delta$ for $t\in[0,T)$ and $\delta_1<\delta<\delta_2$. We argue by contradiction. Assume that there exist a $\delta_0\in(\delta_1,\delta_2)$ and $t_0\in(0,T)$ such that $(u(t_0),v(t_0))\notin W_{\delta_0}$, which means that $E(u(t_0),v(t_0))\ge d(\delta_0)$ or $K_{\delta_0}(u(t_0),v(t_0))\le 0$ and $(u(t_0),v(t_0))\neq(0,0)$.

If $K_{\delta_0}(u(t_0),v(t_0))=0$ and $(u(t_0),v(t_0))\neq(0,0)$, then $(u(t_0),v(t_0))\in\mathcal{N}_{\delta_0}$, by the definition of $\mathcal{N}_\delta$ itself. From the definition of $d(\delta)$, we can obtain that $E(u(t_0),v(t_0))\ge d(\delta_0)$.

If $K_{\delta_0}(u(t_0),v(t_0))<0$ and $(u(t_0),v(t_0))\neq(0,0)$, from the time continuity of $K_\delta(u,v)$ and $K_{\delta_0}(u_0,v_0)>0$, we can obtain that there exists at least one $s\in(0,t_0)$ such that $K_{\delta_0}(u(s),v(s))=0$. Put

$$t^*:=\sup\{s\in(0,t_0)|K_{\delta_0}(u(s),v(s))=0\}.$$

Consequently, $K_{\delta_0}(u(t^*),v(t^*))=0$ and $K_{\delta_0}(u(t),v(t))<0$ for $t\in(t^*,t_0)$.

We have two cases to consider:

Case 1 $(u(t^*),v(t^*))\neq(0,0)$. In this case, $(u(t^*),v(t^*))\in\mathcal{N}_{\delta_0}$, by the definition of $\mathcal{N}_\delta$ itself. From the definition of $d(\delta)$, we can obtain that $E(u(t^*),v(t^*))\ge d(\delta_0)$.

By recalling the conservation of energy (1.9), we note that

$$E(u(t),v(t))\le E(u_0,v_0)<d(\delta)$$

for $t\in[0,T)$ and $\delta_1<\delta<\delta_2$. Hence, $E(u(t_0),v(t_0))\ge d(\delta)$ is impossible for any $\delta_1<\delta<\delta_2$.

Case 2 $(u(t^*),v(t^*))=(0,0)$. In this case, we must have $K_{\delta_0}(u(t),v(t))<0$ for $t\in(t^*,t_0)$ and $\lim\limits_{t\to t^{*+}}(u(t),v(t))=0$. Thus, from Lemma 3.3, we can obtain that

$$\|(\nabla_{\mathbb{M}}u(t),\nabla_{\mathbb{M}}v(t))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}>r(\delta_0)$$

for $t\in(t^*,t_0)$, which is in contradiction with $\lim\limits_{t\to t^{*+}}(u(t),v(t))=0$. □

Proposition 3.13 Assume that $u_0, v_0 \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Let $0<\delta<\frac{p+1}{2}$. Suppose that $0<e<d$, $\delta_1<\delta_2$ are the two roots of equation $d(\delta)=e$. Then all solutions of problem (1.1) with $0<E(u_0,v_0)\le e, K(u_0,v_0)<0$ belong to Z_δ for $\delta_1<\delta<\delta_2$.

Proof Let $(u(t),v(t))$ be any solution of problem (1.1) with $E(u_0,v_0)=e, K(u_0,v_0)<0$, and that T is the maximal existence time of $(u(t),v(t))$. From Lemma 3.6, it follows that

$$0<E(u_0,v_0)\le e=d(\delta_1)=d(\delta_2)<d(\delta)<d$$

for $\delta_1<\delta<\delta_2$. From Lemma 3.7, it follows that $K_\delta(u_0,v_0)<0$ for $\delta_1<\delta<\delta_2$. Hence, we can obtain that $(u_0,v_0)\in Z_\delta$ for $\delta_1<\delta<\delta_2$.

Our goal is to prove that $(u(t),v(t))\in Z_\delta$ for $t\in[0,T)$ and $\delta_1<\delta<\delta_2$. We argue by contradiction. Assume that there exist a $\delta_0\in(\delta_1,\delta_2)$ and $t_0\in(0,T)$ such that $(u(t_0),v(t_0))\notin Z_{\delta_0}$, which means that $E(u(t_0),v(t_0))\geq d(\delta_0)$ or $K_{\delta_0}(u(t_0),v(t_0))\geq 0$.

If $K_{\delta_0}(u(t_0),v(t_0))\geq 0$, from the time continuity of $K_\delta(u,v)$ and $K_{\delta_0}(u_0,v_0)<0$, we can obtain that there exists at least one $s\in(0,t_0)$ such that $K_{\delta_0}(u(s),v(s))=0$. Put

$$t^*:=\inf\{s\in(0,t_0)|K_{\delta_0}(u(s),v(s))=0\}.$$

Consequently, $K_{\delta_0}(u(t^*),v(t^*))=0$ and $K_{\delta_0}(u(t),v(t))<0$ for $t\in(0,t^*)$.

We have two cases to consider:

Case 1 $(u(t^*),v(t^*))\neq(0,0)$. In this case, $(u(t^*),v(t^*))\in\mathcal{N}_{\delta_0}$, by the definition of $\mathcal{N}_\delta$ itself. From the definition of $d(\delta)$, we can obtain that $E(u(t^*),v(t^*))\geq d(\delta_0)$.

By recalling the conservation of energy (1.9), we note that

$$E(u(t),v(t))\leq E(u_0,v_0)<d(\delta)$$

for $t\in[0,T)$ and $\delta_1<\delta<\delta_2$. Hence, $E(u(t_0),v(t_0))\geq d(\delta)$ is impossible for any $\delta_1<\delta<\delta_2$.

Case 2 $(u(t^*),v(t^*))=(0,0)$. In this case, we must have $K_{\delta_0}(u(t),v(t))<0$ for $t\in(0,t^*)$ and $\lim\limits_{t\to t^{*-}}(u(t),v(t))=0$. Thus, from Lemma 3.3, we can obtain that

$$\|(\nabla_{\mathbb{M}}u(t),\nabla_{\mathbb{M}}v(t))\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}>r(\delta_0)$$

for $t\in(0,t^*)$, which is in contradiction with $\lim\limits_{t\to t^{*-}}(u(t),v(t))=0$. $\square$

The above propositions indicate the invariance of W_δ and Z_δ, respectively. Moreover, concerning their intersections with respect to δ, we have

Proposition 3.14 Assume that $u_0,v_0\in\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Let $0<\delta<\frac{p+1}{2}$. Suppose that $0<e<d$, $\delta_1<\delta_2$ are the two roots of equation $d(\delta)=e$. Then, for any $\delta\in(\delta_1,\delta_2)$, both sets

$$W_{\delta_1\delta_2}=\bigcup_{\delta_1<\delta<\delta_2}W_\delta\quad\text{and}\quad Z_{\delta_1\delta_2}=\bigcup_{\delta_1<\delta<\delta_2}Z_\delta$$

are invariant under the flow of problem (1.1), provided that $0<E(u_0,v_0)\leq e$.

The following proposition shows that between these two invariance manifolds, $W_{\delta_1\delta_2}$ and $Z_{\delta_1\delta_2}$, there exists a so called vacuum region, for which no solution exists:

Proposition 3.15 Assume that $u_0,v_0\in\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Let $0<\delta<\frac{p+1}{2}$. Suppose that $0<e<d$, $\delta_1<\delta_2$ are the two roots of equation $d(\delta)=e$. Then for all solutions of problem (1.1) with $0<E(u_0,v_0)\leq e$, we have

$$(u,v)\notin\mathcal{N}_{\delta_1\delta_2}=\bigcup_{\delta_1<\delta<\delta_2}\mathcal{N}_\delta.$$

Remark 3.16 The vacuum region becomes bigger and bigger with the decreasing of e. As the limit case, we obtain

$$\mathcal{N}_0=\bigcup_{0<\delta<\frac{p+1}{2}}\mathcal{N}_\delta.$$

4 Sharp Threshold for Global Existence and Blow-up of Solutions

In this section we prove the main results by making use of the family of potential wells introduced above. First we have the following lemma of Komornik [15], which plays a critical role in the study of the exponential asymptotic behavior for global solutions of problem (1.1):

Lemma 4.1 Let $y(t):\mathbb{R}^+\to\mathbb{R}^+$ be a non-increasing function, and assume that there is a constant $A>0$ such that

$$\int_s^{+\infty} y(t)\mathrm{d}t \le Ay(s), \quad 0\le s<+\infty.$$

Then for all $t\ge 0$, we have

$$y(t)\le y(0)\mathrm{e}^{1-\frac{t}{A}}.$$

Proof of Theorem 1.3 We divide the proof in three steps.

Step 1 Proof of the global existence for the low initial energy case.

If $E(u_0,v_0)<d$, from the definition of d we can obtain that $K(u_0,v_0)=0$ is impossible, hence we only need to consider the cases $E(u_0,v_0)<d$ and $K(u_0,v_0)>0$.

From Proposition 2.9, we can choose $\{\varphi_j(r,x,w)\}$ as the orthonormal basis of $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Construct the approximate solutions of the problem (1.1) as follows:

$$\begin{aligned} u_m(t,r,x,w)&=\sum_{j=1}^m g_{jm}(t)\varphi_j(r,x,w), \quad m=1,2,\cdots\\ v_m(t,r,x,w)&=\sum_{j=1}^m h_{jm}(t)\varphi_j(r,x,w), \quad m=1,2,\cdots, \end{aligned} \tag{4.1}$$

satisfying

$$\begin{aligned} &\int_{\mathbb{M}} r\partial_t u_m\cdot\varphi_k\mathrm{d}\sigma+\int_{\mathbb{M}} r\nabla_{\mathbb{M}}u_m\cdot\nabla_{\mathbb{M}}\varphi_k\mathrm{d}\sigma-\int_{\mathbb{M}} r\mu V_1u_m\cdot\varphi_k\mathrm{d}\sigma=\int_{\mathbb{M}} rf_1(u_m,v_m)\cdot\varphi_k\mathrm{d}\sigma,\\ &\int_{\mathbb{M}} r\partial_t v_m\cdot\varphi_k\mathrm{d}\sigma+\int_{\mathbb{M}} r\nabla_{\mathbb{M}}v_m\cdot\nabla_{\mathbb{M}}\varphi_k\mathrm{d}\sigma-\int_{\mathbb{M}} r\mu V_2v_m\cdot\varphi_k\mathrm{d}\sigma=\int_{\mathbb{M}} rf_2(u_m,v_m)\cdot\varphi_k\mathrm{d}\sigma \end{aligned} \tag{4.2}$$

for $k=1,2,\cdots,m$ and, as $m\to+\infty$,

$$\begin{aligned} u_m(0,r,x,w)&=\sum_{j=1}^m g_{jm}(0)\varphi_j(r,x,w)\to u_0(r,x,w), \quad \text{in } \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}),\\ v_m(0,r,x,w)&=\sum_{j=1}^m h_{jm}(0)\varphi_j(r,x,w)\to v_0(r,x,w), \quad \text{in } \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}). \end{aligned} \tag{4.3}$$

Multiplying (4.2) by $g'_{km}(t)$ and $h'_{km}(t)$ and then summing for k, we have

$$\begin{aligned} &\int_{\mathbb{M}} r\partial_t u_m\cdot\partial_t u_m\mathrm{d}\sigma+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{M}} r\nabla_{\mathbb{M}}u_m\cdot\nabla_{\mathbb{M}}u_m\mathrm{d}\sigma-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{M}} r\mu V_1|u_m|^2\mathrm{d}\sigma\\ =&\frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{M}} rf_1(u_m,v_m)u_m\mathrm{d}\sigma,\\ &\int_{\mathbb{M}} r\partial_t v_m\cdot\partial_t v_m\mathrm{d}\sigma+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{M}} r\nabla_{\mathbb{M}}v_m\cdot\nabla_{\mathbb{M}}v_m\mathrm{d}\sigma-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{M}} r\mu V_2|v_m|^2\mathrm{d}\sigma\\ =&\frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{M}} rf_2(u_m,v_m)v_m\mathrm{d}\sigma. \end{aligned} \tag{4.4}$$

By (4.3) we can get $E(u_m(0),v_m(0))\to E(u_0,v_0)<d$ as $m\to\infty$. Then, integrating (4.4) with respect to t, we can obtain that for sufficiently large m,

$$\int_0^t \|(\partial_\tau u_m,\partial_\tau v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\mathrm{d}\tau+E(u_m,v_m)=E(u_m(0),v_m(0))<d,\quad 0\le t<T. \tag{4.5}$$

Next, we prove that $(u_m(t),v_m(t))\in W$ for sufficiently large m and $0\le t<T$. In fact, if this is false, from the time continuity of $K(u,v)$, there exists $t_0>0$ such that $(u_m(t_0),v_m(t_0))\in\partial W$. Then $E(u_m(t_0),v_m(t_0))=d$ or $K(u_m(t_0),v_m(t_0))=0$ and $(u_m(t_0),v_m(t_0))\neq(0,0)$. From (4.5) we can obtain that $E(u_m(t_0),v_m(t_0))\neq d$. If $K(u_m(t_0),v_m(t_0))=0$ and $(u_m(t_0),v_m(t_0))\neq(0,0)$, then $(u_m(t_0),v_m(t_0))\in\mathcal{N}$. By the definition of $d=\inf\{E(u,v);(u,v)\in\mathcal{N}\}$, we can obtain that $E(u_m(t_0),v_m(t_0))\ge d$, which contradicts (4.5).

Hence, from (4.5) and

$$\begin{aligned}E(u_m,v_m)&=\frac{p-1}{2(p+1)}\bigg[\int_{\mathbb{M}}r(|\nabla_{\mathbb{M}}u_m|^2+|\nabla_{\mathbb{M}}v_m|^2)\mathrm{d}\sigma\\&\quad-\int_{\mathbb{M}}r\mu(V_1|u_m|^2+V_2|v_m|^2)\mathrm{d}\sigma\bigg]+\frac{1}{p+1}K(u_m,v_m)\\&\ge\frac{p-1}{2(p+1)}(1-\mu C^{*2})\|(\nabla_{\mathbb{M}}u_m,\nabla_{\mathbb{M}}v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})},\end{aligned}$$

it follows that

$$\int_0^t \|(\partial_\tau u_m,\partial_\tau v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\mathrm{d}\tau+\frac{p-1}{2(p+1)}(1-\mu C^{*2})\|(\nabla_{\mathbb{M}}u_m,\nabla_{\mathbb{M}}v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<d,$$

for $0\le t<T$ and sufficiently large m which yields

$$\begin{aligned}&\|\nabla_{\mathbb{M}}u_m\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<\frac{2(p+1)}{p-1}(1-\mu C^{*2})^{-1}d,\quad 0\le t<T,\\&\|\nabla_{\mathbb{M}}v_m\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<\frac{2(p+1)}{p-1}(1-\mu C^{*2})^{-1}d,\quad 0\le t<T,\\&\int_0^t\|\partial_\tau u_m\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\mathrm{d}\tau<d,\quad 0\le t<T,\\&\int_0^t\|\partial_\tau v_m\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\mathrm{d}\tau<d,\quad 0\le t<T,\end{aligned}$$

and from Proposition 2.7, we can obtain that

$$\begin{aligned}&\|u_m\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\le d_{\mathbb{M}}^2\|\nabla_{\mathbb{M}}u_m\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<d_{\mathbb{M}}^2\frac{2(p+1)}{p-1}(1-\mu C^{*2})^{-1}d,\quad 0\le t<T,\\&\|v_m\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\le d_{\mathbb{M}}^2\|\nabla_{\mathbb{M}}v_m\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<d_{\mathbb{M}}^2\frac{2(p+1)}{p-1}(1-\mu C^{*2})^{-1}d,\quad 0\le t<T.\end{aligned}$$

It follows that there exist u,v and subsequences $\{u_\nu\}$ and $\{v_\nu\}$ of $\{u_m\}$ and $\{v_m\}$ such that, as $\nu\to\infty$,

$$\begin{aligned}&u_\nu\to u\quad\text{ in } L^\infty(0,\infty;\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))\text{ weakly star and a.e. in }\mathbb{M}\times[0,\infty),\\&v_\nu\to v\quad\text{ in } L^\infty(0,\infty;\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))\text{ weakly star and a.e. in }\mathbb{M}\times[0,\infty),\\&\partial_t u_\nu\to\partial_t u\quad\text{ in } L^2(0,\infty;\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M}))\text{ weakly},\\&\partial_t v_\nu\to\partial_t v\quad\text{ in } L^2(0,\infty;\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M}))\text{ weakly}.\end{aligned}$$

From the continuity of $f_i(u,v), i=1,2$, in (4.2), for fixed k, letting $m=\nu\to\infty$, we obtain that

$$\int_{\mathbb{M}} r\partial_t u\cdot\varphi_k \mathrm{d}\sigma+\int_{\mathbb{M}} r\nabla_{\mathbb{M}}u\cdot\nabla_{\mathbb{M}}\varphi_k\mathrm{d}\sigma-\int_{\mathbb{M}} r\mu V_1 u\cdot\varphi_k\mathrm{d}\sigma=\int_{\mathbb{M}} rf_1(u,v)\cdot\varphi_k\mathrm{d}\sigma,$$
$$\int_{\mathbb{M}} r\partial_t v\cdot\varphi_k \mathrm{d}\sigma+\int_{\mathbb{M}} r\nabla_{\mathbb{M}}v\cdot\nabla_{\mathbb{M}}\varphi_k\mathrm{d}\sigma-\int_{\mathbb{M}} r\mu V_2 v\cdot\varphi_k\mathrm{d}\sigma=\int_{\mathbb{M}} rf_2(u,v)\cdot\varphi_k\mathrm{d}\sigma$$

for $k=1,2,\cdots,m$, and

$$\int_{\mathbb{M}} r\partial_t u\cdot\psi \mathrm{d}\sigma+\int_{\mathbb{M}} r\nabla_{\mathbb{M}}u\cdot\nabla_{\mathbb{M}}\psi\mathrm{d}\sigma-\int_{\mathbb{M}} r\mu V_1 u\cdot\psi\mathrm{d}\sigma=\int_{\mathbb{M}} rf_1(u,v)\cdot\psi\mathrm{d}\sigma,$$
$$\int_{\mathbb{M}} r\partial_t v\cdot\psi \mathrm{d}\sigma+\int_{\mathbb{M}} r\nabla_{\mathbb{M}}v\cdot\nabla_{\mathbb{M}}\psi\mathrm{d}\sigma-\int_{\mathbb{M}} r\mu V_2 v\cdot\psi\mathrm{d}\sigma=\int_{\mathbb{M}} rf_2(u,v)\cdot\psi\mathrm{d}\sigma$$

for any $\psi\in\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$ and $t\in(0,\infty)$. On the other hand, from the time continuity of the weak solution and (4.3), we can obtain that $u(r,x,w,0)=u_0(r,x,w)$ and $v(r,x,w,0)=v_0(r,x,w)$ for $(r,x,w)\in\mathbb{M}$. By the density, we can obtain that $u,v\in L^\infty(0,\infty;\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))$, with $u_t,v_t\in L^2(0,\infty;\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M}))$ being a global weak solution of problem (1.1).

Step 2 Proof of the global existence for the critical initial energy case.

First $E(u_0,v_0)=d$ implies that $(u_0,v_0)\neq 0$. Pick a sequence $\{\lambda_m\}$ such that $0<\lambda_m<1$, $m=1,2,\cdots$ and $\lambda_m\to 1$ as $m\to\infty$. Let $u_{0m}=\lambda_m u_0$ and $v_{0m}=\lambda_m v_0$ for $m=1,2,\cdots$. Consider the initial conditions of the following problem:

$$\begin{cases}\partial_t u-\Delta_{\mathbb{M}}u-\mu V_1 u=F_u(u,v), \text{ in } \mathbb{M}_0\times(0,T),\\ \partial_t v-\Delta_{\mathbb{M}}v-\mu V_2 v=F_v(u,v), \text{ in } \mathbb{M}_0\times(0,T),\\ u(0,z)=u_{0m}, v(0,z)=v_{0m}, \quad \text{in } \mathbb{M}_0,\\ u=0, v=0, \quad \text{on } \partial\mathbb{M}\times(0,T).\end{cases}\tag{4.6}$$

From $K(u_0,v_0)\geq 0$ and Lemma 3.1, we have $\lambda^*=\lambda^*(u_0,v_0)\geq 1$, $K(u_{0m},v_{0m})=K(\lambda_m u_0,\lambda_m v_0)>0$ and $E(u_{0m},v_{0m})=E(\lambda_m u_0,\lambda_m v_0)<E(u_0,v_0)=d$. Thus, for each $m\in\mathbb{N}_+$, it follows from step 1 and Proposition 3.12 that problem (4.6) admits a global weak solution $u_m,v_m\in L^\infty(0,\infty;\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))$ with $u_{mt},v_{mt}\in L^2(0,\infty;\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M}))$ and $(u_m(t),v_m(t))\in W$ for $0\leq t<\infty$. Therefore, we can obtain that

$$\int_0^t \|(\partial_\tau u_m,\partial_\tau v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\mathrm{d}\tau+E(u_m,v_m)=E(u_{0m},v_{0m})<d,\quad 0\leq t<\infty.\tag{4.7}$$

From (4.7) and $K(u_m(t),v_m(t))>0$ for $0\leq t<\infty$, which means that

$$\begin{aligned}E(u_m,v_m)&=\frac{p-1}{2(p+1)}\Big[\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}}u_m|^2+|\nabla_{\mathbb{M}}v_m|^2)\mathrm{d}\sigma\\&\quad-\int_{\mathbb{M}} r(V_1|u_m|^2+V_2|v_m|^2)\mathrm{d}\sigma\Big]+\frac{1}{p+1}K(u_m,v_m)\\&\geq\frac{p-1}{2(p+1)}(1-\mu C^{*2})\|(\nabla_{\mathbb{M}}u_m,\nabla_{\mathbb{M}}v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})},\end{aligned}$$

and we can obtain that

$$\int_0^t \|(\partial_\tau u_m,\partial_\tau v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\mathrm{d}\tau+\frac{p-1}{2(p+1)}(1-\mu C^{*2})\|(\nabla_{\mathbb{M}}u_m,\nabla_{\mathbb{M}}v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}<d$$

for $0\leq t<\infty$. The remainder of the proof is similar to that of the proof of Step 1.

Step 3 Proof of asymptotic behavior.

If $E(u_0,v_0)<d$ and $K(u_0,v_0)>0$, from Proposition 3.12 we can obtain that $(u(t),v(t))\in W$, i.e., $E(u(t),v(t))<d, K(u(t),v(t))>0$ for $0\le t<\infty$. If $E(u_0,v_0)=d$ and $K(u_0,v_0)\ge 0$, from the approximative solution $(u_m(t),v_m(t))\in W$, we can obtain that $(u(t),v(t))\in \bar{W}$, i.e., $E(u(t),v(t))\le d, K(u(t),v(t))\ge 0$ for $0\le t<\infty$. Hence, the definition of C^* implies that

$$\begin{aligned}
d &\ge E(u,v)\\
&=\frac{p-1}{2(p+1)}\left[\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}}u|^2+|\nabla_{\mathbb{M}}v|^2)\mathrm{d}\sigma-\mu\int_{\mathbb{M}} r(V_1|u|^2+V_2|v|^2)\mathrm{d}\sigma\right]+\frac{1}{p+1}K(u,v)\\
&\ge \frac{p-1}{2(p+1)}(1-\mu C^{*2})\|(\nabla_{\mathbb{M}}u_m,\nabla_{\mathbb{M}}v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}.
\end{aligned}$$

Therefore, from the definition of C_*, or Proposition 2.6, we can obtain that

$$\begin{aligned}
&\alpha(p+1)\int_{\mathbb{M}} r|u+v|^{p+1}\mathrm{d}\sigma+2\beta(p+1)\int_{\mathbb{M}} r|uv|^{\frac{p+1}{2}}\mathrm{d}\sigma\\
&\le (p+1)(2^p\alpha+\beta)\int_{\mathbb{M}} r(|u|^{p+1}+|v|^{p+1})\mathrm{d}\sigma\\
&\le (p+1)(2^p\alpha+\beta)C_*^{p+1}(\|\nabla_{\mathbb{M}}u\|^{p+1}_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}+\|\nabla_{\mathbb{M}}v\|^{p+1}_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})})\\
&\le (p+1)(2^p\alpha+\beta)C_*^{p+1}(\|\nabla_{\mathbb{M}}u\|^{2}_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}+\|\nabla_{\mathbb{M}}v\|^{2}_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})})^{\frac{p+1}{2}}\\
&\le (p+1)(2^p\alpha+\beta)C_*^{p+1}\left[\frac{2(p+1)}{p-1}d(1-\mu C^{*2})^{-1}\right]^{\frac{p-1}{2}}\|(\nabla_{\mathbb{M}}u_m,\nabla_{\mathbb{M}}v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}.
\end{aligned}$$

Furthermore, the definition of C^* implies that

$$\mu\int_{\mathbb{M}} r(V_1|u|^2+V_2|v|^2)\mathrm{d}\sigma\le \mu C^{*2}\|(\nabla_{\mathbb{M}}u_m,\nabla_{\mathbb{M}}v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}.$$

We set

$$\sigma:=2(p+1)(2^p\alpha+\beta)C_*^{p+1}\left[\frac{2(p+1)}{p-1}d(1-\mu C^{*2})^{-1}\right]^{\frac{p-1}{2}}+\mu C^{*2},$$

and from Lemma 3.5 we have $0<\sigma<1$. Hence,

$$\mu\int_{\mathbb{M}} r(V_1|u|^2+V_2|v|^2)\mathrm{d}\sigma+(p+1)\int_{\mathbb{M}} rF(u,v)\mathrm{d}\sigma\le \sigma\|(\nabla_{\mathbb{M}}u_m,\nabla_{\mathbb{M}}v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}.$$

For $0<\gamma=1-\sigma<1$, we get

$$\mu\int_{\mathbb{M}} r(V_1|u|^2+V_2|v|^2)\mathrm{d}\sigma+(p+1)\int_{\mathbb{M}} rF(u,v)\mathrm{d}\sigma\le (1-\gamma)\|(\nabla_{\mathbb{M}}u_m,\nabla_{\mathbb{M}}v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})},$$

i.e.,

$$\gamma\|(\nabla_{\mathbb{M}}u_m,\nabla_{\mathbb{M}}v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\le K(u,v). \tag{4.8}$$

Let $T>0$ be a fixed time. Since $(u(t),v(t))\in\bar{W}$ for $0\le t<\infty$ is a global weak solution of problem (1.1), we can obtain that

$$\begin{aligned}
\int_{\mathbb{M}} ru_t\cdot u\mathrm{d}\sigma+\int_{\mathbb{M}} r\nabla_{\mathbb{M}}u\cdot\nabla_{\mathbb{M}}u\mathrm{d}\sigma-\int_{\mathbb{M}} r\mu V_1u\cdot u\mathrm{d}\sigma=\int_{\mathbb{M}} rf_1(u,v)\cdot u\mathrm{d}\sigma,\\
\int_{\mathbb{M}} rv_t\cdot v\mathrm{d}\sigma+\int_{\mathbb{M}} r\nabla_{\mathbb{M}}v\cdot\nabla_{\mathbb{M}}v\mathrm{d}\sigma-\int_{\mathbb{M}} r\mu V_2v\cdot v\mathrm{d}\sigma=\int_{\mathbb{M}} rf_2(u,v)\cdot v\mathrm{d}\sigma,
\end{aligned}$$

which means that
$$\frac{\mathrm{d}}{\mathrm{d}t}\|(u(t),v(t))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} = -2K(u(t),v(t)) \leq 0,$$
i.e., $y(t) = \|(u(t),v(t))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}$ is a non-increasing function for $0 \leq t < \infty$. Hence, from Proposition 2.7 and (4.8), we can obtain that
$$\begin{aligned}\int_s^T \|(u(t),v(t))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\mathrm{d}t &\leq \int_s^T d^2_{\mathbb{M}}\|(\nabla_{\mathbb{M}}u_m,\nabla_{\mathbb{M}}v_m)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\mathrm{d}t \\ &\leq \int_s^T \frac{d^2_{\mathbb{M}}}{\gamma}K(u(t),v(t))\mathrm{d}t \\ &\leq \frac{d^2_{\mathbb{M}}}{2\gamma}\Big[\|(u(s),v(s))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} - \|(u(T),v(T))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\Big] \\ &\leq \frac{d^2_{\mathbb{M}}}{2\gamma}\|(u(s),v(s))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}.\end{aligned}$$
Then, by the arbitrary nature of $T > 0$, it follows that
$$\int_s^\infty \|(u(t),v(t))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\mathrm{d}t \leq \frac{d^2_{\mathbb{M}}}{2\gamma}\|(u(s),v(s))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}$$
for $0 \leq s < \infty$, which means, from Lemma 4.1, that
$$\|(u(t),v(t))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} \leq \|(u_0,v_0)\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}\mathrm{e}^{1-\frac{t}{A}}$$
for $0 \leq t < \infty$, where $A = \frac{d^2_{\mathbb{M}}}{2\gamma} > 0$. Therefore, there exist constants $C > 0$ and $\lambda > 0$ such that
$$\|(u(t),v(t))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} \leq C\mathrm{e}^{-\lambda t} \quad \text{for} \quad 0 \leq t < \infty.$$
□

Next, we show that the global existence also holds for the case of potential well family.

Corollary 4.2 Let $u_0, v_0 \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Assume that $E(u_0,v_0) \leq d$ and $K_{\delta_2}(u_0,v_0) \geq 0$, where $\delta_1 < \delta_2$ are two roots of equation $d(\delta) = E(u_0,v_0)$. Then problem (1.1) admits a global weak solution $u, v \in C(0,\infty;\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))$ with $u_t, v_t \in L^2(0,\infty;\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))$ and $(u(t),v(t)) \in W_\delta$ for $\delta_1 < \delta < \delta_2, t \in [0,\infty)$.

Proof From Theorem 1.3 and Proposition 3.12, we note that to prove Corollary 4.2, it is sufficient to show that $K(u_0,v_0) > 0$, from $K_{\delta_2}(u_0,v_0) > 0$. Indeed, if this is false, then there exists a $\bar{\delta} \in [1,\delta_2)$ such that $K_{\bar{\delta}}(u_0,v_0) = 0$. Combining the fact that $(u_0,v_0) \neq (0,0)$, because of $K_{\delta_2}(u_0,v_0) > 0$, we get that $E(u_0,v_0) \geq d(\bar{\delta})$. However, from Lemma 3.6, we have $E(u_0,v_0) = d(\delta_1) = d(\delta_2) < d(\delta)$ for $\delta \in (\delta_1,\delta_2)$. This is a contradiction to $K_{\bar{\delta}}(u_0,v_0) = 0$, and thus we have the proof. □

Instead of considering the global existence result that depends on $K(u_0,v_0)$, we study the global existence of problem (1.1) with initial data u_0, v_0, relying on the $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$ norm.

Corollary 4.3 Let $u_0, v_0 \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Assume that $E(u_0,v_0) \leq d$ and
$$\|(\nabla_{\mathbb{M}}u_0,\nabla_{\mathbb{M}}v_0)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} < r(\delta_2),$$

where $\delta_1 < \delta_2$ are two roots of equation $d(\delta) = E(u_0, v_0)$. Then problem (1.1) admits a global weak solution $u, v \in C(0, \infty; \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))$ with $u_t, v_t \in L^2(0, \infty; \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))$ satisfying

$$\|(\nabla_{\mathbb{M}} u(t), \nabla_{\mathbb{M}} v(t))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} < \frac{E(u_0, v_0)}{a(\delta_1)}, \quad t \in [0, \infty). \tag{4.9}$$

Proof From $\|(\nabla_{\mathbb{M}} u_0, \nabla_{\mathbb{M}} v_0)\|_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})} < r(\delta_2)$, we get that $K_{\delta_2}(u_0, v_0) > 0$, by Lemma 3.2. Hence, the problem (1.1) has a global solution $u, v \in C(0, \infty; \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))$ with $u_t, v_t \in L^2(0, \infty; \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))$ and $(u(t), v(t)) \in W_\delta$ for $\delta_1 < \delta < \delta_2, t \in [0, \infty)$, from Corollary 4.2. Finally, (4.9) follows from Lemma 3.8. □

In view of Theorem 1.4, we need the following property for the depth of potential well d:

Lemma 4.4 For any $\epsilon > 0$, we have

$$d \le d_\epsilon + \frac{2\epsilon}{p+1},$$

where d_ϵ is defined by

$$d_\epsilon := \inf \Big\{ E(u, v) | (u, v) \in (\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2, K(u, v) = -\epsilon \Big\}.$$

Proof First, we show that for any fixed $\epsilon > 0$, $d_\epsilon > -\infty$.

Let $(u, v) \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}) \times \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$ satisfy $K(u, v) = -\epsilon$. From the definitions of C^* (1.3) and C_* (3.4), we can obtain that

$$\begin{aligned}
&(1 - \mu C^{*2}) \int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} u|^2 + |\nabla_{\mathbb{M}} v|^2) \mathrm{d}\sigma \\
\le& \int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} u|^2 + |\nabla_{\mathbb{M}} u|^2) \mathrm{d}\sigma - \int_{\mathbb{M}} r\mu(V_1|\phi|^2 + V_2|\psi|^2) \mathrm{d}\sigma \\
\le& (p+1) \int_{\mathbb{M}} rF(u, v) \mathrm{d}\sigma \\
\le& (2^p\alpha + \beta) \int_{\mathbb{M}} r(|u|^{p+1} + |v|^{p+1}) \mathrm{d}\sigma \\
\le& (2^p\alpha + \beta) C_*^{p+1} \left[\left(\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 \mathrm{d}\sigma \right)^{\frac{p+1}{2}} + \left(\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} v|^2 \mathrm{d}\sigma \right)^{\frac{p+1}{2}} \right] \\
\le& (2^p\alpha + \beta) C_*^{p+1} \left[\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} u|^2 + |\nabla_{\mathbb{M}} v|^2) \mathrm{d}\sigma \right]^{\frac{p+1}{2}}.
\end{aligned}$$

Thus

$$\int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} u|^2 + |\nabla_{\mathbb{M}} v|^2) \mathrm{d}\sigma \ge \left[\frac{1 - \mu C^{*2}}{(2^p\alpha + \beta) C_*^{p+1}} \right]^{\frac{2}{p-1}} > 0$$

and

$$\begin{aligned}
E(u, v) =& \frac{p-1}{2(p+1)} \int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} u|^2 + |\nabla_{\mathbb{M}} v|^2) \mathrm{d}\sigma \\
&- \frac{p-1}{2(p+1)} \int_{\mathbb{M}} r(V_1|u|^2 + V_2|v|^2) \mathrm{d}\sigma + \frac{1}{p+1} K(u, v) \\
\ge& \frac{p-1}{2(p+1)} (1 - \mu C^{*2}) \left[\frac{1 - \mu C^{*2}}{(2^p\alpha + \beta) C_*^{p+1}} \right]^{\frac{2}{p-1}} - \frac{\epsilon}{p+1}.
\end{aligned}$$

Therefore,

$$d_\epsilon \geq \frac{p-1}{2(p+1)}(1-\mu C^{*2})\left[\frac{1-\mu C^{*2}}{(2^p\alpha+\beta)C_*^{p+1}}\right]^{\frac{2}{p-1}} - \frac{\epsilon}{p+1} > -\infty.$$

Now, we choose a sequence $(u_j, v_j) \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}) \times \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}) \setminus \{(0,0)\}$ satisfying that

$$K(u_j, v_j) = -\epsilon, \quad E(u_j, v_j) \to d_\epsilon \text{ as } j \to \infty.$$

Moreover, we also can suppose that $E(u_j, v_j)$ is decreasing.

For each (u_j, v_j), one can choose a $\lambda_j \in \mathbb{R}$ such that $K(\lambda_j u_j, \lambda_j v_j) = 0$. In fact, λ_j can be determined explicitly by

$$\lambda_j^{p-1} = \frac{H_j}{I_j},$$

where

$$H_j = \int_{\mathbb{M}} r(|\nabla_{\mathbb{M}} u_j|^2 + |\nabla_{\mathbb{M}} v_j|^2)\mathrm{d}\sigma - \int_{\mathbb{M}} r(V_1|u_j|^2 + V_2|v_j|^2)\mathrm{d}\sigma,$$
$$I_j = (p+1)\int_{\mathbb{M}} rF(u_j, v_j)\mathrm{d}\sigma.$$

Since $K(u_j, v_j) = -\epsilon$ and $E(u_j, v_j) \to d_\epsilon$ as $j \to \infty$, we have

$$H_j = I_j - \epsilon,$$

and

$$E(u_j, v_j) = \frac{1}{2}H_j - \frac{1}{p+1}I_j = d_\epsilon + \eta_j,$$

where $\eta_j \to 0^+$, as $j \to \infty$.

Then, using d_ϵ and η_j, H_j and I_j can be expressed by

$$H_j = \frac{2(p+1)}{p-1}\Big(d_\epsilon + \eta_j + \frac{2\epsilon}{p+1}\Big), \quad I_j = \frac{2(p+1)}{p-1}\Big(d_\epsilon + \eta_j + \frac{1}{2}\epsilon\Big).$$

Notice that $K(\lambda_j u_j, \lambda_j v_j) = 0$, i.e., $(\lambda_j u_j, \lambda_j v_j) \in \mathcal{N}$.

Thus, for all $j \in \mathbb{N}$, $E(\lambda_j u_j, \lambda_j v_j) \geq d$ and a straightforward calculation yields that

$$\begin{aligned} d &\leq E(\lambda_j u_j, \lambda_j v_j) = \frac{\lambda_j^2}{2}H_j - \frac{\lambda_j^{p+1}}{p+1}I_j = \frac{1}{2}\left(\frac{H_j}{I_j}\right)^{\frac{2}{p-1}} H_j - \frac{1}{p+1}\left(\frac{H_j}{I_j}\right)^{\frac{p+1}{p-1}} I_j \\ &= \frac{p-1}{2(p+1)}\left(\frac{H_j}{I_j}\right)^{\frac{2}{p-1}} H_j = \left(\frac{d_\epsilon + \eta_j + \frac{\epsilon}{p+1}}{d_\epsilon + \eta_j + \frac{1}{2}\epsilon}\right)^{\frac{2}{p-1}} \left(d_\epsilon + \eta_j + \frac{\epsilon}{p+1}\right) \\ &\leq d_\epsilon + \eta_j + \frac{\epsilon}{p+1}. \end{aligned}$$

From $\eta_j \to 0^+$ as $j \to \infty$, we have $0 < \eta_j < \frac{\epsilon}{p+1}$ for sufficiently large j. Therefore,

$$d \leq d_\epsilon + \frac{2\epsilon}{p+1}.$$

□

Now we can give the proof of Theorem 1.4.

Proof of Theorem 1.4 We divide the proof into two steps.

Step 1 Let us first consider the low initial energy case.

If $E(u_0,v_0)<d$, from (3.2) we can obtain that $K(u_0,v_0)=0$ is impossible. Hence, we only need to consider the case where $E(u_0,v_0)<d$ and $K(u_0,v_0)<0$, i.e., $(u_0,v_0)\in Z$.

For $(u_0,v_0)\in Z$, we choose $\epsilon>0$ such that

$$\epsilon<\min\{\frac{1}{2}(d-E(u_0,v_0)),-K(u_0,v_0)\}.$$

From the choice of ϵ, it is obvious that $K(u_0,v_0)<-\epsilon$.

We claim that $K(u(t),v(t))<-\epsilon$ for all $t\in(0,T)$, where $T>0$ is the maximal existence time. Otherwise, from the time continuity of $K(u(t),v(t))$, there is $t_1\in(0,T)$ satisfying $K(u(t_1),v(t_1))=-\epsilon$. By using Lemma 4.4, we know that

$$E(u_0,v_0)<d-\epsilon<d-\frac{2\epsilon}{p+1}\le d_\epsilon.$$

From the conservation of energy (1.9), we have $E(u(t),v(t))<d_\epsilon$ for all $t\in(0,T)$. Thus $E(u(t_1),v(t_1))<d_\epsilon$, which is in contradiction with the definition of d_ϵ.

For any $(u,v)\in(\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}))^2\setminus\{(0,0)\}$, let

$$L(t)=\frac{1}{2}\|(u(t),v(t))\|^2_{\mathcal{L}_2^{\frac{N-1}{2},\frac{N}{2}}(\mathbb{M})}.$$

Along the flow generated by problem (1.1), we can compute that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L(t)&=\int_{\mathbb{M}}r(uu_t+vv_t)\mathrm{d}\sigma\\&=-\int_{\mathbb{M}}r[(|\nabla_{\mathbb{M}}u|^2+|\nabla_{\mathbb{M}}v|^2)-\mu(V_1|u|^2+V_2|v|^2)-(p+1)F(u,v)]\mathrm{d}\sigma\\&=-K(u,v).\end{aligned}$$

Since $(u_0,v_0)\in Z$, from Proposition 3.13 we know that $(u(t),v(t))\in Z$. It follows that $\frac{\mathrm{d}}{\mathrm{d}t}L(t)>0$ for $t\ge0$, i.e., $L(t)$ is increasing along the flow generated by problem (1.1).

By the claim, we can obtain that $\frac{\mathrm{d}}{\mathrm{d}t}L(t)=-K(u(t),v(t))>\epsilon$ for any $t\in(0,T)$. Integrating from 0 to t, we have

$$L(t)\ge\int_0^t\epsilon ds+L(0)=\epsilon t+L(0). \tag{4.10}$$

We now argue by contradiction. Assume that $T=+\infty$. Then the above inequality implies that $L(t)\to+\infty$ as $t\to T$. Since $(u(t),v(t))\in Z$, we can obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L(t)&=-2E(u,v)+(p-1)\int_{\mathbb{M}}rF(u,v)\mathrm{d}\sigma\\&>-2d+(p-1)\int_{\mathbb{M}}r(|u|^{p+1}+|v|^{p+1})\mathrm{d}\sigma.\end{aligned}$$

By corner type Hölder inequality (2.4), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L(t)&\ge-2d+(p-1)C(\mathbb{M})\Big[\Big(\int_{\mathbb{M}}r|u|^2\mathrm{d}\sigma\Big)^{\frac{p+1}{2}}+\Big(\int_{\mathbb{M}}r|v|^2\mathrm{d}\sigma\Big)^{\frac{p+1}{2}}\Big]\\&\ge-2d+(p-1)C(\mathbb{M},p)L(t)^{\frac{p+1}{2}}.\end{aligned}$$

We choose $\epsilon>0$ and $t_0>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t)\ge\epsilon L(t)^{\frac{p+1}{2}},\quad t>t_0.$$

From (4.10), we can obtain that $L(t) \geq 0$ for $0 < t < T$. Integrating the above inequality on (t_0, t), we can obtain that

$$0 \leq L(t)^{-\frac{p-1}{2}} \leq -\epsilon \frac{p-1}{2}(t-t_0) + L(t_0)^{-\frac{p-1}{2}}.$$

This is a contradiction, since the right-hand side of the above inequality goes to $-\infty$ as $t \to +\infty$. Therefore, the solution of problem (1.1) blows up in a finite time.

Step 2 Now we consider the critical initial energy case.

Let $(u(t), v(t))$ be a weak solution of the problem (1.1) with $E(u_0, v_0) = d > 0, K(u_0, v_0) < 0$, and T being the maximal existence time of $(u(t), v(t))$. Let us prove $T < \infty$. From the time continuity of $E(u(t), v(t))$ and $K(u(t), v(t))$, we know that there exists a sufficient small $t_1 \in (0, T)$ such that $E(u(t_1), v(t_1)) > 0$ and $K(u(t), v(t)) < 0$ for $0 \leq t \leq t_1$. Thus we can deduce $\int_{\mathbb{M}} r(uu_t + vv_t)\mathrm{d}\sigma = -K(u, v) > 0$ and $\|(u_t, v_t)\|^2_{\mathcal{L}_2^{(\frac{N-1}{2}, \frac{N}{2})}(\mathbb{M})} > 0$ for $0 \leq t \leq t_1$. Therefore $\int_0^t \|(u_\tau, v_\tau)\|^2_{\mathcal{L}_2^{(\frac{N-1}{2}, \frac{N}{2})}(\mathbb{M})} \mathrm{d}\tau$ is strictly increasing for $0 \leq t \leq t_1$, and we can choose t_1 such that

$$0 < d_1 = d - \int_0^{t_1} \|(u_\tau, v_\tau)\|^2_{\mathcal{L}_2^{(\frac{N-1}{2}, \frac{N}{2})}(\mathbb{M})} \mathrm{d}\tau < d.$$

Since $E(u_0, v_0) = d$, from the conservation of energy (1.9) and the above inequality, we can obtain that

$$E(u(t_1), v(t_1)) = E(u_0, v_0) - \int_0^{t_1} \|(u_\tau, v_\tau)\|^2_{\mathcal{L}_2^{(\frac{N-1}{2}, \frac{N}{2})}(\mathbb{M})} \mathrm{d}\tau < d.$$

If we take $t = t_1$ as the initial time, then $E(u(t_1), v(t_1)) < d$ and $K(u(t_1), v(t_1)) < 0$. Hence, Step 1 implies that the maximal existence time T of the weak solution $(u(t), v(t))$ for problem (1.1) is finite, and that

$$\lim_{t \to T^-} \|(u(t), v(t))\|^2_{\mathcal{L}_2^{\frac{N-1}{2}, \frac{N}{2}}(\mathbb{M})} = +\infty.$$

$\square$

References

[1] Ambrosetti A, Rabinowitz P H. Dual variational methods in critical point theory and applications. J Funct Anal, 1973, **14**: 349–381

[2] Alimohammady M, Kalleji M K. Existence result for a class of semilinear totally characteristic hypoelliptic equations with conical degeneration. J Funct Anal, 2013, **265**: 2331–2356

[3] Chen H, Liu N. Asymptotic stability and blow-up of solutions for semi-linear edge-degenerate parabolic equations with singular potentials. Discrete Contin Dyn Syst, 2016, **36**(2): 661–682

[4] Chen H, Liu G. Global existence and nonexistence for semilinear parabolic equations with conical degeneration. J Pseudo-Differ Oper Appl, 2012, **3**(3): 329–349

[5] Chen H, Liu G. Global existence, uniform decay and exponential growth for a class of semi-linear wave equation with strong damping. Acta Math Sci, 2013, **33B**(1): 41–58

[6] Chen H, Liu X, Wei Y. Existence theorem for a class of semilinear totally characteristic elliptic equations with critical cone Sobolev exponents. Ann Global Anal Geom, 2011, **39**: 27–43

[7] Chen H, Liu X, Wei Y. Cone Sobolev inequality and Dirichlet problem for nonlinear elliptic equations on manifold with concial singularities. Calc Var Partial Differ Equ, 2012, **43**: 463–484

[8] Chen H, Liu X, Wei Y. Multiple solutions for semilinear totally characteristic elliptic equations with subcritical or critical cone Sobolev exponents. J Differential Equations, 2012, **252**(7): 4200–4228

[9] Chen H, Wei Y, Zhou B. Existence of solutions for degenerate elliptic equations with singular potential on concial singular manifolds. Math Nachr, 2012, **285**: 1370–1384

[10] Chen H, Liu X, Wei Y. Dirichlet problem for semilinear edge-degenerate elliptic equations with singular potential term. J Differ Equ, 2012, **252**: 4289–4314

[11] Chen H, Liu X C, Wei Y W. Multiple solutions for semi-linear corner degenerate elliptic equations. J Funct Anal, 2014, **266**: 3815–3839

[12] Chen H, Tian S Y, Wei Y W. Multiple sign changing solutions for semi-linear corner degenerate elliptic equations with singular potential. J Funct Anal, 2016, **270**: 1602–1621

[13] Chen H, Tian S Y, Wei Y W. Multiple solutions for semi-linear corner degenerate elliptic equations with singular potential term. Communications in Contemporary Mathematics, DOI: 10.1142/S0219199716500437

[14] Egorov Ju V, Schulze B -W. Pseudo-Differential Operators, Singularities, Appliciations. Oper Theory Adv Appl, Vol 93. Basel: Birkhäuser Verlag, 1997

[15] Komornik V. Exact Controllability and Stabilization. The Multiplier Method. Paris: Mason-John Wiley, 1994

[16] Liu Y, Zhao J. On potential wells and applications to semiliear hyperbolic and parabolic equations. Nonliear Anal, 2006, **64**: 2665–2687

[17] Mazzeo R. Elliptic theory of differential edge operators, I. Comm Partial Differ Equ, 1991, **16**(10): 1616–1664

[18] Melrose R B, Mendoza G A. Elliptic operators of totally characteristic type, preprint. Math Sci Res Institute, 1983, MSRI047-83

[19] Melrose R B, Piazza P. Analytic K-theory on manifolds with corners. Adv Math, 1992, **92**(1): 1–26

[20] Payne L E, Sattinger D H. Saddle points and instability of nonlinear hyperbolic equations. Isr J Math, 1975, **22**: 273–303

[21] Sattinger D H. On global solution of nonlinear hyperbolic equations. Arch Ration Mech Anal, 1968, **30**: 148–172

[22] Schrohe E, Seiler J. Ellipticity and invertibility in the cone algebra on L_p-Sobolev spaces. Integral Equations Operator Theory, 2001, **41**: 93–114

[23] Schulze B -W. Boundary Value Problems and Singular Pseudo-Differential Operators. Chichester: John Wiley, 1998

[24] Schulze B -W. Mellin representations of pseudo-differential operators on manifolds with corners. Ann Global Anal Geom, 1990, **8**(3): 261–297

[25] Willem M. Minimax Theorems. Birkhäuser, 1996