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VAR AND CTE BASED OPTIMAL REINSURANCE FROM A REINSURER'S PERSPECTIVE*

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Abstract In this article, we study optimal reinsurance design. By employing the increasing convex functions as the admissible ceded loss functions and the distortion premium principle, we study and obtain the optimal reinsurance treaty by minimizing the VaR (value at risk) of the reinsurer's total risk exposure. When the distortion premium principle is specified to be the expectation premium principle, we also obtain the optimal reinsurance treaty by minimizing the CTE (conditional tail expectation) of the reinsurer's total risk exposure. The present study can be considered as a complement of that of Cai et al. [5].

Key words optimal reinsurance; value at risk; conditional tail expectation; distortion premium principle; expectation premium principle

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1 Introduction

Reinsurance is an effective risk management tool, which can help an insurer to reduce its risk exposure by means of buying reinsurance contracts. A traditional reinsurance contract must deal with the trade-off between the insurer and the reinsurer. In order to make an optimal reinsurance agreement, three aspects should be taken into account: (1) an optimal objective; (2) an admissible class of ceded loss functions; (3) a principle of reinsurance premium.

There has been a great deal of literature on the subject of optimal reinsurance study; see, for example, [1–31] and the references therein. Among those listed above [1] and [3] are the seminal articles.

Under the expected value premium principle, Arrow [1] showed that stop-loss reinsurance is optimal. In [10] and [11] the authors considered VaR and CVaR risk measures with premium principles which preserve the convex ordering. Lu, et al. [21] studied the optimal reinsurance

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under the VaR and TVaR risk measures, and showed that two-layer reinsurance is always the optimal reinsurance policy under both the VaR and TVaR risk measures. From the perspective of an insurer, Cai et al. [5] studied the optimal reinsurance treaty by means of an approximation approach by minimizing the VaR and CTE of the insurer's liability, where the admissible ceded loss functions were assumed to be the increasing convex functions. This method is quite complicated, as pointed out by Chi and Tan [9]. For any increasing convex ceded loss function $I(x)$, Chi and Tan [9] demonstrated that it can be rewritten as

$$I(x) = c\int_{0-}^{\infty}(x-t)_+\nu(\mathrm{d}t)$$

for some constant $c \in [0,1]$ and a probability measure ν defined on $[0,\infty]$, where $(x)_+ := \max\{x,0\}$. This representation can greatly simplify the proof of Cai et al. [5]. Later, Huang and Yu [17] further studied the optimal safety loading of the reinsurance premium principle based on the work of Cai et al. [5]. However, it is well known that the insurer and the reinsurer have a conflict interest in a few of reinsurance contracts. Therefore, a natural and interesting question arises: with the admissible class of increasing convex ceded loss functions, and from the reinsurer's perspective, what is the corresponding optimal reinsurance treaty?

In the present article, we will investigate the above-mentioned question. Namely, by employing the increasing convex functions as the admissible ceded loss functions, and from the reinsurer's perspective, we study the optimal reinsurance treaty by minimizing the VaR of the reinsurer's liability under the distortion premium principle. When the distortion premium principle is specified to the expected value premium principle, we study the optimal reinsurance treaty by minimizing the CTE of the reinsurer's liability. The optimal reinsurance treaties are provided. It turns out that the obtained optimal reinsurance treaties are quite different from those of Cai et al. [5]. The present study can be considered as a complement of that of Cai et al. [5].

The rest of this article is organized as follows: in Section 2, we briefly introduce the preliminaries. In Section 3, the optimal reinsurance design is studied under VaR, and the optimal reinsurance treaty is obtained in general. In Section 4, the optimal reinsurance design is studied under CTE, and the optimal reinsurance treaty is obtained in general. Finally, a conclusion of this article is provided.

2 Preliminaries

Let a non-negative random variable X with positive and finite expectation $E[X]$ be the loss initially faced by an insurer. We denote by $\mathscr{X}$ all the non-negative random variables with positive and finite expectations. We denote by $F_X(x)$ the distribution function of X, and by $S_X(x) := 1 - F_X(x)$ the survival function of X. Assume that X has a continuous strictly increasing distribution function on $(0,+\infty)$ with a possible jump at 0. Furthermore, we assume that the survival function $S_X(x)$ is differentiable with a negative derivative; that is, $S'_X(x) < 0$ for all $x > 0$. We denote by $\mathscr{I}$ the class of ceded loss functions, which consists of all increasing convex functions $I(x)$ defined on $[0,\infty)$ satisfying $0 \le I(x) \le x$ for $x \ge 0$, but excluding $I(x) \equiv 0$.

Under a reinsurance contract, the insurer cedes part of its loss, say $I(X)$ with $0 \le I(X) \le$

X, to a reinsurer, and thus the retained loss of the insurer is $R_I(X) := X - I(X)$, where the function $I(x)$, satisfying $0 \le I(x) \le x$, is called the ceded loss function, and the function $R_I(x) := x - I(x)$ is called the retained loss function. In exchange, the insurer agrees to pay a reinsurance premium to the reinsurer. Denote by $\pi_I(X)$ the reinsurance premium which corresponds to the ceded loss $I(X)$. By definition, a distortion function $r : [0,1] \to [0,1]$ is a non-decreasing function such that $r(0) = 0, r(1) = 1$. The inverse of r is defined by $r^{-1}(x) := \inf\{y : r(y) \ge x\}$. Corresponding to a distortion function r, the distortion premium principle π is defined by

$$\pi(X) = (1+\rho)\int_0^\infty r(S_X(x))\mathrm{d}x \tag{2.1}$$

for $X \in \mathscr{X}$, where $\rho > 0$ is the safety loading.

If $r(x) = x$, then the distortion premium principle recovers the expected value principle. Throughout this article, we assume that the distortion function $r(x)$ has at least finitely many discontinuous points, with $x = 0$ being a continuous point of r.

Upon the issuing of a reinsurance contract, the total risk exposure of the reinsurer is

$$T_I(X) := I(X) - \pi_I(X). \tag{2.2}$$

Next, we introduce the definitions of VaR and CTE.

Definition 2.1 Let $\alpha \in (0,1)$, and for any random variable Y, the VaR (value at risk) of Y at the confidence level $(1-\alpha)$ is defined by

$$\mathrm{VaR}_\alpha(Y) := \inf\{y \in \mathbb{R} : P\{Y > y\} \le \alpha\}. \tag{2.3}$$

Definition 2.2 Let $\alpha \in (0,1)$, and for any random variable Y, the CTE (conditional tail expectation) of Y at confidence level $(1-\alpha)$ is defined by

$$\mathrm{CTE}_\alpha(Y) := E[Y|Y \ge \mathrm{VaR}_\alpha(Y)]. \tag{2.4}$$

Now, we collect some properties of VaR and CTE.

Proposition 2.3 Let $\alpha \in (0,1)$, and for any random variable Y we have that

(i) For any increasing and continuous function $\phi(y)$,

$$\mathrm{VaR}_\alpha(\phi(Y)) = \phi(\mathrm{VaR}_\alpha(Y)). \tag{2.5}$$

(ii) If that the functions $\psi_n(y)$ and $\psi(y)$ are increasing and continuous, and if $\lim\limits_{n\to\infty}\psi_n(y) = \psi(y)$ for any $y \in \mathbb{R}$, then

$$\lim_{n\to\infty}\mathrm{VaR}_\alpha(\psi_n(Y)) = \mathrm{VaR}_\alpha(\psi(Y)). \tag{2.6}$$

(iii) The relationship between VaR and CTE is

$$\mathrm{CTE}_\alpha(Y) = E[Y|Y \ge \mathrm{VaR}_\alpha(Y)] = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_q(Y)\mathrm{d}q. \tag{2.7}$$

Note that (i) is from (15) of Dhaene et al. [16]. (ii) follows from (i). (iii) is from Porth, Tand and Weng [23].

Next, we state the following lemma, the proof of which can be found in Lemma 2.1 of Cheung and Lo [14]:

Lemma 2.4 For any distortion function r and ceded loss function I in $\mathscr{I}$, we have that

$$\pi_I(X) = \int_0^\infty r(S_{I(X)}(x))\mathrm{d}x = \int_0^\infty r(S_X(x))\mathrm{d}I(x). \tag{2.8}$$

Now, we introduce the optimal reinsurance models with which we are concerned in the present article:

VaR-based optimization model

$$\mathrm{VaR}_\alpha[T_{I^*}(X)] = \min_{I\in\mathscr{I}} \mathrm{VaR}_\alpha[T_I(X)] \tag{2.9}$$

and

CTE-based optimization model

$$\mathrm{CTE}_\alpha[T_{I^*}(X)] = \min_{I\in\mathscr{I}} \mathrm{CTE}_\alpha[T_I(X)], \tag{2.10}$$

where I^* are the resulting optimal ceded loss functions, respectively.

The following formulas can be derived from the properties and definitions of VaR and CTE:

$$\mathrm{VaR}_\alpha[T_I(X)] = \mathrm{VaR}_\alpha[I(X)] - \pi_I(X) \tag{2.11}$$

and

$$\mathrm{CTE}_\alpha[T_I(X)] = \mathrm{CTE}_\alpha[I(X)] - \pi_I(X). \tag{2.12}$$

For notational convenience, we denote

$$\rho^* := \frac{1}{1+\rho}, \quad d_1^* := S_X^{-1}(r^{-1}(\rho^*)), \quad d_2^* := S_X^{-1}(\rho^*), \tag{2.13}$$

$$g_1(x) := x - \frac{1}{\rho^*}\int_0^x r(S_X(q))\mathrm{d}q, \qquad x \geq 0, \tag{2.14}$$

$$g_2(x) := x - \frac{1}{\rho^*}\int_0^x S_X(q)\mathrm{d}q, \qquad x \geq 0, \tag{2.15}$$

$$u_1(x) := x + \frac{1}{\rho^*}\int_x^\infty r(S_X(q))\mathrm{d}q, \qquad x \geq 0,$$

$$u_2(x) := x + \frac{1}{\rho^*}\int_x^\infty S_X(q)\mathrm{d}q, \qquad x \geq 0.$$

We end this section with a proposition concerning the properties of $g_1(x)$ defined by (2.14) and $g_2(x)$ defined by (2.15).

Proposition 2.5 (i) If we let the continuous function $g_1(x)$ be defined as in (2.14), then $g_1(x)$ is decreasing on $(0, d_1^*)$, while increasing on (d_1^*, ∞), and there exists a $\theta_1^* > d_1^*$ such that $g_1(\theta_1^*) = g_1(0)$. If $0 \leq x \leq \theta_1^*$, then $g_1(x) \leq 0$, and if $x \geq \theta_1^*$, then $g_1(x) \geq 0$.

(ii) If we let the continuous function $g_2(x)$ be defined as in (2.15), then $g_2(x)$ is decreasing on $(0, d_2^*)$, while increasing on (d_2^*, ∞), and there exists a $\theta_2^* > d_2^*$ such that $g_2(\theta_2^*) = g_2(0)$. If $0 \leq x \leq \theta_2^*$, then $g_2(x) \leq 0$, and if $x \geq \theta_2^*$, then $g_2(x) \geq 0$.

Proof (i) Note that

$$g_1'(x) = 1 - \frac{1}{\rho^*} r(S_X(x)). \tag{2.16}$$

Hence,

$$g_1'(x) \le 0 \quad \text{if} \quad x < d_1^*,$$

$$g_1'(x) = 0 \quad \text{if} \quad x = d_1^*,$$

$$g_1'(x) \ge 0 \quad \text{if} \quad x > d_1^*.$$

From (2.16) it follows that $\lim_{x\to+\infty} g_1'(x) = 1$, which implies that $g_1(x)$ is strictly increasing for large enough $x > d_1^*$. Note that $g_1(d_1^*) \le g_1(0)$ and $\lim_{x\to+\infty} g_1(x) = +\infty$, therefore, by the intermediate value theorem for the continuous function $g_1(x)$, there exists a $\theta_1^* > d_1^*$ such that $g_1(\theta_1^*) = g_1(0)$. Note again that $g_1(0) = 0$, consequently, if $0 \le x \le \theta_1^*$, then $g_1(x) \le 0$, and if $x \ge \theta_1^*$, then $g_1(x) \ge 0$.

(ii) The proof of (ii) is similar to that of (i). □

3 VaR-based Optimal Reinsurance

In this section, we will derive the optimal solution to model (2.9) under the admissible ceded loss functions. We begin this section with the following lemma, which is from Chi and Tan ([10], Lemma 3.1):

Lemma 3.1 We have

$$\mathscr{I} = \left\{ I(x) = c\int_{0-}^{\infty} (x-t)_+ \nu(\mathrm{d}t) : 0 \le c \le 1 \text{ and } \nu \text{ is a probability measure defined on } [0,\infty] \right\}.$$

For any $I \in \mathscr{I}$, it follows from (2.5), (2.8), (2.11) and Lemma 3.1 that

$$\begin{aligned} \mathrm{VaR}_\alpha(T_I(X)) &= \mathrm{VaR}_\alpha(I(X)) - \pi_I(X) \\ &= I(\mathrm{VaR}_\alpha(X)) - (1+\rho)\int_0^\infty r(S_X(x))\mathrm{d}I(x) \\ &= I(\mathrm{VaR}_\alpha(X)) + (1+\rho)\int_0^\infty I(x)\mathrm{d}r(S_X(x)) \\ &= c\int_{0-}^\infty w(t)\nu(\mathrm{d}t) \end{aligned}$$

for some $0 \le c \le 1$ and probability measure ν, where

$$w(t) := (\mathrm{VaR}_\alpha(X) - t)_+ + (1+\rho)\int_0^\infty (x-t)_+ \mathrm{d}r(S_X(x)).$$

Now, we are in a position to state the main result of this section.

Theorem 3.2 Let $\alpha \in (0,1)$ and $X \in \mathscr{X}$.

(i) Assume that $\mathrm{VaR}_\alpha(X) < \theta_1^*$. If $\mathrm{VaR}_\alpha(X) > u_1(0)$, then

$$I^*(x) = 0.$$

If $\mathrm{VaR}_\alpha(X) = u_1(0)$, then

$$I^*(x) = cx,$$

where c is any number in $[0,1]$. If $\mathrm{VaR}_\alpha(X) < u_1(0)$, then

$$I^*(x) = x.$$

(ii) Assume that $\mathrm{VaR}_\alpha(X)=\theta_1^*$. Then either

$$I^*(x)=(x-\mathrm{VaR}_\alpha(X))_+ \quad \text{or} \quad I^*(x)=0,$$

if $\mathrm{VaR}_\alpha(X)>u_1(0)$,

$$I^*(x)=cx,$$

where c is any number in $[0,1]$, if $\mathrm{VaR}_\alpha(X)=u_1(0)$, and

$$I^*(x)=x,$$

if $\mathrm{VaR}_\alpha(X)<u_1(0)$.

(iii) Assume that $\mathrm{VaR}_\alpha(X)>\theta_1^*$, then $I^*(x)=(x-\mathrm{VaR}_\alpha(X))_+$.

Proof It follows from the above results that to analyze the minimization of $\mathrm{VaR}_\alpha(T_I(X))$, it is sufficient to focus on the minimum value of w. For any $t\geq \mathrm{VaR}_\alpha(X)$, we have that

$$w(t)=(1+\rho)\int_0^\infty (x-t)_+\mathrm{d}r(S_X(x))=-(1+\rho)\int_t^\infty r(S_X(x))\mathrm{d}x. \tag{3.1}$$

This leads to $w'(t)=(1+\rho)r(S_X(t))>0$, so the minimum value of w on $[\mathrm{VaR}_\alpha(X),\infty]$ is attained at $\mathrm{VaR}_\alpha(X)$. On the other hand, when $0\leq t\leq \mathrm{VaR}_\alpha(X)$, we have that

$$\begin{aligned} w(t)&=\mathrm{VaR}_\alpha(X)-t+(1+\rho)\int_0^\infty (x-t)_+\mathrm{d}r(S_X(x)) \\ &=\mathrm{VaR}_\alpha(X)-t-(1+\rho)\int_t^\infty r(S_X(x))\mathrm{d}x; \end{aligned}$$

this leads to $w'(t)=(1+\rho)r(S_X(t))-1$.

If we assume that $d_1^*\geq \mathrm{VaR}_\alpha(X)$, then the minimum value of w on $[0,\mathrm{VaR}_\alpha(X)]$ is attained at 0. Combining this with (3.1) yields

$$\min_{x\in\mathbb{R}_+} w(x)=\min\{w(0),w(\mathrm{VaR}_\alpha(X))\}.$$

Recalling that $g_1(x)$, we have $\mathrm{VaR}_\alpha(X)\leq d_1^*<\theta_1^*$, so $g_1(\mathrm{VaR}_\alpha(X))<0$, and thus

$$\min_{x\in\mathbb{R}_+} w(x)=w(0)=\mathrm{VaR}_\alpha(X)-u_1(0).$$

If $\mathrm{VaR}_\alpha(X)>u_1(0)$, we set $\nu(0)=1$ and $c=0$, so the corresponding optimal ceded loss function becomes $I^*(x)=0$. If $\mathrm{VaR}_\alpha(X)=u_1(0)$, we set $\nu(0)=1$, so the corresponding optimal ceded loss function becomes $I^*(x)=cx$ for some $c\in[0,1]$. If $\mathrm{VaR}_\alpha(X)<u_1(0)$, we set $\nu(0)=1$ and $c=1$, so the corresponding optimal ceded loss function becomes $I^*(x)=x$.

On the other hand, assume that $d_1^*<\mathrm{VaR}_\alpha(X)$. In this case, the minimum value of w on $[0,\mathrm{VaR}_\alpha(X)]$ is attained at 0 or $\mathrm{VaR}_\alpha(X)$. Combining this with (3.1) yields

$$\min_{x\in\mathbb{R}_+} w(x)=\min\{w(0),w(\mathrm{VaR}_\alpha(X))\}.$$

Recalling that $g_1(x)$, and supposing that $\mathrm{VaR}_\alpha(X)<\theta_1^*$, we have $g_1(\mathrm{VaR}_\alpha(X))<0$, so

$$\min_{x\in\mathbb{R}_+} w(x)=w(0)=\mathrm{VaR}_\alpha(X)-u_1(0).$$

If $\mathrm{VaR}_\alpha(X)>u_1(0)$, we set $\nu(0)=1$ and $c=0$, so the corresponding optimal ceded loss function becomes $I^*(x)=0$. If $\mathrm{VaR}_\alpha(X)=u_1(0)$, we set $\nu(0)=1$, so the corresponding optimal ceded loss function becomes $I^*(x) = cx$ for some $c \in [0, 1]$. If $\mathrm{VaR}_\alpha(X) < u_1(0)$, we set $\nu(0) = 1$ and $c = 1$, so the corresponding optimal ceded loss function becomes $I^*(x) = x$.

Supposing that $\mathrm{VaR}_\alpha(X) = \theta_1^*$, we have $g_1(\mathrm{VaR}_\alpha(X)) = 0$, and the minimum value of w occurs at both 0 and $\mathrm{VaR}_\alpha(X)$. In this case, we need only to set ν with support on $\{0, \mathrm{VaR}_\alpha(X)\}$. Then either the corresponding optimal ceded loss function becomes $I^*(x) = (x - \mathrm{VaR}_\alpha(X))_+$ for $c = 1$, or if $\mathrm{VaR}_\alpha(X) > u_1(0)$, we set $\nu(0) = 1$ and $c = 0$, so the corresponding optimal ceded loss function becomes $I^*(x) = 0$; if $\mathrm{VaR}_\alpha(X) = u_1(0)$, we set $\nu(0) = 1$ and the corresponding optimal ceded loss function becomes $I^*(x) = cx$ for some $c \in [0, 1]$; if $\mathrm{VaR}_\alpha(X) < u_1(0)$, we set $\nu(0) = 1$ and $c = 1$, and the corresponding optimal ceded loss function becomes $I^*(x) = x$.

If $\mathrm{VaR}_\alpha(X) > \theta_1^*$, we have $g_1(\mathrm{VaR}_\alpha(X)) > 0$, so $\min\limits_{x\in\mathbb{R}_+} w(x) = w(\mathrm{VaR}_\alpha(X))$. In this case, we set $\nu(\mathrm{VaR}_\alpha(X)) = 1$ and $c = 1$, so the corresponding optimal ceded loss function becomes $I^*(x) = (x - \mathrm{VaR}_\alpha(X))_+$.

Theorem 3.2 is proved. □

Remark 3.3 Theorem 3.2 shows that from the reinsurer's perspective, the optimal reinsurance treaty is different from that of the insurer; see [5] for the optimal reinsurance treaty from the insurer's perspective. These different ideas as to what constitutes the optimal treaty reflects the conflict interest between the insurer and the reinsurer in a reinsurance contract. How to balance both the insurer's and the reinsurer's interests in a reinsurance contract is an interesting issue, and this will be the subject of further work.

In the special case where the distortion function $r(x) = x,\ x \geq 0$,

$$\mathrm{VaR}_\alpha(T_I(X)) = I(\mathrm{VaR}_\alpha(X)) - (1+\rho)\mathbb{E}[I(X)] = c\int_{0-}^{\infty} w_1(t)\nu(\mathrm{d}t)$$

for some $0 \leq c \leq 1$ and probability measure ν, where

$$w_1(t) := (\mathrm{VaR}_\alpha(X) - t)_+ - (1+\rho)\mathbb{E}[(X-t)_+].$$

Therefore, in this special case, by Theorem 3.2, we have the following corollary:

Corollary 3.4 Let $\alpha \in (0, 1)$ and $X \in \mathscr{X}$.

(i) Assume that $\mathrm{VaR}_\alpha(X) < \theta_2^*$. If $\mathrm{VaR}_\alpha(X) > u_2(0)$, then

$$I^*(x) = 0.$$

If $\mathrm{VaR}_\alpha(X) = u_2(0)$, then

$$I^*(x) = cx,$$

where c is any number in $[0, 1]$. If $\mathrm{VaR}_\alpha(X) < u_2(0)$, then

$$I^*(x) = x.$$

(ii) Assume that $\mathrm{VaR}_\alpha(X) = \theta_2^*$. Then either

$$I^*(x) = (x - \mathrm{VaR}_\alpha(X))_+$$

or

$$I^*(x) = 0,$$

if $\mathrm{VaR}_\alpha(X) > u_2(0)$,

$$I^*(x) = cx,$$

where c is any number in $[0,1]$, if $\mathrm{VaR}_\alpha(X)=u_2(0)$, and

$$I^*(x)=x,$$

if $\mathrm{VaR}_\alpha(X)<u_2(0)$.

(iii) Assume that $\mathrm{VaR}_\alpha(X)>\theta_2^*$. Then $I^*(x)=(x-\mathrm{VaR}_\alpha(X))_+$.

4 CTE-based Optimal Reinsurance

In this section, we will derive the optimal solution to model (2.10) under the admissible ceded loss functions with the expectation premium principle. Recall that from (2.7) and (2.11), we obtain

$$\begin{aligned}\mathrm{CTE}_\alpha[T_I(X)]&=\frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_q(T_I(X))\mathrm{d}q\\&=\frac{1}{\alpha}\int_0^\alpha I(\mathrm{VaR}_q(X))\mathrm{d}q-\pi_I(X)\\&=c\int_{0-}^\infty w_2(t)\nu(\mathrm{d}t)\end{aligned}$$

for some $0\le c\le 1$ and probability measure ν, where

$$w_2(t):=\frac{1}{\alpha}\int_0^\alpha(\mathrm{VaR}_q(X)-t)_+\mathrm{d}q-(1+\rho)\mathbb{E}(X-t)_+.$$

Now, by virtue of Lemma 3.1, we obtain the following theorem, which is the main result of this section:

Theorem 4.1 Let $\alpha\in(0,1)$ and $X\in\mathscr{X}$.

(I) Assume that $\alpha\ge\rho^*$,

(i) Suppose that $\mathrm{VaR}_\alpha(X)<\theta_2^*$. If $\mathrm{CTE}_\alpha(X)>u_2(0)$, then $I^*(x)=0$.

If $\mathrm{CTE}_\alpha(X)=u_2(0)$, then

$$I^*(x)=cx,$$

where c is any number in $[0,1]$.

If $\mathrm{CTE}_\alpha(X)<u_2(0)$, then $I^*(x)=x$.

(ii) Suppose that $\mathrm{VaR}_\alpha(X)=\theta_2^*$. If $\mathrm{CTE}_\alpha(X)>u_2(0)=u_2(\mathrm{VaR}_\alpha(X))$, then $I^*(x)=0$.

If $\mathrm{CTE}_\alpha(X)=u_2(0)=u_2(\mathrm{VaR}_\alpha(X))$, then

$$I^*(x)=cx,$$

where c is any number in $[0,1]$, or

$$I^*(x)=c(x-\mathrm{VaR}_\alpha(X))_+,$$

where c is any number in $[0,1]$.

If $\mathrm{CTE}_\alpha(X)<u_2(0)=u_2(\mathrm{VaR}_\alpha(X))$, then

$$I^*(x)=x,$$

or

$$I^*(x)=(x-\mathrm{VaR}_\alpha(X))_+.$$

(iii) Suppose that $\mathrm{VaR}_\alpha(X) > \theta_2^*$. If $\mathrm{CTE}_\alpha(X) > u_2(\mathrm{VaR}_\alpha(X))$, then

$$I^*(x) = 0.$$

If $\mathrm{CTE}_\alpha(X) = u_2(\mathrm{VaR}_\alpha(X))$, then

$$I^*(x) = c(x - \mathrm{VaR}_\alpha(X))_+,$$

where c is any number in $[0, 1]$.

If $\mathrm{CTE}_\alpha(X) < u_2(\mathrm{VaR}_\alpha(X))$, then

$$I^*(x) = (x - \mathrm{VaR}_\alpha(X))_+.$$

(II) Assume that $\alpha < \rho^*$.

(i) Suppose that $\mathrm{VaR}_\alpha(X) < \theta_2^*$. If $\mathrm{CTE}_\alpha(X) > u_2(0)$, then $I^*(x) = 0$.

If $\mathrm{CTE}_\alpha(X) = u_2(0)$, then

$$I^*(x) = cx,$$

where c is any number in $[0, 1]$, or

$$I^*(x) = 0.$$

If $\mathrm{CTE}_\alpha(X) < u_2(0)$, then $I^*(x) = x$.

(ii) Suppose that $\mathrm{VaR}_\alpha(X) = \theta_2^*$. If $\mathrm{CTE}_\alpha(X) > u_2(0) = u_2(\mathrm{VaR}_\alpha(X))$, then $I^*(x) = 0$.

If $\mathrm{CTE}_\alpha(X) = u_2(0) = u_2(\mathrm{VaR}_\alpha(X))$, then

$$I^*(x) = cx,$$

where c is any number in $[0, 1]$, or

$$I^*(x) = c(x - \mathrm{VaR}_\alpha(X))_+,$$

where c is any number in $[0, 1]$, or

$$I^*(x) = 0.$$

If $\mathrm{CTE}_\alpha(X) < u_2(0) = u_2(\mathrm{VaR}_\alpha(X))$, then

$$I^*(x) = x,$$

or

$$I^*(x) = (x - \mathrm{VaR}_\alpha(X))_+.$$

(iii) Suppose that $\mathrm{VaR}_\alpha(X) > \theta_2^*$. If $\mathrm{CTE}_\alpha(X) > u_2(\mathrm{VaR}_\alpha(X))$, then $I^*(x) = 0$.

If $\mathrm{CTE}_\alpha(X) = u_2(\mathrm{VaR}_\alpha(X))$, then

$$I^*(x) = c(x - \mathrm{VaR}_\alpha(X))_+,$$

where c is any number in $[0, 1]$, or

$$I^*(x) = 0.$$

If $\mathrm{CTE}_\alpha(X) < u_2(\mathrm{VaR}_\alpha(X))$, then

$$I^*(x) = (x - \mathrm{VaR}_\alpha(X))_+.$$

Proof It follows from the above results that to analyze the minimization of $\mathrm{CTE}_\alpha(T_I(X))$, it is sufficient to focus on the minimum value of w_2.

Assume that $\alpha \le S_X(t)$, that is, that $0 \le t \le \mathrm{VaR}_\alpha(X)$. Then

$$w_2(t) = \frac{1}{\alpha}\int_0^\alpha (\mathrm{VaR}_q(X) - t)\mathrm{d}q - (1+\rho)\mathbb{E}(X-t)_+,$$

which leads to $w_2'(t) = (1+\rho)S_X(t) - 1$. Suppose that $\mathrm{VaR}_\alpha(X) \le d_2^*$, that is, that $\alpha \ge \rho^*$. Then the minimum value of w_2 on $[0, \mathrm{VaR}_\alpha(X)]$ is attained at 0. Suppose that $\mathrm{VaR}_\alpha(X) > d_2^*$, that is, that $\alpha < \rho^*$. Then the minimum value of w_2 on $[0, \mathrm{VaR}_\alpha(X)]$ is attained at 0 or $\mathrm{VaR}_\alpha(X)$.

Assume that $\alpha \ge S_X(t)$, that is, that $t \ge \mathrm{VaR}_\alpha(X)$. Then

$$w_2(t) = \frac{1}{\alpha}\int_0^{S_X(t)} (\mathrm{VaR}_q(X) - t)\mathrm{d}q - (1+\rho)\mathbb{E}(X-t)_+,$$

which leads to $w_2'(t) = (\frac{1}{\rho^*} - \frac{1}{\alpha})S_X(t)$. If $\frac{1}{\rho^*} < \frac{1}{\alpha}$, that is, $\alpha < \rho^*$, then the minimum value of w_2 on $[\mathrm{VaR}_\alpha(X), \infty]$ is attained at ∞. If $\frac{1}{\rho^*} \ge \frac{1}{\alpha}$, that is, $\alpha \ge \rho^*$, then the minimum value of w_2 on $[\mathrm{VaR}_\alpha(X), \infty]$ is attained at $\mathrm{VaR}_\alpha(X)$.

To sum up: (I) If we have that $\alpha \ge \rho^*$, then

$$\min_{x\in\mathbb{R}_+} w_2(x) = \min\{w_2(0), w_2(\mathrm{VaR}_\alpha(X))\}.$$

(i) If we have that $\mathrm{VaR}_\alpha(X) < \theta_2^*$, then

$$\min_{x\in\mathbb{R}_+} w_2(x) = w_2(0) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_q(X)\mathrm{d}q - u_2(0).$$

If $\mathrm{CTE}_\alpha(X) > u_2(0)$, we set $\nu(0) = 1$ and $c = 0$, so the corresponding optimal ceded loss function becomes $I^*(x) = 0$. If $\mathrm{CTE}_\alpha(X) = u_2(0)$, we set $\nu(0) = 1$, so the corresponding optimal ceded loss function becomes $I^*(x) = cx$ for some $c \in [0,1]$. If $\mathrm{CTE}_\alpha(X) < u_2(0)$, we set $\nu(0) = 1$ and $c = 1$, so the corresponding optimal ceded loss function becomes $I^*(x) = x$.

(ii) If we have that $\mathrm{VaR}_\alpha(X) = \theta_2^*$, then

$$\min_{x\in\mathbb{R}_+} w_2(x) = w_2(0) = w_2(\mathrm{VaR}_\alpha(X)) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_q(X)\mathrm{d}q - u_2(\mathrm{VaR}_\alpha(X)).$$

In this case, we need only to set ν with support on $\{0, \mathrm{VaR}_\alpha(X)\}$. If $\mathrm{CTE}_\alpha(X) > u_2(0) = u_2(\mathrm{VaR}_\alpha(X))$, we set $c = 0$, so the corresponding optimal ceded loss function becomes $I^*(x) = 0$. If $\mathrm{CTE}_\alpha(X) = u_2(0) = u_2(\mathrm{VaR}_\alpha(X))$, then the corresponding optimal ceded loss function becomes $I^*(x) = cx$ for some $c \in [0,1]$ or $I^*(x) = c(x - \mathrm{VaR}_\alpha(X))_+$ for some $c \in [0,1]$. If $\mathrm{CTE}_\alpha(X) < u_2(0) = u_2(\mathrm{VaR}_\alpha(X))$, we set $c = 1$, so the corresponding optimal ceded loss function becomes $I^*(x) = x$ or $I^*(x) = (x - \mathrm{VaR}_\alpha(X))_+$.

(iii) Suppose that $\mathrm{VaR}_\alpha(X) > \theta_2^*$. Then $\min_{x\in\mathbb{R}_+} w_2(x) = w_2(\mathrm{VaR}_\alpha(X))$.

If $\mathrm{CTE}_\alpha(X) > u_2(\mathrm{VaR}_\alpha(X))$, we set $\nu(\mathrm{VaR}_\alpha(X)) = 1$ and $c = 0$, so the corresponding optimal ceded loss function becomes $I^*(x) = 0$. If $\mathrm{CTE}_\alpha(X) = u_2(\mathrm{VaR}_\alpha(X))$, then the corresponding optimal ceded loss function becomes $I^*(x) = c(x - \mathrm{VaR}_\alpha(X))_+$ for some $c \in [0,1]$. If $\mathrm{CTE}_\alpha(X) < u_2(\mathrm{VaR}_\alpha(X))$, we set $\nu(\mathrm{VaR}_\alpha(X)) = 1$ and $c = 1$, so the corresponding optimal ceded loss function becomes $I^*(x) = (x - \mathrm{VaR}_\alpha(X))_+$.

(II) Assuming that $\alpha < \rho^*$,

$$\min_{x\in\mathbb{R}_+} w_2(x) = \min\{w_2(0), w_2(\mathrm{VaR}_\alpha(X)), w_2(\infty)\}.$$

(i) If we have that $\mathrm{VaR}_\alpha(X) < \theta_2^*$, then $g_2(\mathrm{VaR}_\alpha(X)) < 0$; that is, $w_2(0) < w_2(\mathrm{VaR}_\alpha(X))$. If $\mathrm{CTE}_\alpha(X) < u_2(0)$, then

$$\min_{x\in\mathbb{R}_+} w_2(x) = w_2(0).$$

In this case, we set $\nu(0) = 1$ and $c = 1$, so the corresponding optimal ceded loss function becomes $I^*(x) = x$. If $\mathrm{CTE}_\alpha(X) = u_2(0)$, then

$$\min_{x\in\mathbb{R}_+} w_2(x) = w_2(0) = w_2(\infty).$$

In this case, we need only to set ν with support on $\{0, \infty\}$, so the corresponding optimal ceded loss function becomes $I^*(x) = cx$ for some $c \in [0, 1]$ or $I^*(x) = 0$. If $\mathrm{CTE}_\alpha(X) > u_2(0)$, and

$$\min_{x\in\mathbb{R}_+} w_2(x) = w_2(\infty).$$

In this case, we set $\nu(\infty) = 1$, and the corresponding optimal ceded loss function becomes $I^*(x) = 0$.

(ii) If we have that $\mathrm{VaR}_\alpha(X) = \theta_2^*$, then $g_2(\mathrm{VaR}_\alpha(X)) = 0$; that is, $w_2(0) = w_2(\mathrm{VaR}_\alpha(X))$. If $\mathrm{CTE}_\alpha(X) < u_2(0) = u_2(\mathrm{VaR}_\alpha(X))$, then

$$\min_{x\in\mathbb{R}_+} w_2(x) = w_2(0) = w_2(\mathrm{VaR}_\alpha(X)).$$

In this case, we need only to set ν with support on $\{0, \mathrm{VaR}_\alpha(X)\}$, so the corresponding optimal ceded loss function becomes $I^*(x) = x$ for $c = 1$ or $I^*(x) = (x - \mathrm{VaR}_\alpha(X))_+$ for $c = 1$. If $\mathrm{CTE}_\alpha(X) = u_2(0) = u_2(\mathrm{VaR}_\alpha(X))$, then

$$\min_{x\in\mathbb{R}_+} w_2(x) = w_2(0) = w_2(\mathrm{VaR}_\alpha(X)) = w_2(\infty).$$

In this case, we need only to set ν with support on $\{0, \mathrm{VaR}_\alpha(X), \infty\}$, so the corresponding optimal ceded loss function becomes $I^*(x) = cx$ for some $c \in [0, 1]$ or $I^*(x) = c(x - \mathrm{VaR}_\alpha(X))_+$ for some $c \in [0, 1]$ or $I^*(x) = 0$. If $\mathrm{CTE}_\alpha(X) > u_2(0) = u_2(\mathrm{VaR}_\alpha(X))$, then

$$\min_{x\in\mathbb{R}_+} w_2(x) = w_2(\infty).$$

In this case, we set $\nu(\infty) = 1$, so the corresponding optimal ceded loss function becomes $I^*(x) = 0$.

(iii) Suppose that $\mathrm{VaR}_\alpha(X) > \theta_2^*$. Then $g_2(\mathrm{VaR}_\alpha(X)) > 0$; that is, $w_2(0) > w_2(\mathrm{VaR}_\alpha(X))$. If $\mathrm{CTE}_\alpha(X) < u_2(\mathrm{VaR}_\alpha(X))$, then

$$\min_{x\in\mathbb{R}_+} w_2(x) = w_2(\mathrm{VaR}_\alpha(X)).$$

In this case, we set $\nu(\mathrm{VaR}_\alpha(X)) = 1$ and $c = 1$, so the corresponding optimal ceded loss function becomes $I^*(x) = (x - \mathrm{VaR}_\alpha(X))_+$. If $\mathrm{CTE}_\alpha(X) = u_2(\mathrm{VaR}_\alpha(X))$, then

$$\min_{x\in\mathbb{R}_+} w_2(x) = w_2(\mathrm{VaR}_\alpha(X)) = w_2(\infty).$$

In this case, we need only to set ν with support on $\{\mathrm{VaR}_\alpha(X), \infty\}$, so the corresponding optimal ceded loss function becomes $I^*(x) = c(x - \mathrm{VaR}_\alpha(X))_+$ for some $c \in [0, 1]$ or $I^*(x) = 0$. If $\mathrm{CTE}_\alpha(X) > u_2(\mathrm{VaR}_\alpha(X))$, then

$$\min_{x\in\mathbb{R}_+} w_2(x) = w_2(\infty).$$

In this case, we set $\nu(\infty) = 1$, and the corresponding optimal ceded loss function becomes $I^*(x) = 0$.

Theorem 4.1 is proved. □

Remark 4.2 Theorem 4.1 shows that from the reinsurer's perspective, the optimal reinsurance treaty is different from that of the insurer; see [5] for the optimal reinsurance treaty from the insurer's perspective.

Remark 4.3 Under the assumption that the class of increasing convex functions is the admissible ceded loss functions, Huang and Yin [18] studied the optimal reinsurance treaty by an approximation approach under the distortion risk measure and the distortion premium principle from the perspectives of an insurer and a reinsurer. This method is quite complicated, as pointed out by Chi and Tan [9]. For any increasing convex ceded loss function $I(x)$, Chi and Tan [9] demonstrated that it can be rewritten as

$$I(x) = c\int_{0-}^{\infty}(x-t)_+\nu(\mathrm{d}t).$$

In this article, we have used this representation of an increasing convex function to derive the optimal reinsurance treaty, and this method is different from the one used by Huang and Yin [18].

5 Conclusion

In this article, we have studied two optimal reinsurance models from the reinsurer's perspective. Under the assumption that the admissible ceded loss functions consist of all increasing convex functions, when the reinsurance premium principle is calculated by a general distortion premium principle, we obtained the optimal reinsurance treaties by minimizing the VaR of the reinsurer's total risk exposure. When the reinsurance premium principle is calculated by a general expected value premium principle, we obtained the optimal reinsurance treaties by minimizing the CTE of the reinsurer's total risk exposure. It turns out that the optimal reinsurance treaty for the reinsurer is quite different from that of the insurer.

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