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THE CAUCHY PROBLEM FOR THE TWO LAYER VISCOUS SHALLOW WATER EQUATIONS*

Pengcheng MU (穆鹏程)
School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
E-mail: *mupc288@nenu.edu.cn*

Qiangchang JU (琚强昌)†
Institute of Applied Physics and Computational Mathematics, Beijing 100088, China
E-mail: *ju_qiangchang@iapcm.ac.cn*

Abstract In this paper, the Cauchy problem for the two layer viscous shallow water equations is investigated with third-order surface-tension terms and a low regularity assumption on the initial data. The global existence and uniqueness of the strong solution in a hybrid Besov space are proved by using the Littlewood-Paley decomposition and Friedrichs' regularization method.

Key words two layer shallow water equations; global strong solution; hybrid Besov spaces

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1 Introduction

The two layer shallow water equations, which can be used to describe the interaction of Mediterranean and Atlantic water in the strait of Gibraltar [16], are written as follows [21]:

$$\begin{cases} h_{1t} + \mathrm{div}(h_1\mathbf{u}_1) = 0, \\ \rho_1(h_1\mathbf{u}_1)_t + \rho_1\mathrm{div}(h_1\mathbf{u}_1 \otimes \mathbf{u}_1) + g\rho_1 h_1\nabla h_1 + g\rho_2 h_1 \nabla h_2 \\ -\beta_1 h_1 \nabla(\triangle h_1) - \beta_2 h_1 \nabla(\triangle h_2) = 2\nu_1 \mathrm{div}(h_1 \cdot \nabla \mathbf{u}_1), \\ h_{2t} + \mathrm{div}(h_2\mathbf{u}_2) = 0, \\ \rho_2(h_2\mathbf{u}_2)_t + \rho_2\mathrm{div}(h_2\mathbf{u}_2 \otimes \mathbf{u}_2) + g\rho_2 h_2\nabla h_1 + g\rho_2 h_2 \nabla h_2 \\ -\beta_2 h_2 \nabla(\triangle h_1) - \beta_2 h_2 \nabla(\triangle h_2) = 2\nu_2 \mathrm{div}(h_2 \cdot \nabla \mathbf{u}_2), \\ (h_1, \mathbf{u}_1, h_2, \mathbf{u}_2)|_{t=0} = (h_{10}, \mathbf{u}_{10}, h_{20}, \mathbf{u}_{20}), \end{cases} \tag{1.1}$$

where index 1 refers to the deeper layer and index 2 the upper layer of the flow; ρ_1 and ρ_2 denote the densities and $\rho_2 < \rho_1$; ν_1 and ν_2 denote the viscosity coefficients; β_1 and β_2 denote the interface and free surface tension coefficients, respectively; and g is the gravitational

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†Corresponding author: Qiangchang JU.

acceleration. All of these physical coefficients are positive constants. $h_j = h_j(t, \mathbf{x})$ and $\mathbf{u}_j = \mathbf{u}_j(t, \mathbf{x})$ denote the thickness and velocity field of each layer, where $j = 1, 2$.

Distinguished from the single layer model, the two layer shallow water equations capture something of the density stratification of the ocean, and it is a powerful model of many geophysically interesting phenomena, as well as being physically realizable in the laboratory [4, 9, 19]. However, there are only a few mathematical analyse of the two layer model. Zabsonré-Reina [21] obtained the existence of global weak solutions in a periodic domain and Roamba-Zabsonré [18] proved the global existence of weak solutions for the two layer viscous shallow water equations without friction or capillary term. There are other results regarding weak solutions of the two layer shallow water equations in [10, 15]. To the best of our knowledge, there are no results about the strong solution to the 2-D two layer viscous shallow water equations. In the present paper, our aim is to prove the existence and uniqueness of the global strong solution of (1.1) in the whole space $\mathbf{x} \in \mathbb{R}^2$.

Dividing the second and the fourth equations in (1.1) by $\rho_1 h_1$ and $\rho_2 h_2$, respectively, we have

$$\begin{cases} h_{1t} + \mathrm{div}(h_1\mathbf{u}_1) = 0, \\ \mathbf{u}_{1t} + \mathbf{u}_1 \cdot \nabla\mathbf{u}_1 + g\nabla h_1 + g\dfrac{\rho_2}{\rho_1}\nabla h_2 - \dfrac{\beta_1}{\rho_1}\nabla(\triangle h_1) - \dfrac{\beta_2}{\rho_1}\nabla(\triangle h_2) = 2\dfrac{\nu_1}{\rho_1}\dfrac{\mathrm{div}(h_1 \cdot \nabla\mathbf{u}_1)}{h_1}, \\ h_{2t} + \mathrm{div}(h_2\mathbf{u}_2) = 0, \\ \mathbf{u}_{2t} + \mathbf{u}_2 \cdot \nabla\mathbf{u}_2 + g\nabla h_1 + g\nabla h_2 - \dfrac{\beta_2}{\rho_2}\nabla(\triangle h_1) - \dfrac{\beta_2}{\rho_2}\nabla(\triangle h_2) = 2\dfrac{\nu_2}{\rho_2}\dfrac{\mathrm{div}(h_2 \cdot \nabla\mathbf{u}_2)}{h_2}, \\ (h_1, \mathbf{u}_1, h_2, \mathbf{u}_2)|_{t=0} = (h_{10}, \mathbf{u}_{10}, h_{20}, \mathbf{u}_{20}). \end{cases} \tag{1.2}$$

We seek the solution of (1.2) near the equilibrium state $(h_1, \mathbf{u}_1, h_2, \mathbf{u}_2) = (1, \mathbf{0}, 1, \mathbf{0})$. To this end, putting the transform $h_j = 1 + \tilde{h}_j,\ h_{j0} = 1 + \tilde{h}_{j0},\ j = 1, 2$ into the above equations, and dropping the tilde, we have

$$\begin{cases} h_{1t} + \mathrm{div}(h_1\mathbf{u}_1) + \mathrm{div}\mathbf{u}_1 = 0, \\ \mathbf{u}_{1t} + \mathbf{u}_1 \cdot \nabla\mathbf{u}_1 + g\nabla h_1 + \dfrac{\rho_2}{\rho_1}g\nabla h_2 - \dfrac{\beta_1}{\rho_1}\nabla(\triangle h_1) - \dfrac{\beta_2}{\rho_1}\nabla(\triangle h_2) = 2\dfrac{\nu_1}{\rho_1}\triangle\mathbf{u}_1 + 2\dfrac{\nu_1}{\rho_1}\dfrac{\nabla h_1 \cdot \nabla\mathbf{u}_1}{1 + h_1}, \\ h_{2t} + \mathrm{div}(h_2\mathbf{u}_2) + \mathrm{div}\mathbf{u}_2 = 0, \\ \mathbf{u}_{2t} + \mathbf{u}_2 \cdot \nabla\mathbf{u}_2 + g\nabla h_1 + g\nabla h_2 - \dfrac{\beta_2}{\rho_2}\nabla(\triangle h_1) - \dfrac{\beta_2}{\rho_2}\nabla(\triangle h_2) = 2\dfrac{\nu_2}{\rho_2}\triangle\mathbf{u}_2 + 2\dfrac{\nu_2}{\rho_2}\dfrac{\nabla h_2 \cdot \nabla\mathbf{u}_2}{1 + h_2}, \\ (h_1, \mathbf{u}_1, h_2, \mathbf{u}_2)|_{t=0} = (h_{10}, \mathbf{u}_{10}, h_{20}, \mathbf{u}_{20}). \end{cases} \tag{1.3}$$

Now we state the main result of this paper. For convenience, we set

$$\begin{aligned} E_s = \Big\{&(h_1, \mathbf{u}_1, h_2, \mathbf{u}_2) \in \Big(\mathcal{C}_b(\mathbb{R}^+; \tilde{B}_{2,1}^{s-1,s}) \cap L^1(\mathbb{R}^+; \tilde{B}_{2,1}^{s+1,s+2}) \\ &\times \Big(\mathcal{C}_b(\mathbb{R}^+; \dot{B}_{2,1}^{s-1}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{s+1})\Big)^2\Big)^2\Big\}, \end{aligned}$$

$$\begin{aligned} E(0) &= \|(h_{10}, \mathbf{u}_{10}, h_{20}, \mathbf{u}_{20})\|_{(\tilde{B}_{2,1}^{0,1} \times (\dot{B}_{2,1}^{0})^2)^2}, \\ E(h_1, \mathbf{u}_1, h_2, \mathbf{u}_2, T) &= \|(h_1, \mathbf{u}_1, h_2, \mathbf{u}_2)\|_{(L_T^\infty(\tilde{B}_{2,1}^{0,1}) \times (L_T^\infty(\dot{B}_{2,1}^{0}))^2)^2} \\ &\quad + \|(h_1, \mathbf{u}_1, h_2, \mathbf{u}_2\|_{(L_T^1(\tilde{B}_{2,1}^{2,3}) \times (L_T^1(\dot{B}_{2,1}^{2}))^2)^2}, \end{aligned}$$

where $\mathcal{C}_b(\mathbb{R}^+;X)$ is the subset of functions of $L^\infty(\mathbb{R}^+;X)$ which are continuous and bounded on $\mathbb{R}^+$ with values in X and $L^p_T(X)=L^p(0,T;X)$. Then we have

Theorem 1.1 Assume that $(h_{10},\mathbf{u}_{10},h_{20},\mathbf{u}_{20})\in(\tilde{B}^{0,1}_{2,1}\times(\dot{B}^0_{2,1})^2)^2$. For any positive constants ρ_1, β_1 ν_1 and ν_2, if ρ_2 and β_2 satisfy the following conditions:

$$\beta_2<\min\left\{\nu_2\left(\sqrt{\frac{\nu_1^2}{\rho_1^2}+\frac{2\nu_1\beta_1}{\nu_2\rho_1}}-\frac{\nu_1}{\rho_1}\right),\frac{\nu_1}{\rho_1(\frac{1}{2\nu_1}+\frac{\nu_1}{\beta_1\rho_1})}\right\},\tag{1.4}$$

$$\rho_2<\min\left\{\rho_1,\frac{\nu_2}{\nu_1}\rho_1\right\},\tag{1.5}$$

there exist positive constants α and M such that if

$$E(0)\le\alpha,$$

then the Cauchy problem (1.3) admits a unique solution $(h_1,\mathbf{u}_1,h_2,\mathbf{u}_2)\in E_1$ and the following estimate holds:

$$E(h_1,\mathbf{u}_1,h_2,\mathbf{u}_2,+\infty)\le ME(0),$$

where α and M depend only on the physical coefficients $\rho_1,\rho_2,\nu_1,\nu_2,\beta_1,\beta_2$ and g.

Remark 1.2 The conditions (1.4) and (1.5) could imply

$$\begin{aligned}&\max\left\{\frac{\nu_1\rho_2}{\rho_1^2},\frac{\beta_2}{\beta_1}\left(\frac{\beta_2}{2\nu_2}+\frac{\nu_1}{\rho_1}\right)\right\}<\frac{\nu_1}{\rho_1},\\&\max\left\{\frac{\rho_2\nu_1}{\rho_1^2},\beta_2\left(\frac{1}{2\nu_1}+\frac{\nu_1}{\rho_1\beta_1}\right)\right\}<\frac{\nu_1}{\rho_1}<\frac{\nu_2}{\rho_2},\end{aligned}\tag{1.6}$$

which are used to deal with the linear terms in the a priori estimates in Section 3.2.

In the proof of Theorem 1.1, the main difficulties arise from the complexity of the system and the coupling of pressure terms in momentum equations. We could not directly split the system into the two independent parts, and the method for the single layer shallow water equations in [12, 13] couldn't work in our case. To dispose of the coupling, we perform a careful combination of the elementary estimates (see (3.19)). However, some other nonlinear coupling terms appear in the new combination estimates for which Lemma 6.2 in [7], used in [3, 12, 13] to deal with the convection terms in single layer shallow water system, cannot be applied. Therefore, we construct a proposition in the Appendix to estimate these troublesome terms in our problem, which is a standard generalization of Lemma 6.2 in [7].

In this paper, the Littlewood-Paley decomposition will be used to construct the a priori estimates in a hybrid Besov space and the solution of (1.3) is obtained by the Friedrichs' regularization method. Finally, the uniqueness of the solution will be proved directly with the help of the estimates we construct. Some ideas of this paper are motivated by Danchin [6].

We also mention some results regarding the Cauchy problem for the 2-D single layer shallow water equations. For example, Wang-Xu [20] obtained the local solution for any initial data and obtained the global solution for small initial data in Sobolev space $H^{2+s}(\mathbb{R}^2)$ with $s>0$. Then, Chen-Miao-Zhang [3] improved the result of Wang-Xu by getting the global existence in time for small initial data $h_0-\bar{h}_0\in\dot{B}^0_{2,1}\cap\dot{B}^1_{2,1}$ and $\mathbf{u}_0\in\dot{B}^0_{2,1}$. In [13], Haspot considered the compressible Navier-Stokes equation with density dependent viscosity coefficients and a term of capillarity, and obtained the global existence and uniqueness in critical space. In [12], Hao-Hsiao-Li studied the single layer viscous equations with both rotation and capillary term, and also obtained the global well-posedness in Besov space. In addition, there are some researchs on the global weak solutions of the single layer shallow water equations in [2, 11, 14]. Other results related to the regularities of the oceanic flow can be found in [5, 17].

Throughout this paper, the subscript j takes on the values $j = 1, 2$, and we omit this for the sake of convenience. We denote the Fourier transform of f by $\mathcal{F}f$ or $\hat{f}$, and denote the inverse Fourier transform by $\mathcal{F}^{-1}f$. The notation $A \approx B$ means that $A \leq CB$, while $B \leq CA$ for some 'irrelevant' constant C. $\langle \cdot, \cdot \rangle$ is the L^2 inner product. The integral $\int f$ means $\int_{\mathbb{R}^N} f(x)\mathrm{d}x$, and $\|u\|_0 = (\int |u|^2)^{\frac{1}{2}}$ is the L^2 norm of u.

The paper is arranged as follows: in Section 2 we recall the definitions and some properties of Besov spaces; in Section 3 we give the a priori estimates of the linear system of (1.3) in a hybrid Besov space; in Section 4 we prove Theorem 1.1 by the classical Friedrichs' regularization method. Finally in appendix we give a proposition that is used in this paper.

2 Littlewood-Paley Theory and Besov Spaces

In this section, we recall the definitions and some properties of Littlewood-Paley decomposition and Besov spaces. The details can be found in [1, 7, 8].

2.1 Littlewood-Paley decomposition

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, supported in the annular $\mathcal{C} = \{\xi \in \mathbb{R}^N : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that

$$\sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1, \text{ if } \xi \neq 0.$$

Setting $h = \mathcal{F}^{-1}\varphi$, and defining

$$\triangle_k u(x) = \varphi(2^{-k}D)u(x) = 2^{kN}\int_{\mathbb{R}^N} h(2^k y)u(x-y)\mathrm{d}y \text{ and } S_k u = \sum_{p\leq k-1} \triangle_p u,$$

we have the properties

$$\triangle_p\triangle_q u \equiv 0 \text{ if } |p-q| \geq 2,$$
$$\triangle_p(S_{q-1}u\triangle_q v) \equiv 0 \text{ if } |p-q| \geq 4,$$

and the homogeneous Littlewood-Paley decomposition

$$u = \sum_{k\in\mathbb{Z}} \triangle_k u. \tag{2.1}$$

However, the right-hand side in (2.1) does not necessarily converge in $\mathcal{S}'(\mathbb{R}^N)$. Even if it does, the equality is not always true in $\mathcal{S}'(\mathbb{R}^N)$ (consider the case $u = 1$). Hence we will define the homogeneous Besov spaces in the following way:

2.2 Homogenous Besov spaces

Definition 2.1 Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$, for $u \in \mathcal{S}'(\mathbb{R}^N)$, and define

$$\|u\|_{\dot{B}^s_{p,r}} = \left(\sum_{k\in\mathbb{Z}} (2^{ks}\|\triangle_k u\|_{L^p})^r\right)^{\frac{1}{r}}.$$

Indeed, $\|\cdot\|_{\dot{B}^s_{p,r}}$ cannot be a norm in $\mathcal{S}'(\mathbb{R}^N)$ since $\|u\|_{\dot{B}^s_{p,r}} = 0$ means that u is a polynomial. This forces us to adopt the following definition for homogenous Besov space [7]:

Definition 2.2 Let s be a real number and (p,r) be in $[1,\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ consists of those distributions u in $\mathcal{S}'_h(\mathbb{R}^N)$ such that $\|u\|_{\dot{B}^s_{p,r}} < \infty$.

We emphasize that the definition of $\dot{B}^s_{p,r}$ does not depend on the choice of the function φ. We have the following properties of homogenous Besov spaces.

Proposition 2.3 (1) Density: If $p < +\infty$, $1 \le r < +\infty$ and $|s| \le \frac{N}{p}$, then C_0^∞ is dense in $\dot{B}^s_{p,r}$;

(2) Derivatives: There exists a universal constant C such that

$$\frac{1}{C}\|u\|_{\dot{B}^s_{p,r}} \le \|\nabla u\|_{\dot{B}^{s-1}_{p,r}} \le C\|u\|_{\dot{B}^s_{p,r}};$$

(3) Sobolev embedding: If $p_1 \le p_2$ and $r_1 \le r_2$, then $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$;

(4) Algebra properties: For $s > 0$, $\dot{B}^s_{p,r} \cap L^\infty$ is an algebra. Moreover, for any $p \in [1,+\infty]$, $\dot{B}^{\frac{N}{p}}_{p,1} \hookrightarrow \dot{B}^{\frac{N}{p}}_{p,+\infty} \cap L^\infty$, and $\dot{B}^{\frac{N}{p}}_{p,1}$ is an algebra if p is finite.

2.3 Hybrid Besov spaces

In this paper, we will use the following hybrid Besov spaces:

Definition 2.4 Let $s, t \in \mathbb{R}$, $1 \le p, r \le +\infty$, and define

$$\|u\|_{\tilde{B}^{s,t}_{p,r}} = \left(\sum_{k\le 0}(2^{ks}\|\triangle_k u\|_{L^p})^r + \sum_{k>0}(2^{kt}\|\triangle_k u\|_{L^p})^r\right)^{\frac{1}{r}},$$

and

$$\tilde{B}^{s,t}_{p,r} = \left\{u \in \mathcal{S}'_h(\mathbb{R}^N) : \|u\|_{\tilde{B}^{s,t}_{p,r}} < \infty\right\}.$$

Some embedding properties, interpolation inequalities and the action of multiplication on hybrid Besov spaces are involved in the following propositions:

Proposition 2.5 Some properties of $\tilde{B}^{s,t}_{p,r}$:

(1) $\tilde{B}^{s,s}_{p,r} = \dot{B}^s_{p,r}$;

(2) If $s \le t$, then $\tilde{B}^{s,t}_{p,r} = \dot{B}^s_{p,r} \cap \dot{B}^t_{p,r}$, and if $s > t$, then $\tilde{B}^{s,t}_{p,r} = \dot{B}^s_{p,r} + \dot{B}^t_{p,r}$;

(3) If $s_1 \le s_2$ and $t_1 \ge t_2$, then $\tilde{B}^{s_1,t_1}_{p,r} \hookrightarrow \tilde{B}^{s_2,t_2}_{p,r}$.

For the convenience of notation, we set $L^\rho_T(X) = L^\rho(0,T;X)$, and if $T = +\infty$, we set $L^\rho(X) = L^\rho_T(X)$.

Proposition 2.6 Let $s,\ t,\ s_1,\ s_2,\ t_1,\ t_2 \in \mathbb{R}$, $r,\ \rho,\ \rho_1,\ \rho_2 \in [1,+\infty]$. We have the following interpolation property:

$$\|u\|_{L^\rho_T(\tilde{B}^{s,t}_{p,r})} \le \|u\|^\theta_{L^{\rho_1}_T(\tilde{B}^{s_1,t_1}_{p,r})}\|u\|^{1-\theta}_{L^{\rho_2}_T(\tilde{B}^{s_2,t_2}_{p,r})},$$

with $\frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2}$, $s = \theta s_1 + (1-\theta)s_2$, and $t = \theta t_1 + (1-\theta)t_2$.

Proposition 2.7 Let $p,\ r \in [1,+\infty]$. We have for some universal constant C that

(1) If we let $s > 0$, $t > 0$, $\frac{1}{\rho_2} + \frac{1}{\rho_3} = \frac{1}{\rho_1} + \frac{1}{\rho_4} = \frac{1}{\rho} \le 1$, $u \in L^{\rho_3}_T(\tilde{B}^{s,t}_{p,r}) \cap L^{\rho_1}_T(L^\infty)$ and $v \in L^{\rho_4}_T(\tilde{B}^{s,t}_{p,r}) \cap L^{\rho_2}_T(L^\infty)$, then $uv \in L^\rho_T(\tilde{B}^{s,t}_{p,r})$ and

$$\|uv\|_{L^\rho_T(\tilde{B}^{s,t}_{p,r})} \le C\|u\|_{L^{\rho_1}_T(L^\infty)}\|v\|_{L^{\rho_4}_T(\tilde{B}^{s,t}_{p,r})} + \|v\|_{L^{\rho_2}_T(L^\infty)}\|u\|_{L^{\rho_3}_T(\tilde{B}^{s,t}_{p,r})};$$

(2) If $s_1,\ s_2,\ t_1,\ t_2 \le \frac{N}{p}$, $s_1+s_2>0$, $t_1+t_2>0$, $\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{1}{\rho}\le 1$, $u\in L^{\rho_1}_T(\tilde{B}^{s_1,t_1}_{p,r})$ and $v\in L^{\rho_2}_T(\tilde{B}^{s_2,t_2}_{p,r})$, then $uv\in L^{\rho}_T(\tilde{B}^{s_1+s_2-\frac{N}{p},t_1+t_2-\frac{N}{p}}_{p,r})$ and

$$\|uv\|_{L^{\rho}_T(\tilde{B}^{s_1+s_2-\frac{N}{p},t_1+t_2-\frac{N}{p}}_{p,r})}\le C\|u\|_{L^{\rho_1}_T(\tilde{B}^{s_1,t_1}_{p,r})}\|v\|_{L^{\rho_2}_T(\tilde{B}^{s_2,t_2}_{p,r})}.$$

2.4 Some estimates

The action of some smooth functions on Besov spaces is involved in the following proposition [7].

Proposition 2.8 (1) Let $s>0$ and $F\in W^{[s]+2,\infty}_{\rm loc}(\mathbb{R}^N)$ such that $F(0)=0$. Then there exists a function of one variable C_0 depending only on $s,\ N$ and F, such that

$$\|F(u)\|_{\dot{B}^s_{2,1}}\le C_0(\|u\|_{L^\infty})\|u\|_{\dot{B}^s_{2,1}};$$

(2) Let $s\in(\frac{N}{2},\frac{N}{2}]$ and $G\in W^{[\frac{N}{2}]+3,\infty}_{\rm loc}(\mathbb{R}^N)$ such that $G'(0)=0$. Then there exists a function of two variables C_1 which only depends on $s,\ N$ and G such that

$$\|G(u)-G(v)\|_{\dot{B}^s_{2,1}}\le C_1(\|u\|_{L^\infty},\|v\|_{L^\infty})(\|u\|_{\dot{B}^{\frac{N}{2}}_{2,1}}+\|v\|_{\dot{B}^{\frac{N}{2}}_{2,1}})\|u-v\|_{\dot{B}^s_{2,1}}.$$

For $m\in\mathbb{R}$, we define $\Lambda^m f=\mathcal{F}^{-1}(|\xi|^m\hat{f})$. Λ^m is a well-known pseudo-differential operator of degree m, and has been widely used. The following two basic properties for Λ^m will be frequently used in this paper:

Proposition 2.9 (1) Λ^m and $\triangle_k$ are commutative, i.e., $\triangle_k(\Lambda^m u)=\Lambda^m(\triangle_k u)$;
(2) Λ^m is self-adjoint, i.e., $\langle\Lambda^m f,g\rangle=\langle f,\Lambda^m g\rangle$;
(3) $\|\triangle_k(\Lambda^m u)\|_0\approx 2^{km}\|\triangle_k u\|_0$.

We will also use the following proposition to estimate the convection terms in the equations, for the proof and the more general state of the proposition, we refer to Lemma 6.2 in [7]:

Proposition 2.10 For $-\frac{N}{2}<s,t\le 1+\frac{N}{2}$, $m\ge 0$, we have

$$\int\Lambda^m\big(\triangle_k(\mathbf{v}\cdot\nabla a)\big)\triangle_k(\Lambda^m a)\le C\varepsilon_k 2^{-k(s-m)}\|\mathbf{v}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\|a\|_{\dot{B}^s_{2,1}}\|\triangle_k(\Lambda^m a)\|_0,$$

$$\begin{aligned}&\int\Lambda^m\big(\triangle_k(\mathbf{v}\cdot\nabla a)\big)\triangle_k b+\int\triangle_k(\mathbf{v}\cdot\nabla b)\triangle_k(\Lambda^m a)\\ \le\,& C\varepsilon_k\|\mathbf{v}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\times\Big(2^{-kt}\|\triangle_k(\Lambda^m a)\|_0\|b\|_{\dot{B}^t_{2,1}}+2^{-k(s-m)}\|a\|_{\dot{B}^s_{2,1}}\|\triangle_k b\|_0\Big),\end{aligned}$$

where $\sum\limits_{k\in\mathbb{Z}}\varepsilon_k\le 1$.

3 A Priori Estimates

3.1 Preparation for the estimates

Let $c_j=\Lambda^{-1}{\rm div}\mathbf{u}_j$, $d_j=\Lambda^{-1}{\rm div}^{\perp}\mathbf{u}_j$, where ${\rm div}^{\perp}\mathbf{u}_j=\nabla^{\perp}\cdot\mathbf{u}_j$, $\nabla^{\perp}=(-\partial_2,\partial_1)$ and Λ^{-1} is defined in Section 2.4. It is easy to check that

$$\mathbf{u}_j=-\Lambda^{-1}\nabla c_j-\Lambda^{-1}\nabla^{\perp}d_j.$$

For convenience, we set

$$\gamma=\frac{\rho_2}{\rho_1},\ \mu_1=\frac{\nu_1}{\rho_1},\ \mu_2=\frac{\nu_2}{\rho_2},\ \alpha_1=\frac{\beta_1}{\rho_1},\ \alpha_2=\frac{\beta_2}{\rho_1},\ \alpha_3=\frac{\beta_2}{\rho_2}.\tag{3.1}$$

With these notations, system (1.3) could be changed into the following:

$$\begin{cases} h_{1t} + \mathbf{u}_1 \cdot \nabla h_1 + \Lambda c_1 = F_1, \\ c_{1t} + \mathbf{u}_1 \cdot \nabla c_1 + 2\mu_1 \Lambda^2 c_1 - g\Lambda h_1 - \gamma g \Lambda h_2 - \alpha_1 \Lambda^3 h_1 - \alpha_2 \Lambda^3 h_2 = G_1, \\ d_{1t} + 2\mu_1 \Lambda^2 d_1 = \Lambda^{-1} \mathrm{div}^{\perp} H_1, \\ h_{2t} + \mathbf{u}_2 \cdot \nabla h_2 + \Lambda c_2 = F_2, \\ c_{2t} + \mathbf{u}_2 \cdot \nabla c_2 + 2\mu_2 \Lambda^2 c_2 - g\Lambda h_1 - g\Lambda h_2 - \alpha_3 \Lambda^3 h_1 - \alpha_3 \Lambda^3 h_2 = G_2, \\ d_{2t} + 2\mu_2 \Lambda^2 d_2 = \Lambda^{-1} \mathrm{div}^{\perp} H_2, \\ \mathbf{u}_j = -\Lambda^{-1} \nabla c_j - \Lambda^{-1} \nabla^{\perp} d_j, \end{cases}$$

where

$$F_j = -h_j \mathrm{div} \mathbf{u}_j,$$
$$G_j = \mathbf{u}_j \cdot \nabla c_j + \Lambda^{-1} \mathrm{div} H_j,$$
$$H_j = -\mathbf{u}_j \cdot \nabla \mathbf{u}_j + 2\mu_j \frac{\nabla h_j \cdot \nabla \mathbf{u}_j}{1 + h_j}.$$

To begin with, we study the following linear system:

$$\begin{cases} h_{1t} + \mathbf{v}_1 \cdot \nabla h_1 + \Lambda c_1 = F_1, \\ c_{1t} + \mathbf{v}_1 \cdot \nabla c_1 + 2\mu_1 \Lambda^2 c_1 - g\Lambda h_1 - \gamma g \Lambda h_2 - \alpha_1 \Lambda^3 h_1 - \alpha_2 \Lambda^3 h_2 = G_1, \\ d_{1t} + 2\mu_1 \Lambda^2 d_1 = P_1, \\ h_{2t} + \mathbf{v}_2 \cdot \nabla h_2 + \Lambda c_2 = F_2, \\ c_{2t} + \mathbf{v}_2 \cdot \nabla c_2 + 2\mu_2 \Lambda^2 c_2 - g\Lambda h_1 - g\Lambda h_2 - \alpha_3 \Lambda^3 h_1 - \alpha_3 \Lambda^3 h_2 = G_2, \\ d_{2t} + 2\mu_2 \Lambda^2 d_2 = P_2, \end{cases} \tag{3.2}$$

where $\mathbf{v}_j,\ F_j,\ G_j,\ P_j$ are regarded as given functions of $(t, \mathbf{x})$. Set

$$V(t) = \int_0^t \|(\mathbf{v}_1, \mathbf{v}_2)(\tau)\|_{\dot{B}^2_{2,1}} \mathrm{d}\tau.$$

Proposition 3.1 Assume that the physical coefficients $\rho_1, \rho_2, \nu_1, \nu_2, \beta_1, \beta_2$ satisfy the conditions in Theorem 1.1, and that $(h_1, c_1, d_1, h_2, c_2, d_2)$ is a solution of (3.2) on $[0, T)$, $T > 0$. Then for any $s \in (0, 2]$, we have

$$\begin{aligned} &\|(h_1, c_1, d_1, h_2, c_2, d_2)(t)\|_{(\tilde{B}^{s-1,s}_{2,1} \times \dot{B}^{s-1}_{2,1} \times \dot{B}^{s-1}_{2,1})^2} \\ &\quad + \int_0^t \|(h_1, c_1, d_1, h_2, c_2, d_2)(\tau)\|_{(\tilde{B}^{s+1,s+2}_{2,1} \times \dot{B}^{s+1}_{2,1} \times \dot{B}^{s+1}_{2,1})^2} \mathrm{d}\tau \\ &\leq A\mathrm{e}^{KV(t)} \Big(\|(h_1, c_1, d_1, h_2, c_2, d_2)(0)\|_{(\tilde{B}^{s-1,s}_{2,1} \times \dot{B}^{s-1}_{2,1} \times \dot{B}^{s-1}_{2,1})^2} \\ &\quad + \int_0^t \|(F_1, G_1, P_1, F_2, G_2, P_2)(\tau)\|_{(\tilde{B}^{s-1,s}_{2,1} \times \dot{B}^{s-1}_{2,1} \times \dot{B}^{s-1}_{2,1})^2} \mathrm{d}\tau \Big), \end{aligned} \tag{3.3}$$

where A and K are constants which depend on $s,\ g,\ \rho_1,\ \rho_2,\ \nu_1,\ \nu_2,\ \beta_1$ and β_2.

The next part of this section is devoted to proving Proposition 3.1.

For $K > 0$ to be determined, take the transform

$$(h_j, c_j, d_j, F_j, G_j, P_j) = \mathrm{e}^{KV(t)} (\tilde{h}_j, \tilde{c}_j, \tilde{d}_j, \tilde{F}_j, \tilde{G}_j, \tilde{P}_j).$$

Then (3.2) can be changed into

$$\begin{cases} \tilde{h}_{1t} + \mathbf{v}_1 \cdot \nabla \tilde{h}_1 + \Lambda \tilde{c}_1 = \tilde{F}_1 - KV'(t)\tilde{h}_1, \\ \tilde{c}_{1t} + \mathbf{v}_1 \cdot \nabla \tilde{c}_1 + 2\mu_1 \Lambda^2 \tilde{c}_1 - g\Lambda \tilde{h}_1 - \gamma g \Lambda \tilde{h}_2 - \alpha_1 \Lambda^3 \tilde{h}_1 - \alpha_2 \Lambda^3 \tilde{h}_2 = \tilde{G}_1 - KV'(t)\tilde{c}_1, \\ \tilde{d}_{1t} + 2\mu_1 \Lambda^2 \tilde{d}_1 = \tilde{P}_1 - KV'(t)\tilde{d}_1, \\ \tilde{h}_{2t} + \mathbf{v}_2 \cdot \nabla \tilde{h}_2 + \Lambda \tilde{c}_2 = \tilde{F}_2 - KV'(t)\tilde{h}_2, \\ \tilde{c}_{2t} + \mathbf{v}_2 \cdot \nabla \tilde{c}_2 + 2\mu_2 \Lambda^2 \tilde{c}_2 - g\Lambda \tilde{h}_1 - g\Lambda \tilde{h}_2 - \alpha_3 \Lambda^3 \tilde{h}_1 - \alpha_3 \Lambda^3 \tilde{h}_2 = \tilde{G}_2 - KV'(t)\tilde{c}_2, \\ \tilde{d}_{2t} + 2\mu_2 \Lambda^2 \tilde{d}_2 = \tilde{P}_2 - KV'(t)\tilde{d}_2. \end{cases} \tag{3.4}$$

Applying $\triangle_k$ to (3.4), we have

$$\begin{cases} \triangle_k \tilde{h}_{1t} + \triangle_k(\mathbf{v}_1 \cdot \nabla \tilde{h}_1) + \triangle_k(\Lambda \tilde{c}_1) = \triangle_k \tilde{F}_1 - KV'(t)\triangle_k \tilde{h}_1, \\ \triangle_k \tilde{c}_{1t} + \triangle_k(\mathbf{v}_1 \cdot \nabla \tilde{c}_1) + 2\mu_1 \triangle_k(\Lambda^2 \tilde{c}_1) - g\triangle_k(\Lambda \tilde{h}_1) - \gamma g \triangle_k(\Lambda \tilde{h}_2) \\ -\alpha_1 \triangle_k(\Lambda^3 \tilde{h}_1) - \alpha_2 \triangle_k(\Lambda^3 \tilde{h}_2) = \triangle_k \tilde{G}_1 - KV'(t)\triangle_k \tilde{c}_1, \\ \triangle_k \tilde{d}_{1t} + 2\mu_1 \triangle_k(\Lambda^2 \tilde{d}_1) = \triangle_k \tilde{P}_1 - KV'(t)\triangle_k \tilde{d}_1, \\ \triangle_k \tilde{h}_{2t} + \triangle_k(\mathbf{v}_2 \cdot \nabla \tilde{h}_2) + \triangle_k(\Lambda \tilde{c}_2) = \triangle_k \tilde{F}_2 - KV'(t)\triangle_k \tilde{h}_2, \\ \triangle_k \tilde{c}_{2t} + \triangle_k(\mathbf{v}_2 \cdot \nabla \tilde{c}_2) + 2\mu_2 \triangle_k(\Lambda^2 \tilde{c}_2) - g\triangle_k(\Lambda \tilde{h}_1) - g\triangle_k(\Lambda \tilde{h}_2) \\ -\alpha_3 \triangle_k(\Lambda^3 \tilde{h}_1) - \alpha_3 \triangle_k(\Lambda^3 \tilde{h}_2) = \triangle_k \tilde{G}_2 - KV'(t)\triangle_k \tilde{c}_2, \\ \triangle_k \tilde{d}_{2t} + 2\mu_2 \triangle_k(\Lambda^2 \tilde{d}_2) = \triangle_k \tilde{P}_2 - KV'(t)\triangle_k \tilde{d}_2. \end{cases} \tag{3.5}$$

Now we deduce some identities which will be used later. Taking the L^2 scalar product of the equations in (3.5) with $\triangle_k \tilde{h}_1,\ \triangle_k \tilde{c}_1,\ \triangle_k \tilde{d}_1,\ \triangle_k \tilde{h}_2,\ \triangle_k \tilde{c}_2$ and $\triangle_k \tilde{d}_2$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle_k \tilde{h}_1\|_0^2 + \int \triangle_k(\Lambda \tilde{c}_1)\triangle_k \tilde{h}_1 = \int \triangle_k(\tilde{F}_1 - \mathbf{v}_1 \cdot \nabla \tilde{h}_1 - KV'(t)\tilde{h}_1)\triangle_k \tilde{h}_1, \tag{3.6}$$

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle_k \tilde{c}_1\|_0^2 + 2\mu_1\|\triangle_k(\Lambda \tilde{c}_1)\|_0^2 - g\int \triangle_k(\Lambda \tilde{h}_1)\triangle_k \tilde{c}_1 \\ &- \gamma g \int \triangle_k(\Lambda \tilde{h}_2)\triangle_k \tilde{c}_1 - \alpha_1 \int \triangle_k(\Lambda^3 \tilde{h}_1)\triangle_k \tilde{c}_1 - \alpha_2 \int \triangle_k(\Lambda^3 \tilde{h}_2)\triangle_k \tilde{c}_1 \\ = &\int \triangle_k(\tilde{G}_1 - \mathbf{v}_1 \cdot \nabla \tilde{c}_1 - KV'(t)\tilde{c}_1)\triangle_k \tilde{c}_1, \end{aligned} \tag{3.7}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle_k \tilde{d}_1\|_0^2 + 2\mu_1\|\triangle_k(\Lambda \tilde{d}_1)\|_0^2 = \int \triangle_k(\tilde{P}_1 - KV'(t)\tilde{d}_1)\triangle_k \tilde{d}_1, \tag{3.8}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle_k \tilde{h}_2\|_0^2 + \int \triangle_k(\Lambda \tilde{c}_2)\triangle_k \tilde{h}_2 = \int \triangle_k(\tilde{F}_2 - \mathbf{v}_2 \cdot \nabla \tilde{h}_2 - KV'(t)\tilde{h}_2)\triangle_k \tilde{h}_2, \tag{3.9}$$

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle_k \tilde{c}_2\|_0^2 + 2\mu_2\|\triangle_k(\Lambda \tilde{c}_2)\|_0^2 - g\int \triangle_k(\Lambda \tilde{h}_1)\triangle_k \tilde{c}_2 \\ &- g \int \triangle_k(\Lambda \tilde{h}_2)\triangle_k \tilde{c}_2 - \alpha_3 \int \triangle_k(\Lambda^3 \tilde{h}_1)\triangle_k \tilde{c}_2 - \alpha_3 \int \triangle_k(\Lambda^3 \tilde{h}_2)\triangle_k \tilde{c}_2 \\ = &\int \triangle_k(\tilde{G}_2 - \mathbf{v}_2 \cdot \nabla \tilde{c}_2 - KV'(t)\tilde{c}_2)\triangle_k \tilde{c}_2, \end{aligned} \tag{3.10}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle_k \tilde{d}_2\|_0^2 + 2\mu_2\|\triangle_k(\Lambda \tilde{d}_2)\|_0^2 = \int \triangle_k(\tilde{P}_2 - KV'(t)\tilde{d}_2)\triangle_k \tilde{d}_2. \tag{3.11}$$

Then taking the L^2 scale product of the first equation in (3.5) with $\triangle_k(\Lambda^2\tilde{h}_1)$, of the second equation with $\triangle_k(\Lambda\tilde{h}_1)$, of the fourth equation with $\triangle_k(\Lambda^2\tilde{h}_2)$, and of the fifth equation with $\triangle_k(\Lambda\tilde{h}_2)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2+\int\triangle_k(\Lambda\tilde{c}_1)\triangle_k(\Lambda^2\tilde{h}_1)=\int\triangle_k(\tilde{F}_1-\mathbf{v}_1\cdot\nabla\tilde{h}_1-KV'(t)\tilde{h}_1)\triangle_k(\Lambda^2\tilde{h}_1), \tag{3.12}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\triangle_k\tilde{c}_1\triangle_k(\Lambda\tilde{h}_1)+2\mu_1\int\triangle_k(\Lambda^2\tilde{c}_1)\triangle_k(\Lambda\tilde{h}_1)-\alpha_2\int\triangle_k(\Lambda^3\tilde{h}_2)\triangle_k(\Lambda\tilde{h}_1)\\&-\alpha_1\|\triangle_k(\Lambda^2\tilde{h}_1)\|_0^2-g\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2-\gamma g\int\triangle_k(\Lambda\tilde{h}_2)\triangle_k(\Lambda\tilde{h}_1)+\|\triangle_k(\Lambda\tilde{c}_1)\|_0^2\\=&\int\triangle_k(\tilde{G}_1-\mathbf{v}_1\cdot\nabla\tilde{c}_1-KV'(t)\tilde{c}_1)\triangle_k(\Lambda\tilde{h}_1)+\int\triangle_k(\tilde{F}_1-\mathbf{v}_1\cdot\nabla\tilde{h}_1-KV'(t)\tilde{h}_1)\triangle_k(\Lambda\tilde{c}_1),\end{aligned} \tag{3.13}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2+\int\triangle_k(\Lambda\tilde{c}_2)\triangle_k(\Lambda^2\tilde{h}_2)=\int\triangle_k(\tilde{F}_2-\mathbf{v}_2\cdot\nabla\tilde{h}_2-KV'(t)\tilde{h}_2)\triangle_k(\Lambda^2\tilde{h}_2), \tag{3.14}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\triangle_k\tilde{c}_2\triangle_k(\Lambda\tilde{h}_2)+2\mu_2\int\triangle_k(\Lambda^2\tilde{c}_2)\triangle_k(\Lambda\tilde{h}_2)-\alpha_3\int\triangle_k(\Lambda^3\tilde{h}_1)\triangle_k(\Lambda\tilde{h}_2)\\&-\alpha_3\|\triangle_k(\Lambda^2\tilde{h}_2)\|_0^2-g\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2-g\int\triangle_k(\Lambda\tilde{h}_2)\triangle_k(\Lambda\tilde{h}_1)+\|\triangle_k(\Lambda\tilde{c}_2)\|_0^2\\=&\int\triangle_k(\tilde{G}_2-\mathbf{v}_2\cdot\nabla\tilde{c}_2-KV'(t)\tilde{c}_2)\triangle_k(\Lambda\tilde{h}_2)+\int\triangle_k(\tilde{F}_2-\mathbf{v}_2\cdot\nabla\tilde{h}_2-KV'(t)\tilde{h}_2)\triangle_k(\Lambda\tilde{c}_2).\end{aligned} \tag{3.15}$$

Finally, taking the L^2 scale product of the first equation in (3.5) with $\triangle_k\tilde{h}_2$, and of the fourth equation with $\triangle_k\tilde{h}_1$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\triangle_k\tilde{h}_1\triangle_k\tilde{h}_2+\int\triangle_k(\Lambda\tilde{c}_1)\triangle_k\tilde{h}_2+\int\triangle_k(\Lambda\tilde{c}_2)\triangle_k\tilde{h}_1\\=&\int\triangle_k(\tilde{F}_1-\mathbf{v}_1\cdot\nabla\tilde{h}_1-KV'(t)\tilde{h}_1)\triangle_k\tilde{h}_2+\int\triangle_k(\tilde{F}_2-\mathbf{v}_2\cdot\nabla\tilde{h}_2-KV'(t)\tilde{h}_2)\triangle_k\tilde{h}_1.\end{aligned} \tag{3.16}$$

Now we take K_1, K_2 such that

$$\begin{aligned}&\max\left\{\frac{\nu_1\rho_2}{\rho_1^2},\frac{\beta_2}{\beta_1}\left(\frac{\beta_2}{2\nu_2}+\frac{\nu_1}{\rho_1}\right)\right\}<K_1<\frac{\nu_1}{\rho_1},\\&\max\left\{\frac{\rho_2\nu_1}{\rho_1^2},\beta_2\left(\frac{1}{2\nu_1}+\frac{\nu_1}{\rho_1\beta_1}\right)\right\}<K_2<\frac{\nu_1}{\rho_1}<\frac{\nu_2}{\rho_2}.\end{aligned} \tag{3.17}$$

The reason for the choice of K_1, K_2 will be explicit. We first see from (1.6) that the constants K_1, K_2 in the above inequalities do exist, and also that (3.17) could imply

$$\begin{aligned}&\max\left\{\frac{\rho_2}{\rho_1}K_2,\frac{\beta_2}{\beta_1}\left(\frac{\beta_2}{2\nu_2}+K_2\right)\right\}<\max\left\{\frac{\nu_1\rho_2}{\rho_1^2},\frac{\beta_2}{\beta_1}\left(\frac{\beta_2}{2\nu_2}+\frac{\nu_1}{\rho_1}\right)\right\}<K_1<\frac{\nu_1}{\rho_1},\\&\max\left\{\frac{\rho_2}{\rho_1}K_1,\beta_2\left(\frac{1}{2\nu_1}+\frac{K_1}{\beta_1}\right)\right\}<\max\left\{\frac{\rho_2\nu_1}{\rho_1^2},\beta_2\left(\frac{1}{2\nu_1}+\frac{\nu_1}{\rho_1\beta_1}\right)\right\}<K_2<\frac{\nu_2}{\rho_2}.\end{aligned} \tag{3.18}$$

Taking the flowing combination of the identities obtained above, i.e.,

$$\begin{aligned}&\Big(g\times(3.6)+(3.7)-K_1\times(3.13)\Big)+\gamma\times\Big(g\times(3.9)+(3.10)-K_2\times(3.15)\Big)\\&+\gamma g\times(3.16)+\Big(\alpha_1+2\mu_1K_1\Big)\times(3.12)+\gamma\times\Big(\alpha_3+2\mu_2K_2\Big)\times(3.14),\end{aligned} \tag{3.19}$$

we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}A_k^2 + B_k^2 + KV'(t)A_k^2 = L_k + N_k, \tag{3.20}$$

where

$$\begin{aligned}
A_k^2 =& g\|\triangle_k\tilde{h}_1\|_0^2 + \|\triangle_k\tilde{c}_1\|_0^2 + (\alpha_1 + 2\mu_1K_1)\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2 + \gamma g\|\triangle_k\tilde{h}_2\|_0^2\\
&+ \gamma\|\triangle_k\tilde{c}_2\|_0^2 + \gamma(\alpha_3 + 2\mu_2K_2)\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2 - 2K_1\int\triangle_k\tilde{c}_1\triangle_k(\Lambda\tilde{h}_1)\\
&- 2\gamma K_2\int\triangle_k\tilde{c}_2\triangle_k(\Lambda\tilde{h}_2) + 2\gamma g\int\triangle_k\tilde{h}_1\triangle_k\tilde{h}_2,\\
B_k^2 =& (2\mu_1 - K_1)\|\triangle_k(\Lambda\tilde{c}_1)\|_0^2 + gK_1\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2 + \alpha_1K_1\|\triangle_k(\Lambda^2\tilde{h}_1)\|_0^2\\
&+ \gamma(2\mu_2 - K_2)\|\triangle_k(\Lambda\tilde{c}_2)\|_0^2 + \gamma gK_2\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2 + \gamma\alpha_3K_2\|\triangle_k(\Lambda^2\tilde{h}_2)\|_0^2,\\
L_k =& \alpha_2\int\triangle_k(\Lambda^3\tilde{h}_2)\triangle_k\tilde{c}_1 - \gamma gK_1\int\triangle_k(\Lambda\tilde{h}_1)\triangle_k(\Lambda\tilde{h}_2)\\
&- \alpha_2K_1\int\triangle_k(\Lambda^3\tilde{h}_2)\triangle_k(\Lambda\tilde{h}_1) + \gamma\alpha_3\int\triangle_k(\Lambda^3\tilde{h}_1)\triangle_k\tilde{c}_2\\
&- \gamma gK_2\int\triangle_k(\Lambda\tilde{h}_1)\triangle_k(\Lambda\tilde{h}_2) - \gamma\alpha_3K_2\int\triangle_k(\Lambda^3\tilde{h}_1)\triangle_k(\Lambda\tilde{h}_2),\\
N_k =& g\int\triangle_k(\tilde{F}_1 - \mathbf{v}_1\cdot\nabla\tilde{h}_1)\triangle_k\tilde{h}_1 + \int\triangle_k(\tilde{G}_1 - \mathbf{v}_1\cdot\nabla\tilde{c}_1)\triangle_k\tilde{c}_1\\
&- K_1\int\triangle_k(\tilde{G}_1 - \mathbf{v}_1\cdot\nabla\tilde{c}_1)\triangle_k(\Lambda\tilde{h}_1) - K_1\int\triangle_k(\tilde{F}_1 - \mathbf{v}_1\cdot\nabla\tilde{h}_1)\triangle_k(\Lambda\tilde{c}_1)\\
&+ \gamma g\int\triangle_k(\tilde{F}_2 - \mathbf{v}_2\cdot\nabla\tilde{h}_2)\triangle_k\tilde{h}_2 + \gamma\int\triangle_k(\tilde{G}_2 - \mathbf{v}_2\cdot\nabla\tilde{c}_2)\triangle_k\tilde{c}_2\\
&- \gamma K_2\int\triangle_k(\tilde{G}_2 - \mathbf{v}_2\cdot\nabla\tilde{c}_2)\triangle_k(\Lambda\tilde{h}_2) - \gamma K_2\int\triangle_k(\tilde{F}_2 - \mathbf{v}_2\cdot\nabla\tilde{h}_2)\triangle_k(\Lambda\tilde{c}_2)\\
&+ \gamma g\int\triangle_k(\tilde{F}_1 - \mathbf{v}_1\cdot\nabla\tilde{h}_1)\triangle_k\tilde{h}_2 + \gamma g\int\triangle_k(\tilde{F}_2 - \mathbf{v}_2\cdot\nabla\tilde{h}_2)\triangle_k\tilde{h}_1\\
&+ (\alpha_1 + 2\mu_1K_1)\int\triangle_k(\tilde{F}_1 - \mathbf{v}_1\cdot\nabla\tilde{h}_1)\triangle_k(\Lambda^2\tilde{h}_1)\\
&+ \gamma(\alpha_3 + 2\mu_2K_2)\int\triangle_k(\tilde{F}_2 - \mathbf{v}_2\cdot\nabla\tilde{h}_2)\triangle_k(\Lambda^2\tilde{h}_2).
\end{aligned}$$

Since $0 < K_1 < \frac{\nu_1}{\rho_1} < 2\mu_1$, $0 < K_2 < \frac{\nu_1}{\rho 1} < \frac{\nu_2}{\rho_2} < 2\mu_2$, we have

$$B_k^2 \approx \|\triangle_k(\Lambda\tilde{h}_1, \Lambda^2\tilde{h}_1, \Lambda\tilde{c}_1, \Lambda\tilde{h}_2, \Lambda^2\tilde{h}_2, \Lambda\tilde{c}_2)\|_0^2. \tag{3.21}$$

3.2 Estimates of linear terms

By Hölder's inequality, we have

$$\begin{aligned}
A_k^2 \le& g\|\triangle_k\tilde{h}_1\|_0^2 + \|\triangle_k\tilde{c}_1\|_0^2 + (\alpha_1 + 2\mu_1K_1)\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2 + \gamma g\|\triangle_k\tilde{h}_2\|_0^2 + \gamma\|\triangle_k\tilde{c}_2\|_0^2\\
&+ \gamma(\alpha_3 + 2\mu_2K_2)\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2 + 2K_1\Big(\frac{1}{4\mu_1}\|\triangle_k\tilde{c}_1\|_0^2 + \mu_1\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2\Big)\\
&+ 2\gamma K_2\Big(\frac{1}{4\mu_2}\|\triangle_k\tilde{c}_2\|_0^2 + \mu_2\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2\Big) + 2\gamma g\Big(\frac{\gamma}{1+\gamma}\|\triangle\tilde{h}_2\|_0^2 + \frac{1+\gamma}{4\gamma}\|\triangle_k\tilde{h}_1\|_0^2\Big)\\
=& g\Big(1 + \frac{1+\gamma}{2}\Big)\|\triangle_k\tilde{h}_1\|_0^2 + \Big(1 + \frac{1}{2\mu_1}K_1\Big)\|\triangle_k\tilde{c}_1\|_0^2 + (\alpha_1 + 4\mu_1K_1)\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2\\
&+ \gamma g\Big(1 + \frac{2\gamma}{\gamma+1}\Big)\|\triangle_k\tilde{h}_2\|_0^2 + \gamma\Big(1 + \frac{1}{2\mu_2}K_2\Big)\|\triangle_k\tilde{c}_2\|_0^2 + \gamma(\alpha_3 + 4\mu_2K_2)\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2.
\end{aligned}$$

We could also have

$$\begin{aligned}
A_k^2 \geq\, & g\|\triangle_k\tilde{h}_1\|_0^2+\|\triangle_k\tilde{c}_1\|_0^2+(\alpha_1+2\mu_1K_1)\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2+\gamma g\|\triangle_k\tilde{h}_2\|_0^2+\gamma\|\triangle_k\tilde{c}_2\|_0^2\\
&+\gamma(\alpha_3+2\mu_2K_2)\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2-2K_1\Big(\frac{1}{4\mu_1}\|\triangle_k\tilde{c}_1\|_0^2+\mu_1\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2\Big)\\
&-2\gamma K_2\Big(\frac{1}{4\mu_2}\|\triangle_k\tilde{c}_2\|_0^2+\mu_2\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2\Big)-2\gamma g\Big(\frac{\gamma}{1+\gamma}\|\triangle\tilde{h}_2\|_0^2+\frac{1+\gamma}{4\gamma}\|\triangle_k\tilde{h}_1\|_0^2\Big)\\
=\, & g\Big(1-\frac{1+\gamma}{2}\Big)\|\triangle_k\tilde{h}_1\|_0^2+\Big(1-\frac{1}{2\mu_1}K_1\Big)\|\triangle_k\tilde{c}_1\|_0^2+\alpha_1\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2\\
&+\gamma g\Big(1-\frac{2\gamma}{\gamma+1}\Big)\|\triangle_k\tilde{h}_2\|_0^2+\gamma\Big(1-\frac{1}{2\mu_2}K_2\Big)\|\triangle_k\tilde{c}_2\|_0^2+\gamma\alpha_3\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2.
\end{aligned}$$

Since $\gamma=\frac{\rho_2}{\rho_1}<1$, $0<K_1<2\mu_1$, $0<K_2<2\mu_2$, we have

$$1-\frac{1+\gamma}{2}>0,\ 1-\frac{2\gamma}{\gamma+1}>0,\ 1-\frac{1}{2\mu_1}K_1>0,\ 1-\frac{1}{2\mu_2}K_2>0.$$

Hence the above equations imply that

$$A_k^2\approx\|\triangle_k(\tilde{h}_1,\Lambda\tilde{h}_1,\tilde{c}_1,\tilde{h}_2,\Lambda\tilde{h}_2,\tilde{c}_2)\|_0^2. \tag{3.22}$$

By noting (3.21) and Proposition 2.9 we also obtain

$$B_k^2\approx 2^{2k}A_k^2. \tag{3.23}$$

Now we estimate L_k. Invoking the definitions in (3.1), the commutativity of Λ and $\triangle_k$, and applying Hölder's inequality, we obtain

$$\begin{aligned}
L_k \leq\, & \alpha_2\bigg(\frac{\nu_1}{\beta_2}\|\triangle_k(\Lambda\tilde{c}_1)\|_0^2+\frac{\beta_2}{4\nu_1}\|\triangle_k(\Lambda^2\tilde{h}_2)\|_0^2\bigg)\\
&+\gamma gK_1\bigg(\frac{\rho_1}{2\rho_2}\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2+\frac{\rho_2}{2\rho_1}\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2\bigg)\\
&+\alpha_2K_1\bigg(\frac{\beta_1}{2\beta_2}\|\triangle_k(\Lambda^2\tilde{h}_1)\|_0^2+\frac{\beta_2}{2\beta_1}\|\triangle_k(\Lambda^2\tilde{h}_2)\|_0^2\bigg)\\
&+\gamma\alpha_3\bigg(\frac{\nu_2}{\beta_2}\|\triangle_k(\Lambda\tilde{c}_2)\|_0^2+\frac{\beta_2}{4\nu_2}\|\triangle_k(\Lambda^2\tilde{h}_1)\|_0^2\bigg)\\
&+\gamma gK_2\bigg(\frac{1}{2}\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2+\frac{1}{2}\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2\bigg)\\
&+\gamma\alpha_3K_2\bigg(\frac{1}{2}\|\triangle_k(\Lambda^2\tilde{h}_1)\|_0^2+\frac{1}{2}\|\triangle_k(\Lambda^2\tilde{h}_2)\|_0^2\bigg)\\
=\, & \frac{\nu_1}{\rho_1}\|\triangle_k(\Lambda\tilde{c}_1)\|_0^2+\frac{1}{2}g\bigg(K_1+\frac{\rho_2}{\rho_1}K_2\bigg)\|\triangle_k(\Lambda\tilde{h}_1)\|_0^2\\
&+\frac{1}{2\rho_1}\bigg(\beta_1K_1+\frac{\beta_2^2}{2\nu_2}+\beta_2K_2\bigg)\|\triangle_k(\Lambda^2\tilde{h}_1)\|_0^2+\frac{\nu_2}{\rho_1}\|\triangle_k(\Lambda\tilde{c}_2)\|_0^2\\
&+\frac{\rho_2}{2\rho_1}g\bigg(\frac{\rho_2}{\rho_1}K_1+K_2\bigg)\|\triangle_k(\Lambda\tilde{h}_2)\|_0^2+\frac{\beta_2}{2\rho_1}\bigg(\frac{\beta_2}{2\nu_1}+\frac{\beta_2}{\beta_1}K_1+K_2\bigg)\|\triangle_k(\Lambda^2\tilde{h}_2)\|_0^2.
\end{aligned}$$

Due to (3.18), we have

$$\begin{aligned}
&\frac{\nu_1}{\rho_1}<2\frac{\nu_1}{\rho_1}-K_1,\\
&\frac{1}{2}g\Big(K_1+\frac{\rho_2}{\rho_1}K_2\Big)<gK_1,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2\rho_1}\Big(\beta_1 K_1+\frac{\beta_2^2}{2\nu_2}+\beta_2 K_2\Big)<\frac{\beta_1}{\rho_1}K_1,\\
&\frac{\nu_2}{\rho_1}<\frac{\rho_2}{\rho_1}\Big(\frac{2\nu_2}{\rho_2}-K_2\Big),\\
&\frac{\rho_2}{2\rho_1}g\Big(\frac{\rho_2}{\rho_1}K_1+K_2\Big)<\frac{\rho_2}{\rho_1}gK_2,\\
&\frac{\beta_2}{2\rho_1}\Big(\frac{\beta_2}{2\nu_1}+\frac{\beta_2}{\beta_1}K_1+K_2\Big)<\frac{\beta_2}{\rho_1}K_2,
\end{aligned}$$

hence there is a positive constant C independent of k such that

$$B_k^2-L_k\geq\frac{1}{C}B_k^2.$$

Therefore we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}A_k^2+\frac{1}{C}B_k^2+KV'(t)A_k^2<N_k. \tag{3.24}$$

Combining (3.24) with (3.23), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}A_k^2+\frac{1}{C}2^{2k}A_k^2+KV'(t)A_k^2<N_k. \tag{3.25}$$

3.3 Estimates of nonlinear terms

Now we estimate the nonlinear terms in N_k. Assume that $s\in(0,2]$. We are not going to estimate all of the terms in N_k, but only some of them. For example, by Proposition 2.3, Proposition 2.9 and Proposition 2.10, we have

$$g\int\triangle_k(\mathbf{v}_1\cdot\nabla\tilde{h}_1)\triangle_k\tilde{h}_1\leq C\varepsilon_k 2^{-k(s-1)}\|\mathbf{v}_1\|_{\dot{B}^2_{2,1}}\|\tilde{h}_1\|_{\dot{B}^{s-1}_{2,1}}\|\triangle_k\tilde{h}_1\|_0,$$

where $\sum\limits_{k\in\mathbb{Z}}\varepsilon_k\leq 1$, and

$$\begin{aligned}
&K_1\int\triangle_k(\mathbf{v}_1\cdot\nabla\tilde{h}_1)\triangle_k(\Lambda\tilde{c}_1)+K_1\int\triangle_k(\mathbf{v}_1\cdot\nabla\tilde{c}_1)\triangle_k(\Lambda\tilde{h}_1)\\
&=K_1\int\Lambda\big(\triangle_k(\mathbf{v}_1\cdot\nabla\tilde{h}_1)\big)\triangle_k\tilde{c}_1+K_1\int\triangle_k(\mathbf{v}_1\cdot\nabla\tilde{c}_1)\triangle_k(\Lambda\tilde{h}_1)\\
&\leq C\varepsilon_k\|\mathbf{v}_1\|_{\dot{B}^2_{2,1}}\times\Big(2^{-k(s-1)}\|\triangle_k(\Lambda\tilde{h}_1)\|_0\|\tilde{c}_1\|_{\dot{B}^{s-1}_{2,1}}+2^{-k(s-1)}\|\tilde{h}_1\|_{\dot{B}^{s}_{2,1}}\|\triangle_k\tilde{c}_1\|_0\Big)\\
&\leq C\varepsilon_k 2^{-k(s-1)}\|\mathbf{v}_1\|_{\dot{B}^2_{2,1}}\times\Big(\|\triangle_k(\Lambda\tilde{h}_1)\|_0\|\tilde{c}_1\|_{\dot{B}^{s-1}_{2,1}}+\|\Lambda\tilde{h}_1\|_{\dot{B}^{s-1}_{2,1}}\|\triangle_k\tilde{c}_1\|_0\Big),
\end{aligned}$$

$$\begin{aligned}
(\alpha_1+2\mu_1K_1)\int\triangle_k(\mathbf{v}_1\cdot\nabla\tilde{h}_1)\triangle_k(\Lambda^2\tilde{h}_1)&=(\alpha_1+2\mu_1K_1)\int\Lambda\big(\triangle_k(\mathbf{v}_1\cdot\nabla\tilde{h}_1)\big)\triangle_k(\Lambda\tilde{h}_1)\\
&\leq C\varepsilon_k 2^{-k(s-1)}\|\mathbf{v}_1\|_{\dot{B}^2_{2,1}}\|\tilde{h}_1\|_{\dot{B}^{s}_{2,1}}\|\triangle_k(\Lambda\tilde{h}_1)\|_0\\
&\leq C\varepsilon_k 2^{-k(s-1)}\|\mathbf{v}_1\|_{\dot{B}^2_{2,1}}\|\Lambda\tilde{h}_1\|_{\dot{B}^{s-1}_{2,1}}\|\triangle_k(\Lambda\tilde{h}_1)\|_0.
\end{aligned}$$

The following estimate could not be obtained by Proposition 2.10 or Lemma 6.2 in [7], however, by applying Proposition 4.4 in the Appendix, we have:

$$\gamma g\int\triangle_k(\mathbf{v}_1\nabla\tilde{h}_1)\triangle_k\tilde{h}_2\leq C\varepsilon_k 2^{-k(s-1)}\|\mathbf{v}_1\|_{\dot{B}^2_{2,1}}\|\tilde{h}_1\|_{\dot{B}^{s-1}_{2,1}}\big(\|\triangle_k\tilde{h}_2\|_0+\|\triangle_k(\Lambda\tilde{h}_2)\|_0\big).$$

Other terms in N_k could be estimated in the same way. Combine these estimates with (3.22), and note that

$$V(t)=\int_0^t\|(\mathbf{v}_1,\mathbf{v}_2)(\tau)\|_{\dot{B}^2_{2,1}}\mathrm{d}\tau,$$

we obtain the following estimate for N_k:

$$N_k \le CA_k\Big(\|\triangle_k(\tilde F_1,\tilde G_1,\Lambda\tilde F_1,\tilde F_2,\tilde G_2,\Lambda\tilde F_2)\|_0 + \varepsilon_k 2^{-k(s-1)}V'(t)\|(\tilde h_1,\Lambda\tilde h_1,\tilde c_1,\tilde h_2,\Lambda\tilde h_2,\tilde c_2)\|_{(\dot B^{s-1}_{2,1})^6}\Big).$$

Taking the estimate for N_k into (3.25) and dividing by A_k, we obtain

$$\frac{\mathrm d}{\mathrm dt}A_k+\frac1C 2^{2k}A_k+KV'(t)A_k \le C\Big(\|\triangle_k(\tilde F_1,\tilde G_1,\Lambda\tilde F_1,\tilde F_2,\tilde G_2,\Lambda\tilde F_2)\|_0+\varepsilon_k 2^{-k(s-1)}V'(t)\|(\tilde h_1,\Lambda\tilde h_1,\tilde c_1,\tilde h_2,\Lambda\tilde h_2,\tilde c_2)\|_{(\dot B^{s-1}_{2,1})^6}\Big).$$

Noting that $A_k\approx \varepsilon_k 2^{-k(s-1)}\|(\tilde h_1,\Lambda\tilde h_1,\tilde c_1,\tilde h_2,\Lambda\tilde h_2,\tilde c_2)\|_{(\dot B^{s-1}_{2,1})^6}$ for $k\in\mathbb Z$, we have, by taking K large enough, that

$$\frac{\mathrm d}{\mathrm dt}A_k+\frac1C 2^{2k}A_k\le C\|\triangle_k(\tilde F_1,\tilde G_1,\Lambda\tilde F_1,\tilde F_2,\tilde G_2,\Lambda\tilde F_2)\|_0,\quad k\in\mathbb Z, \tag{3.26}$$

where C is independent of k.

3.4 Accomplishment of the proof of Proposition 3.1

Multiplying (3.26) by $2^{k(s-1)}$, taking the summation on k, and integrating the result over $[0,t]$, we have

$$\sum_{k\in\mathbb Z}2^{k(s-1)}A_k(t)+\int_0^t\sum_{k\in\mathbb Z}2^{k(s+1)}A_k(\tau)\mathrm d\tau \le C\sum_{k\in\mathbb Z}2^{k(s-1)}A_k(0)+C\int_0^t\sum_{k\in\mathbb Z}2^{k(s-1)}\|\triangle_k(\tilde F_1,\tilde G_1,\Lambda\tilde F_1,\tilde F_2,\tilde G_2,\Lambda\tilde F_2)(\tau)\|_0\mathrm d\tau. \tag{3.27}$$

Since $A_k\approx\|\triangle_k(\tilde h_1,\Lambda\tilde h_1,\tilde c_1,\tilde h_2,\Lambda\tilde h_2,\tilde c_2)\|_0$, we have

$$\sum_{k\in\mathbb Z}2^{k(s-1)}A_k(t)\approx\|(\tilde h_1,\Lambda\tilde h_1,\tilde c_1,\tilde h_2,\Lambda\tilde h_2,\tilde c_2)(t)\|_{\dot B^{s-1}_{2,1}}.$$

Due to Proposition 2.3 and Proposition 2.5, we have, for $j=1,2$, that

$$\begin{aligned}\|(\tilde h_j,\Lambda\tilde h_j)(t)\|_{\dot B^{s-1}_{2,1}}&\approx\|\tilde h_j(t)\|_{\dot B^{s-1}_{2,1}}+\|\Lambda\tilde h_j(t)\|_{\dot B^{s-1}_{2,1}}\\&\approx\|\tilde h_j(t)\|_{\dot B^{s-1}_{2,1}}+\|\tilde h_j(t)\|_{\dot B^{s}_{2,1}}\\&\approx\|\tilde h_j(t)\|_{\tilde B^{s-1,s}_{2,1}}.\end{aligned}$$

Hence

$$\sum_{k\in\mathbb Z}2^{k(s-1)}A_k(t)\approx\|(\tilde h_1,\tilde c_1,\tilde h_2,\tilde c_2)(t)\|_{(\tilde B^{s-1,s}_{2,1}\times\dot B^{s-1}_{2,1})^2}.$$

Applying the discussions above to other terms in (3.27), we obtain

$$\begin{aligned}&\|(\tilde h_1,\tilde c_1,\tilde h_2,\tilde c_2)(t)\|_{(\tilde B^{s-1,s}_{2,1}\times\dot B^{s-1}_{2,1})^2}+\int_0^t\|(\tilde h_1,\tilde c_1,\tilde h_2,\tilde c_2)(\tau)\|_{(\tilde B^{s+1,s+2}_{2,1}\times\dot B^{s+1}_{2,1})^2}\mathrm d\tau\\&\le C\Big(\|(\tilde h_1,\tilde c_1,\tilde h_2,\tilde c_2)(0)\|_{(\tilde B^{s-1,s}_{2,1}\times\dot B^{s-1}_{2,1})^2}+\int_0^t\|(\tilde F_1,\tilde G_1,\tilde F_2,\tilde G_2)(t)\|_{(\tilde B^{s-1,s}_{2,1}\times\dot B^{s-1}_{2,1})^2}\mathrm d\tau\Big).\end{aligned} \tag{3.28}$$

To complete the proof, it suffices to show the estimate of $\tilde d_j$. From (3.8), we have that

$$\frac12\frac{\mathrm d}{\mathrm dt}\|\triangle_k\tilde d_1\|_0^2+2\mu_1\|\triangle_k(\Lambda\tilde d_1)\|_0^2+KV'(t)\|\triangle_k\tilde d_1\|_0^2\le\|\triangle_k\tilde d_1\|_0\|\triangle_k\tilde P_1\|_0,$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle_k\tilde{d}_1\|_0+\frac{1}{C}2^{2k}\|\triangle_k\tilde{d}_1\|_0\le\|\triangle_k\tilde{P}_1\|_0. \tag{3.29}$$

Multiplying (3.29) by $2^{k(s-1)}$ and taking sum on $k\in\mathbb{Z}$ and integral over $[0,t]$, we have

$$\|\tilde{d}_1(t)\|_{\dot{B}^{s-1}_{2,1}}+\int_0^t\|\tilde{d}_1(\tau)\|_{\dot{B}^{s+1}_{2,1}}\mathrm{d}\tau\le C\bigg(\|\tilde{d}_1(0)\|_{\dot{B}^{s-1}_{2,1}}+\int_0^t\|\tilde{P}_1(\tau)\|_{\dot{B}^{s-1}_{2,1}}\mathrm{d}\tau\bigg). \tag{3.30}$$

We also have the same estimate about $\tilde{d}_2$:

$$\|\tilde{d}_2(t)\|_{\dot{B}^{s-1}_{2,1}}+\int_0^t\|\tilde{d}_2(\tau)\|_{\dot{B}^{s+1}_{2,1}}\mathrm{d}\tau\le C\bigg(\|\tilde{d}_2(0)\|_{\dot{B}^{s-1}_{2,1}}+\int_0^t\|\tilde{P}_2(\tau)\|_{\dot{B}^{s-1}_{2,1}}\mathrm{d}\tau\bigg). \tag{3.31}$$

Combining (3.30), (3.31) with (3.28), we have

$$\begin{aligned}&\|(\tilde{h}_1,\tilde{c}_1,\tilde{d}_1,\tilde{h}_2,\tilde{c}_2,\tilde{d}_2)(t)\|_{(\tilde{B}^{s-1,s}_{2,1}\times\dot{B}^{s-1}_{2,1}\times\dot{B}^{s-1}_{2,1})^2}\\&\quad+\int_0^t\|(\tilde{h}_1,\tilde{c}_1,\tilde{d}_1,\tilde{h}_2,\tilde{c}_2,\tilde{d}_2)(\tau)\|_{(\tilde{B}^{s+1,s+2}_{2,1}\times\dot{B}^{s+1}_{2,1}\times\dot{B}^{s+1}_{2,1})^2}\mathrm{d}\tau\\&\le C\bigg(\|(\tilde{h}_1,\tilde{c}_1,\tilde{d}_1,\tilde{h}_2,\tilde{c}_2,\tilde{d}_2)(0)\|_{(\tilde{B}^{s-1,s}_{2,1}\times\dot{B}^{s-1}_{2,1}\times\dot{B}^{s-1}_{2,1})^2}\\&\quad+\int_0^t\|(\tilde{F}_1,\tilde{G}_1,\tilde{P}_1,\tilde{F}_2,\tilde{G}_2,\tilde{P}_2)(\tau)\|_{(\tilde{B}^{s-1,s}_{2,1}\times\dot{B}^{s-1}_{2,1}\times\dot{B}^{s-1}_{2,1})^2}\mathrm{d}\tau\bigg).\end{aligned}$$

Finally, taking the inverse transform

$$(\tilde{h}_j,\tilde{c}_j,\tilde{d}_j,\tilde{F}_j,\tilde{G}_j,\tilde{P}_j)=\mathrm{e}^{-KV(t)}(h_j,c_j,d_j,F_j,G_j,P_j),$$

we get the following estimate for the original functions:

$$\begin{aligned}&\|(h_1,c_1,d_1,h_2,c_2,d_2)(t)\|_{(\tilde{B}^{s-1,s}_{2,1}\times\dot{B}^{s-1}_{2,1}\times\dot{B}^{s-1}_{2,1})^2}\\&\quad+\mathrm{e}^{KV(t)}\int_0^t\mathrm{e}^{-KV(\tau)}\|(h_1,c_1,d_1,h_2,c_2,d_2)(\tau)\|_{(\tilde{B}^{s+1,s+2}_{2,1}\times\dot{B}^{s+1}_{2,1}\times\dot{B}^{s+1}_{2,1})^2}\mathrm{d}\tau\\&\le C\bigg(\mathrm{e}^{KV(t)}\|(h_1,c_1,d_1,h_2,c_2,d_2)(0)\|_{(\tilde{B}^{s-1,s}_{2,1}\times\dot{B}^{s-1}_{2,1}\times\dot{B}^{s-1}_{2,1})^2}\\&\quad+\int_0^t\mathrm{e}^{KV(t)-KV(\tau)}\|(F_1,G_1,P_1,F_2,G_2,P_2)(\tau)\|_{(\tilde{B}^{s-1,s}_{2,1}\times\dot{B}^{s-1}_{2,1}\times\dot{B}^{s-1}_{2,1})^2}\mathrm{d}\tau\bigg),\end{aligned}$$

which leads to (3.3). Now we have completed the proof of Proposition 3.1.

4 Existence and Uniqueness

With Proposition 3.1 in hand, we can prove the existence of the solution of (1.3) through the classical Friedrichs' regularization method, which was used in [12, 13] and the references therein.

4.1 Construction of the approximate solutions

Define the operators $\{J_n\}_{n\in\mathbb{N}}$ by

$$J_nf=\mathcal{F}^{-1}\mathbf{1}_{B(\frac{1}{n},n)}\hat{f},$$

and consider the approximate system

$$\begin{cases} h^n_{1t} + J_n(J_n\mathbf{u}^n_1 \cdot \nabla J_n h^n_1) + \Lambda J_n c^n_1 = F^n_1, \\ c^n_{1t} + J_n(J_n\mathbf{u}^n_1 \cdot \nabla J_n c^n_1) + 2\mu_1\Lambda^2 J_n c^n_1 - g\Lambda J_n h^n_1 \\ -\gamma g\Lambda J_n h^n_2 - \alpha_1\Lambda^3 J_n h^n_1 - \alpha_2\Lambda^3 J_n h^n_2 = G^n_1, \\ d^n_{1t} + 2\mu_1\Lambda^2 J_n d^n_1 = J_n\Lambda^{-1}\mathrm{div}^\perp H^n_1, \\ h^n_{2t} + J_n(J_n\mathbf{u}^n_2 \cdot \nabla J_n h^n_2) + \Lambda J_n c^n_2 = F^n_2, \\ c^n_{2t} + J_n(J_n\mathbf{u}^n_2 \cdot \nabla J_n c^n_2) + 2\mu_2\Lambda^2 J_n c^n_2 - g\Lambda J_n h^n_1 \\ -g\Lambda J_n h^n_2 - \alpha_3\Lambda^3 J_n h^n_1 - \alpha_3\Lambda^3 J_n h^n_2 = G^n_2, \\ d^n_{2t} + 2\mu_2\Lambda^2 J_n d^n_2 = J_n\Lambda^{-1}\mathrm{div}^\perp H^n_2, \\ \mathbf{u}^n_j = -\Lambda^{-1}\nabla c^n_j - \Lambda^{-1}\nabla^\perp d^n_j, \\ (h^n_j, c^n_j, d^n_j)(0) = (h^n_{j0}, c^n_{j0}, d^n_{j0}) = J_n(h_{j0}, \Lambda^{-1}\mathrm{div}\mathbf{u}_{j0}, \Lambda^{-1}\mathrm{div}^\perp\mathbf{u}_{j0}), \end{cases} \tag{4.1}$$

where

$$\begin{aligned} F^n_j &= -J_n(J_n h^n_j \mathrm{div} J_n\mathbf{u}^n_j), \\ G^n_j &= J_n(J_n\mathbf{u}^n_j \cdot \nabla J_n c^n_j) + \Lambda^{-1}\mathrm{div} J_n H^n_j, \\ H^n_j &= -J_n\mathbf{u}^n_j \cdot \nabla J_n\mathbf{u}^n_j + 2\mu_j \frac{\nabla J_n h^n_j \cdot \nabla J_n\mathbf{u}^n_j}{\theta(J_n h^n_j) + 1}, \end{aligned}$$

with $\theta(s)$ a smooth function satisfying

$$\theta(s) = \begin{cases} -\dfrac{3}{4}, & s < -\frac{3}{4}; \\ s, & |s| \le \dfrac{1}{2}; \\ \dfrac{3}{4}, & s > \dfrac{3}{4}; \\ \text{smooth}, & \text{otherwise}. \end{cases}$$

The existence of the solution of (4.1) in some time interval could be obtained by the Cauchy-Lipschitz theorem. Indeed, by setting $X(t) = (h^n_1, c^n_1, d^n_1, h^n_2, c^n_2, d^n_2)(t)$, (4.1) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \mathscr{P}(X). \tag{4.2}$$

We regard (4.2) as an initial data problem of the ODE system in $(L^2)^6$. Due to the Cauchy-Lipschitz theory, it suffices to check that $\mathscr{P}(X)$ is local Lipschitz to X in $(L^2)^6$. For example, we have

$$\begin{aligned} &\left\| J_n\Lambda^{-1}\mathrm{div}\left(\frac{\nabla J_n h^n_{j,1} \cdot \nabla J_n\mathbf{u}^n_{j,1}}{\theta(J_n h^n_{j,1}) + 1} - \frac{\nabla J_n h^n_{j,2} \cdot \nabla J_n\mathbf{u}^n_{j,2}}{\theta(J_n h^n_{j,2}) + 1} \right) \right\|_{L^2} \\ =& \left\| \mathbf{1}_{B(\frac{1}{n},n)} |\xi|^{-1}(\xi_1, \xi_2) \cdot \mathcal{F}\left(\frac{\nabla J_n h^n_{j,1} \cdot \nabla J_n\mathbf{u}^n_{j,1}}{\theta(J_n h^n_{j,1}) + 1} - \frac{\nabla J_n h^n_{j,2} \cdot \nabla J_n\mathbf{u}^n_{j,2}}{\theta(J_n h^n_{j,2}) + 1} \right) \right\|_{L^2} \\ \le& \left\| \frac{\nabla J_n h^n_{j,1} \cdot \nabla J_n\mathbf{u}^n_{j,1}}{\theta(J_n h^n_{j,1}) + 1} - \frac{\nabla J_n h^n_{j,2} \cdot \nabla J_n\mathbf{u}^n_{j,2}}{\theta(J_n h^n_{j,2}) + 1} \right\|_{L^2} \\ \le& \left\| \frac{\nabla J_n(h^n_{j,1} - h^n_{j,2}) \cdot \nabla J_n\mathbf{u}^n_{j,1}}{\theta(J_n h^n_{j,1}) + 1} \right\|_{L^2} + \left\| \frac{\nabla J_n h^n_{j,2} \cdot \nabla J_n(\mathbf{u}^n_{j,1} - \mathbf{u}^n_{j,2})}{\theta(J_n h^n_{j,1}) + 1} \right\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&\quad + \left\| \nabla J_n h^n_{j,2} \cdot \nabla J_n \mathbf{u}^n_{j,2} \frac{\theta(J_n h^n_{j,2}) - \theta(J_n h^n_{j,1})}{[\theta(J_n h^n_{j,1})+1][\theta(J_n h^n_{j,2})+1]} \right\|_{L^2} \\
&\le \left\| \frac{\nabla J_n \mathbf{u}^n_{j,1}}{\theta(J_n h^n_{j,1})+1} \right\|_{L^\infty} \|\nabla J_n(h^n_{j,1} - h^n_{j,2})\|_{L^2} + \left\| \frac{\nabla J_n h^n_{j,2}}{\theta(J_n h^n_{j,1})+1} \|_{L^\infty} \|\nabla J_n(\mathbf{u}^n_{j,1} - \mathbf{u}^n_{j,2}) \right\|_{L^2} \\
&\quad + \left\| \frac{\nabla J_n h^n_{j,2} \cdot \nabla J_n \mathbf{u}^n_{j,2}}{[\theta(J_n h^n_{j,1})+1][\theta(J_n h^n_{j,2})+1]} \right\|_{L^\infty} \|\theta(J_n h^n_{j,2}) - \theta(J_n h^n_{j,1})\|_{L^2} \\
&\le C(n) \||\xi| \mathbf{1}_{B(\frac{1}{n},n)} \mathcal{F} \mathbf{u}^n_{j,1}\|_{L^1} \|h^n_{j,1} - h^n_{j,2}\|_{L^2} + C(n) \||\xi| \mathbf{1}_{B(\frac{1}{n},n)} \mathcal{F} h^n_{j,2}\|_{L^1} \|\mathbf{u}^n_{j,1} - \mathbf{u}^n_{j,2}\|_{L^2} \\
&\quad + C(n) \|\xi| \mathbf{1}_{B(\frac{1}{n},n)} \mathcal{F} h^n_{j,2}\|_{L^1} \||\xi| \mathbf{1}_{B(\frac{1}{n},n)} \mathcal{F} \mathbf{u}^n_{j,2}\|_{L^1} \|J_n h^n_{j,2} - J_n h^n_{j,1}\|_{L^2} \\
&\le C(n) \big(\|\mathbf{u}^n_{j,1}\|_{L^2} \|h^n_{j,1} - h^n_{j,2}\|_{L^2} + \|h^n_{j,2}\|_{L^2} \|\mathbf{u}^n_{j,1} - \mathbf{u}^n_{j,2}\|_{L^2} \\
&\quad + \|h^n_{j,2}\|_{L^2} \|\mathbf{u}^n_{j,2}\|_{L^2} \|h^n_{j,2} - h^n_{j,1}\|_{L^2} \big),
\end{aligned}
$$

where we have used Plancherel theorem, Hausdorff-Young's inequality, Hölder's inequality and the smoothness and boundedness of $\theta(s)$ above. Since other terms in $\mathscr{P}(X)$ are either linear or bilinear, the verification of the local Lipschitz in $(L^2)^6$ of these terms are simpler. Therefore, by the Cauchy-Lipschitz theorem, we conclude that (4.1) admits a unique solution $(h^n_j, c^n_j, d^n_j) \in C([0,T_n),(L^2)^6)$ for some $T_n > 0$. Since $J_n^2 = J_n$, it is easy to verify that $J_n(h^n_j, c^n_j, d^n_j)$ is also a solution of (4.1). Hence, uniqueness implies that $J_n(h^n_j, c^n_j, d^n_j) = (h^n_j, c^n_j, d^n_j)$. So (h^n_j, c^n_j, d^n_j) is also a solution of the system

$$
\begin{cases}
h^n_{1t} + J_n(\mathbf{u}^n_1 \cdot \nabla h^n_1) + \Lambda c^n_1 = \bar{F}^n_1, \\
c^n_{1t} + J_n(\mathbf{u}^n_1 \cdot \nabla c^n_1) + 2\mu_1 \Lambda^2 c^n_1 - g\Lambda h^n_1 - \gamma g \Lambda h^n_2 - \alpha_1 \Lambda^3 h^n_1 - \alpha_2 \Lambda^3 h^n_2 = \bar{G}^n_1, \\
d^n_{1t} + 2\mu_1 \Lambda^2 d^n_1 = J_n \Lambda^{-1} \mathrm{div}^\perp \bar{H}^n_1, \\
h^n_{2t} + J_n(\mathbf{u}^n_2 \cdot \nabla h^n_2) + \Lambda c^n_2 = \bar{F}^n_2, \\
c^n_{2t} + J_n(\mathbf{u}^n_2 \cdot \nabla c^n_2) + 2\mu_2 \Lambda^2 c^n_2 - g\Lambda h^n_1 - g\Lambda h^n_2 - \alpha_3 \Lambda^3 h^n_1 - \alpha_3 \Lambda^3 h^n_2 = \bar{G}^n_2, \\
d^n_{2t} + 2\mu_2 \Lambda^2 d^n_2 = J_n \Lambda^{-1} \mathrm{div}^\perp \bar{H}^n_2, \\
\mathbf{u}^n_j = -\Lambda^{-1} \nabla c^n_j - \Lambda^{-1} \nabla^\perp d^n_j, \\
(h^n_j, c^n_j, d^n_j)(0) = (h^n_{j0}, c^n_{j0}, d^n_{j0}) = J_n(h_{j0}, \Lambda^{-1} \mathrm{div} \mathbf{u}_{j0}, \Lambda^{-1} \mathrm{div}^\perp \mathbf{u}_{j0}),
\end{cases}
\tag{4.3}
$$

where

$$
\begin{aligned}
\bar{F}^n_j &= -J_n(h^n_j \mathrm{div} \mathbf{u}^n_j), \\
\bar{G}^n_j &= J_n(\mathbf{u}^n_j \cdot \nabla c^n_j) + \Lambda^{-1} \mathrm{div} J_n \bar{H}^n_j, \\
\bar{H}^n_j &= -\mathbf{u}^n_j \cdot \nabla \mathbf{u}^n_j + 2\mu_j \frac{\nabla h^n_j \cdot \nabla \mathbf{u}^n_j}{\theta(h^n_j) + 1}.
\end{aligned}
$$

Note that (4.3) is an ODE system in the space

$$
\mathcal{L}^2_n := \left\{ f \in L^2(\mathbb{R}^2) : \mathrm{supp} \hat{f} \subset B(\frac{1}{n}, n) \right\}. \tag{4.4}
$$

According to the Cauchy-Lipschitz theorem, (4.3) admits a unique solution in $C([0,T_n^*),(\mathcal{L}^2_n)^6)$ where$[0,T_n^*)$ is the maximal time interval in which the solution exists.

4.2 Uniform estimates

In this subsection, we derive the uniform estimates for $\{(h^n_j, c^n_j, d^n_j)\}_{n\in\mathbb{N}}$ and prove the global existence of the approximate solutions.

Setting

$$E^{(n)}=\|(h^n_{10},\mathbf{u}^n_{10},h^n_{20},\mathbf{u}^n_{20})\|_{(\tilde{B}^{0,1}_{2,1}\times(\dot{B}^0_{2,1})^2)^2},$$

we have, for n large enough, that

$$E^{(n)}\leq 2E(0).$$

Hence, from Proposition 3.1, we have that

$$\begin{aligned}E(h^n_1,\mathbf{u}^n_1,h^n_2,\mathbf{u}^n_2,t)\leq& A\mathrm{e}^{K\|(\mathbf{u}^n_1,\mathbf{u}^n_2)\|_{(L^1_t(\dot{B}^2_{2,1}))^2}}\Big(E^{(n)}\\&+\|(\bar{F}^n_1,\bar{G}^n_1,\bar{H}^n_1,\bar{F}^n_2,\bar{G}^n_2,\bar{H}^n_2)\|_{(L^1_t(\tilde{B}^{0,1}_{2,1})\times L^1_t(\dot{B}^0_{2,1})\times L^1_t(\dot{B}^0_{2,1}))^2}\Big)\\\leq& A\mathrm{e}^{K\|(\mathbf{u}^n_1,\mathbf{u}^n_2)\|_{(L^1_t(\dot{B}^2_{2,1}))^2}}\Big(2E(0)\\&+\|(\bar{F}^n_1,\bar{G}^n_1,\bar{H}^n_1,\bar{F}^n_2,\bar{G}^n_2,\bar{H}^n_2)\|_{(L^1_t(\tilde{B}^{0,1}_{2,1})\times L^1_t(\dot{B}^0_{2,1})\times L^1_t(\dot{B}^0_{2,1}))^2}\Big).\end{aligned}$$

Assume that $A>1$, otherwise we can substitute A by $A+1$ in the above inequality. Denote

$$\tilde{T}_n=\sup\Big\{t\in[0,T^*_n):E(h^n_1,\mathbf{u}^n_1,h^n_2,\mathbf{u}^n_2,t)\leq 3AE(0)\Big\}.$$

Since $3A>1$, we have $\tilde{T}_n>0$ by continuity. Assume that $6C_1AE(0)\leq 1$, where C_1 is the continuity modulus of $\dot{B}^1_{2,1}\hookrightarrow L^\infty$. For any $T<\tilde{T}_n$, we have

$$\|h^n_j\|_{L^\infty_T(L^\infty)}\leq C_1\|h^n_j\|_{L^\infty_T(\dot{B}^1_{2,1})}\leq C_1\|h^n_j\|_{L^\infty_T(\tilde{B}^{0,1}_{2,1})}\leq 3C_1AE(0)\leq\frac{1}{2},$$

hence

$$\begin{aligned}\Big\|\frac{\nabla h^n_j\cdot\nabla\mathbf{u}^n_j}{\theta(h^n_j)+1}\Big\|_{L^1_T(\dot{B}^0_{2,1})}=&\Big\|\frac{\nabla h^n_j\cdot\nabla\mathbf{u}^n_j}{h^n_j+1}\Big\|_{L^1_T(\dot{B}^0_{2,1})}\\\leq&\|\nabla h^n_j\cdot\nabla\mathbf{u}^n_j\|_{L^1_T(\dot{B}^0_{2,1})}+\Big\|\frac{h^n_j\cdot\nabla h^n_j\cdot\nabla\mathbf{u}^n_j}{h^n_j+1}\Big\|_{L^1_T(\dot{B}^0_{2,1})}\\\leq& C\|\nabla h^n_j\|_{L^\infty_T(\dot{B}^0_{2,1})}\|\nabla\mathbf{u}^n_j\|_{L^1_T(\dot{B}^1_{2,1})}\\&+C\Big\|\frac{h^n_j\cdot\nabla h^n_j}{h^n_j+1}\Big\|_{L^\infty_T(\dot{B}^0_{2,1})}\|\nabla\mathbf{u}^n_j\|_{L^1_T(\dot{B}^1_{2,1})}\\\leq& C\|\nabla h^n_j\|_{L^\infty_T(\dot{B}^0_{2,1})}\|\nabla\mathbf{u}^n_j\|_{L^1_T(\dot{B}^1_{2,1})}\bigg(1+\Big\|\frac{h^n_j}{h^n_j+1}\Big\|_{L^\infty_T(\dot{B}^1_{2,1})}\bigg)\\\leq& C\|h^n_j\|_{L^\infty_T(\tilde{B}^{0,1}_{2,1})}\|\mathbf{u}^n_j\|_{L^1_T(\dot{B}^2_{2,1})}\Big(1+C\|h^n_j\|_{L^\infty_T(\tilde{B}^{0,1}_{2,1})}\Big).\end{aligned}$$

Similarly, we can deduce that

$$\begin{aligned}&\|(\bar{F}^n_j,\bar{G}^n_j,\bar{H}^n_j)\|_{L^1_T(\tilde{B}^{0,1}_{2,1})\times L^1_T(\dot{B}^0_{2,1})\times L^1_T(\dot{B}^0_{2,1})}\\\leq& C\|h^n_j\|_{L^\infty_T(\tilde{B}^{0,1}_{2,1})}\|\mathbf{u}^n_j\|_{L^1_T(\dot{B}^2_{2,1})}\Big(1+C\|h^n_j\|_{L^\infty_T(\tilde{B}^{0,1}_{2,1})}\Big)+C\|\mathbf{u}^n_j\|_{L^\infty_T(\dot{B}^0_{2,1})}\|\mathbf{u}^n_j\|_{L^1_T(\dot{B}^2_{2,1})}\\\leq& 9CA^2E(0)^2(1+3CAE(0)),\end{aligned}$$

hence we have

$$E(h^n_1,\mathbf{u}^n_1,h^n_2,\mathbf{u}^n_2,T)\leq A\mathrm{e}^{3KAE(0)}(2E(0)+9CA^2E(0)^2(1+3CAE(0))).$$

Therefore, by taking $E(0)$ small enough, we can obtain the key estimate

$$E(h^n_1,\mathbf{u}^n_1,h^n_2,\mathbf{u}^n_2,T)\leq\frac{5}{2}AE(0)<3AE(0)$$

for any $T < \tilde{T}_n$. Hence $\tilde{T}_n = T_n^*$. Indeed, due to the arbitrariness of $T < \tilde{T}_n$, and continuity, we know that $E(h_1^n, \mathbf{u}_1^n, h_2^n, \mathbf{u}_2^n, \tilde{T}_n) \leq \frac{5}{2}AE(0)$. Hence, there is a $\epsilon > 0$ such that $E(h_1^n, \mathbf{u}_1^n, h_2^n, \mathbf{u}_2^n, \tilde{T}_n + \epsilon) \leq 3AE(0)$, which is in contradiction to the definition of $\tilde{T}_n$.

Finally, we claim that $\tilde{T}_n = T_n^* = +\infty$. Indeed, if $T_n^* < +\infty$, then, due to the discussion above, we have that $\|(h_1^n, \mathbf{u}_1^n, h_2^n, \mathbf{u}_2^n)\|_{L^\infty_{T_n^*}(L^2_n)} < +\infty$, which is in contradiction to the definition of T_n^*. Therefore $\tilde{T}_n = T_n^* = +\infty$, and hence

$$E(h_1^n, \mathbf{u}_1^n, h_2^n, \mathbf{u}_2^n, +\infty) \leq \frac{5}{2}AE(0) < +\infty, \ \forall n \in \mathbb{N}. \tag{4.5}$$

4.3 Existence of the solution

In the previous subsections, we have proved that when the initial data is small enough, (4.3) admits a solution $(h_1^n, \mathbf{u}_1^n, h_2^n, \mathbf{u}_2^n)$ which is global in time. Now we prove that, up to an extraction, the sequence $\{(h_1^n, \mathbf{u}_1^n, h_2^n, \mathbf{u}_2^n)\}_{n\in\mathbb{N}}$ converges to a solution of (1.3) in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^2)$. Firstly, we give a lemma.

Lemma 4.1 $\{(h_1^n, \mathbf{u}_1^n, h_2^n, \mathbf{u}_2^n)\}_{n\in\mathbb{N}}$ is uniformly bounded in

$$(C^{\frac{1}{2}}(\mathbb{R}^+; \dot{B}^0_{2,1}) \times (C^{\frac{1}{4}}(\mathbb{R}^+; \dot{B}^{-\frac{1}{2}}_{2,1}))^2)^2.$$

Proof We use u.b for that which is uniformly bounded. It suffices to show that $\frac{\partial}{\partial t}h_j^n$ is u.b in $L^2(\dot{B}^0_{2,1})$, $\frac{\partial}{\partial t}c_j^n$ is u.b in $L^{\frac{4}{3}}(\dot{B}^{-\frac{1}{2}}_{2,1}) + L^4(\dot{B}^{-\frac{1}{2}}_{2,1})$, and $\frac{\partial}{\partial t}d_j^n$ is u.b in $L^{\frac{4}{3}}(\dot{B}^{-\frac{1}{2}}_{2,1})$. Indeed,

$$\|\Lambda c_j^n\|_{L^2(\dot{B}^0_{2,1})} \leq \|c_j^n\|_{L^2(\dot{B}^1_{2,1})} \leq C\|c_j^n\|^{\frac{1}{2}}_{L^1(\dot{B}^2_{2,1})}\|c_j^n\|^{\frac{1}{2}}_{L^\infty(\dot{B}^0_{2,1})},$$

$$\|J_n \mathrm{div}(\mathbf{u}_j^n \cdot h_j^n)\|_{L^2(\dot{B}^0_{2,1})} \leq \|\mathbf{u}_j^n \cdot h_j^n\|_{L^2(\dot{B}^1_{2,1})} \leq C\|\mathbf{u}_j^n\|_{L^2(\dot{B}^1_{2,1})}\|h_j^n\|_{L^\infty(\tilde{B}^{0,1}_{2,1})},$$

$$\|\Lambda^2 c_j^n\|_{L^{\frac{4}{3}}(\dot{B}^{-\frac{1}{2}}_{2,1})} \leq C\|c_j^n\|_{L^{\frac{4}{3}}(\dot{B}^{\frac{3}{2}}_{2,1})} \leq C\|c_j^n\|^{\frac{3}{4}}_{L^1(\dot{B}^2_{2,1})}\|c_j^n\|^{\frac{1}{4}}_{L^\infty(\dot{B}^0_{2,1})},$$

$$\begin{aligned}\|J_n\Lambda^{-1}\mathrm{div}^\perp(\mathbf{u}_j^n \cdot \nabla\mathbf{u}_j^n)\|_{L^{\frac{4}{3}}(\dot{B}^{-\frac{1}{2}}_{2,1})} &\leq C\|\mathbf{u}_j^n \cdot \nabla\mathbf{u}_j^n\|_{L^{\frac{4}{3}}(\dot{B}^{-\frac{1}{2}}_{2,1})}\\
&\leq C\|\mathbf{u}_j^n\|_{L^\infty(\dot{B}^0_{2,1})}\|\nabla\mathbf{u}_j^n\|_{L^{\frac{4}{3}}(\dot{B}^{\frac{1}{2}}_{2,1})}\\
&\leq C\|\mathbf{u}_j^n\|_{L^\infty(\dot{B}^0_{2,1})}\|\mathbf{u}_j^n\|^{\frac{1}{2}}_{L^1(\dot{B}^2_{2,1})}\|\mathbf{u}_j^n\|^{\frac{1}{2}}_{L^2(\dot{B}^1_{2,1})}\\
&\leq C\|\mathbf{u}_j^n\|^{\frac{5}{4}}_{L^\infty(\dot{B}^0_{2,1})}\|\mathbf{u}_j^n\|^{\frac{3}{4}}_{L^1(\dot{B}^2_{2,1})},\end{aligned}$$

$$\begin{aligned}\left\|J_n\Lambda^{-1}\mathrm{div}^\perp\left(\frac{\nabla h_j^n \cdot \nabla\mathbf{u}_j^n}{h_j^n+1}\right)\right\|_{L^{\frac{4}{3}}(\dot{B}^{-\frac{1}{2}}_{2,1})} &\leq C\left\|\frac{\nabla h_j^n \cdot \nabla\mathbf{u}_j^n}{h_j^n+1}\right\|_{L^{\frac{4}{3}}(\dot{B}^{-\frac{1}{2}}_{2,1})}\\
&\leq C\left\|\frac{\nabla h_j^n}{h_j^n+1}\right\|_{L^\infty(\dot{B}^0_{2,1})}\|\nabla\mathbf{u}_j^n\|_{L^{\frac{4}{3}}(\dot{B}^{\frac{1}{2}}_{2,1})},\end{aligned}$$

where

$$\begin{aligned}\left\|\frac{\nabla h_j^n}{h_j^n+1}\right\|_{L^\infty(\dot{B}^0_{2,1})} &\leq \|\nabla h_j^n\|_{L^\infty(\dot{B}^0_{2,1})} + \left\|\frac{h_j^n\nabla h_j^n}{h_j^n+1}\right\|_{L^\infty(\dot{B}^0_{2,1})}\\
&\leq C\|h_j^n\|_{L^\infty(\tilde{B}^{0,1}_{2,1})} + \|\nabla h_j^n\|_{L^\infty(\dot{B}^0_{2,1})}\left\|\frac{h_j^n}{h_j^n+1}\right\|_{L^\infty(\dot{B}^1_{2,1})}\\
&\leq C\|h_j^n\|_{L^\infty(\tilde{B}^{0,1}_{2,1})}(1 + C\|h_j^n\|_{L^\infty(\tilde{B}^{0,1}_{2,1})}),\end{aligned}$$

and

$$\begin{aligned}
\|\nabla \mathbf{u}_j^n\|_{L^{\frac{4}{3}}(\dot{B}_{2,1}^{\frac{1}{2}})} &\le C\|\mathbf{u}_j^n\|_{L^{\frac{4}{3}}(\dot{B}_{2,1}^{\frac{3}{2}})} \le C\|\mathbf{u}_j^n\|^{\frac{1}{2}}_{L^2(\dot{B}_{2,1}^1)}\|\mathbf{u}_j^n\|^{\frac{1}{2}}_{L^1(\dot{B}_{2,1}^2)}\\
&\le C\|\mathbf{u}_j^n\|^{\frac{3}{4}}_{L^1(\dot{B}_{2,1}^2)}\|\mathbf{u}_j^n\|^{\frac{1}{4}}_{L^\infty(\dot{B}_{2,1}^0)},\\
\|\Lambda^3 h_j^n\|_{L^{\frac{4}{3}}(\dot{B}_{2,1}^{-\frac{1}{2}})} &\le C\|h_j^n\|_{L^{\frac{4}{3}}(\dot{B}_{2,1}^{\frac{5}{2}})} \le C\|h_j^n\|^{\frac{3}{4}}_{L^1(\dot{B}_{2,1}^3)}\|h_j^n\|^{\frac{1}{4}}_{L^\infty(\dot{B}_{2,1}^1)}\\
&\le C\|h_j^n\|^{\frac{3}{4}}_{L^1(\tilde{B}_{2,1}^{2,3})}\|h_j^n\|^{\frac{1}{4}}_{L^\infty(\tilde{B}_{2,1}^{0,1})},\\
\|\Lambda h_j^n\|_{L^4(\dot{B}_{2,1}^{-\frac{1}{2}})} &\le C\|h_j^n\|_{L^4(\dot{B}_{2,1}^{\frac{1}{2}})} \le C\|h_j^n\|^{\frac{1}{2}}_{L^2(\dot{B}_{2,1}^1)}\|h_j^n\|^{\frac{1}{2}}_{L^\infty(\dot{B}_{2,1}^0)}\\
&\le C\|h_j^n\|^{\frac{1}{4}}_{L^1(\tilde{B}_{2,1}^{2,3})}\|h_j^n\|^{\frac{3}{4}}_{L^\infty(\tilde{B}_{2,1}^{0,1})}.
\end{aligned}$$

Other terms in (4.3) could be verified similarly. The lemma could be proved by collecting the estimates above and the uniform bound (4.5). □

Now we prove the existence of the solution of (1.3). We need the following Ascoli theorem:

Theorem 4.2 Let X be a compact metric space and Y a complete metric space. Let A be an equicontinuous part of $C(X, Y)$. Then we have the following two equivalent propositions:

1. A is relatively compact in $C(X, Y)$;
2. $A(x) = \{f(x) : f \in A\}$ is relatively compact in Y.

Let $\{\chi_p\}_{p\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^2)$ be cut-off functions such that $\text{supp}\chi_p \subset B(0, p+1)$, $\chi_p = 1$ in $B(0, p)$. Due to Lemma 4.1, we have that for any $p \in \mathbb{N}$, $\{(\chi_p h_j^n, \chi_p \mathbf{u}_j^n)\}_{n\in\mathbb{N}}$ is equicontinuous in $C(\mathbb{R}^+; \dot{B}_{2,1}^0 \times (\dot{B}_{2,1}^{-\frac{1}{2}})^2)$. Therefore, for any $\varepsilon > 0$, $\exists \delta > 0$ such that for any t_1, $t_2 \in \mathbb{R}^+$, and $|t_1 - t_2| < \delta$, we have

$$\sup_{n\in\mathbb{N}} \|\chi_p h_j^n(t_1) - \chi_p h_j^n(t_2)\|_{\dot{H}^0} \le \sup_{n\in\mathbb{N}} \|\chi_p h_j^n(t_1) - \chi_p h_j^n(t_2)\|_{\dot{B}_{2,1}^0} \le \varepsilon,$$

where in the first inequality we have used the fact that $\|u\|_{\dot{H}^0} = \|u\|_{\dot{B}_{2,2}^0} \le \|u\|_{\dot{B}_{2,1}^0}$, and in the second inequality we have used the equicontinuity of $\{\chi_p h_j^n\}_{n\in\mathbb{N}}$ in $C(\mathbb{R}^+; \dot{B}_{2,1}^0)$. Therefore, we obtain that $\{\chi_p h_j^n\}_{n\in\mathbb{N}}$ is equicontinuous in $C(\mathbb{R}^+; \dot{H}^0)$, and hence equicontinuous in $C([0, p]; \dot{H}^0)$. In a similar way we could also show that $\{\chi_p \mathbf{u}_j^n\}_{n\in\mathbb{N}}$ is equicontinuous in $C([0, p]; \dot{H}^{-\frac{1}{2}})$.

On the other hand, according to (4.5), Lemma 4.1 and Proposition 2.5 we could obtain that $\{(h_j^n(t), \mathbf{u}_j^n(t))\}_{n\in\mathbb{N}}$ is uniformly bounded in $\tilde{B}_{2,1}^{0,1} \times \tilde{B}_{2,1}^{-\frac{1}{2},0}$. Since the application $u \to \chi_p u$ is compact from $\tilde{B}_{2,1}^{0,1}$ into $\dot{H}^0$, and from $\tilde{B}_{2,1}^{-\frac{1}{2},0}$ into $\dot{H}^{-\frac{1}{2}}$, by Theorem 4.2 we know that for any $p \in \mathbb{N}$, the sequence $\{(\chi_p h_j^n, \chi_p \mathbf{u}_j^n)\}_{n\in\mathbb{N}}$ is compact in $C([0, p], \dot{H}^0 \times (\dot{H}^{-\frac{1}{2}})^2)$. By a standard diagonal process, we obtain a distribution $(h_j, \mathbf{u}_j) \in C(\mathbb{R}^+, \dot{H}^0 \times (\dot{H}^{-\frac{1}{2}})^2)$ and a subsequence which is still labeled by $\{(h_j^n, \mathbf{u}_j^n)\}_{n\in\mathbb{N}}$ such that, for any $p \in \mathbb{N}$, we have

$$(\chi_p h_j^n, \chi_p \mathbf{u}_j^n) \to (\chi_p h_j, \chi_p \mathbf{u}_j) \text{ in } C([0, p], \dot{H}^0 \times (\dot{H}^{-\frac{1}{2}})^2), \text{ as } n \to +\infty.$$

Hence $(h_j^n, \mathbf{u}_j^n)$ tends to $(h_j, \mathbf{u}_j)$ in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^2)$.

The next part of the discussion is to show that $(h_1, \mathbf{u}_1, h_2, \mathbf{u}_2)$ is a solution of (1.3). First, due to the property of J_n, $(h_{j0}^n, \mathbf{u}_{j0}^n)$ tends to $(h_{j0}, \mathbf{u}_{j0})$ in $\dot{B}_{2,1}^0 \times (\dot{B}_{2,1}^{-\frac{1}{2}})^2$, where $\mathbf{u}_{j0}^n = J_n \mathbf{u}_{j0}$. To show that the limit $(h_1, \mathbf{u}_1, h_2, \mathbf{u}_2)$ solves (1.3), we should pass to the limit of the nonlinear terms of (4.3) in $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^2)$. We only prove that

$$J_n\frac{\nabla h_j^n\cdot\mathrm{div}\mathbf{u}_j^n}{h_j^n+1}\to\frac{\nabla h_j\cdot\mathrm{div}\mathbf{u}_j}{h_j+1},\ \text{in }\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^2),\ \text{as } n\to+\infty, \tag{4.6}$$

for other terms could be verified similarly. Indeed,

$$\begin{aligned}&J_n\frac{\nabla h_j^n\cdot\mathrm{div}\mathbf{u}_j^n}{h_j^n+1}-\frac{\nabla h_j\cdot\mathrm{div}\mathbf{u}_j}{h_j+1}\\
&=J_n\Big(\frac{\nabla h_j^n}{h_j^n+1}\mathrm{div}(\mathbf{u}_j^n-\mathbf{u}_j)\Big)+J_n\Big(\frac{1}{h_j^n+1}\nabla(h_j^n-h_j)\mathrm{div}\mathbf{u}_j\Big)\\
&\quad+J_n\Big(\Big(\frac{1}{h_j^n+1}-\frac{1}{h_j+1}\Big)\nabla h_j\mathrm{div}\mathbf{u}_j\Big)+(J_n-Id)\frac{\nabla h_j\cdot\mathrm{div}\mathbf{u}_j}{h_j+1}\\
&\triangleq I_1+I_2+I_3+I_4.\end{aligned}$$

—For any $\psi\in C_0^\infty(\mathbb{R}^+\times\mathbb{R}^2)$, we take $\varphi\in C_0^\infty(\mathbb{R}^+\times\mathbb{R}^2)$, $\varphi=1$ on a neighborhood of suppψ. We then have, for n large enough, that

$$\begin{aligned}\langle I_1,\psi\rangle&=\Big\langle\frac{\nabla h_j^n}{h_j^n+1}\mathrm{div}(\mathbf{u}_j^n-\mathbf{u}_j),J_n\psi\Big\rangle=\Big\langle\frac{\nabla h_j^n}{h_j^n+1}\mathrm{div}(\varphi\mathbf{u}_j^n-\varphi\mathbf{u}_j),J_n\psi\Big\rangle\\
&\le C\Big\|\frac{\nabla h_j^n}{h_j^n+1}\mathrm{div}(\varphi\mathbf{u}_j^n-\varphi\mathbf{u}_j)\Big\|_{L^1(\dot H^{-\frac12})}\|J_n\psi\|_{L^\infty(\dot H^{\frac12})}\\
&\le C\Big\|\frac{\nabla h_j^n}{h_j^n+1}\Big\|_{L^\infty(\dot H^0)}\|\mathrm{div}(\varphi\mathbf{u}_j^n-\varphi\mathbf{u}_j)\|_{L^1(\dot H^{\frac12})}\|J_n\psi\|_{L^\infty(\dot H^{\frac12})}\\
&\le C\|h_j^n\|_{L^\infty(\tilde B^{0,1}_{2,1})}(1+C\|h_j^n\|_{L^\infty(\tilde B^{0,1}_{2,1})})\|\varphi(\mathbf{u}_j^n-\mathbf{u}_j)\|_{L^1(\dot H^{\frac32})}\|J_n\psi\|_{L^\infty(\dot H^{\frac12})}\\
&\le C\|h_j^n\|_{L^\infty(\tilde B^{0,1}_{2,1})}(1+C\|h_j^n\|_{L^\infty(\tilde B^{0,1}_{2,1})})\|\varphi(\mathbf{u}_j^n-\mathbf{u}_j)\|^{\frac15}_{L^1(\dot H^{-\frac12})}\\
&\quad\cdot\|\varphi(\mathbf{u}_j^n-\mathbf{u}_j)\|^{\frac45}_{L^1(\dot H^2)}\|J_n\psi\|_{L^\infty(\dot H^{\frac12})}\\
&\le C\|h_j^n\|_{L^\infty(\tilde B^{0,1}_{2,1})}(1+C\|h_j^n\|_{L^\infty(\tilde B^{0,1}_{2,1})})\|\varphi(\mathbf{u}_j^n-\mathbf{u}_j)\|^{\frac15}_{L^\infty(\dot H^{-\frac12})}\\
&\quad\cdot\|\varphi(\mathbf{u}_j^n-\mathbf{u}_j)\|^{\frac45}_{L^1(\dot B^2_{2,1})}\|J_n\psi\|_{L^\infty(\dot H^{\frac12})},\end{aligned}$$

$$\begin{aligned}\langle I_2,\psi\rangle&=\Big\langle\frac{1}{h_j^n+1}\nabla(h_j^n-h_j)\mathrm{div}\mathbf{u}_j,J_n\psi\Big\rangle=-\Big\langle\varphi(h_j^n-h_j),\nabla\Big(\frac{1}{h_j^n+1}\mathrm{div}\mathbf{u}_jJ_n\psi\Big)\Big\rangle\\
&\le C\|\varphi(h_j^n-h_j)\|_{L^\infty(\dot H^0)}\Big\|\frac{1}{h_j^n+1}\mathrm{div}\mathbf{u}_jJ_n\psi\Big\|_{L^1(\dot H^1)}\\
&\le C\|\varphi(h_j^n-h_j)\|_{L^\infty(\dot H^0)}\Big\|\frac{1}{h_j^n+1}\Big\|_{L^\infty(\dot H^1)}\|\mathrm{div}\mathbf{u}_jJ_n\psi\|_{L^1(\dot H^1)}\\
&\le C\|\varphi(h_j^n-h_j)\|_{L^\infty(\dot H^0)}\Big\|\frac{1}{h_j^n+1}\Big\|_{L^\infty(\dot H^1)}\|\mathrm{div}\mathbf{u}_j\|_{L^1(\dot H^1)}\|J_n\psi\|_{L^\infty(\dot H^1)}\\
&\le C\|\varphi(h_j^n-h_j)\|_{L^\infty(\dot H^0)}\Big(\|h_j^n\|_{L^\infty(\dot H^1)}+\Big\|\frac{(h_j^n)^2}{h_j^n+1}\Big\|_{L^\infty(\dot H^1)}\Big)\|\mathbf{u}_j\|_{L^1(\dot H^2)}\|J_n\psi\|_{L^\infty(\dot H^1)}\\
&\le C\|\varphi(h_j^n-h_j)\|_{L^\infty(\dot H^0)}\|h_j^n\|_{L^\infty(\tilde B^{0,1}_{2,1})}\big(1+C\|h_j^n\|_{L^\infty(\tilde B^{0,1}_{2,1})}\big)\|\mathbf{u}_j\|_{L^1(\dot B^2_{2,1})}\|J_n\psi\|_{L^\infty(\dot H^1)},\end{aligned}$$

$$\langle I_3,\psi\rangle=\Big\langle\Big(\frac{1}{h_j^n+1}-\frac{1}{h_j+1}\Big)\nabla h_j\mathrm{div}\mathbf{u}_j,J_n\psi\Big\rangle=\Big\langle\frac{\varphi(h_j-h_j^n)}{(h_j^n+1)(h_j+1)}\nabla h_j\mathrm{div}\mathbf{u}_j,J_n\psi\Big\rangle$$

$$
\begin{aligned}
&\le C\Big\|\frac{1}{(h_j^n+1)(h_j+1)}\Big\|_{L^\infty(\dot H^1)}\|\varphi(h_j-h_j^n)\|_{L^\infty(\dot H^0)}\|\nabla h_j\cdot \mathrm{div}\mathbf{u}_j\|_{L^1(\dot H^0)}\|J_n\psi\|_{L^\infty(L^\infty)}\\
&\le C\Big\|\frac{1}{h_j^n+1}\Big\|_{L^\infty(\dot H^1)}\Big\|\frac{1}{h_j+1}\Big\|_{L^\infty(\dot H^1)}\|\varphi(h_j-h_j^n)\|_{L^\infty(\dot H^0)}\\
&\quad\cdot\|\nabla h_j\|_{L^\infty(\dot H^0)}\|\mathrm{div}\mathbf{u}_j\|_{L^1(\dot H^1)}\|J_n\psi\|_{L^\infty(L^\infty)}\\
&\le C\|h_j^n\|_{L^\infty(\dot H^1)}(1+C\|h_j^n\|_{L^\infty(\dot H^1)})\|h_j\|_{L^\infty(\dot H^1)}(1+C\|h_j\|_{L^\infty(\dot H^1)})\\
&\quad\cdot\|\varphi(h_j-h_j^n)\|_{L^\infty(\dot H^0)}\|h_j\|_{L^\infty(\dot H^1)}\|\mathbf{u}_j\|_{L^1(\dot H^2)}\|J_n\psi\|_{L^\infty(L^\infty)}\\
&\le C\|h_j^n\|_{L^\infty(\tilde B^{0,1}_{2,1})}(1+C\|h_j^n\|_{L^\infty(\tilde B^{0,1}_{2,1})})\|h_j\|_{L^\infty(\tilde B^{0,1}_{2,1})}(1+C\|h_j\|_{L^\infty(\tilde B^{0,1}_{2,1})})\\
&\quad\cdot\|\varphi(h_j-h_j^n)\|_{L^\infty(\dot H^0)}\|h_j\|_{L^\infty(\tilde B^{0,1}_{2,1})}\|\mathbf{u}_j\|_{L^1(\dot B^2_{2,1})}\|J_n\psi\|_{L^\infty(L^\infty)}.
\end{aligned}
$$

The convergence of $\langle I_4,\psi\rangle$ is just a consequence of the property of J_n. Combining the above estimates with (4.5), (4.6) can be proved.

Hence, we have found a global solution $(h_1,\mathbf{u}_1,h_2,\mathbf{u}_2)$ of (1.3) which obviously satisfies the estimate

$$E(h_1,\mathbf{u}_1,h_2,\mathbf{u}_2,+\infty)\le \frac{5}{2}AE(0)<+\infty.$$

4.4 Uniqueness of the solution

To finish the proof of Theorem 1.1, it suffices to prove the uniqueness of the solution. Assume that $(\acute h_1,\acute{\mathbf{u}}_1,\acute h_2,\acute{\mathbf{u}}_2)\in E_1$ and $(\grave h_1,\grave{\mathbf{u}}_1,\grave h_2,\grave{\mathbf{u}}_2)\in E_1$ are two different solutions of (1.3), and set

$$(h_1,\mathbf{u}_1,h_2,\mathbf{u}_2)=(\acute h_1,\acute{\mathbf{u}}_1,\acute h_2,\acute{\mathbf{u}}_2)-(\grave h_1,\grave{\mathbf{u}}_1,\grave h_2,\grave{\mathbf{u}}_2).$$

Then $(h_1,\mathbf{u}_1,h_2,\mathbf{u}_2)$ solves the following equations:

$$
\begin{cases}
h_{1t}+\acute{\mathbf{u}}_1\cdot\nabla h_1+\mathrm{div}\mathbf{u}_1=\dot F_1,\\
\mathbf{u}_{1t}-2\mu_1\triangle\mathbf{u}_1+\acute{\mathbf{u}}_1\cdot\nabla\mathbf{u}_1+g\nabla h_1+\gamma g\nabla h_2-\alpha_1\nabla(\triangle h_1)-\alpha_2\nabla(\triangle h_2)=\dot G_1,\\
h_{2t}+\acute{\mathbf{u}}_2\cdot\nabla h_2+\mathrm{div}\mathbf{u}_2=\dot F_2,\\
\mathbf{u}_{2t}-2\mu_2\triangle\mathbf{u}_2+\acute{\mathbf{u}}_2\cdot\nabla\mathbf{u}_2+g\nabla h_1+g\nabla h_2-\alpha_3\nabla(\triangle h_1)-\alpha_3\nabla(\triangle h_2)=\dot G_2,\\
(h_1,\mathbf{u}_1,h_2,\mathbf{u}_2)|_{t=0}=\mathbf{0},
\end{cases}
$$

where

$$
\begin{aligned}
&\dot F_j=-\grave h_j\mathrm{div}\mathbf{u}_j-\mathbf{u}\cdot\nabla\grave h_j-h_j\mathrm{div}\acute{\mathbf{u}}_j,\\
&\dot G_j=-\mathbf{u}_j\cdot\nabla\grave{\mathbf{u}}_j+2\mu_j\Big(\frac{\nabla\acute h_j\cdot\nabla\mathbf{u}_j}{\acute h_j+1}+\frac{\nabla h_j\cdot\nabla\grave{\mathbf{u}}_j}{\acute h_j+1}\Big)+2\mu_j\Big(\frac{1}{\acute h_j+1}-\frac{1}{\grave h_j+1}\Big)\nabla\grave h_j\cdot\nabla\grave{\mathbf{u}}_j.
\end{aligned}
$$

Due to Proposition 3.1, we have

$$
\begin{aligned}
\bar E&:=E(h_1,\mathbf{u}_1,h_2,\mathbf{u}_2,+\infty)\\
&\le A\mathrm{e}^{K\|(\acute{\mathbf{u}}_1,\acute{\mathbf{u}}_2)\|_{(L^1(\dot B^2_{2,1}))^2}}\Big(\|(\dot F_1,\dot G_1,\dot F_2,\dot G_2)\|_{(L^1(\tilde B^{0,1}_{2,1})\times(L^1(\dot B^0_{2,1}))^2)^2}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\acute E&:=E(\acute h_1,\acute{\mathbf{u}}_1,\acute h_2,\acute{\mathbf{u}}_2,+\infty)\le ME(0),\\
\grave E&:=E(\grave h_1,\grave{\mathbf{u}}_1,\grave h_2,\grave{\mathbf{u}}_2,+\infty)\le ME(0).
\end{aligned}
$$

Similarly to the previous subsections, we have

$$\begin{aligned}
\|\dot{F}_j\|_{L^1(\dot{B}^0_{2,1})} &\le C\Big(\|\grave{h}_j\|_{L^\infty(\dot{B}^0_{2,1})}\|\mathbf{u}_j\|_{(L^1(\dot{B}^2_{2,1}))^2} + \|\mathbf{u}_j\|_{(L^\infty(\dot{B}^0_{2,1}))^2}\|\grave{h}_j\|_{L^1(\dot{B}^2_{2,1})} \\
&\quad + \|h_j\|_{L^\infty(\dot{B}^0_{2,1})}\|\acute{\mathbf{u}}_j\|_{(L^1(\dot{B}^2_{2,1}))^2}\Big) \\
&\le C\bar{E}(\acute{E}+\grave{E}) \le C\bar{E}E(0), \\
\|\dot{F}_j\|_{L^1(\dot{B}^1_{2,1})} &\le C\Big(\|\grave{h}_j\|_{L^\infty(\dot{B}^1_{2,1})}\|\mathbf{u}_j\|_{(L^1(\dot{B}^2_{2,1}))^2} + \|\mathbf{u}_j\|_{(L^2(\dot{B}^1_{2,1}))^2}\|\grave{h}_j\|_{L^2(\dot{B}^2_{2,1})} \\
&\quad + \|h_j\|_{L^\infty(\dot{B}^1_{2,1})}\|\acute{\mathbf{u}}_j\|_{(L^1(\dot{B}^2_{2,1}))^2}\Big) \\
&\le C\Big(\|\grave{h}_j\|_{L^\infty(\tilde{B}^{0,1}_{2,1})}\|\mathbf{u}_j\|_{(L^1(\dot{B}^2_{2,1}))^2} + \|\mathbf{u}_j\|^{\frac12}_{(L^1(\dot{B}^2_{2,1}))^2}\|\mathbf{u}_j\|^{\frac12}_{(L^\infty(\dot{B}^0_{2,1}))^2} \\
&\quad \cdot \|\grave{h}_j\|^{\frac12}_{L^1(\tilde{B}^{2,3}_{2,1})}\|\grave{h}_j\|^{\frac12}_{L^\infty(\tilde{B}^{0,1}_{2,1})} + \|h_j\|_{L^\infty(\tilde{B}^{0,1}_{2,1})}\|\acute{\mathbf{u}}_j\|_{(L^1(\dot{B}^2_{2,1}))^2}\Big) \\
&\le C\bar{E}(\acute{E}+\grave{E}) \le C\bar{E}E(0), \\
\|\dot{G}_j\|_{(L^1(\dot{B}^0_{2,1}))^2} &\le C\bigg(\|\mathbf{u}_j\|_{(L^\infty(\dot{B}^0_{2,1}))^2}\|\grave{\mathbf{u}}_j\|_{(L^1(\dot{B}^2_{2,1}))^2} + \Big\|\frac{\nabla\acute{h}_j}{\acute{h}_j+1}\Big\|_{(L^\infty(\dot{B}^0_{2,1}))^2}\|\mathbf{u}_j\|_{(L^1(\dot{B}^2_{2,1}))^2} \\
&\quad + \Big\|\frac{\nabla h_j}{\acute{h}_j+1}\Big\|_{(L^\infty(\dot{B}^0_{2,1}))^2}\|\grave{\mathbf{u}}_j\|_{(L^1(\dot{B}^2_{2,1}))^2} \\
&\quad + \Big\|\frac{1}{\acute{h}_j+1}-\frac{1}{\grave{h}_j+1}\Big\|_{L^\infty(\dot{B}^1_{2,1})}\|\nabla\grave{h}_j\cdot\nabla\grave{\mathbf{u}}_j\|_{(L^1(\dot{B}^0_{2,1}))^2}\bigg) \\
&\le C\bigg(\|\mathbf{u}_j\|_{(L^\infty(\dot{B}^0_{2,1}))^2}\|\grave{\mathbf{u}}_j\|_{(L^1(\dot{B}^2_{2,1}))^2} + \|\acute{h}_j\|_{L^\infty(\tilde{B}^{0,1}_{2,1})}(1+C\|\acute{h}_j\mid_{L^\infty(\tilde{B}^{0,1}_{2,1})}) \\
&\quad \cdot\|\mathbf{u}_j\|_{(L^1(\dot{B}^2_{2,1}))^2} + \|h_j\|_{L^\infty(\tilde{B}^{0,1}_{2,1})}(1+C\|\acute{h}_j\|_{L^\infty(\tilde{B}^{0,1}_{2,1})})\|\grave{\mathbf{u}}_j\|_{(L^1(\dot{B}^2_{2,1}))^2} \\
&\quad + \bigg(\Big\|\Big(\frac{1}{\acute{h}_j+1}+\acute{h}_j\Big)-\Big(\frac{1}{\grave{h}_j+1}+\grave{h}_j\Big)\Big\|_{L^\infty(\dot{B}^1_{2,1})} + \|h_j\|_{L^\infty(\dot{B}^1_{2,1})}\bigg) \\
&\quad \cdot\|\grave{h}_j\|_{L^\infty(\dot{B}^1_{2,1})}\|\grave{\mathbf{u}}_j\|_{(L^1(\dot{B}^2_{2,1}))^2}\bigg),
\end{aligned}$$

where

$$\begin{aligned}
&\Big\|\Big(\frac{1}{\acute{h}_j+1}+\acute{h}_j\Big)-\Big(\frac{1}{\grave{h}_j+1}+\grave{h}_j\Big)\Big\|_{L^\infty(\dot{B}^1_{2,1})} \\
&\le C\Big(\|\acute{h}_j\|_{L^\infty(\dot{B}^1_{2,1})}+\|\grave{h}_j\|_{L^\infty(\dot{B}^1_{2,1})}\Big)\|h_j\|_{L^\infty(\dot{B}^1_{2,1})},
\end{aligned}$$

and hence

$$\begin{aligned}
\|\dot{G}_j\|_{(L^1(\dot{B}^0_{2,1}))^2} &\le C\bar{E}\Big((\acute{E}+\grave{E})+(\acute{E}+\grave{E})^2+(\acute{E}+\grave{E})^3\Big) \\
&\le C\bar{E}\Big(E(0)+E(0)^2+E(0)^3\Big).
\end{aligned}$$

Collecting the above estimates, we obtain

$$\bar{E} \le C\mathrm{e}^{CE(0)}\bar{E}\Big(E(0)+E(0)^2+E(0)^3\Big),$$

where the constant C is independent of the initial data. Hence, when $E(0)$ is small enough, we have $\bar{E}=0$, which means that $(h_1,\mathbf{u}_1,h_2,\mathbf{u}_2)=\mathbf{0}$. Therefore,

$$(\acute{h}_1,\acute{\mathbf{u}}_1,\acute{h}_2,\acute{\mathbf{u}}_2)=(\grave{h}_1,\grave{\mathbf{u}}_1,\grave{h}_2,\grave{\mathbf{u}}_2),$$

and hence the solution of (1.3) is unique.

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Appendix

This section is devoted to proving a proposition which is a supplement of Lemma 6.2 in [7] and has been used in Section 3. We need the following Bony's decomposition (modulo a polynomial):

$$uv = T_u v + T_v u + R(u,v),$$

where

$$T_u v = \sum_{k\in\mathbb{Z}} S_{k-1}u\triangle_k v,$$
$$R(u,v) = \sum_{k\in\mathbb{Z}} \triangle_k u(\triangle_{k-1}v + \triangle_k v + \triangle_{k+1}v).$$

Proposition A.1 For any $-\frac{N}{2} < s \le \frac{N}{2}+1$, we have

$$\int \triangle_k(\mathbf{u}\cdot\nabla f)\triangle_k g \le C\varepsilon_k 2^{-ks}\|\mathbf{u}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\|f\|_{\dot{B}^s_{2,1}}\Big(\|\triangle_k g\|_0 + \|\triangle_k(\Lambda g)\|_0\Big), \tag{A.1}$$

where C is a constant independent of k, and $\sum\limits_{k\in\mathbb{Z}}\varepsilon_k \le 1$.

Proof Due to Bony's decomposition,

$$\int \triangle_k(\mathbf{u}\cdot\nabla f)\triangle_k g = \int \triangle_k(T_{\mathbf{u}}\nabla f + T_{\nabla f}\mathbf{u} + R(\nabla f,\mathbf{u}))\triangle_k g,$$

where

$$\begin{aligned}\int \triangle_k(T_{\mathbf{u}}\nabla f)\triangle_k g &= \sum_{|q-k|\le\tilde N}\int \triangle_k(S_{q-1}\mathbf{u}\triangle_q(\nabla f))\triangle_k g\\ &= \sum_{|q-k|\le\tilde N}\int ([\triangle_k, S_{q-1}\mathbf{u}]\triangle_q(\nabla f))\triangle_k g\\ &\quad + \sum_{|q-k|\le\tilde N}\int (S_{q-1}\mathbf{u} - S_{k-1}\mathbf{u})\triangle_k\triangle_q(\nabla f)\triangle_k g\\ &\quad + \sum_{|q-k|\le\tilde N}\int S_{k-1}\mathbf{u}\triangle_k\triangle_q(\nabla f)\triangle_k g\\ &\triangleq I_1 + I_2 + I_3\end{aligned}$$

for some integer $\tilde N$. For the estimate of I_1, we have

$$\begin{aligned}[\triangle_k, S_{q-1}\mathbf{u}]\triangle_q(\nabla f)(x) &= 2^{Nk}\int_{\mathbb{R}^N} h(2^k(x-y))(S_{q-1}\mathbf{u}(y) - S_{q-1}\mathbf{u}(x))\triangle_q(\nabla f)(y)\mathrm{d}y\\ &\le 2^{Nk}\|S_{q-1}(\nabla\mathbf{u})\|_{L^\infty}\int_{\mathbb{R}^N}|h(2^k(x-y))(x-y)\triangle_q(\nabla f)(y)|dy\\ &\le 2^{Nk}\|S_{q-1}(\nabla\mathbf{u})\|_{L^\infty}\Big(H*|\triangle_q(\nabla f)|(x)\Big),\end{aligned}$$

where $H(x) = |h(2^k x)||x|$. Hence,

$$\begin{aligned}\|[\triangle_k, S_{q-1}\mathbf{u}]\triangle_q(\nabla f)\|_0 &\le C2^{Nk}\|S_{q-1}(\nabla\mathbf{u})\|_{L^\infty}\|H\|_{L^1}\|\triangle_q(\nabla f)\|_0\\ &\le C2^{q-k}\|\nabla\mathbf{u}\|_{L^\infty}\|\triangle_q f\|_0\\ &\le C2^{q-k}2^{-qs}\varepsilon_q\|\mathbf{u}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\|f\|_{\dot{B}^s_{2,1}},\end{aligned}$$

and

$$\begin{aligned}I_1 &\le \sum_{|q-k|\le\tilde N}\|[\triangle_k, S_{q-1}\mathbf{u}]\triangle_q(\nabla f)\|_0\|\triangle_k g\|_0\\ &\le C2^{-ks}\varepsilon_k\|\mathbf{u}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\|f\|_{\dot{B}^s_{2,1}}\|\triangle_k g\|_0.\end{aligned}$$

To estimate I_2, noting that when $|q-k| \le \tilde{N}$, $S_{q-1}\mathbf{u} - S_{k-1}\mathbf{u}$ is supported in an annular like $\mathcal{C} = \{C_1 2^k \le |\xi| \le C_2 2^k\}$ for some positive constant C_1, C_2, we could easily pass the operator ∇ from f to $\mathbf{u}$ by Bernstein's inequalities. We have

$$\begin{aligned} I_2 &\le C \sum_{|q-k|\le \tilde{N}} \|S_{q-1}\mathbf{u} - S_{k-1}\mathbf{u}\|_{L^\infty} \|\triangle_k \triangle_q (\nabla f)\|_0 \|\triangle_k g\|_0 \\ &\le C \sum_{|q-k|\le \tilde{N}} 2^{q-k} \|\nabla \mathbf{u}\|_{L^\infty} \|\triangle_k f\|_0 \|\triangle_k g\|_0 \\ &\le C 2^{-ks} \varepsilon_k \|\mathbf{u}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \|f\|_{\dot{B}^s_{2,1}} \|\triangle_k g\|_0. \end{aligned}$$

For I_3, we have

$$\begin{aligned} I_3 &= \int S_{k-1}\mathbf{u} \triangle_k (\nabla f) \triangle_k g \le \|S_{k-1}\mathbf{u}\|_{L^\infty} \|\triangle_k (\nabla f)\|_0 \|\triangle_k g\|_0 \\ &\le C 2^{-ks} \varepsilon_k \|\mathbf{u}\|_{\dot{B}^{\frac{N}{2}+1}} \|f\|_{\dot{B}^s_{2,1}} \|\triangle_k (\Lambda g)\|_0. \end{aligned} \tag{A.2}$$

We emphasize that I_2 and I_3 could be estimated together by the method used in (A.2), however, we split them so as to see which one is responsible for the derivative term $\|\triangle_k(\Lambda g)\|_0$ in (A.1).

To estimate $\int \triangle_k (T_{\nabla f}\mathbf{u}) \triangle_k g$ and $\int \triangle_k (R(\nabla f, \mathbf{u})) \triangle_k g$, we apply Proposition 6.1 in [7] to obtain

$$\begin{aligned} \int \triangle_k (T_{\nabla f}\mathbf{u}) \triangle_k g &\le \|\triangle_k (T_{\nabla f}\mathbf{u})\|_0 \|\triangle_k g\|_0 \\ &\le C 2^{-ks} \varepsilon_k \|T_{\nabla f}\mathbf{u}\|_{\dot{B}^s_{2,1}} \|\triangle_k g\|_0 \\ &\le C 2^{-ks} \varepsilon_k \|\mathbf{u}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \|\nabla f\|_{\dot{B}^{s-1}_{2,1}} \|\triangle_k g\|_0 \\ &\le C 2^{-ks} \varepsilon_k \|\mathbf{u}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \|f\|_{\dot{B}^s_{2,1}} \|\triangle_k g\|_0, \end{aligned}$$

and similarly,

$$\int \triangle_k (R(\nabla f, \mathbf{u})) \triangle_k g \le C 2^{-ks} \varepsilon_k \|\mathbf{u}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \|f\|_{\dot{B}^s_{2,1}} \|\triangle_k g\|_0.$$

By collecting the estimates above, we obtain (A.1). □

Remark A.2 It is noted that there is a loss of one derivative in (A.1), due to the fact that the gradient operator ∇ could not be passed from f to $\mathbf{u}$ in I_3.