Acta Mathematica Scientia, 2020, **40B**(5): 1289–1315
https://doi.org/10.1007/s10473-020-0509-7

Mathematica Acta Scientia
数学物理学报
http://actams.wipm.ac.cn

ON SINGULAR EQUATIONS INVOLVING FRACTIONAL LAPLACIAN*

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Abstract We study the existence and the regularity of solutions for a class of nonlocal equations involving the fractional Laplacian operator with singular nonlinearity and Radon measure data.

Key words fractional Laplacian; singular elliptic equations; measure data

2010 MR Subject Classification 35R11; 35J62; 46E35; 35A15

1 Introduction

Lately, problems involving nonlocal operators and singular terms have recently received considerable attention in the literature. A good amount of investigations have focused on the existence and/or regularity of solutions to such problems governed by the fractional Laplacian with a singularity due to a negative power of the unknown or described by a potential, see for instance, [1, 2, 4, 7, 8, 12] and related papers.

A prototype of nonlocal operators is the fractional Laplacian operator of the form $(-\Delta)^s$, $0 < s < 1$, which is actually the infinitesimal generator of the radially symmetric and s-stable Lévy processes [6]. Fractional Laplacian operators naturally arise from a wide range of applications. They appear, for instance, in thin obstacle problems [14], crystal dislocation [18], phase transition [30] and others.

In this paper, we are interested in the existence and regularity of solutions to the following

*Received February 18, 2019; revised September 7, 2019.
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Dirichlet problem

$$\begin{cases}(-\Delta)^s u = \dfrac{f(x)}{u^\gamma} + \mu & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \mathrm{R}^N \backslash \Omega,\end{cases} \tag{1.1}$$

where Ω is an open bounded subset in R^N, $N > 2s$, of class $\mathcal{C}^{0,1}$, $s \in (0,1)$, $\gamma > 0$, f is a non-negative function on Ω, μ is a non-negative bounded Radon measure on Ω and $(-\Delta)^s$ is the fractional Laplacian operator of order $2s$ defined by

$$(-\Delta)^s u = \alpha(N,s)\mathrm{P.V.}\int_{\mathrm{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y,$$

where "P.V." stands for the integral in the principal value sense and $\alpha(N,s)$ is a positive renormalizing constant, depending only on N and s, given by

$$\alpha(N,s) = \frac{4^s\Gamma(\frac{N}{2}+s)}{\pi^{\frac{N}{2}}}\frac{s}{\Gamma(1-s)}$$

so that the identity $(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u)$, $\xi \in \mathrm{R}^N, s \in (0,1)$ and $u \in \mathcal{S}(\mathrm{R}^N)$ holds, where $\mathcal{F}u$ stands for the Fourier transform of u belonging to the Schwartz class $\mathcal{S}(\mathrm{R}^N)$ (cf. [23]). More details on the operator $(-\Delta)^s$ and the asymptotic behaviour of $\alpha(N,s)$ can be found in [17].

The case $s = 1$ corresponds to the classical Laplacian operator. If further $\mu = 0$, an important result is due to Lazer-McKenna [24]. Under regularity assumptions on Ω and f, the authors present an obstruction to the existence of an energy solution. In fact, such a solution lying in $H_0^1(\Omega)$ should exists if and only if $\gamma < 3$ while it is not in $\mathcal{C}^1(\overline{\Omega})$ if $\gamma > 1$. As far as problem with L^1-data are concerned, the threshold 3 essentially due to the boundedness of the datum was sharpened in [32] while in [11] the existence of a distributional solution u is proved. In fact, it is proved in [11] that if $\gamma < 1$ and $f \in L^m(\Omega)$, $1 \le m < \left(\frac{2^*}{1-\gamma}\right)'$, then $u \in W_0^{1,q}(\Omega)$ where $q = \frac{Nm(\gamma+1)}{N-m(1-\gamma)}$ while $u \in H_0^1(\Omega)$ if $f \in L^m(\Omega)$ with $m = \left(\frac{2^*}{1-\gamma}\right)'$. In the case where $f \in L^1(\Omega)$, if $\gamma = 1$ then $u \in H_0^1(\Omega)$; while $u \in H_{\mathrm{loc}}^1(\Omega)$ if $\gamma > 1$. We note that in the latter case, the boundary datum is only assumed in a weaker sense than the usual one of traces, that is $u^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$. Let us point out here that solutions with infinite energy may exist if $\gamma > 1$ even for smooth data ([24]).

The nonhomogeneous case (i.e., $\mu \neq 0$) has been considered. In [26] the authors studied the existence of weak solutions for the problem

$$-\Delta u = \frac{f(x)}{u^\gamma} + \mu, \tag{1.2}$$

where $f \in L^1(\Omega)$ and μ is a bounded Radon measure. They prove the existence of a weak solution u of the problem (1.2) such that $u \in W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ when $\gamma \le 1$ while if $\gamma > 1$, $u \in W_{\mathrm{loc}}^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ with the regularity $\left(T_k(u)\right)^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$, T_k being the truncation function at levels $\pm k$. Other related singular equations can be found for instance in [13, 15, 21, 27, 31].

Regarding nonlocal problems, the study of (1.2) with $\mu = 0$ was extended in [7, 12] where the Laplacian is substituted by the fractional Laplacian $(-\Delta)_p^s$, $0 < s < 1$ and $p > 1$. The authors obtain some existence and regularity results for the solutions depending on the summability of the datum f and γ (splitting in the cases $\gamma < 1$, $\gamma = 1$, $\gamma > 1$). Some fractional equations with measure data are studied in [5, 20, 28].

It is our purpose in this paper, to consider the problem (1.1) in the nonlocal framework and prove existence results of solutions to problem (1.1) with μ a bounded Radon measure and data $f \in L^1(\Omega)$. We use an approximation method that consists in analyzing the sequence of approximated problems truncating the datum f and the singular term $\frac{1}{u^\gamma}$ and approximating μ by smooth functions, obtaining non singular problems with L^∞-data whose approximated solutions u_n can be obtained by a direct application of the Schauder fixed point theorem. We faced many difficulties in dealing with the nonlocal problem (1.1), but the main one is how to get estimations in appropriate fractional Sobolev spaces.

Observe that in the local setting, if the approximated solutions are such that the sequence $\{\nabla u_n\}_n$ is uniformly bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{N}{N-1}}(\Omega)$, then we conclude that the sequence $\{u_n\}_n$ is uniformly bounded in the Sobolev spaces $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ (see [9]).

However, we underline here that given the fractional structure of the operator of the principal part, we can not retrieve the gradient of the approximate solutions and so appears the problem of getting a priori estimates in some fractional Sobolev spaces. To overcome this difficulty, we first prove the key result Lemma 4.1 (Section 4) and use suitable test functions and algebraic inequalities that enable us to get appropriate a priori estimates in both cases $\gamma \leq 1$ and $\gamma > 1$.

The paper is organized as follows: in Section 2 we give some basic notations and necessary results that we will use in the accomplishment of the paper. We also give the main results. In Section 3, we construct a series of approximate problems to which we show the existence and uniqueness of the approximate solution. In Section 4, we prove some a priori estimates of the approximate solutions in fractional Sobolev spaces. Section 5, is devoted to the proof of the main results (Theorems 2.7 and 2.8). While in Section 6 we give a regularity result. Finally, in Appendix we expose and prove the technical and functional results that we used in the previous sections.

2 Some Useful Notations and Main Results

In this section we provide some basic facts about fractional Sobolev spaces. We refer to [10, 16, 17, 29] for more details. Let Ω be an open subset in R^N and let $\mathcal{C}\Omega := \mathrm{R}^N \backslash \Omega$. For any $0 < s < 1$ and for any $1 \leq q < +\infty$, the fractional Sobolev space $W^{s,q}(\Omega)$ is defined as the set of all functions (equivalence class) u in $L^q(\Omega)$ such that

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^q}{|x - y|^{N+qs}} \mathrm{d}x\mathrm{d}y < \infty.$$

$W^{s,q}(\Omega)$, also known as Aronszajn, Gagliardo or Slobodeckij spaces, is a Banach space when equipped with the natural norm

$$\|u\|_{W^{s,q}(\Omega)} = \left(\|u\|_{L^q(\Omega)}^q + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^q}{|x - y|^{N+qs}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{q}}. \tag{2.1}$$

It can be regarded as an intermediate space between $L^q(\Omega)$ and $W^{1,q}(\Omega)$. Recall that the space $W^{s,q}(\Omega)$ is reflexive for all $q > 1$ (see [22, Theorem 6.8.4]). We point out that if $0 < s \leq s' < 1$

then $W^{s',q}(\Omega)$ is continuously embedded in $W^{s,q}(\Omega)$ (see [17, Proposition 2.1]). Let us define $W_0^{s,q}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{s,q}(\Omega)$ with respect to the norm defined in (2.1) where

$$\mathcal{C}_0^\infty(\Omega) = \Big\{ f : \mathrm{R}^N \to \mathrm{R} / f \in \mathcal{C}^\infty(\mathrm{R}^N), \mathrm{Supp}\, f \text{ is compact and } \mathrm{Supp}\, f \subset \Omega \Big\}.$$

Here and in the sequel Supp f stands for the support of the function f. $W_0^{s,q}(\Omega)$ is a Banach space under the norm $\|u\|_{W^{s,q}(\Omega)}$.

If Ω is bounded and is of class $\mathcal{C}^{0,1}$, we can give a fractional version of the Poincaré inequality in $W_0^{s,q}(\Omega)$, $1 \le q < +\infty$, whose proof in the case where $q = 2$ can be found in [3]. For the convenience of the reader, we are giving the proof here.

Lemma 2.1 (fractional Poincaré-type inequality) Let Ω be a bounded open subset of R^N of class $\mathcal{C}^{0,1}$, $1 \le q < +\infty$ and let $0 < s < 1$. Then there exists a constant $C(N, s, \Omega)$ such that for any $\varphi \in W_0^{s,q}(\Omega)$ one has

$$\|\varphi\|^q_{L^q(\Omega)} \le C(N, s, \Omega) \int_\Omega \int_\Omega \frac{|\varphi(x) - \varphi(y)|^q}{|x-y|^{N+qs}} \mathrm{d}x\mathrm{d}y.$$

Proof Let $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Observe first that the above inequality holds if $\varphi = 0$. Assume that $\varphi \ne 0$ and set

$$\lambda(\Omega) = \inf_{\{\varphi \in \mathcal{C}_0^\infty(\Omega), \varphi \ne 0\}} \frac{\int_\Omega \int_\Omega \frac{|\varphi(x)-\varphi(y)|^q}{|x-y|^{N+qs}} \mathrm{d}x\mathrm{d}y}{\int_\Omega |\varphi(x)|^q \mathrm{d}x}.$$

We shall prove that $\lambda(\Omega) > 0$. To do so, we argue by contradiction assuming that $\lambda(\Omega) = 0$. Thus, there exists a sequence $\{\varphi_n\}$ of $\mathcal{C}_0^\infty(\Omega)$ such that

$$\int_\Omega |\varphi_n(x)|^q \mathrm{d}x = 1 \quad \text{and} \quad \int_\Omega \int_\Omega \frac{|\varphi_n(x) - \varphi_n(y)|^q}{|x-y|^{N+qs}} \mathrm{d}x\mathrm{d}y \to 0 \text{ as } n \to \infty.$$

It follows that

$$\|\varphi_n\|_{W^{s,q}(\Omega)} \le C.$$

By virtue of [17, Corollary 7.2], there exists a function f and a subsequence of $\{\varphi_n\}$, still indexed by n, such that

$$\varphi_n \to f \text{ in norm in } L^q(\Omega),$$
$$\varphi_n \to f \text{ a.e. in } \Omega.$$

Therefore,

$$\int_\Omega |f(x)|^q \mathrm{d}x = 1 \text{ and } \frac{|\varphi_n(x) - \varphi_n(y)|^q}{|x-y|^{N+qs}} \to \frac{|f(x) - f(y)|^q}{|x-y|^{N+qs}} \text{ a.e. in } \Omega \times \Omega.$$

Applying Fatou's lemma, we get

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^q}{|x-y|^{N+qs}} \mathrm{d}x\mathrm{d}y \le \liminf_{n\to\infty} \int_\Omega \int_\Omega \frac{|\varphi_n(x) - \varphi_n(y)|^q}{|x-y|^{N+qs}} \mathrm{d}x\mathrm{d}y \to 0$$

and thus

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^q}{|x-y|^{N+qs}} \mathrm{d}x\mathrm{d}y = 0. \tag{2.2}$$

Thus, we have $f \in W^{s,q}(\Omega)$. On the other hand, in view of (2.2) we can write

$$\begin{aligned} &\int_\Omega \int_\Omega \frac{|(\varphi_n(x) - f(x)) - (\varphi_n(y) - f(y))|^q}{|x-y|^{N+qs}} \mathrm{d}x\mathrm{d}y \\ &\le 2^{q-1} \int_\Omega \int_\Omega \frac{|\varphi_n(x) - \varphi_n(y)|^q}{|x-y|^{N+qs}} \mathrm{d}x\mathrm{d}y + 2^{q-1} \int_\Omega \int_\Omega \frac{|f(y) - f(x)|^q}{|x-y|^{N+qs}} \mathrm{d}x\mathrm{d}y \end{aligned}$$

$$= 2^{q-1}\int_\Omega\int_\Omega \frac{|\varphi_n(x)-\varphi_n(y)|^q}{|x-y|^{N+qs}}\mathrm{d}x\mathrm{d}y \to 0.$$

Hence, $\varphi_n \to f$ in $W^{s,q}(\Omega)$ and so $f \in W_0^{s,q}(\Omega)$. By (2.2), the function f has a constant value on Ω. The only possible value is $f \equiv 0$ which yields a contradiction with the fact that

$$\int_\Omega |f(x)|^q \mathrm{d}x = 1.$$

So, we get

$$\|\varphi\|^q_{L^q(\Omega)} \le C(N,s,\Omega)\int_\Omega\int_\Omega \frac{|\varphi(x)-\varphi(y)|^q}{|x-y|^{N+qs}}\mathrm{d}x\mathrm{d}y,\ \forall \varphi \in \mathcal{C}_0^\infty(\Omega). \tag{2.3}$$

Now, for every $\varphi \in W_0^{s,q}(\Omega)$, there exists a sequence $\{\varphi_n\}$ of $\mathcal{C}_0^\infty(\Omega)$ functions such that

$$\varphi_n \to \varphi \text{ in norm in } W^{s,q}(\Omega).$$

Applying the inequality (2.3) for φ_n and passing to the limit, we conclude the result. □

Under the same assumptions of Lemma 2.1, the Banach space $W_0^{s,q}(\Omega)$ can be endowed with the norm

$$\|u\|_{W_0^{s,q}(\Omega)} = \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^q}{|x-y|^{N+qs}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}}$$

which is equivalent to $\|u\|_{W^{s,q}(\Omega)}$. Now, we define the space

$$W^{s,q}_{\mathrm{loc}}(\Omega) = \Big\{u:\Omega\to \mathrm{R} : u \in L^q(K),\ \int_K\int_K \frac{|u(x)-u(y)|^q}{|x-y|^{N+qs}}\mathrm{d}x\mathrm{d}y < \infty,$$
$$\text{for every compact } K \subset \Omega\Big\}.$$

In the case where $q=2$, we note $W^{s,2}(\Omega) = H^s(\Omega)$ and $W_0^{s,2}(\Omega) = H_0^s(\Omega)$. Endowed with the inner product

$$\langle u, v\rangle_{H_0^s(\Omega)} = \int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y,$$

$(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)})$ is a Hilbert space.

It is worth recalling that for any u and φ belonging to $H^s(\mathrm{R}^N)$, we have the following duality product

$$\int_{\mathrm{R}^N}(-\Delta)^s u\varphi \mathrm{d}x = \frac{\alpha(N,s)}{2}\int_{\mathrm{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.$$

Thus, it can be seen that

$$(-\Delta)^s : H^s(\mathrm{R}^N) \to H^{-s}(\mathrm{R}^N)$$

is a continuous and symmetric operator defined on $H^s(\mathrm{R}^N)$.

In the particular case, if u and φ belong to $H^s(\mathrm{R}^N)$ with $u=\varphi=0$, on $\mathcal{C}\Omega$, we have

$$\int_{\mathrm{R}^N}(-\Delta)^s u\varphi \mathrm{d}x = \frac{\alpha(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x,$$

where $Q := \mathrm{R}^{2N}\backslash(\mathcal{C}\Omega\times\mathcal{C}\Omega)$. For $N > 2s$ we define the fractional Sobolev critical exponent $2_s^* = \frac{2N}{N-2s}$. The following result is a fractional version of the Sobolev inequality which provides a continuous embedding of $H_0^s(\Omega)$ in the critical Lebesgue space $L^{2_s^*}(\Omega)$. The proof can be found in [17, 29].

Theorem 2.2 (Fractional Sobolev embedding) Let $0 < s < 1$ be such that $N > 2s$. Then, there exists a constant $S(N, s)$ depending only on N and s, such that for all $f \in \mathcal{C}_0^\infty(\mathrm{R}^N)$

$$\|f\|^2_{L^{2^*_s}(\mathrm{R}^N)} \le S(N, s) \int_{R^N} \int_{R^N} \frac{|f(x) - f(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

Remark 2.3 In particular, if Ω is an open bounded subset in R^N of class $\mathcal{C}^{0,1}$ with $N > 2s$ and $0 < s < 1$ and $f \in \mathcal{C}_0^\infty(\Omega)$ we have

$$\|f\|^2_{L^{2^*_s}(\Omega)} \le S(N, s, \Omega) \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

Indeed, by [17, Theorem 5.4] we can write

$$\int_{R^N} \int_{R^N} \frac{|f(x) - f(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y \le \|f\|^2_{H^s(\mathrm{R}^N)} \le C\|f\|^2_{H^s(\Omega)}$$
$$= C\|f\|^2_{L^2(\Omega)} + C \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

The result follows then by Theorem 2.2 and Lemma 2.1.

We will prove some estimates in the usual Marcinkiewicz space $\mathcal{M}^q(\Omega)$, $0 < q < \infty$, which consists of all measurable functions $u : \Omega \to \mathrm{R}$ such that there exists a constant $c = c(u) > 0$ satisfying

$$t^q \mathrm{meas}(\{x : |u(x)| > t\}) \le c$$

for every $t > 0$. Here and in what follows, meas(E) denotes the Lebesgue measure of a measurable subset E of Ω. It is worth recalling the following connection between Marcinkiewicz and Lebesgue spaces

$$L^q(\Omega) \hookrightarrow M^q(\Omega) \hookrightarrow L^{q-\varepsilon}(\Omega)$$

for every $1 < q < \infty$ and $0 < \varepsilon \le q - 1$ (see for instance [22]). We will also use the following truncation functions T_k and G_k, $k > 0$, defined for every $s \in \mathrm{R}$ by

$$T_k(s) = \max\{-k; \min\{k, s\}\} \text{ and } G_k(s) = s - T_k(s).$$

We denote by $\mathcal{M}_b(\Omega)$ the space of all bounded Radon measures on Ω. The norm of a measure $\mu \in \mathcal{M}_b(\Omega)$ is given by $\|\mu\|_{\mathcal{M}_b(\Omega)} = \int_\Omega \mathrm{d}|\mu|$.

Definition 2.4 We say that the sequence of measurable functions $\{\mu_n\}$ is converging weakly to μ in the sense of the measures if

$$\lim_{n\to\infty} \int_\Omega \varphi(x)\mu_n(x)\mathrm{d}x = \int_\Omega \varphi \mathrm{d}\mu, \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega).$$

In what follows we make use of the following technical algebraic inequalities.

Lemma 2.5 i) Let $\alpha > 0$. For every x, $y \ge 0$ one has

$$(x - y)(x^\alpha - y^\alpha) \ge \frac{4\alpha}{(\alpha + 1)^2}(x^{\frac{\alpha+1}{2}} - y^{\frac{\alpha+1}{2}})^2.$$

ii) Let $0 < \alpha < 1$. For every x, $y \ge 0$ with $x \ne y$ one has

$$\frac{x - y}{x^\alpha - y^\alpha} \le \frac{1}{\alpha}(x^{1-\alpha} + y^{1-\alpha}).$$

iii) Let $\alpha \ge 1$. Then

$$|x + y|^{\alpha-1}|x - y| \le c_\alpha |x^\alpha - y^\alpha|,$$

where c_α is a constant depending only on α.

Taking into account that less regular data are involved, the classical notion of finite energy solution cannot be used. Instead, we shall consider the notion of weak solution whose meaning is defined as follows.

Definition 2.6 Let $f \in L^1(\Omega)$ and let μ be a non-negative bounded Radon measure. By a weak solution of problem (1.1), we mean a measurable function u satisfying

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0\ :\ u(x) \geq c_\omega > 0,\ \text{in } \omega$$

and

$$\frac{\alpha(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y = \int_\Omega \frac{f\varphi}{u^\gamma}\mathrm{d}x + \int_\Omega \varphi \mathrm{d}\mu$$

for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

Theorem 2.7 Let Ω be an open bounded subset in R^N of class $\mathcal{C}^{0,1}$ with $N > 2s$ and $0 < s < 1$. Let $0 < \gamma \leq 1$ and let $f \in L^1(\Omega)$. Then the problem (1.1) admits a weak solution $u \in W_0^{s_1,q}(\Omega)$ for every $1 < q < \frac{N}{N-s}$ and for every $s_1 < s$.

Theorem 2.8 Let Ω be an open bounded subset in R^N of class $\mathcal{C}^{0,1}$ with $N > 2s$ and $0 < s < 1$. Let $\gamma > 1$ and let $f \in L^1(\Omega)$. Then the problem (1.1) admits a weak solution $u \in W_{\mathrm{loc}}^{s_1,q}(\Omega)$ for every $1 < q < \frac{N}{N-s}$, for all $s_1 < s$. Furthermore, $T_k^{\frac{\gamma+1}{2}}(u) \in H_0^s(\Omega)$ for every $k > 0$.

We point out that the inclusion $W_0^{s_1,q}(\Omega) \subset W_0^{s_2,q}(\Omega)$ holds for any $s_2 < s_1$ (see [17]). Therefore, the range of s_1 in both Theorem 2.7 and Theorem 2.8 can be that of the set of the exponents s_1 close to s. Indeed, we can consider s_1 to be such that $\frac{s}{2-s} \leq s_1 < s$. So that when s tends to 1 one has also s_1 tends to 1^-. In addition, letting s tends to 1^- the operator $(-\Delta)^s$ is nothing but the standard Laplacian. So that the equation in (1.1) becomes

$$-\Delta u = \frac{f(x)}{u^\gamma} + \mu$$

and then the results in both Theorem 2.7 and Theorem 2.8 covers those obtained in [26].

3 Approximated Problems: Existence Result and Comparison Principle

Consider the sequence of approximate problems

$$\begin{cases} (-\Delta)^s u_n = \dfrac{f_n}{(u_n+\frac{1}{n})^\gamma} + \mu_n & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \mathrm{R}^N \backslash \Omega, \end{cases} \tag{3.1}$$

where $f_n = T_n(f)$ is the truncation at level n of f and μ_n is a sequence of bounded non-negative smooth functions in $L^1(\Omega)$ converging weakly to μ in the sense of the measures.

We shall prove that for every fixed integer $n \in \mathrm{N}$, the problem (3.1) admits a unique weak solution u_n in the following sense :

$$\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y = \int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}\mathrm{d}x + \int_\Omega \mu_n\varphi\mathrm{d}x$$

for any $\varphi \in X_0^s(\Omega)$.

Lemma 3.1 For each integer $n \in \mathrm{N}$, the problem (3.1) admits a non-negative weak solution $u_n \in H_0^s(\Omega) \cap L^\infty(\Omega)$.

Proof Let $n \in \mathrm{N}$ be fixed and let $v \in L^2(\Omega)$. We define the map

$$\begin{aligned} S : L^2(\Omega) &\to L^2(\Omega), \\ v \quad &\mapsto S(v), \end{aligned}$$

where $w = S(v)$ is the weak solution to the following problem

$$\begin{cases} (-\Delta)^s w = \dfrac{f_n}{(|v| + \frac{1}{n})^\gamma} + \mu_n & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \mathrm{R}^N \backslash \Omega. \end{cases} \tag{3.2}$$

The existence of w can be derived by classical minimization argument. Indeed, since $\frac{f_n}{(|v|+\frac{1}{n})^\gamma} + \mu_n \in L^\infty(\Omega)$, we already know (see [12, Lemma 2.1]) that problem (3.2) has a unique weak solution $w \in X_0^s(\Omega)$, where

$$X_0^s(\Omega) = \Big\{ \varphi \in H^s(\mathrm{R}^N) \text{ such that } \varphi = 0 \text{ a.e. in } \mathrm{R}^N \backslash \Omega \Big\},$$

in the following sense

$$\frac{\alpha(N,s)}{2} \int_Q \frac{(w(x) - w(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \int_\Omega \frac{f_n \varphi}{(|v| + \frac{1}{n})^\gamma} \mathrm{d}x + \int_\Omega \mu_n \varphi \mathrm{d}x$$

for any $\varphi \in X_0^s(\Omega)$. Since Ω is regular enough, by [19, Theorem 6] the linear space $X_0^s(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H^s(\mathrm{R}^N)} = \left(\|u\|^2_{L^2(\mathrm{R}^N)} + \int_{\mathrm{R}^N} \int_{\mathrm{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

Hence, by density arguments it follows that $X_0^s(\Omega) \subset H_0^s(\Omega)$. Thus, $w \in H_0^s(\Omega)$. As regards the uniqueness of w in $H_0^s(\Omega)$, we suppose there exist two solutions $w_1, w_2 \in H_0^s(\Omega)$ of (3.2). Summing up the both equations satisfied by w_1 and w_2 respectively, we get $(-\Delta)^s(w_1 - w_2) = 0$. Thus, taking $(w_1 - w_2)$ as a test function in this last equation and then integrating over R^N, we obtain

$$0 \leq \|w_1 - w_2\|^2_{H_0^s(\Omega)} \leq \int_Q \frac{|(w_1(x) - w_2(x)) - (w_1(y) - w_2(y))|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = 0.$$

So we get $w_1(x) = w_2(x)$, for almost every $x \in \Omega$. Since $w_1 = w_2 = 0$ on $\mathrm{R}^N \backslash \Omega$, we get $w_1(x) = w_2(x)$ for almost every $x \in \mathrm{R}^N$. Furthermore, by the comparison principle [8, Lemma 2.1] we get $w \geq 0$. Now, inserting w as a test function in (3.2) we obtain

$$\begin{aligned} \frac{\alpha(N,s)}{2} \int_Q \frac{(w(x) - w(y))^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x &= \int_\Omega \frac{f_n w}{(|v| + \frac{1}{n})^\gamma} \mathrm{d}x + \int_\Omega w \mu_n \mathrm{d}x \\ &\leq n^{\gamma+1} \int_\Omega w \mathrm{d}x + C(n) \int_\Omega w \mathrm{d}x. \end{aligned}$$

By the Hölder inequality and the Sobolev embedding, we get

$$\|w\|_{H_0^s(\Omega)} \leq C'(n^{\gamma+1} + C(n)), \tag{3.3}$$

with C' and $C(n,s,N,\Omega)$ are independent of v, so that the ball of radius $C'(n^{\gamma+1}+C(n))$ is invariant under S in $H_0^s(\Omega)$.

Now, using the Schauder's fixed point theorem over S to prove the existence and uniqueness of solution of (3.1), we need to verify the continuity and compactness of S as an operator from $H_0^s(\Omega)$ to $H_0^s(\Omega)$.

First, we go to prove the continuity of S as an operator from $L^2(\Omega)$ to $L^2(\Omega)$. Let us consider a sequence v_k that converges to v in $L^2(\Omega)$, then up to a subsequence, we have

$$v_k \to v \text{ a.e. in } \Omega. \tag{3.4}$$

Denoting $w_k = S(v_k)$ and $w = S(v)$, we have

$$(-\Delta)^s w_k = \frac{f_n}{(|v_k|+\frac{1}{n})^\gamma} + \mu_n. \tag{3.5}$$

$$(-\Delta)^s w = \frac{f_n}{(|v|+\frac{1}{n})^\gamma} + \mu_n. \tag{3.6}$$

Taking $w_k(x)-w(x) \in X_0^s(\Omega)$ as a test function in (3.5) and (3.6) respectively, then subtracting term at term the both resulting equations and using Hölder's inequality we arrive at

$$\begin{aligned}
&\frac{\alpha(N,s)}{2}\int_Q \frac{\Big(w_k(x)-w(x)-\big(w_k(y)-w(y)\big)\Big)^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&=\int_\Omega \left(\frac{f_n}{(|v_k|+\frac{1}{n})^\gamma}-\frac{f_n}{(|v|+\frac{1}{n})^\gamma}\right)(w_k(x)-w(x))\mathrm{d}x\\
&\le \|w_k-w\|_{L^{2_s^*}(\Omega)}\left(\int_\Omega\left(\frac{f_n}{(|v_k|+\frac{1}{n})^\gamma}-\frac{f_n}{(|v|+\frac{1}{n})^\gamma}\right)^{(2_s^*)'}\mathrm{d}x\right)^{\frac{1}{(2_s^*)'}}.
\end{aligned}$$

Applying the fractional Sobolev embedding and Hölder's inequality with the exponents $\frac{2_s^*}{2}$ and $\frac{N}{2s}$, we get

$$\|w_k-w\|_{L^2(\Omega)} \le \frac{2S(N,s,\Omega)}{\alpha(N,s)}|\Omega|^{\frac{s}{N}}\left(\int_\Omega\left(\frac{f_n}{(|v_k|+\frac{1}{n})^\gamma}-\frac{f_n}{(|v|+\frac{1}{n})^\gamma}\right)^{(2_s^*)'}\mathrm{d}x\right)^{\frac{1}{(2_s^*)'}}.$$

Since

$$\left|\frac{f_n}{(|v_k|+\frac{1}{n})^\gamma}-\frac{f_n}{(|v|+\frac{1}{n})^\gamma}\right|^{(2_s^*)'} \le 2^{(2_s^*)'}n^{(\gamma+1)(2_s^*)'} \tag{3.7}$$

and

$$\frac{f_n}{(|v_k|+\frac{1}{n})^\gamma}-\frac{f_n}{(|v|+\frac{1}{n})^\gamma} \to 0 \text{ a.e. in } \Omega,$$

then by the dominated convergence theorem we conclude that

$$\|w_k-w\|_{L^2(\Omega)} \to 0 \text{ as } k\to+\infty.$$

So S is continuous from $L^2(\Omega)$ to $L^2(\Omega)$ and it follows that S is continuous from $H_0^s(\Omega)$ to $H_0^s(\Omega)$.

Now, we prove that S is compact from $H_0^s(\Omega)$ to $H_0^s(\Omega)$, let us consider a sequence $\{v_k\}_{k\in\mathrm{N}}$ such that $\|v_k\|_{H_0^s(\Omega)} \le C$, then by the compact embedding $H_0^s(\Omega)$ in $L^r(\Omega)$ for every $1\le r<2_s^*$ (see [17, Corollary 7.2]), we have

$$\begin{aligned}
&v_k \rightharpoonup v \text{ weakly in } H_0^s(\Omega),\\
&v_k \to v \text{ in norm in } L^2(\Omega).
\end{aligned}$$

Denoting $w_k = S(v_k)$ and $w = S(v)$, by (3.3) we have

$$\|w_k\|_{H_0^s(\Omega)} \le C,$$

where C is a constant not depending on k, then by the previous compact embedding and by the continuity of S on $L^2(\Omega)$ we get

$$\begin{aligned} &S(v_k) = w_k \rightharpoonup \overline{w},\ \text{weakly in } H_0^s(\Omega),\\ &S(v_k) = w_k \to S(v) = w,\ \text{in norm in } L^2(\Omega). \end{aligned} \tag{3.8}$$

So, by the uniqueness of the limit we have $\overline{w} = w$. In view of the previous equations (3.5) and (3.6) we have

$$(-\Delta)^s(w_k - w) = \frac{f_n}{(|v_k| + \frac{1}{n})^\gamma} - \frac{f_n}{(|v| + \frac{1}{n})^\gamma}.$$

Taking $w_k - w$ as a test function in the previous equation, using Hölder's inequality and (3.7) we obtain

$$\frac{\alpha(N,s)}{2}\|S(v_k) - S(v)\|^2_{H_0^s(\Omega)} \le 2n^{\gamma+1}C(\Omega)\|S(v_k) - S(v)\|_{L^2(\Omega)}.$$

It follows that

$$\lim_{k\to+\infty}\|S(v_k) - S(v)\|_{H_0^s(\Omega)} = 0.$$

Hence, S is a compact operator from $H_0^s(\Omega)$ to $H_0^s(\Omega)$ and therefore by Schauder's fixed point theorem there exists $u_n \in H_0^s(\Omega)$ such that $u_n = S(u_n)$. This means that u_n is a weak solution to the problem

$$\begin{cases} (-\Delta)^s u_n = \dfrac{f_n}{(u_n + \frac{1}{n})^\gamma} + \mu_n & \text{in } \Omega,\\ u_n > 0 & \text{in } \Omega,\\ u_n = 0 & \text{on } \mathrm{R}^N\backslash\Omega. \end{cases}$$

In addition, since the right hand side of belongs to $L^\infty(\Omega)$ by [25] we obtain $u_n \in L^\infty(\Omega)$. □

Lemma 3.2 (comparison principle) The sequence $\{u_n\}_{n\in\mathrm{N}}$ is such that for every subset $\omega \subset\subset \Omega$ there exists a positive constant c_ω, independent on n, such that

$$u_n(x) \ge c_\omega > 0, \quad \text{for every } x \in \omega \text{ and for every } n \in \mathrm{N}.$$

Proof Consider the following problem

$$\begin{cases} (-\Delta)^s v_n = \dfrac{f_n}{(v_n + \frac{1}{n})^\gamma} & \text{in } \Omega,\\ v_n > 0 & \text{in } \Omega,\\ v_n = 0 & \text{on } \mathrm{R}^N\backslash\Omega. \end{cases} \tag{3.9}$$

In [7], the authors proved the existence of a weak solution v_n of (3.9) such that

$$\forall\omega \subset\subset \Omega,\ \exists c_\omega > 0\ :\ v_n(x) \ge c_\omega > 0,$$

for every $x \in \omega$ and for every $n \in \mathrm{N}$. Here the constant c_ω is independent on n. On the other hand, we have

$$(-\Delta)^s v_n = \frac{f_n}{(v_n + \frac{1}{n})^\gamma}$$

and

$$(-\Delta)^s u_n = \frac{f_n}{(u_n + \frac{1}{n})^\gamma} + \mu_n.$$

Then

$$(-\Delta)^s (v_n - u_n) = f_n\left(\frac{1}{(v_n + \frac{1}{n})^\gamma} - \frac{1}{(u_n + \frac{1}{n})^\gamma}\right) - \mu_n.$$

Hence

$$(-\Delta)^s (v_n - u_n) = f_n\left(\frac{(u_n + \frac{1}{n})^\gamma - (v_n + \frac{1}{n})^\gamma}{(v_n + \frac{1}{n})^\gamma (u_n + \frac{1}{n})^\gamma}\right) - \mu_n. \tag{3.10}$$

Since

$$\left((u_n + \frac{1}{n})^\gamma - (v_n + \frac{1}{n})^\gamma\right)(v_n - u_n)^+ \leq 0,$$

we obtain the following inequality

$$f_n\left(\frac{(u_n + \frac{1}{n})^\gamma - (v_n + \frac{1}{n})^\gamma}{(v_n + \frac{1}{n})^\gamma (u_n + \frac{1}{n})^\gamma}\right)(v_n - u_n)^+ - \mu_n (v_n - u_n)^+ \leq 0.$$

Now, taking $(v_n - u_n)^+$ as a test function in (3.10) and then integrating over R^N, we get

$$\int_{\mathrm{R}^N} (-\Delta)^s (v_n - u_n)(v_n - u_n)^+ \mathrm{d}x \leq 0.$$

Observe that for any function $g : \mathrm{R}^N \to \mathrm{R}$ the following inequality

$$(g(x) - g(y))(g^+(x) - g^+(y)) \geq (g^+(x) - g^+(y))^2$$

holds true for every x, $y \in \mathrm{R}^N$, where $g^+ = \max(g, 0)$. Therefore, we obtain

$$0 \leq \|(v_n - u_n)^+\|^2_{H^s_0(\Omega)} \leq 0$$

which implies that $u_n \geq v_n$ in Ω and so

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 \ :\ u_n(x) \geq c_\omega > 0$$

for every $x \in \omega$ and for every $n \in \mathrm{N}$. □

Remark 3.3 Lemma 3.2 shows that the problem (3.1) has a unique solution. Indeed, if u_n and w_n are two solutions of problem (3.1), then as above taking $(u_n - w_n)^+$ as a test function in the problem satisfied by $(u_n - w_n)$, we conclude that $u_n \leq w_n$ in Ω and again taking $(w_n - u_n)^+$ as a test function we get $w_n \leq u_n$ in Ω. Hence, follows $u_n = w_n$ in Ω.

4 A Priori Estimates in Fractional Sobolev Spaces

In order to prove the existence of solutions for problem (1.1), we first need some a priori estimates on u_n. We start by proving the following lemma that we will use in both cases $\gamma \leq 1$ and $\gamma > 1$.

Lemma 4.1 Let $v_n \in H^s_0(\Omega)$ be a sequence that satisfies the following assumptions

1) The sequence $\{v_n\}_n$ is uniformly bounded in $L^r(\Omega)$, for all $r < \frac{N}{N-2s}$.

2) For any sufficient small $\theta \in (0,1)$

$$\int_\Omega \int_\Omega \frac{|w_n(x) - w_n(y)|}{|x-y|^{N+2s}} \frac{|w_n^\theta(x) - w_n^\theta(y)|}{w_n^\theta(x) w_n^\theta(y)} \mathrm{d}y\mathrm{d}x \leq C,$$

where C is a constant not depending on n and $w_n = v_n + 1$. Then the sequence $\{v_n\}_n$ is uniformly bounded in the fractional Sobolev space $W_0^{s_1,q}(\Omega)$ for every $q < \frac{N}{N-s}$ and for all $s_1 < s$.

Proof We shall prove that the sequence $\{v_n\}$ is uniformly bounded in the fractional Sobolev space $W_0^{s_1,q}(\Omega)$ for every $q < \frac{N}{N-s}$ and for all $s_1 < s$. That is there is a constant C not depending on n such that

$$\int_\Omega\int_\Omega \frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x \le C \quad \text{for all } q < \frac{N}{N-s} \text{ and for all } s_1 < s. \tag{4.1}$$

To this aim, let $q < 2$ which will be chosen in a few lines. We can write

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x &= \int_\Omega\int_\Omega \frac{|w_n(x)-w_n(y)|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x\\ &= \int_\Omega\int_{\{y\in\Omega: w_n(y)\neq w_n(x)\}} \frac{|w_n(x)-w_n(y)|^q}{|x-y|^{\frac{q}{2}N+qs}}\\ &\quad\times \frac{(w_n^\theta(x)-w_n^\theta(y))}{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}\\ &\quad\times \frac{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}{(w_n^\theta(x)-w_n^\theta(y))|x-y|^{\frac{2-q}{2}N-q(s-s_1)}}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Pointing out that the quantity in the middle of the product inside the integral can be written as follows

$$\begin{aligned}&\frac{(w_n^\theta(x)-w_n^\theta(y))}{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}\\ &= \left(\frac{(w_n^\theta(x)-w_n^\theta(y))}{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}\right)^{\frac{q}{2}}\left(\frac{(w_n^\theta(x)-w_n^\theta(y))}{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}\right)^{1-\frac{q}{2}}\end{aligned}$$

and using Hölder's inequality, we obtain

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x\\ &\le \left[\int_\Omega\int_{\{y\in\Omega: w_n(y)\neq w_n(x)\}} \frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}\frac{|w_n^\theta(x)-w_n^\theta(y)|}{|w_n(x)-w_n(y)|(w_n^\theta(x)w_n^\theta(y))}\mathrm{d}y\mathrm{d}x\right]^{\frac{q}{2}}\\ &\quad\times\left[\int_\Omega\int_{\{y\in\Omega: w_n(y)\neq w_n(x)\}}\left(\frac{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}{(w_n^\theta(x)-w_n^\theta(y))}\right)^{\frac{2}{2-q}}\right.\\ &\quad\left.\times\frac{(w_n^\theta(x)-w_n^\theta(y))}{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}\frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\beta}}\right]^{\frac{2-q}{2}},\end{aligned}$$

where $\beta = \frac{2q(s-s_1)}{2-q} > 0$. Using Lemma 2.5, we get

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x\\ &\le C^{\frac{q}{2}}\left(\int_\Omega\int_{\{y\in\Omega: w_n(y)\neq w_n(x)\}}\left(\frac{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}{[w_n^\theta(x)-w_n^\theta(y)]}\right)^{\frac{q}{2-q}}\frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\beta}}\right)^{\frac{2-q}{2}}\\ &\le \left(\frac{C}{\theta}\right)^{\frac{q}{2}}\left(\int_\Omega\int_\Omega\left((w_n^{1-\theta}(x)+w_n^{1-\theta}(y))w_n^\theta(x)w_n^\theta(y)\right)^{\frac{q}{2-q}}\frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\beta}}\right)^{\frac{2-q}{2}}\end{aligned}$$

$$= \left(\frac{C}{\theta}\right)^{\frac{q}{2}} \left(\int_\Omega \int_\Omega \left((w_n(x) w_n^\theta(y) + w_n(y) w_n^\theta(x)) \right)^{\frac{q}{2-q}} \frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\beta}} \right)^{\frac{2-q}{2}}.$$

Applying the Young inequality with the exponents $\frac{\theta+1}{\theta}$ and $\theta+1$, we have

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+qs_1}} \mathrm{d}y\mathrm{d}x \\
&\le \left(\frac{C}{\theta}\right)^{\frac{q}{2}} \left(\int_\Omega \int_\Omega \left(w_n^{1+\theta}(x) + w_n^{1+\theta}(y) \right)^{\frac{q}{2-q}} \frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\beta}} \right)^{\frac{2-q}{2}} \\
&\le 2^{\frac{2(q-1)}{2-q}} \left(\frac{C}{\theta}\right)^{\frac{q}{2}} \left(\int_\Omega \int_\Omega \left(w_n^{\frac{q(1+\theta)}{2-q}}(x) + w_n^{\frac{q(1+\theta)}{2-q}}(y) \right) \frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\beta}} \right)^{\frac{2-q}{2}} \\
&\le 2^{\frac{2(q-1)}{2-q}} \left(\frac{C}{\theta}\right)^{\frac{q}{2}} \left(\int_\Omega w_n^{\frac{q(1+\theta)}{2-q}}(x) \left[\int_\Omega \frac{\mathrm{d}y}{|x-y|^{N-\beta}} \right] \mathrm{d}x \right)^{\frac{2-q}{2}} \\
&\quad + 2^{\frac{2(q-1)}{2-q}} \left(\frac{C}{\theta}\right)^{\frac{q}{2}} \left(\int_\Omega w_n^{\frac{q(1+\theta)}{2-q}}(y) \left[\int_\Omega \frac{\mathrm{d}x}{|x-y|^{N-\beta}} \right] \mathrm{d}y \right)^{\frac{2-q}{2}}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\int_\Omega \frac{\mathrm{d}y}{|x-y|^{N-\beta}} &= \int_{\Omega\cap|x-y|>1} \frac{\mathrm{d}y}{|x-y|^{N-\beta}} + \int_{\Omega\cap|x-y|\le 1} \frac{\mathrm{d}y}{|x-y|^{N-\beta}} \\
&\le |\Omega| + \int_{|z|\le 1} \frac{\mathrm{d}z}{|z|^{N-\beta}} = |\Omega| + \frac{|S^{N-1}|}{\beta}. \qquad (4.2)
\end{aligned}$$

Here, $|S^{N-1}|$ stands for the Lebesgue measure of the unit sphere in R^N. By x/y symmetry, there exists a constant C, not depending on n, such that

$$\int_\Omega \int_\Omega \frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+qs_1}} \mathrm{d}y\mathrm{d}x \le C \left(\int_\Omega w_n^{\frac{q(1+\theta)}{2-q}}(y)\mathrm{d}y \right)^{\frac{2-q}{2}}.$$

Now we choose $\theta > 0$ in order to get $\frac{q(1+\theta)}{2-q} < \frac{N}{N-2s}$. That is $\theta < \frac{2N-2q(N-s)}{q(N-2s)}$. To ensure the existence of θ we must have $2N - 2q(N-s) > 0$ which yields $q < \frac{N}{N-s}$. We then conclude that (4.1) is fulfilled and the sequence $\{v_n\}$ is uniformly bounded in $W_0^{s_1,q}(\Omega)$ for every $q < \frac{N}{N-s}$ and for all $s_1 < s$. □

4.1 The case $\gamma \le 1$

Lemma 4.2 Let $u_n \in H_0^s(\Omega)$ be the solution of the problem (3.1). If $0 < \gamma \le 1$, then the sequence $\{u_n\}$ is uniformly bounded in $W_0^{s_1,q}(\Omega)$ for every $q < \frac{N}{N-s}$ and for all $s_1 < s$.

Proof Let $k \ge 1$ be fixed. By Lemma 6.4 (in Appendix) the function $T_k(u_n)$ is an admissible test function in (3.1). Thus, inserting it in (3.1) we obtain

$$\begin{aligned}
&\frac{\alpha(N,s)}{2} \int_Q \frac{(u_n(x)-u_n(y))(T_k(u_n(x)) - T_k(u_n(y)))}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\
&= \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\gamma} T_k(u_n)\mathrm{d}x + \int_\Omega \mu_n T_k(u_n)\mathrm{d}x.
\end{aligned}$$

By using Proposition 6.2 (in Appendix), we get

$$\frac{\alpha(N,s)}{2} \int_Q \frac{|T_k(u_n(x)) - T_k(u_n(y))|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \le k^{1-\gamma}\|f\|_{L^1(\Omega)} + k\|\mu_n\|_{L^1(\Omega)} \le Ck,$$

where $C = \|f\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}_b(\Omega)}$ is a constant not depending on n. Applying the Sobolev embedding theorem we get

$$\frac{1}{S}\left(\int_\Omega |T_k(u_n)(x)|^{2^*_s}\mathrm{d}x\right)^{\frac{2}{2^*_s}} \le Ck.$$

For the left hand side, observing that on the set $\{u_n \ge k\}$, we have $T_k(u_n) = k$, we get

$$\frac{1}{S}k^2(\mathrm{meas}(\{u_n \ge k\}))^{\frac{2}{2^*_s}} \le Ck,$$

which yields

$$\mathrm{meas}(\{u_n \ge k\}) \le \frac{C}{k^{\frac{N}{N-2s}}}. \tag{4.3}$$

Thus, the sequence $\{u_n\}$ is uniformly bounded in $M^{\frac{N}{N-2s}}(\Omega)$ and then so it is in $L^r(\Omega)$, for all $r < \frac{N}{N-2s}$. Let $s_1 \in (0,s)$ be fixed. For every $x \ge 0$ we define the function

$$\phi(x) = 1 - \frac{1}{(1+x)^\theta},\ \text{where } 0 < \theta \le 1. \tag{4.4}$$

Observe that the function ϕ satisfies

$$\phi(x) \le 1 \text{ and } \phi(x) \le x^\gamma \text{ for any } 0 < \theta \le \gamma \le 1.$$

The function $\phi(u_n)$ is an admissible test function in (3.1). So that inserting it as a test function in (3.1) we obtain

$$\begin{aligned}&\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))(\phi(u_n)(x)-\phi(u_n)(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ =&\int_\Omega \frac{f_n(x)\phi(u_n)}{(u_n+\frac{1}{n})^\gamma} + \int_\Omega \mu_n(x)\phi(u_n)\mathrm{d}x \le \|f\|_{L^1(\Omega)} + \|\mu_n\|_{L^1(\Omega)} \le C.\end{aligned}$$

Being ϕ non-decreasing and $\Omega\times\Omega \subset Q$, the integral in the left-hand side can be treated as follows

$$\begin{aligned}&\int_Q \frac{(u_n(x)-u_n(y))(\phi(u_n)(x)-\phi(u_n)(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ \ge&\int_\Omega\int_\Omega \frac{(u_n(x)-u_n(y))}{|x-y|^{N+2s}}\frac{(u_n(x)+1)^\theta-(u_n(y)+1)^\theta}{(u_n(x)+1)^\theta(u_n(y)+1)^\theta}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

So that we obtain

$$\int_\Omega\int_\Omega \frac{(w_n(x)-w_n(y))}{|x-y|^{N+2s}}\frac{(w_n(x))^\theta-(w_n(y))^\theta}{(w_n(x))^\theta(w_n(y))^\theta}\mathrm{d}y\mathrm{d}x \le \frac{2C}{\alpha(N,s)},$$

where we have set $w_n = u_n + 1$. Therefore, by Lemma 4.1 with $0 < \theta \le \gamma$ the sequence $\{u_n\}$ is uniformly bounded in $W_0^{s_1,q}(\Omega)$ for every $q < \frac{N}{N-s}$ and for all $s_1 < s$. □

4.2 The case $\gamma > 1$

Lemma 4.3 Let $f \in L^1(\Omega)$ and let u_n be the solution of (3.1). For $k > 0$ and $\gamma > 1$ the sequence $\left\{T_k^{\frac{\gamma+1}{2}}(u_n)\right\}_n$ is uniformly bounded in $H_0^s(\Omega)$.

Proof Let us fix $k > 0$. Inserting $T_k^\gamma(u_n)$ as a test function in (3.1), we get

$$\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))(T_k^\gamma(u_n(x))-T_k^\gamma(u_n(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x$$

$$
\begin{aligned}
&= \int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma} T_k^\gamma(u_n)\mathrm{d}x + \int_\Omega \mu_n T_k^\gamma(u_n)\mathrm{d}x \\
&\le \|f\|_{L^1(\Omega)} + k^\gamma \|\mu_n\|_{L^1(\Omega)} \le C_1,
\end{aligned}
$$

where $C_1 = \|f\|_{L^1(\Omega)} + k^\gamma \|\mu\|_{\mathcal{M}_b(\Omega)}$ is a constant not depending on n. By applying Proposition 6.2 (in Appendix) and Lemma 2.5, we have

$$
\begin{aligned}
&\int_Q \frac{(u_n(x)-u_n(y))(T_k^\gamma(u_n(x)) - T_k^\gamma(u_n(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x \\
\ge &\int_Q \frac{(T_k(u_n(x)) - T_k(u_n(y)))(T_k^\gamma(u_n(x)) - T_k^\gamma(u_n(y)))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x \\
\ge &\frac{4\gamma}{(\gamma+1)^2}\int_Q \frac{|T_k^{\frac{\gamma+1}{2}}(u_n(x)) - T_k^{\frac{\gamma+1}{2}}(u_n(y))|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.
\end{aligned}
$$

Therefore, we obtain

$$
\|T_k^{\frac{\gamma+1}{2}}(u_n)\|^2_{H_0^s(\Omega)} \le \int_Q \frac{|T_k^{\frac{\gamma+1}{2}}(u_n(x)) - T_k^{\frac{\gamma+1}{2}}(u_n(y))|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x \le \frac{(\gamma+1)^2}{4\gamma}\frac{2}{\alpha(N,s)}C_1.
$$

The proof is then achieved. □

Lemma 4.4 Let u_n be the solution of the problem (3.1). If $\gamma > 1$, then the sequence $\{u_n\}$ is uniformly bounded in $W^{s_1,q}_{\mathrm{loc}}(\Omega)$ for every $q < \frac{N}{N-s}$ and for all $s_1 < s$.

Proof For every $\omega \subset\subset \Omega$, for all $q < \frac{N}{N-s}$ and for all $s_1 < s$, we shall prove that there exists a constant $C = C(q, s_1, w)$, not depending on n, such that

$$
\int_\omega\int_\omega \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x \le C \quad \text{and} \quad \int_\omega |u_n|^q \mathrm{d}x \le C. \tag{4.5}
$$

We begin by proving the left estimate in (4.5). Let $k_0 \ge 1$ be fixed. Let $q < 2$ and $s_1 < s$. Using the fact that $u_n = T_{k_0}(u_n) + G_{k_0}(u_n)$, we can write

$$
\begin{aligned}
&\int_\omega\int_\omega \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x \\
= &\int_\omega\int_\omega \frac{|T_{k_0}(u_n(x)) + G_{k_0}(u_n(x)) - T_{k_0}(u_n(y)) - G_{k_0}(u_n(y))|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x \\
\le &2^{q-1}\int_\omega\int_\omega \frac{|T_{k_0}(u_n(x)) - T_{k_0}(u_n(y))|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x \\
&+2^{q-1}\int_\Omega\int_\Omega \frac{|G_{k_0}(u_n(x)) - G_{k_0}(u_n(y))|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x.
\end{aligned}
$$

Applying the Hölder inequality, we get

$$
\begin{aligned}
&\int_\omega\int_\omega \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x \\
\le &2^{q-1}\bigg(\int_\omega\int_\omega \frac{|T_{k_0}(u_n(x)) - T_{k_0}(u_n(y))|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\bigg)^{\frac{q}{2}}\bigg(\int_\Omega\int_\Omega \frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\beta}}\bigg)^{\frac{2-q}{2}} \\
&+2^{q-1}\int_\Omega\int_\Omega \frac{|G_{k_0}(u_n(x)) - G_{k_0}(u_n(y))|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x,
\end{aligned}
$$

where $\beta = \frac{2q(s-s_1)}{2-q} > 0$. Thanks to (4.2), we have

$$
2^{q-1}\bigg(\int_\Omega\int_\Omega \frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\beta}}\bigg)^{\frac{2-q}{2}} \le C_3 := 2^{q-1}\bigg(|\Omega| + \frac{|S^{N-1}|}{\beta}\bigg)^{\frac{2-q}{2}}
$$

which implies

$$\int_\omega\int_\omega \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x \le C_3\bigg(\int_\omega\int_\omega \frac{|T_{k_0}(u_n(x))-T_{k_0}(u_n(y))|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\bigg)^{\frac{q}{2}} + 2^{q-1}\int_\Omega\int_\Omega \frac{|G_{k_0}(u_n(x))-G_{k_0}(u_n(y))|^q}{|x-y|^{N+qs_1}}\mathrm{d}y\mathrm{d}x.$$

So, it is sufficient to prove that $\{G_{k_0}(u_n)\}_n$ and $\{T_{k_0}(u_n)\}_n$ are uniformly bounded in $W_0^{s_1,q}(\Omega)$ and $H^s_{\rm loc}(\Omega)$ respectively. We begin by proving that $G_{k_0}(u_n)$ is uniformly bounded in $W_0^{s_1,q}(\Omega)$ for all $q<\frac{N}{N-s}$ and for all $s_1<s$. To do so, for $k>k_0$ we take $T_k(G_{k_0}(u_n))$ as a test function in (3.1) and use the fact that $G_{k_0}(u_n)=0$ on $\{u_n\le k_0\}$, we obtain

$$\begin{aligned}&\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))[T_k(G_{k_0}(u_n(x)))-T_k(G_{k_0}(u_n(y)))]}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ =&\int_\Omega \frac{f_nT_k(G_{k_0}(u_n))}{(u_n+\frac1n)^\gamma}\mathrm{d}x+\int_\Omega \mu_nT_k(G_{k_0}(u_n))\mathrm{d}x\\ \le& k\int_{\{u_n>k_0\}}\frac{f}{(u_n+\frac1n)^\gamma}\mathrm{d}x+k\|\mu_n\|_{L^1(\Omega)}\le C_1k,\end{aligned}$$

where $C_1=k_0^{-\gamma}\|f\|_{L^1(\Omega)}+\|\mu\|_{\mathcal{M}_b(\Omega)}$, is a constant not depending on n. Using the decomposition of u_n as $u_n=T_{k_0}(u_n)+G_{k_0}(u_n)$, we can write

$$\begin{aligned}&\int_Q \frac{(u_n(x)-u_n(y))[T_k(G_{k_0}(u_n(x)))-T_k(G_{k_0}(u_n(y)))]}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ =&\int_Q \frac{(T_{k_0}(u_n(x))-T_{k_0}(u_n(y)))[T_k(G_{k_0}(u_n(x)))-T_k(G_{k_0}(u_n(y)))]}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ &+\int_Q \frac{(G_{k_0}(u_n(x))-G_{k_0}(u_n(y)))[T_k(G_{k_0}(u_n(x)))-T_k(G_{k_0}(u_n(y)))]}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Let us observe that since T_{k_0} and $T_k(G_{k_0})$ are non-decreasing functions, we get

$$(T_{k_0}(u_n(x))-T_{k_0}(u_n(y)))[T_k(G_{k_0}(u_n(x)))-T_k(G_{k_0}(u_n(y)))]\ge 0 \text{ a.e. in Q}.$$

Hence, it follows

$$\begin{aligned}&\int_Q \frac{(u_n(x)-u_n(y))[T_k(G_{k_0}(u_n(x)))-T_k(G_{k_0}(u_n(y)))]}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ \ge&\int_Q \frac{(G_{k_0}(u_n(x))-G_{k_0}(u_n(y)))[T_k(G_{k_0}(u_n(x)))-T_k(G_{k_0}(u_n(y)))]}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

In the right-hand side of the above inequality, we decompose $G_{k_0}(u_n)$ as follows $G_{k_0}(u_n(x))= G_k(G_{k_0}(u_n(x)))+T_k(G_{k_0}(u_n(x)))$ and we apply Proposition 6.2 (in Appendix) with $\alpha=1$ obtaining

$$\int_\Omega\int_\Omega \frac{|T_k(G_{k_0}(u_n(x)))-T_k(G_{k_0}(u_n(y)))|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\le \frac{2kC_1}{\alpha(N,s)}.$$

Hence, using the fractional Sobolev inequality, we get again the inequality (4.3) for the function $G_{k_0}(u_n)$ that is

$$\text{meas}(\{G_{k_0}(u_n)\ge k\})\le Ck^{-\frac{N}{N-2s}}$$

which implies that $\{G_{k_0}(u_n)\}_n$ is uniformly bounded in $L^r(\Omega)$ for every $r<\frac{N}{N-2s}$.

Let ϕ be the function defined in (4.4). Observe that for every $0 < \theta < 1$ the function ϕ enjoys the following properties

$$\phi(x) \leq x \text{ and } \phi(x) \leq 1.$$

Inserting $\phi(G_{k_0}(u_n))$ as a test function in (3.1) we get

$$\begin{aligned}
&\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))\Big(\phi(G_{k_0}(u_n(x)))-\phi(G_{k_0}(u_n(y)))\Big)}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&= \int_{\{u_n\geq k_0\}} \frac{f_n\phi(G_{k_0}(u_n))}{(u_n+\frac{1}{n})^\gamma}\mathrm{d}x + \int_\Omega \mu_n(x)\phi(G_{k_0}(u_n))\mathrm{d}x\\
&\leq \int_{\{u_n\geq k_0\}} \frac{f_n G_{k_0}(u_n)}{(u_n+\frac{1}{n})^\gamma}\mathrm{d}x + \|\mu_n\|_{L^1(\Omega)}\\
&\leq \int_{\{u_n\geq k_0\}} \frac{|f|}{(u_n+\frac{1}{n})^{\gamma-1}}\mathrm{d}x + \|\mu_n\|_{L^1(\Omega)}\\
&\leq C_2 := k_0^{1-\gamma}\|f\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}_b(\Omega)}.
\end{aligned}$$

Then, writing the decomposition $u_n = T_{k_0}(u_n) + G_{k_0}(u_n)$ and using the fact that T_{k_0} and $\phi(G_{k_0})$ are non-decreasing functions, we obtain

$$\int_\Omega\int_\Omega \frac{(G_{k_0}(u_n)(x)-G_{k_0}(u_n)(y))\Big(\phi(G_{k_0}(u_n(x)))-\phi(G_{k_0}(u_n(y)))\Big)}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x \leq \frac{2C_2}{\alpha(N,s)}$$

which yields

$$\int_\Omega\int_\Omega \frac{(w_n(x)-w_n(y))}{|x-y|^{N+2s}}\frac{(w_n(x))^\theta-(w_n(y))^\theta}{(w_n(x))^\theta(w_n(y))^\theta}\mathrm{d}y\mathrm{d}x \leq C_3 := \frac{2C_2}{\alpha(N,s)},$$

where we have set $w_n = G_{k_0}(u_n)+1$. Thus, Lemma 4.1 ensures that the sequence $\{G_{k_0}(u_n)\}$ is uniformly bounded in $W_0^{s_1,q}(\Omega)$ for all $q < \frac{N}{N-s}$ and for all $s_1 < s$.

Now, we shall prove that $\{T_{k_0}(u_n)\}_n$ is uniformly bounded in $H^{s_1}_{\mathrm{loc}}(\Omega)$. To do so, we insert $T^\gamma_{k_0}(u_n)$ as a test function in (3.1) obtaining

$$\begin{aligned}
&\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))\Big(T^\gamma_{k_0}(u_n(x))-T^\gamma_{k_0}(u_n(y))\Big)}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&= \int_\Omega \frac{f_nT^\gamma_{k_0}(u_n)}{(u_n+\frac{1}{n})^\gamma}\mathrm{d}x + \int_\Omega \mu_nT^\gamma_{k_0}(u_n)\mathrm{d}x \leq C_4 := \|f\|_{L^1(\Omega)} + k_0^\gamma\|\mu\|_{\mathcal{M}_b(\Omega)}.
\end{aligned}$$

By Lemma 2.5 (item iii)) there exists a constant $c_\gamma > 0$, depending only on γ such that

$$\begin{aligned}
&\int_Q \frac{(u_n(x)-u_n(y))\Big(T^\gamma_{k_0}(u_n(x))-T^\gamma_{k_0}(u_n(y))\Big)}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&\geq \int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)||T^\gamma_{k_0}(u_n(x))-T^\gamma_{k_0}(u_n(y))|}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&\geq \frac{1}{c_\gamma}\int_\Omega\int_\Omega \frac{\Big|(u_n(x)-u_n(y))\Big(T_{k_0}(u_n(x))-T_{k_0}(u_n(y))\Big)\Big|}{|x-y|^{N+2s}}\\
&\quad\times(T_{k_0}(u_n(x))+T_{k_0}(u_n(y)))^{\gamma-1}\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Let now ω be a compact subset in Ω. By Proposition 6.2 (in Appendix) we can write

$$\int_Q \frac{(u_n(x)-u_n(y))[T^\gamma_{k_0}(u_n(x))-T^\gamma_{k_0}(u_n(y))]}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x$$

$$\geq \frac{1}{c_\gamma}\int_\omega\int_\omega \frac{|T_{k_0}(u_n(x))-T_{k_0}(u_n(y))|^2(T_{k_0}(u_n(x))+T_{k_0}(u_n(y)))^{\gamma-1}}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.$$

Pointing out that by Lemma 3.2 we have $T_{k_0}(u_n(x))\geq \min(k_0,c_\omega)$ for every $x\in\omega$, we obtain

$$\begin{aligned}&\int_Q \frac{(u_n(x)-u_n(y))[T_{k_0}^\gamma(u_n(x))-T_{k_0}^\gamma(u_n(y))]}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ &\geq \frac{1}{c_\gamma}(2\min(k_0,c_\omega))^{\gamma-1}\int_\omega\int_\omega \frac{|T_{k_0}(u_n(x))-T_{k_0}(u_n(y))|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\end{aligned}$$

which proves that $\{T_{k_0}(u_n)\}_n$ is uniformly bounded in $H^s_{\mathrm{loc}}(\Omega)$.

We now prove the second estimate in (4.5). For $q<\frac{N}{N-s}$ and $s_1<s$, writing

$$\begin{aligned}\int_\omega |u_n|^q\mathrm{d}x &\leq 2^{q-1}\int_\omega |T_{k_0}(u_n)|^q\mathrm{d}x+2^{q-1}\int_\omega |G_{k_0}(u_n)|^q\mathrm{d}x\\ &\leq 2^{q-1}k_0^q|\omega|+2^{q-1}\|G_{k_0}(u_n)\|^q_{L^q(\Omega)}\end{aligned}$$

we conclude the result. In fact, for every $\gamma>0$ the sequence $\{u_n\}$ is uniformly bounded in $L^q(\Omega)$ for all $1\leq q<\frac{N}{N-2s}$. □

5 Proof of the Main Results

In this section, we show that in both cases $\gamma\leq 1$ and $\gamma>1$, the problem (1.1) has a weak solution obtained as the limit of approximate solutions $\{u_n\}_n$ of the problem (3.1).

5.1 The case $\gamma\leq 1$

Proof of Theorem 2.7. By virtue of Lemma 4.2 and the compact embedding of $W_0^{s_1,q}(\Omega)$ in $L^1(\Omega)$ (see [17, Corollary 7.2]), there exist a subsequence of $\{u_n\}_n$ still indexed by n and a measurable function $v\in W_0^{s_1,q}(\Omega)$ such that

$$\begin{aligned}&u_n\rightharpoonup v \text{ weakly in } W_0^{s_1,q}(\Omega),\\ &u_n\to v \text{ in norm in } L^1(\Omega),\\ &u_n\to v \text{ a.e. in } \Omega.\end{aligned}$$

Let u the function such that $u=v$ in Ω and $u=0$ in $\mathrm{R}^N\setminus\Omega$. Thus, $u_n\to u$ a.e. in R^N which implies

$$\frac{|u_n(x)-u_n(y)|}{|x-y|^{N+2s}}\to\frac{|u(x)-u(y)|}{|x-y|^{N+2s}} \text{ a.e. in } Q.$$

Let $\rho>0$ be a small enough real number that we will choose later. For any $\varphi\in C_0^\infty(\Omega)$ we have

$$\begin{aligned}&\int_\Omega\int_\Omega\left[\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\right]^{1+\rho}\mathrm{d}y\mathrm{d}x\\ &\leq \int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{1+\rho}(\|D\varphi\|_{L^\infty(\Omega)}|x-y|)^{1+\rho}}{|x-y|^{N+(1+\rho)s_1}}\frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{\rho N+(1+\rho)(2s-s_1)}}\\ &\leq \|D\varphi\|^{1+\rho}_{L^\infty(\Omega)}\int_\Omega\int_\Omega\frac{|u_n(x)-u_n(y)|^{1+\rho}|x-y|^{(1+\rho)(1+s_1-2s)-\rho N}}{|x-y|^{N+(1+\rho)s_1}}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

We need that the term $|x-y|^{\rho N+(1+\rho)(2s-s_1)}$ vanishes from within the integral. To get this, it is sufficient to have $(1+\rho)(1+s_1-2s)-\rho N\geq 0$. To this aim, we consider s_1 to be very close of s. Precisely, we impose on s_1 the condition

$$\max(0,1-3s) < s-s_1 < 1-s.$$

We point out that with this range of values of s_1 and with the assumption $N>2s$, we easily get

$$1+s_1-2s>0 \quad \text{ and } \quad N-1-s_1+2s>0.$$

Thus, the fact that $(1+\rho)(1+s_1-2s)-\rho N \geq 0$ is equivalent to $0<\rho\leq \frac{1+s_1-2s}{N-1-s_1+2s}$. Hence, we get

$$\begin{aligned}&\int_\Omega\int_\Omega\left[\frac{|u_n(x)-u_n(y)||\varphi(x)-\varphi(y)|}{|x-y|^{N+2s}}\right]^{1+\rho}\mathrm{d}y\mathrm{d}x\\ &\leq \|D\varphi\|^{1+\rho}_{L^\infty(\Omega)}\mathrm{diam}(\Omega)^{(1+\rho)(1+s_1-2s)-\rho N}\int_\Omega\int_\Omega\frac{|u_n(x)-u_n(y)|^{1+\rho}}{|x-y|^{N+(1+\rho)s_1}}\mathrm{d}y\mathrm{d}x,\end{aligned}$$

where $\mathrm{diam}(\Omega)$ stands for the diameter of Ω. Now we have to make a choice of ρ which enables us to use the uniform boundedness of $\{u_n\}_n$ in $W^{s_1,q}_0(\Omega)$ for every $q<\frac{N}{N-s}$. This is the case if $1+\rho<\frac{N}{N-s}$. Finally, we choose ρ to be such that

$$0<\rho<\min\Big(\frac{s}{N-s},\frac{1+s_1-2s}{N-1-s_1+2s}\Big).$$

Therefore, there is a constant $C>0$ not depending on n such that

$$\sup_n\int_\Omega\int_\Omega\left[\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\right]^{1+\rho}\mathrm{d}y\mathrm{d}x\leq C.$$

Consequently by De La Valle Poussin and Dunford-Pettis theorems the sequence

$$\Big\{\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\Big\}$$

is equi-integrable in $L^1(\Omega\times\Omega)$. Now, taking $\varphi\in C^\infty_0(\Omega)$ as a test function in (3.1) we get

$$\frac{\alpha(N,s)}{2}\int_Q\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x=\int_\Omega\frac{f_n\varphi}{(u_n+\frac1n)^\gamma}\mathrm{d}x+\int_\Omega\varphi\mu_n\mathrm{d}x. \tag{5.1}$$

We split the integral in left-hand side into three integrals as follows

$$\begin{aligned}\int_Q\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x&=\int_\Omega\int_\Omega\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ &\quad+\int_\Omega\int_{\mathcal{C}\Omega}\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ &\quad+\int_{\mathcal{C}\Omega}\int_\Omega\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ &=I_1+I_2+I_3.\end{aligned} \tag{5.2}$$

By Vitali's lemma we have

$$\lim_{n\to\infty}\int_\Omega\int_\Omega\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x=\int_\Omega\int_\Omega\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.$$

For the second integral I_2 in (5.2), we start noticing that since $u_n(y)=\varphi(y)=0$ for every $y\in\mathcal{C}\Omega$ we can write

$$\Big|\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\Big|=\frac{|u_n(x)\varphi(x)|}{|x-y|^{N+2s}}\quad \text{for every } (x,y)\in\Omega\times\mathcal{C}\Omega.$$

Since $\operatorname{Supp}\varphi$ is a compact subset in Ω, we have

$$|x-y| \geq d_1 := \operatorname{dist}(\operatorname{Supp}\varphi, \partial\Omega) > 0 \text{ for every } (x,y) \in \operatorname{Supp}\varphi \times \mathcal{C}\Omega.$$

Therefore, an easy computation leads to

$$\int_{\mathcal{C}\Omega} \frac{\mathrm{d}y}{|x-y|^{N+2s}} \leq \int_{d_1}^{+\infty} \frac{\mathrm{d}z}{|z|^{N+2s}} \leq \frac{|S^{N-1}|}{2sd_1^{2s}}. \tag{5.3}$$

As a consequence of the convergence in norm of the sequence $\{u_n\}$ in $L^1(\Omega)$ there exist a subsequence of $\{u_n\}$ still indexed by n and a positive function g in $L^1(\Omega)$ such that

$$|u_n(x)| \leq g(x) \text{ a.e. in } \Omega,$$

which enables us to get

$$\frac{|(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}} \leq \frac{|g(x)\varphi(x)|}{|x-y|^{N+2s}} \text{ a.e. in } (x,y) \in \Omega \times \mathcal{C}\Omega.$$

We observe that by (5.3) the function $(x,y) \to \frac{|g(x)\varphi(x)|}{|x-y|^{N+2s}}$ belongs to $L^1(\Omega \times \mathcal{C}\Omega)$

$$\int_\Omega \int_{\mathcal{C}\Omega} \frac{|g(x)\varphi(x)|}{|x-y|^{N+2s}} = \int_{\operatorname{Supp}\varphi} \int_{\mathcal{C}\Omega} \frac{|g(x)\varphi(x)|}{|x-y|^{N+2s}} \leq \frac{|S^{N-1}| \|\varphi\|_{L^\infty(\Omega)} \|g\|_{L^1(\Omega)}}{2sd_1^{2s}}.$$

Thus, by the dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_\Omega \int_{\mathcal{C}\Omega} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x = \int_\Omega \int_{\mathcal{C}\Omega} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x.$$

For the third integral I_3 in (5.2), we can follow exactly the same lines as above using the x/y symmetry. We then conclude that

$$\lim_{n\to\infty} \int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x = \int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Now, for what concerns the right-hand side of (5.1), by virtue of lemma 3.2, for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$ with $Supp\,\varphi = \omega$, there exists a constant $c_\omega > 0$ not depending on n such that

$$0 \leq \left| \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} \right| \leq \frac{|f||\varphi|}{c_\omega^\gamma} \in L^1(\Omega)$$

obtaining by the dominated convergence theorem

$$\lim_{n\to\infty} \int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma} \mathrm{d}x = \int_\Omega \frac{f\varphi}{u^\gamma} \mathrm{d}x$$

and in the last term in (5.1), by the convergence of μ_n to μ we have

$$\lim_{n\to\infty} \int_\Omega \varphi(x)\mu_n(x)\mathrm{d}x = \int_\Omega \varphi(x)\mathrm{d}\mu.$$

Finally, passing to the limit as $n \to +\infty$, we obtain

$$\frac{\alpha(N,s)}{2} \int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \int_\Omega \frac{f\varphi}{u^\gamma} \mathrm{d}x + \int_\Omega \varphi \mathrm{d}\mu$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Therefore, u is a weak solution of (1.1). □

5.2 The case $\gamma > 1$

Proof of Theorem 2.8. By virtue of Lemma 4.4, there exist a subsequence of $\{u_n\}_n$ still indexed by n and a measurable function $v \in W^{s_1,q}_{\rm loc}(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup v \text{ in } W^{s_1,q}_{\rm loc}(\Omega),\\ &u_n \to v \text{ in } L^1_{\rm loc}(\Omega),\\ &u_n \to v \text{ a.e. in } \Omega. \end{aligned}$$

So that defining the function u by $u = v$ in Ω and $u = 0$ in $\mathrm{R}^N\backslash\Omega$, one has

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } W^{s_1,q}_{\rm loc}(\Omega),\\ &u_n \to u \text{ in } L^1_{\rm loc}(\Omega),\\ &u_n \to u \text{ a.e. in } \mathrm{R}^N,\\ &T_k^{\frac{\gamma+1}{2}}(u_n) \to T_k^{\frac{\gamma+1}{2}}(u) \text{ a.e. in } \Omega. \end{aligned}$$

Then for $\varphi \in \mathcal{C}_0^\infty(\Omega)$, we have

$$\frac{|(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}} \to \frac{|(u(x)-u(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}} \text{ a.e. in } Q.$$

Inserting $\varphi \in C_0^\infty(\Omega)$ as a test function in (3.1), we have

$$\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x = \int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}\mathrm{d}x + \int_\Omega \varphi\mu_n \mathrm{d}x. \tag{5.4}$$

Let K be a compact subset of Ω such that $\mathrm{Supp}\,\varphi \subset K$ and $\mathrm{dist}(\mathrm{Supp}\,\varphi, \partial K) > 0$. The integral in the left-hand side of the previous equality can be splitted as

$$\begin{aligned} \int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x &= \int_{\mathrm{R}^N}\int_{\mathrm{R}^N} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ &= \int_K\int_K \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ &\quad + \int_K\int_{\mathcal{C}K} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\ &\quad + \int_{\mathcal{C}K}\int_K \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x. \end{aligned}$$

As in the proof of the Theorem 2.7, the same ideas allow to obtain

$$\lim_{n\to\infty}\int_K\int_K \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x = \int_K\int_K \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x,$$

$$\lim_{n\to\infty}\int_K\int_{\mathcal{C}K} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x = \int_K\int_{\mathcal{C}K} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x$$

and

$$\lim_{n\to\infty}\int_{\mathcal{C}K}\int_K \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x = \int_{\mathcal{C}K}\int_K \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.$$

Then we then conclude that

$$\lim_{n\to\infty}\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x = \int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$. For what concerns the right-hand side of (5.4), it is exactly the same term in Theorem 2.7. Finally, passing to the limit as $n \to +\infty$, we obtain

$$\frac{\alpha(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x = \int_\Omega \frac{f\varphi}{u^\gamma}\mathrm{d}x + \int_\Omega \varphi\mathrm{d}\mu$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$, So u is a weak solution to (1.1). Now, by virtue of lemma 4.3, and Fatou's lemma, we have for every $\omega \subset\subset \Omega$

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|T_k^{\frac{\gamma+1}{2}}(u(x)) - T_k^{\frac{\gamma+1}{2}}(u(y))|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y &\le \liminf_{n\to+\infty}\int_\Omega\int_\Omega \frac{|T_k^{\frac{\gamma+1}{2}}(u_n(x)) - T_k^{\frac{\gamma+1}{2}}(u_n(y))|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &\le C.\end{aligned}$$

It follows that $T_k^{\frac{\gamma+1}{2}}(u) \in H_0^s(\Omega)$, for every $k > 0$. □

6 Regularity of Solutions

Now, we prove some regularities of the solution u of the problem (1.1).

Proposition 6.1 Assume that μ is a Radon measure, $f \in L^1(\Omega)$ and $0 < \gamma \le 1$. Then the solution u of the problem (1.1) obtained by approximation is such that

$$u \in L^r(\Omega), \quad \forall\, r \in \Big(1, \frac{N}{N-2s}\Big).$$

$$|(-\Delta)^{\frac{s}{2}}u| \in L^r(\Omega), \quad \forall\, r \in \Big(1, \frac{N}{N-s}\Big).$$

Proof We follow closely the lines in [25]. By (4.3) and Theorem 2.7, we can apply Fatou's Lemma, we conclude that $u \in L^r(\Omega)$, for every $1 < r < \frac{N}{N-2s}$. Now, we will prove that $|(-\Delta)^{\frac{s}{2}}u_n|$ is bounded in the Marcinkiewicz space $M^{\frac{N}{N-s}}(\Omega)$. We fix $\beta > 0$ and for any positive $k \ge 1$, we have

$$\begin{aligned}\{|(-\Delta)^{\frac{s}{2}}u_n| \ge \beta\} &= \{|(-\Delta)^{\frac{s}{2}}u_n| \ge \beta\ u_n < k\} \cup \{|(-\Delta)^{\frac{s}{2}}u_n| \ge \beta\ u_n \ge k\} \\ &\subset \{|(-\Delta)^{\frac{s}{2}}u_n| \ge \beta\ u_n < k\} \cup \{u_n \ge k\}.\end{aligned}$$

Then

$$\mathrm{meas}(\{|(-\Delta)^{\frac{s}{2}}u_n| \ge \beta,\ u_n < k\}) \le \frac{1}{\beta^2}\int_{\{u_n<k\}} |(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x.$$

By using [25, Corollary 1] and Lemma 4.2, we get

$$\begin{aligned}\mathrm{meas}(\{|(-\Delta)^{\frac{s}{2}}u_n| \ge \beta,\ u_n < k\}) &\le \frac{1}{\beta^2}\int_{\{u_n<k\}} |(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x \\ &\le \frac{1}{\beta^2}\int_{\mathrm{R}^N} |(-\Delta)^{\frac{s}{2}}T_k(u_n)|^2\mathrm{d}x \\ &\le \frac{C(N,s)}{\beta^2}\int_Q \frac{|T_k(u_n)(x) - T_k(u_n)(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \le C\frac{k}{\beta^2}.\end{aligned}$$

By using (4.3), we have

$$\begin{aligned}\mathrm{meas}(\{|(-\Delta)^{\frac{s}{2}}u_n| \ge \beta\}) &\le \mathrm{meas}(\{|(-\Delta)^{\frac{s}{2}}u_n| \ge \beta,\ u_n < k\}) + \mathrm{meas}(\{u_n \ge k\}) \\ &\le C\frac{k}{\beta^2} + \frac{C}{k^{\frac{N}{N-2s}}}.\end{aligned}$$

Choosing $k=\beta^{\frac{N-2s}{N-s}}$, we get

$$\text{meas}(\{|(-\Delta)^{\frac{s}{2}}u_n|\geq\beta\})\leq\frac{C}{\beta^{\frac{N}{N-s}}}.$$

This implies that $|(-\Delta)^{\frac{s}{2}}u_n|$ is bounded in the Marcinkiewicz space $M^{\frac{N}{N-s}}(\Omega)$. So, by the converges almost everywhere in the proof of Theorem 2.7, we can apply Fatou's Lemma, we conclude the result. □

Appendix

In this Appendix we give the functional and technical results we have used in the previous sections. We start with the following inequality whose proof in the cases where $\alpha=1$ can be found [25]. Here we give a simple proof based on the monotony of the truncation functions.

Proposition 6.2 Let $\alpha\geq 1$ and let $v:\mathrm{R}^N\to\mathrm{R}$ be a positive measurable function. Then for every $k>0$ and for every $(x,y)\in\mathrm{R}^N\times\mathrm{R}^N$

$$\Big(G_k(v(x))-G_k(v(y))\Big)\Big(T_k(v(x))^\alpha-T_k(v(y))^\alpha\Big)\geq 0.$$

Proof Let $x,\ y\in\mathrm{R}^N$ be arbitrary. Without loss of generality we can assume that $v(x)\geq v(y)$. Since the functions $s\mapsto T_k(s)$ and $s\mapsto G_k(s)$ are non-decreasing on R, we have

$$T_k(v(x))^\alpha\geq T_k(v(y))^\alpha \text{ and } G_k(v(x))\geq G_k(v(y)).$$

Then

$$(G_k(v(x))-G_k(v(y)))(T_k(v(x))^\alpha-T_k(v(y))^\alpha)\geq 0.$$

□

The next result, well known in classical Sobolev spaces, provides a necessary condition for a function to belong to the fractional Sobolev space $W_0^{s,p}(\Omega)$.

Lemma 6.3 Let Ω be an open set in R^N of class $\mathcal{C}^{0,1}$ with bounded boundary, $1\leq p<+\infty$ and let $0<s<1$. If $u\in W^{s,p}(\Omega)$ with $\text{Supp}\, u$ is a compact set in Ω, then $u\in W_0^{s,p}(\Omega)$.

Proof Let $u\in W^{s,p}(\Omega)$ be a function with $\text{Supp}\, u$ be a compact subset included in Ω. Then there exists an open set ω such that

$$\text{Supp}\, u\subset\omega \text{ and } \overline{\omega}\subset\Omega.$$

Then by [17, Corollary 5.5], there exists a sequence $\{u_n\}_n$ of functions $u_n\in\mathcal{C}_0^\infty(\mathrm{R}^N)$ such that

$$u_n\to u \text{ in norm in } W^{s,p}(\Omega).$$

Let $\varphi\in\mathcal{C}_0^\infty(\omega)$ be such that

$$\varphi=1 \text{ on } \text{Supp}\, u \text{ and } 0\leq\varphi\leq 1, \text{ a.e in } \omega.$$

It is clear that $\varphi u_n\in\mathcal{C}_0^\infty(\omega)$. Therefore, it sufficient to prove that

$$\varphi u_n\to u \text{ in } W^{s,p}(\Omega).$$

Using the fact that $\varphi u=u$ on Ω, we obtain

$$\int_\Omega|\varphi u_n-u|^p\mathrm{d}x=\int_\Omega|\varphi u_n-\varphi u|^p\mathrm{d}x\leq\int_\Omega|u_n-u|^p\mathrm{d}x\to 0.$$

For the second part of the norm $\|\varphi u_n - u\|_{W^{s,p}(\Omega)}$, we can write it as follows.

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{|(\varphi(x)u_n(x)-\varphi(y)u_n(y))-(u(x)-u(y))|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\ =&\int_\Omega\int_\Omega \left|\frac{\varphi(x)u_n(x)-\varphi(y)u_n(y)}{|x-y|^{\frac{N+ps}{p}}}-\frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}}\right|^p\mathrm{d}x\mathrm{d}y\\ =&\int_\Omega\int_\Omega |F_n(x,y)-F(x,y)|^p\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where we have set

$$F_n(x,y)=\frac{\varphi(x)u_n(x)-\varphi(y)u_n(y)}{|x-y|^{\frac{N+ps}{p}}} \text{ and } F(x,y)=\frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}}.$$

Thus, in order to prove that φu_n converges to u in $W^{s,p}(\Omega)$, it is sufficient to prove that up to a subsequence, $\{F_n(x,y)\}$ converges to $F(x,y)$ in norm in $L^p(\Omega\times\Omega)$. Since, up to a subsequence still indexed by n, u_n converges almost everywhere to u, we obtain

$$F_n(x,y)=\frac{\varphi(x)u_n(x)-\varphi(y)u_n(y)}{|x-y|^{\frac{N+ps}{p}}}\to\frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}}=F(x,y) \text{ a.e in } \Omega\times\Omega.$$

The norm convergence of u_n to u in $W^{s,p}(\Omega)$, yields

$$\frac{u_n(x)-u_n(y)}{|x-y|^{\frac{N+ps}{p}}}\to\frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}} \text{ in norm in } L^p(\Omega\times\Omega). \tag{6.1}$$

According to (6.1) and the norm convergence of $\{u_n\}$ in $L^p(\Omega)$, there exist a subsequence of $\{u_n\}$ still indexed by n and two positive functions g in $L^1(\Omega\times\Omega)$ and h in $L^1(\Omega)$ such that

$$\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\le g(x,y) \text{ a.e in } \Omega\times\Omega$$

and

$$|u_n(x)|^p\le h(x) \text{ a.e in } \Omega.$$

So that writing

$$\begin{aligned}|F_n(x,y)|^p&=\frac{\Big|\varphi(x)u_n(x)-\varphi(x)u_n(y)+\varphi(x)u_n(y)-\varphi(y)u_n(y)\Big|^p}{|x-y|^{N+ps}}\\ &\le 2^{p-1}\frac{\Big|u_n(x)-u_n(y)\Big|^p}{|x-y|^{N+ps}}+2^{p-1}\frac{|u_n(y)|^p|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}},\end{aligned}$$

we obtain

$$|F_n(x,y)|^p\le 2^{p-1}|g(x,y)|+2^{p-1}\frac{|h(y)||\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}. \tag{6.2}$$

We need to prove that the function in the second term in the right-hand side in (6.2) belongs to $L^1(\Omega\times\Omega)$. To do so we can write

$$\begin{aligned}\int_\Omega\int_\Omega\frac{|h(y)||\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y=&\int_\Omega|h(y)|\Big[\int_{\Omega\cap|x-y|<1}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\Big]\mathrm{d}y\\ &+\int_\Omega|h(y)|\Big[\int_{\Omega\cap|x-y|\ge1}\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\Big]\mathrm{d}y.\end{aligned}$$

Since φ belongs at least to $\mathcal{C}_0^1(\Omega)$ and $0\le\varphi\le1$, a.e. in ω we have

$$\int_\Omega\int_\Omega\frac{|h(y)||\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\le C_{\rm lip}^p\int_\Omega|h(y)|\Big[\int_{\Omega\cap|z|<1}\frac{\mathrm{d}z}{|z|^{N+p(s-1)}}\Big]\mathrm{d}y$$

$$
\begin{aligned}
&+2^p\int_\Omega |h(y)|\Big[\int_{\Omega\cap|z|\geq 1}\frac{\mathrm{d}z}{|z|^{N+ps}}\Big]\mathrm{d}y\\
&\leq 2\max\Big(C_{\rm lip}^p\frac{|\mathcal{S}^{N-1}|}{p(1-s)},2^p\frac{|\mathcal{S}^{N-1}|}{ps}\Big)\int_\Omega |h(y)|\mathrm{d}y\\
&<+\infty,
\end{aligned}
$$

where $C_{\rm lip}$ stands for the Lipschitz constant of φ and $|\mathcal{S}^{N-1}|$ stands for the Lebesgue measure of the surface area of the unit N-sphere $\mathcal{S}^{N-1}$ of R^N. Applying the dominated convergence theorem, we conclude our claim and thus follows $u\in W_0^{s,p}(\Omega)$. □

Lemma 6.4 Let Ω be an open set in R^N of class $\mathcal{C}^{0,1}$ with bounded boundary, $1\leq p<+\infty$ and let $0<s<1$. Let $\phi:\mathrm{R}\to\mathrm{R}$ be a uniformly Lipschitz function, with $\phi(0)=0$. Then for every $u\in W_0^{s,p}(\Omega)$ one has $\phi(u)\in W_0^{s,p}(\Omega)$.

Proof Let us denote by K the Lipschitz constant of ϕ and let $u\in W_0^{s,p}(\Omega)$. There exists a sequence $\{u_n\}$ of $C_0^\infty(\Omega)$ functions which converges to u in norm in $W^{s,p}(\Omega)$. That is there exists $n_0\in\mathrm{N}$ such that for all $n\in\mathrm{N}$ with $n\geq n_0$ one has

$$\|u_n-u\|_{W^{s,p}(\Omega)}<1.$$

Defining $v_n=\phi(u_n)$, $G_n(x,y)=u_n(x)-u_n(y)$ and $G(x,y)=u(x)-u(y)$, we can write for every $n\geq n_0$

$$
\begin{aligned}
\int_\Omega\int_\Omega\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y&=\int_\Omega\int_\Omega\frac{|\phi(u_n)(x)-\phi(u_n)(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\
&\leq K^p\int_\Omega\int_\Omega\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\
&=K^p\int_\Omega\int_\Omega\frac{|G_n(x,y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\
&\leq 2^{p-1}K^p\int_\Omega\int_\Omega\frac{|G_n(x,y)-G(x,y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\
&\quad+2^{p-1}K^p\int_\Omega\int_\Omega\frac{|G(x,y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\
&=2^{p-1}K^p\|u_n-u\|^p_{W^{s,p}(\Omega)}+2^{p-1}K^p\|u\|^p_{W^{s,p}(\Omega)}\\
&\leq C_0
\end{aligned}
$$

and

$$\|v_n\|_{L^p(\Omega)}\leq K\|u_n\|_{L^p(\Omega)}\leq K\|u_n-u\|_{W^{s,p}(\Omega)}+K\|u\|_{W^{s,p}(\Omega)}\leq C_1,$$

C_0 and C_1 are two constants not depending on n. Thus, $\{v_n\}$ is uniformly bounded in $W^{s,p}(\Omega)$. Since by $\phi(0)=0$ the function v_n is compactly supported in Ω, so that by Lemma 6.3 we obtain $v_n\in W_0^{s,p}(\Omega)$. Now, we prove that

$$v_n\to\phi(u)\text{ in }W^{s,p}(\Omega).$$

Since the sequence $\{u_n\}$ converges to u in norm in $W^{s,p}(\Omega)$, then for a subsequence of $\{u_n\}$, still indexed by n, we have

$$u_n\to u\text{ a.e. in }\Omega.$$

Then, it follows

$$v_n=\phi(u_n)\to\phi(u)\text{ a.e. in }\Omega.$$

Furthermore,

$$\|v_n - \phi(u)\|_{L^p(\Omega)} = \|\phi(u_n) - \phi(u)\|_{L^p(\Omega)} \leq K\|u_n - u\|_{L^p(\Omega)} \to 0.$$

On the other hand we can write

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{|(v_n(x) - \phi(u)(x) - (v_n(y) - \phi(u)(y))|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\ =& \int_\Omega\int_\Omega \left|\frac{v_n(x) - v_n(y)}{|x-y|^{\frac{N+ps}{p}}} - \frac{\phi(u(x)) - \phi(u(y))}{|x-y|^{\frac{N+ps}{p}}}\right|^p \mathrm{d}x\mathrm{d}y\\ =& \int_\Omega\int_\Omega |F_n(x,y) - F(x,y)|^p\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where we noted

$$F_n(x,y) = \frac{v_n(x) - v_n(y)}{|x-y|^{\frac{N+ps}{p}}} \quad \text{and} \quad F(x,y) = \frac{\phi(u(x)) - \phi(u(y))}{|x-y|^{\frac{N+ps}{p}}}.$$

In order to show that v_n converges to $\phi(u)$ in $W^{s,p}(\Omega)$, it sufficient to prove that for a subsequence of $\{F_n(x,y)\}_{n\geq 1}$, still denoted by $\{F_n(x,y)\}_{n\geq 1}$, $\|F_n(x,y) - F(x,y)\|_{L^p(\Omega\times\Omega)} \to 0$. By the almost everywhere convergence of v_n to $\phi(u)$, we have

$$F_n(x,y) = \frac{v_n(x) - v_n(y)}{|x-y|^{\frac{N+ps}{p}}} \to \frac{\phi(u)(x) - \phi(u)(y)}{|x-y|^{\frac{N+ps}{p}}} = F(x,y), \quad \text{a.e. in } \Omega\times\Omega.$$

Observe that the norm convergence of u_n to u in $W^{s,p}(\Omega)$ implies

$$\frac{u_n(x) - u_n(y)}{|x-y|^{\frac{N+ps}{p}}} \to \frac{u(x) - u(y)}{|x-y|^{\frac{N+ps}{p}}} \text{ in norm in } L^p(\Omega\times\Omega).$$

So that since

$$|F_n(x,y)| \leq K\frac{|u_n(x) - u_n(y)|}{|x-y|^{\frac{N+ps}{p}}},$$

the sequence $\{|F_n(x,y)|^p\}_n$ is then equi-integrable. Applying Vitali's theorem we get $\|F_n(x,y) - F(x,y)\|_{L^p(\Omega\times\Omega)} \to 0$ which in turn implies $\|v_n - \phi(u)\|_{W^{s,p}(\Omega)}$ as $n \to +\infty$. Since the sequence $\{v_n\}$ belongs to the closed space $W_0^{s,p}(\Omega)$ forces the limit $\phi(u)$ to belong to $W_0^{s,p}(\Omega)$. □

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