Acta Mathematica Scientia, 2020, **40B**(4): 1105–1115
https://doi.org/10.1007/s10473-020-0416-y

Mathematica Acta Scientia
数学物理学报
http://actams.wipm.ac.cn

ON SELECTIONS OF SET-VALUED EULER-LAGRANGE INCLUSIONS WITH APPLICATIONS*

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Abstract We discuss the set-valued dynamics related to the theory of functional equations. We look for selections of convex set-valued functions satisfying set-valued Euler-Lagrange inclusions. We improve and extend upon some of the results in [13, 20], but under weaker assumptions. Some applications of our results are also provided.

Key words set-valued dynamics; Euler-Lagrange inclusion; composite operator; selection

2010 MR Subject Classification 54C60; 54C65; 28B20; 39B82; 52A07

1 Introduction

Functional inclusions play a significant role in various branches of mathematics and they are a tool for defining many notions of set-valued analysis, e.g., linear, affine, convex, concave, subadditive, superadditive, subquadratic and superquadratic set-valued functions. Finding a selection of such set-valued functions, with some special properties, is one of the main problems of set-valued analysis (see [1, 3, 4, 8, 10, 16, 18, 19, 23, 24, 26, 27]).

Park et al. [15, 20] investigated the stability of some set-valued functional equations. Brzdęk and Piszczek [4–6, 21, 22] obtained many results on selections of some set-valued functional equations satisfying some inclusions and on the approximation of those inclusions. Let us recall that the notion of stability for functional equations was motivated by a problem of Ulam [28] and a paper of Hyers [12] in which was published a solution to it (for further information, see the recent monograph [7]).

In the rest of this paper, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, K stands for a commutative group, Y is a real Banach space and $\ell, \alpha_1, \cdots, \alpha_n$ are fixed positive integers, unless explicitly stated otherwise.

The main goal of this paper is to obtain some results on selections of a convex set-valued

*Received March 19, 2019; revised August 30, 2019.

function $F : K \to 2^Y$ satisfying one of the following Euler-Lagrange functional inclusions:

$$\begin{aligned}&\biguplus_{\alpha_2x_2,\cdots,\alpha_nx_n}^{n-1} F(\alpha_1x_1) + 2^{n-1}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2F(x_1)\\ &\subseteq 2^{n-2}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2\biguplus_{x_i}F(x_1) + 2^{n-1}\alpha_1^{\ell}F(x_1)\end{aligned} \tag{1.1}$$

and

$$\begin{aligned}&\biguplus_{\alpha_2x_2,\cdots,\alpha_nx_n}^{n-1} F(\alpha_1x_1) + 2^{n-1}\sum_{i=1}^{n}\alpha_i^2\Big(\sum_{j=1,j\neq i}^{n}\alpha_j^2\Big)F(x_i)\\ &\subseteq 2^{n-2}\sum_{1\leq i<j\leq n}\alpha_i^2\alpha_j^2\biguplus_{x_j}F(x_i) + 2^{n-1}\sum_{i=1}^{n}\alpha_i^4F(x_i)\end{aligned} \tag{1.2}$$

for all $x_1,\cdots,x_n \in K$. Here, the operator $\biguplus_{x_2} F(x_1)$ is defined by the formula $\biguplus_{x_2} F(x_1) = F(x_1+x_2)+F(x_1-x_2)$. The composite operator $\biguplus_{x_2,\cdots,x_{n+1}}^{n} F(x_1)$ is defined by $\biguplus_{x_2,\cdots,x_{n+1}}^{n} F(x_1) = \biguplus_{x_{n+1}}\Big(\biguplus_{x_2,\cdots,x_n}^{n-1} F(x_1)\Big)$ for all $n \in \mathbb{N}\setminus\{1\}$. Note that

$$\biguplus_{x_2,x_3,0,\cdots,0}^{n-1} F(x_1) = 2^{n-3}\biguplus_{x_2,x_3}^{2} F(x_1) = 2^{n-3}\biguplus_{x_3,x_2}^{2} F(x_1)$$

and

$$\biguplus_{x_2,\cdots,x_{n+1}}^{n} F(x_1) = \biguplus_{x_2,\cdots,x_n}^{n-1} F(x_1+x_{n+1}) + \biguplus_{x_2,\cdots,x_n}^{n-1} F(x_1-x_{n+1}).$$

In particular, the inclusion (1.1) includes the n-dimensional Euler-Lagrange cubic inclusion for $\ell = 3$ (see [13]), the cubic inclusion for $n = 2$, $\ell = 3$, $\alpha_1 = 2$ and $\alpha_2 = 1$ (see [20]), the quadratic inclusion for $n = 2$, $\ell = 2$, $\alpha_1 = 2$ and $\alpha_2 = 1$ (the equation related to this inclusion has been studied in [9]) and the Cauchy inclusion for $n = 2$, $\ell = 1$, $\alpha_1 = 2$ and $\alpha_2 = 1$ (the equation related to this inclusion has been studied in [2]). Also, the inclusion (1.2) includes the n-dimensional quartic inclusion for $\alpha_1 = \cdots = \alpha_n = 1$ (the equation related to this inclusion has been studied in [14]) and the quartic inclusion for $n = 2$, $\alpha_1 = 2$ and $\alpha_2 = 1$ (see [20]).

By some results in [17, 21, 25], we extend the conclusions of [13, 20], but under weaker assumptions. Furthermore, a few applications of our results to the stability of some functional equations are given. Our results can be regarded as an important extension of stability results corresponding to single-valued functional equations.

2 Euler-Lagrange Functional Inclusions in Several Variables

We start this section by recalling some basic concepts. In a real normed space Y, we denote by $n(Y)$ the family of all nonempty subsets of Y and we define the following families of sets:

$$\begin{aligned}&\mathrm{ccl}(Y) := \{A \in n(Y) :\ A \text{ is a closed and convex set}\},\\ &\mathrm{cclz}(Y) := \{A \in n(Y) :\ A \text{ is a closed and convex set containing zero}\},\\ &\mathrm{ccz}(Y) := \{A \in n(Y) :\ A \text{ is a compact and convex set containing zero}\}.\end{aligned}$$

For $A, B \in n(Y)$ and $\lambda \in \mathbb{R}$, the Minkowski addition is defined as $A+B = \{x+y :\ x \in A,\ y \in B\}$ and the scalar multiplication as $\lambda A = \{\lambda x :\ x \in A\}$. We say that a set C is the

Hukuhara difference of A and B, i.e., $C = A - B$, whenever $A = B + C$. If this difference exists, then it is unique (by Lemma 2.1 below). The number $\delta(A) := \sup\{\|a-b\| :\ a, b \in A\}$ is said to be the diameter of $A \in n(Y)$. Let D be a nonempty set. Any function $f : D \to Y$ such that $f(x) \in F(x)$ for all $x \in K$ is said to be a selection of the set-valued function $F : D \to n(Y)$.

Lemma 2.1 (see [17, 25]) Let Y be a real normed space, $\beta, \gamma \in \mathbb{R}$ and $A, B, C \in n(Y)$. Then,

(i) $\beta(A+B) = \beta A + \beta B$ and $(\beta+\gamma)A \subseteq \beta A + \gamma A$. If, additionally, A is convex and $\beta \geq \gamma \geq 0$, then we have $(\beta+\gamma)A = \beta A + \gamma A$;

(ii) If $B \in \mathrm{ccl}(Y)$, C is bounded and $A + C \subseteq B + C$, then $A \subseteq B$. If, additionally, $A \in ccl(Y)$ and $A + C = B + C$, then $A = B$.

Now we deal with some results corresponding to inclusions in a single variable and applications to the inclusions in several variables.

Theorem 2.2 (see [21]) Let $\mu \in (0, +\infty)$, S be a nonempty set, (X, d) be a metric space, $\tau : S \to S$ be a function defined by $\tau^0(x) = x$ for $x \in S$ and $\tau^{n+1} = \tau^n \circ \tau$ for $n \in \mathbb{N}_0$, $F : S \to n(X)$, $\Gamma : X \to X$, $d(\Gamma(x), \Gamma(y)) \leq \mu d(x, y)$ for all $x, y \in X$, and $\lim\limits_{n\to\infty} \mu^n \delta(F(\tau^n(x))) = 0$ for all $x \in S$. Then,

(i) If X is complete and $\Gamma(F(\tau(x))) \subseteq F(x)$ for all $x \in S$, then the limit $\lim\limits_{n\to\infty} \mathrm{cl}\,\Gamma^n \circ F \circ \tau^n(x) =: f(x)$ exists for each $x \in S$, and f is a unique selection of the multifunction $\mathrm{cl}\,F$ such that $\Gamma \circ f \circ \tau = f$, where $\mathrm{cl}\,F$ is defined by $(\mathrm{cl}\,F)(x) = \mathrm{cl}\,F(x)$, $x \in S$;

(ii) If $F(x) \subseteq \Gamma(F(\tau(x)))$ for all $x \in S$, then F is single-valued and $\Gamma \circ F \circ \tau = F$.

We are now going to deal with the inclusion (1.1).

Theorem 2.3 Suppose $F : K \to \mathrm{cclz}(Y)$ is a set-valued function such that $\sup\limits_{x\in K} \delta(F(x)) < +\infty$ and there exists a $\jmath \in \{2, \cdots, n\}$ such that $\alpha_1 \neq \alpha_\jmath = 1$. Then,

(i) If F satisfies the inclusion (1.1), there exists a unique selection $f : K \to Y$ of F such that, for all $x_1, x_\jmath \in K$,

(1) $f(x_1 + x_\jmath) = f(x_1) + f(x_\jmath)$ when $\ell = 1$,

(2) $\biguplus_{x_\jmath} f(x_1) = 2f(x_1) + 2f(x_\jmath)$ when $\ell = 2$,

(3) $\biguplus_{x_\jmath} f(2x_1) = 2\biguplus_{x_\jmath} f(x_1) + 12f(x_1)$ when $\ell = 3$;

(ii) If

$$
\begin{aligned}
&2^{n-2}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2 \biguplus_{x_i} F(x_1) + 2^{n-1}\alpha_1^{\ell}F(x_1) \\
&\subseteq \biguplus_{\alpha_2x_2,\cdots,\alpha_nx_n}^{n-1} F(\alpha_1x_1) + 2^{n-1}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2 F(x_1)
\end{aligned}
\tag{2.1}
$$

for all $x_1, \cdots, x_n \in K$, F is single-valued.

Proof (i) Setting $x_1 = x$ and $x_i = 0$ for $(i = 2, 3, \cdots, n)$ in (1.1), we have

$$
\begin{aligned}
&\Big(\underbrace{F(\alpha_1x) + \cdots + F(\alpha_1x)}_{2^{n-1}\,\text{times}}\Big) + 2^{n-1}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2F(x) \\
&\subseteq 2^{n-2}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2\left(F(x) + F(x)\right) + 2^{n-1}\alpha_1^{\ell}F(x)
\end{aligned}
$$

for all $x \in K$. Since the set $F(x)$ is convex, we can conclude from Lemma 2.1 (i) that

$$2^{n-1}F(\alpha_1 x) + 2^{n-1}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2 F(x) \subseteq 2^{n-1}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2 F(x) + 2^{n-1}\alpha_1^{\ell}F(x)$$

for all $x \in K$. Using Lemma 2.1 (ii), one obtains

$$\alpha_1^{-\ell}F(\alpha_1 x) \subseteq F(x)$$

for all $x \in K$. Next, by Theorem 2.2, with

$$\Gamma(x) = \alpha_1^{-\ell}x, \qquad \tau(x) = \alpha_1 x, \qquad x \in K,$$

for every $x \in K$ there exists the limit

$$\lim_{m\to+\infty}\Gamma^m\left(F\left(\tau^m(x)\right)\right) = \lim_{m\to+\infty}\alpha_1^{-\ell m}F(\alpha_1^m x) = f(x),$$

and moreover

$$f(x) \in F(x), \qquad x \in K.$$

Thus, in view of (1.1), for every $x_1, \cdots, x_n \in K$, $m \in \mathbb{N}$ one has

$$\begin{aligned}
&\alpha_1^{-\ell m}\biguplus_{\alpha_2\alpha_1^m x_2,\cdots,\alpha_n\alpha_1^m x_n}^{n-1} F(\alpha_1^{m+1}x_1) + 2^{n-1}\alpha_1^{\ell(1-m)-2}\sum_{i=2}^{n}\alpha_i^2 F(\alpha_1^m x_1)\\
&\subseteq 2^{n-2}\alpha_1^{\ell(1-m)-2}\sum_{i=2}^{n}\alpha_i^2\biguplus_{\alpha_1^m x_i} F(\alpha_1^m x_1) + 2^{n-1}\alpha_1^{\ell(1-m)}F(\alpha_1^m x_1),
\end{aligned}$$

and letting $m \to \infty$, we observe that

$$\begin{aligned}
&\biguplus_{\alpha_2 x_2,\cdots,\alpha_n x_n}^{n-1} f(\alpha_1 x_1) + 2^{n-1}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2 f(x_1)\\
&= 2^{n-2}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2\biguplus_{x_i} f(x_1) + 2^{n-1}\alpha_1^{\ell}f(x_1)
\end{aligned} \tag{2.2}$$

for all $x_1, \cdots, x_n \in K$. Putting $x_i = 0$ for $(i = 1, 2, \cdots, n)$ in (2.2), we get $f(0) = 0$, since $\alpha_1 \neq 1$. Setting $x_i = 0$ for $(i = 2, \cdots, n$ and $i \neq \jmath)$ in (2.2) and using $f(0) = 0$, we have

$$2^{n-2}\biguplus_{\alpha_\jmath x_\jmath} f(\alpha_1 x_1) = 2^{n-2}\alpha_1^{\ell-2}\alpha_\jmath^2\biguplus_{x_\jmath} f(x_1) + 2^{n-1}\alpha_1^{\ell-2}(\alpha_1^2 - \alpha_\jmath^2)f(x_1)$$

for all $x_1, x_\jmath \in K$. Since $\alpha_\jmath = 1$, we can conclude that

$$\biguplus_{x_\jmath} f(\alpha_1 x_1) = \alpha_1^{\ell-2}\biguplus_{x_\jmath} f(x_1) + 2\alpha_1^{\ell-2}(\alpha_1^2 - 1)f(x_1)$$

for all $x_1, x_\jmath \in K$. Then it follows from Theorem 2.1 of [11] that, for all $x_1, x_\jmath \in K$, if $\ell = 1$, then $f(x_1 + x_\jmath) = f(x_1) + f(x_\jmath)$, if $\ell = 2$, then $\biguplus_{x_\jmath} f(x_1) = 2f(x_1) + 2f(x_\jmath)$ (i.e., f is quadratic) and if $\ell = 3$, then $\biguplus_{x_\jmath} f(2x_1) = 2\biguplus_{x_\jmath} f(x_1) + 12f(x)$ (i.e., f is cubic).

Next, let us prove the uniqueness of f. Suppose that f and ρ are selections of F. We have $(\alpha_1 m)^\ell f(x) = f(\alpha_1 mx) \in F(\alpha_1 mx)$ and $(\alpha_1 m)^\ell \rho(x) = \rho(\alpha_1 mx) \in F(\alpha_1 mx)$ for all $x \in K$ and $m \in \mathbb{N}$. Thus

$$\begin{aligned}
(\alpha_1 m)^\ell \|f(x) - \rho(x)\| &= \left\|(\alpha_1 m)^\ell f(x) - (\alpha_1 m)^\ell \rho(x)\right\|\\
&= \|f(\alpha_1 mx) - \rho(\alpha_1 mx)\| \leq 2\delta\left(F(\alpha_1 mx)\right)
\end{aligned}$$

for all $x \in K$ and $m \in \mathbb{N}$. It follows from $\sup\limits_{x\in K} \delta(F(x)) < +\infty$ that $f(x) = \rho(x)$ for all $x \in K$.

(ii) Letting $x_1 = x$ and $x_i = 0$ for $(i = 2, 3, \cdots, n)$ in (2.1) and using the convexity of $F(x)$ and Lemma 2.1, we obtain

$$F(x) \subseteq \alpha_1^{-\ell} F(\alpha_1 x)$$

for all $x \in K$. Therefore, using Theorem 2.2 with Γ and τ defined as in the previous case, we deduce that if $\ell = 1$, then F is single-valued and additive, if $\ell = 2$, then F is single-valued and quadratic and if $\ell = 3$, then F is single-valued and cubic. □

We are now going to deal with the inclusion (1.2).

Theorem 2.4 Suppose that $F : K \to \text{cclz}(Y)$ is a set-valued function such that $\sup\limits_{x\in K} \delta(F(x)) < +\infty$, $\sum\limits_{i=1}^{n} \alpha_i^2 \big(\alpha_i^2 - \sum\limits_{j=i+1}^{n} \alpha_j^2\big) > 1$ and there exists a $\jmath \in \{2, \cdots, n\}$ such that $\alpha_1 \neq \alpha_\jmath = 1$. Then,

(i) If F satisfies the inclusion (1.2), there exists a unique selection $f : K \to Y$ of F such that $\biguplus_{x_\jmath} f(2x_1) + 6f(x_\jmath) = 4 \biguplus_{x_\jmath} f(x_1) + 24 f(x_1)$ for all $x_1, x_\jmath \in K$;

(ii) If

$$\begin{aligned} &2^{n-2} \sum_{1\le i<j\le n} \alpha_i^2 \alpha_j^2 \biguplus_{x_j} F(x_i) + 2^{n-1} \sum_{i=1}^{n} \alpha_i^4 F(x_i) \\ &\subseteq \biguplus_{\alpha_2 x_2, \cdots, \alpha_n x_n}^{n-1} F(\alpha_1 x_1) + 2^{n-1} \sum_{i=1}^{n} \alpha_i^2 \Big(\sum_{j=1, j\neq i}^{n} \alpha_j^2 \Big) F(x_i) \end{aligned} \tag{2.3}$$

for all $x_1, \cdots, x_n \in K$, then F is single-valued.

Proof (i) Letting $x_i = 0$ for $i = 1, 2, \cdots, n$ in (1.2), we get

$$\begin{aligned} &\Big(\underbrace{F(0) + \cdots + F(0)}_{2^{n-1}\,\text{times}} \Big) + 2^{n-1} \sum_{i=1}^{n} \alpha_i^2 \Big(\sum_{j=1, j\neq i}^{n} \alpha_j^2 \Big) F(0) \\ &\subseteq 2^{n-2} \sum_{1\le i<j\le n} \alpha_i^2 \alpha_j^2 \left(F(0) + F(0)\right) + 2^{n-1} \sum_{i=1}^{n} \alpha_i^4 F(0). \end{aligned}$$

Now it follows from the convexity of $F(0)$ and Lemma 2.1 that

$$F(0) + \sum_{1\le i<j\le n} \alpha_j^2 \alpha_i^2 F(0) \subseteq \sum_{i=1}^{n} \alpha_i^4 F(0).$$

Using $\sum\limits_{i=1}^{n} \alpha_i^2 \big(\alpha_i^2 - \sum\limits_{j=i+1}^{n} \alpha_j^2\big) > 1$ and Lemma 2.1 (ii), one obtains

$$\{0\} \subseteq F(0). \tag{2.4}$$

Letting $x_1 = x$ and $x_i = 0$ for $(i = 2, 3, \cdots, n)$ in (1.2), we have

$$\begin{aligned} &\Big(\underbrace{F(\alpha_1 x) + \cdots + F(\alpha_1 x)}_{2^{n-1}\,\text{times}} \Big) + 2^{n-1} \alpha_1^2 \sum_{j=2}^{n} \alpha_j^2 F(x) + 2^{n-1} \sum_{i=2}^{n} \alpha_i^2 \Big(\sum_{j=1, j\neq i}^{n} \alpha_j^2 \Big) F(0) \\ &\subseteq 2^{n-2} \alpha_1^2 \sum_{j=2}^{n} \alpha_j^2 \left(F(x) + F(x)\right) + 2^{n-2} \sum_{2\le i<j\le n} \alpha_i^2 \alpha_j^2 \left(F(0) + F(0)\right) \end{aligned}$$

$$+ 2^{n-1}\alpha_1^4 F(x) + 2^{n-1}\sum_{i=2}^{n}\alpha_i^4 F(0)$$

for all $x \in K$. Hence, from the convexity of $F(x)$ and Lemma 2.1, we see that

$$F(\alpha_1 x) + \sum_{1\le i<j\le n}\alpha_j^2\alpha_i^2 F(0) \subseteq \alpha_1^4 F(x) + \sum_{i=2}^{n}\alpha_i^4 F(0) \tag{2.5}$$

for all $x \in K$. It follows from (2.4), (2.5) and Lemma 2.1 (i) that

$$\begin{aligned} F(\alpha_1 x) &\subseteq F(\alpha_1 x) + \Big(\sum_{1\le i<j\le n}\alpha_j^2\alpha_i^2 - \sum_{i=2}^{n}\alpha_i^4\Big)F(0) \\ &\subseteq F(\alpha_1 x) + \sum_{1\le i<j\le n}\alpha_j^2\alpha_i^2 F(0) - \sum_{i=2}^{n}\alpha_i^4 F(0) \\ &\subseteq \alpha_1^4 F(x) \end{aligned}$$

for all $x \in K$. Next, by Theorem 2.2, with

$$\Gamma(x) = \alpha_1^{-4}x, \qquad \tau(x) = \alpha_1 x, \qquad x \in K,$$

for every $x \in K$ there exists the limit

$$\lim_{m\to+\infty}\Gamma^m\left(F\left(\tau^m(x)\right)\right) = \lim_{m\to+\infty}\alpha_1^{-4m}F(\alpha_1^m x) = f(x),$$

and moreover,

$$f(x) \in F(x), \qquad x \in K.$$

From (1.2) we see that, for every $x_1, \cdots, x_n \in K$, $m \in \mathbb{N}$,

$$\begin{aligned} &\alpha_1^{-4m}\biguplus_{\alpha_2\alpha_1^m x_2,\cdots,\alpha_n\alpha_1^m x_n}^{n-1} F(\alpha_1^{m+1}x_1) + 2^{n-1}\alpha_1^{-4m}\sum_{i=1}^{n}\alpha_i^2\Big(\sum_{j=1,j\ne i}^{n}\alpha_j^2\Big)F(\alpha_1^m x_i) \\ &\subseteq 2^{n-2}\alpha_1^{-4m}\sum_{1\le i<j\le n}\alpha_i^2\alpha_j^2\biguplus_{\alpha_1^m x_j}F(\alpha_1^m x_i) + 2^{n-1}\alpha_1^{-4m}\sum_{i=1}^{n}\alpha_i^4 F(\alpha_1^m x_i), \end{aligned}$$

and letting $m \to \infty$, we observe that

$$\begin{aligned} &\biguplus_{\alpha_2 x_2,\cdots,\alpha_n x_n}^{n-1} f(\alpha_1 x_1) + 2^{n-1}\sum_{i=1}^{n}\alpha_i^2\Big(\sum_{j=1,j\ne i}^{n}\alpha_j^2\Big)f(x_i) \\ &= 2^{n-2}\sum_{1\le i<j\le n}\alpha_i^2\alpha_j^2\biguplus_{x_j}f(x_i) + 2^{n-1}\sum_{i=1}^{n}\alpha_i^4 f(x_i) \end{aligned} \tag{2.6}$$

for all $x_1, \cdots, x_n \in K$. Putting $x_i = 0$ for $(i = 1, 2, \cdots, n)$ in (2.6), we obtain

$$2^{n-1}\Big(\sum_{i=1}^{n}\alpha_i^2\Big(\alpha_i^2 - \sum_{j=i+1}^{n}\alpha_j^2\Big) - 1\Big)f(0) = 0,$$

so, $f(0) = 0$. Setting $x_i = 0$ for $(i = 2, \cdots, n$ and $i \ne \jmath)$ in (2.2) and using $f(0) = 0$, we have

$$2^{n-2}\biguplus_{\alpha_\jmath x_\jmath}f(\alpha_1 x_1) = 2^{n-2}\alpha_1^2\alpha_\jmath^2\biguplus_{x_\jmath}f(x_1) + 2^{n-1}\alpha_1^2(\alpha_1^2 - \alpha_\jmath^2)f(x_1) + 2^{n-1}\alpha_\jmath^2(\alpha_\jmath^2 - \alpha_1^2)f(x_\jmath)$$

for all $x_1, x_\jmath \in K$. Since $\alpha_\jmath = 1$, we can conclude that

$$\biguplus_{x_\jmath}f(\alpha_1 x_1) = \alpha_1^2\biguplus_{x_\jmath}f(x_1) + 2\alpha_1^2(\alpha_1^2 - 1)f(x_1) + 2(1 - \alpha_1^2)f(x_\jmath)$$

for all $x_1, x_j \in K$. Then it follows from Theorem 2.1 of [11] that, for all $x_1, x_j \in K$, $\biguplus_{x_j} f(2x_1) + 6f(x_j) = 4\biguplus_{x_j} f(x_1) + 24f(x_1)$ (i.e., f is quartic).

Also the uniqueness of f can be easily deduced from Theorem 2.3.

(ii) Letting $x_1 = x$ and $x_i = 0$ for $(i = 2, 3, \cdots, n)$ in (2.3) and using the convexity of $F(x)$ and Lemma 2.1, we obtain

$$\alpha_1^4 F(x) + \sum_{i=2}^{n} \alpha_i^4 F(0) \subseteq F(\alpha_1 x) + \sum_{1\le i<j\le n} \alpha_j^2\alpha_i^2 F(0) \tag{2.7}$$

for all $x \in K$, which, by replacing x by 0, yields

$$\sum_{i=1}^{n} \alpha_i^4 F(0) \subseteq F(0) + \sum_{1\le i<j\le n} \alpha_i^2\alpha_j^2 F(0),$$

and thus we get the inclusion

$$F(0) \subseteq \{0\}. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\alpha_1^4 F(x) \subseteq F(\alpha_1 x) + \left(\sum_{1\le i<j\le n} \alpha_i^2\alpha_j^2 - \sum_{i=2}^{n} \alpha_i^4\right) F(0) \subseteq F(\alpha_1 x)$$

for all $x \in K$. Hence

$$F(x) \subseteq \alpha_1^{-4} F(\alpha_1 x)$$

for all $x \in K$. Thus, using Theorem 2.2 with Γ and τ defined as in the previous case, we deduce that F must be single-valued and quartic. □

Theorem 2.4 with $n = 2$ implies the following:

Corollary 2.5 Suppose $F : K \to \mathrm{cclz}(Y)$ is a set-valued function such that $\sup\limits_{x\in K} \delta(F(x)) < +\infty$. Then,

(i) If F satisfies the inclusion (1.2) with $n = 2$ and $\alpha_1 \neq \alpha_2 = 1$, there exists a unique selection $f : K \to Y$ of F such that f is quartic;

(ii) If

$$\alpha_1^2 \biguplus_{x_2} F(x_1) + 2\left(\alpha_1^4 F(x_1) + F(x_2)\right) \subseteq \biguplus_{x_2} F(\alpha_1 x_1) + 2\alpha_1^2 \left(F(x_1) + F(x_2)\right)$$

for all $x_1, x_2 \in K$, F is single-valued.

From Theorems 2.3 and 2.4, we can deduce the same conclusions as in [13, 20], but under weaker assumptions. Corollary 2.5 with $\alpha_1 = 2$ implies.

Corollary 2.6 (see [20, Theorem 5.1]) Let $F : K \to \mathrm{cclz}(Y)$ be a set-valued function such that $\sup\limits_{x\in K} \delta(F(x)) < +\infty$. If F satisfies the inclusion (1.2) with $n = 2$, $\alpha_1 = 2$ and $\alpha_2 = 1$, then there exists a unique selection $f : K \to Y$ of F such that f is quartic.

As a consequence of Theorem 2.4, we obtain the following result:

Corollary 2.7 (see [13, Theorem 2.5]) Let $F : K \to \mathrm{cclz}(Y)$ be a set-valued function such that $\sup\limits_{x\in K} \delta(F(x)) < +\infty$. If F satisfies the inclusion (1.2) with $\alpha_1 \neq \alpha_n = 1$ and $\sum\limits_{i=1}^{n-1} \alpha_i^2(\alpha_i^2 - \sum\limits_{j=i+1}^{n-1} \alpha_j^2) > \sum\limits_{i=1}^{n-1} \alpha_i^2$, then there exists a unique selection $f : K \to Y$ of F such that f is quartic.

From Theorem 2.3, we easily obtain the following results:

Corollary 2.8 (see [13, Theorem 2.3]) Let $F : K \to \text{cclz}(Y)$ be a set-valued function such that $\sup_{x\in K} \delta(F(x)) < +\infty$. If F satisfies the inclusion (1.1) with $\ell = 3$ and $\alpha_1 \neq \alpha_n = 1$, then there exists a unique selection $f : K \to Y$ of F such that f is cubic.

Corollary 2.9 (see [20, Theorem 4.1]) Let $F : K \to \text{cclz}(Y)$ be a set-valued function such that $\sup_{x\in K} \delta(F(x)) < +\infty$. If F satisfies the inclusion (1.1) with $n = 2$, $\alpha_1 = 2$ and $\alpha_2 = 1$, then there exists a unique selection $f : K \to Y$ of F such that f is cubic.

3 Applications

Some significant applications follow on from these results. We assume throughout this section that $W \in \text{ccz}(Y)$ and that there exists a $\jmath \in \{2, \cdots, n\}$ such that $\alpha_1 \neq \alpha_\jmath = 1$.

Theorem 3.1 If $f : K \to Y$ satisfies

$$\biguplus_{\alpha_2x_2,\cdots,\alpha_nx_n}^{n-1} f(\alpha_1x_1) - 2^{n-2}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2\biguplus_{x_i} f(x_1) + 2^{n-1}\alpha_1^{\ell}\left(\alpha_1^{-2}\sum_{i=2}^{n}\alpha_i^2 - 1\right)f(x_1) \in W \quad (3.1)$$

for all $x_1, \cdots, x_n \in K$, then there exists a unique function $\omega : K \to Y$ such that ω satisfies (2.2) and for all $x \in K$,

$$\omega(x) - f(x) \in \frac{1}{2^{n-1}\left(\alpha_1^{\ell} - 1\right)}W.$$

Proof Let $F(x) := f(x) + \left(2^{n-1}\alpha_1^{\ell} - 2^{n-1}\right)^{-1}W$ for $x \in K$. Then

$$\begin{aligned}
&\biguplus_{\alpha_2x_2,\cdots,\alpha_nx_n}^{n-1} F(\alpha_1x_1) + 2^{n-1}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2F(x_1)\\
&= \biguplus_{\alpha_2x_2,\cdots,\alpha_nx_n}^{n-1} f(\alpha_1x_1) + 2^{n-1}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2f(x_1) + \frac{1 + \alpha_1^{\ell-2}\sum\limits_{i=2}^{n}\alpha_i^2}{\alpha_1^{\ell} - 1}W\\
&\subseteq 2^{n-2}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2\biguplus_{x_i} f(x_1) + 2^{n-1}\alpha_1^{\ell}f(x_1) + \frac{1 + \alpha_1^{\ell-2}\sum\limits_{i=2}^{n}\alpha_i^2}{\alpha_1^{\ell} - 1}W + W\\
&= \alpha_1^{\ell-2}\alpha_2^2\left(2^{n-2}\biguplus_{x_2} f(x_1) + \frac{1}{\alpha_1^{\ell} - 1}W\right) + \alpha_1^{\ell-2}\alpha_3^2\left(2^{n-2}\biguplus_{x_3} f(x_1) + \frac{1}{\alpha_1^{\ell} - 1}W\right)\\
&\quad + \cdots + \alpha_1^{\ell-2}\alpha_n^2\left(2^{n-2}\biguplus_{x_n} f(x_1) + \frac{1}{\alpha_1^{\ell} - 1}W\right) + \alpha_1^{\ell}\left(2^{n-1}f(x_1) + \frac{1}{\alpha_1^{\ell} - 1}W\right)\\
&= 2^{n-2}\alpha_1^{\ell-2}\sum_{i=2}^{n}\alpha_i^2\biguplus_{x_i} F(x_1) + 2^{n-1}\alpha_1^{\ell}F(x_1)
\end{aligned}$$

for all $x_1, \cdots, x_n \in K$. Now, according to Theorem 2.3, there exists a unique function $\omega : K \to Y$ such that ω satisfies (2.2) and $\omega(x) \in F(x)$ for all $x \in K$. □

Corollary 3.2 Suppose that $f : K \to Y$ satisfies (3.1). Then there exists a unique function $\omega : K \to Y$ such that (i) ω is additive when $\ell = 1$; (ii) ω is quadratic when $\ell = 2$; (iii) ω is cubic when $\ell = 3$, and for all $x \in K$, $\omega(x) - f(x) \in 2^{1-n}\left(\alpha_1^{\ell} - 1\right)^{-1}W$.

Theorem 3.3 If $\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)>1$ and $f:K\to Y$ satisfies

$$\biguplus_{\alpha_2x_2,\cdots,\alpha_nx_n}^{n-1} f(\alpha_1x_1)-2^{n-2}\sum_{1\le i<j\le n}\alpha_i^2\alpha_j^2\biguplus_{x_j} f(x_i)+2^{n-1}\sum_{i=1}^{n}\Big(\alpha_i^2\sum_{j=1,j\ne i}^{n}\alpha_j^2-\alpha_i^4\Big)f(x_i)\in W \tag{3.2}$$

for all $x_1,\cdots,x_n\in K$, then there exists a unique function $\omega:K\to Y$ such that ω satisfies (2.6) and for all $x\in K$,

$$\omega(x)-f(x)\in\frac{1}{2^{n-1}\Big(\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)-1\Big)}W.$$

Proof Let $F(x):=f(x)+\Big(2^{n-1}\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)-2^{n-1}\Big)^{-1}W$ for $x\in K$. Then

$$\begin{aligned}
&\biguplus_{\alpha_2x_2,\cdots,\alpha_nx_n}^{n-1} F(\alpha_1x_1)+2^{n-1}\sum_{i=1}^{n}\alpha_i^2\Big(\sum_{j=1,j\ne i}^{n}\alpha_j^2\Big)F(x_i)\\
&=\biguplus_{\alpha_2x_2,\cdots,\alpha_nx_n}^{n-1} f(\alpha_1x_1)+2^{n-1}\sum_{i=1}^{n}\alpha_i^2\Big(\sum_{j=1,j\ne i}^{n}\alpha_j^2\Big)f(x_i)+\frac{1+\sum\limits_{i=1}^{n}\alpha_i^2\Big(\sum\limits_{j=1,j\ne i}^{n}\alpha_j^2\Big)}{\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)-1}W\\
&\subseteq 2^{n-2}\sum_{1\le i<j\le n}\alpha_i^2\alpha_j^2\biguplus_{x_j} f(x_i)+2^{n-1}\sum_{i=1}^{n}\alpha_i^4f(x_i)+\frac{1+\sum\limits_{i=1}^{n}\alpha_i^2\Big(\sum\limits_{j=1,j\ne i}^{n}\alpha_j^2\Big)}{\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)-1}W+W\\
&=\alpha_1^2\alpha_2^2\left(2^{n-2}\biguplus_{x_2} f(x_1)+\frac{1}{\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)-1}W\right)\\
&\quad+\cdots+\alpha_1^2\alpha_n^2\left(2^{n-2}\biguplus_{x_n} f(x_1)+\frac{1}{\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)-1}W\right)\\
&\quad+\alpha_2^2\alpha_3^2\left(2^{n-2}\biguplus_{x_3} f(x_2)+\frac{1}{\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)-1}W\right)\\
&\quad+\cdots+\alpha_2^2\alpha_n^2\left(2^{n-2}\biguplus_{x_n} f(x_2)+\frac{1}{\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)-1}W\right)\\
&\quad+\cdots+\alpha_{n-1}^2\alpha_n^2\left(2^{n-2}\biguplus_{x_n} f(x_{n-1})+\frac{1}{\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)-1}W\right)
\end{aligned}$$

$$
\begin{aligned}
&+\alpha_1^4\left(2^{n-1}f(x_1)+\frac{1}{\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)-1}W\right)\\
&+\cdots+\alpha_n^4\left(2^{n-1}f(x_n)+\frac{1}{\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)-1}W\right)\\
=\,&2^{n-2}\sum_{1\le i<j\le n}\alpha_i^2\alpha_j^2\biguplus_{x_j}F(x_i)+2^{n-1}\sum_{i=1}^{n}\alpha_i^4F(x_i)
\end{aligned}
$$

for all $x_1,\cdots,x_n\in K$. Now, according to Theorem 2.4, there exists a unique function $\omega:K\to Y$ such that ω satisfies (2.6) and $\omega(x)\in F(x)$ for all $x\in K$. □

Corollary 3.4 Suppose that $\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)>1$ and $f:K\to Y$ satisfies (3.2). Then there exists a unique quartic function $\omega:K\to Y$ such that, for all $x\in K$, $\omega(x)-f(x)\in 2^{1-n}\Big(\sum\limits_{i=1}^{n}\alpha_i^2\big(\alpha_i^2-\sum\limits_{j=i+1}^{n}\alpha_j^2\big)-1\Big)^{-1}W$.

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