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GROUND STATE SOLUTIONS OF NEHARI-POHOZAEV TYPE FOR A FRACTIONAL SCHRÖDINGER-POISSON SYSTEM WITH CRITICAL GROWTH*

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Abstract We study the following nonlinear fractional Schrödinger-Poisson system with critical growth:

$$\begin{cases} (-\Delta)^s u + u + \phi u = f(u) + |u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{0.1}$$

where $0 < s, t < 1$, $2s + 2t > 3$ and $2_s^* = \frac{6}{3-2s}$ is the critical Sobolev exponent in $\mathbb{R}^3$. Under some more general assumptions on f, we prove that (0.1) admits a nontrivial ground state solution by using a constrained minimization on a Nehari-Pohozaev manifold.

Key words fractional Schrödinger-Poisson system; Nehari-Pohozaev manifold; ground state solutions; critical growth

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1 Introduction and Main Result

In this paper, we consider the following nonlinear fractional Schrödinger-Poisson system:

$$\begin{cases} (-\Delta)^s u + u + \phi u = f(u) + |u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $0 < s, t < 1$, $2s+2t > 3$, $2_s^* = \frac{6}{3-2s}$ is the critical Sobolev exponent in $\mathbb{R}^3$ and $f \in C(\mathbb{R}, \mathbb{R})$ is a subcritical perturbation. The fractional Laplacian $(-\Delta)^\alpha$, $\alpha = s, t \in (0, 1)$ is a nonlocal operator defined in the Schwartz class $\mathscr{S}(\mathbb{R}^3)$ as

$$(-\Delta)^\alpha u(x) = C(\alpha)\text{P.V.} \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x - y|^{3+2\alpha}} \mathrm{d}y,$$

where $C(\alpha)$ is a suitable normalized constant and P.V. means the Canchy principle value on the integral.

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When $\phi(x) = 0$, system (1.1) can be reduced to the usual fractional Schrödinger equation

$$(-\Delta)^s u + u = f(u) + |u|^{2_s^*-2}u, \quad x \in \mathbb{R}^3. \tag{1.2}$$

The fractional Schrödinger equation was introduced by Laskin [22, 23] in the context of fractional quantum mechanics, as a result of expanding the Feynman path integral from the Brownian like to the Lévy like quantum mechanical paths. In particular, the fractional Laplacian can be understood as the infinitesimal generator of a stable Lévy diffusion process [2]. It also has various applications in different subjects, such as the thin obstacle problem [26, 32], optimization [17], finance [10], conservation laws [5], minimal surfaces [6, 8] and so on. The non-locality of the fractional Laplacian makes it difficult to study. To overcome this difficulty, Caffarelli and Silvestre [7] introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions. This extension method has been applied successfully and a series of fruitful results have been obtained; we refer the readers to [9, 14–16, 18, 31, 39] and the references therein.

When $s = t = 1$, (1.1) is related to the classical Schrödinger-Poisson system of the form

$$\begin{cases} -\Delta u + u + \phi u = h(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

System (1.3) has great importance for describing the interaction of a charged particle with an electromagnetic field; we refer the readers to [4] for more details on the physical background. In the last decades, system (1.3) has been studied extensively. In the case $h(u) = |u|^{p-2}u$, D'Aprile and Mugnai [12] used the Mountain Pass Theorem to show a radially symmetric solution of (1.3) for $4 \le p < 6$. In the same case, they established in [13] a Pohozaev identity to prove that there do not exist nontrivial solutions of (1.3) for $p \le 2$ and $p \ge 6$. For the case when $h(u) = |u|^{p-2}u$ and $2 < p \le 4$, the variant Ambrosetti-Rabinowitz type condition ((AR) in short) is not satisfied; i.e., for some $\theta > 4$,

$$H(u) = \int_0^u h(\tau)\mathrm{d}\tau \le \frac{1}{\theta} h(u)u \quad \text{for all } u \in \mathbb{R}. \tag{1.4}$$

By constructing a constrained minimization on a new manifold based on the Nehari manifold and the Pohozaev identity, Ruiz in [28], proved that (1.3) admits a positive radial solution if $h(u) = |u|^{p-2}u$ with $3 < p < 6$. Moreover, by using an inequality derived from [24], he showed that problem (1.3) does not admit any nontrivial solution if $2 < p \le 3$. After that, combining a monotone method and a version of the global compactness lemma, the authors of [42] generalized the existence result of [28] to a more general case of replacing u by $V(x)u$. For more existence results for (1.3) under a huge variety of hypotheses on the potential function and the nonlinearity, we refer the readers to [1, 3, 19–21, 33, 40, 43] and the references therein.

As we have mentioned above, there are many papers dealing with problems (1.2) and (1.3) which have only one nonlocal term. However, the case of the variational problem involving double or more nonlocal terms is much more complicated. To the best of our knowledge, systems like (1.1) have been less studied. Recently, Zhang et al. in [41] were concerned with the fractional nonlinear Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + \lambda\phi u = g(u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.5}$$

where g satisfies the Berestycki-Lions type conditions in the subcritical and critical case. By using a perturbation approach, they proved the existence of positive solutions and studied the asymptotic of solutions for a vanishing parameter. In [35], Teng considered the system

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.6}$$

where $s \in (\frac{3}{4}, 1)$ and $2 < p < 2_s^* - 1$. Motivated by the work of Ruiz [28], he first established a new Nehari-Pohozaev manifold which is C^1, and proved that the corresponding limiting problem of (1.6) has a nonnegative ground state solution. Next, under certain assumptions on $V(x)$, the existence of nonnegative ground state solutions for (1.6) was established through the use of a monotone method and a global compactness lemma. Later on, by adapting the same arguments used in [35], Teng in [36] obtained the existence of ground state solutions for system (1.1) by adding a suitable potential function in front of the term u and replacing $f(u) + |u|^{2_s^*-2}u$ by $\mu|u|^{q-1}u + |u|^{2_s^*-2}u$, where $\frac{3s+t}{s+t} < q < 2_s^* - 1$. We refer also to [25, 27] for more results involving the fractional Schrödinger-Poisson system.

Motivated by the works mentioned above, the main purpose of this paper is to establish the existence of ground state solutions of Nehari-Pohozaev type for problem (1.1) with critical growth and general subcritical perturbation. Compared with [36], we focus on the study of (1.1) with a more general subcritical perturbation f. Moreover, we only assume that f is a continuous function rather than of C^1-class. In some sense, our results generalize and improve the works of [36].

Throughout this paper, we assume that f satisfies the following conditions:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(\tau) \equiv 0$ for all $\tau \in (-\infty, 0)$;

(f_2) $\lim\limits_{\tau \to 0^+} \frac{f(\tau)}{\tau} = 0$ and $\lim\limits_{\tau \to +\infty} \frac{f(\tau)}{\tau^{2_s^*-1}} = 0$;

(f_3) $\frac{(s+t)f(\tau)\tau - 3F(\tau)}{\tau^{\frac{4s+2t}{s+t}}}$ is increasing on $(0, +\infty)$, where $F(\tau) = \int_0^\tau f(\theta)\mathrm{d}\theta$;

(f_4) there exist $\mu > 0$ and $\frac{4s+2t}{s+t} < p < 2_s^*$ such that $f(\tau) \geq \mu\tau^{p-1}$ for all $\tau \geq 0$.

As we can see in Section 2, by the Lax-Milgram theorem, for any $u \in H^s(\mathbb{R}^3)$ there exists a unique $\phi_u^t \in D^{t,2}(\mathbb{R}^3)$ such that $(-\Delta)^t \phi_u^t = u^2$. Substituting ϕ_u^t in (1.1) leads to the following single fractional Schrödinger equation:

$$(-\Delta)^s u + u + \phi_u^t u = f(u) + |u|^{2_s^*-2}u, \quad x \in \mathbb{R}^3. \tag{1.7}$$

The variational functional associated with equation (1.7) is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \big(|(-\Delta)^{\frac{s}{2}}u|^2 + u^2\big)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*}\mathrm{d}x \tag{1.8}$$

for $u \in H^s(\mathbb{R}^3)$. We can prove that $I \in C^1(H^s(\mathbb{R}^3), \mathbb{R})$ and the pair $(u, \phi_u^t) \in H^s(\mathbb{R}^3) \times D^{t,2}(\mathbb{R}^3)$ is a weak solution of (1.1) if $u \in H^s(\mathbb{R}^3)$ is a critical point of I. In the sequel, we say that u is a weak solution to (1.1) for the sake of simplicity.

To establish our main results, we need to introduce the following Nehari-Pohozaev manifold:

$$\mathcal{M} := \Big\{u \in H^s(\mathbb{R}^3)\backslash\{0\} \;\Big|\; G(u) = 0\Big\},$$

where

$$G(u) := \frac{4s+2t-3}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \frac{2s+2t-3}{2}\int_{\mathbb{R}^3} u^2\mathrm{d}x + \frac{4s+2t-3}{4}\int_{\mathbb{R}^3} \phi_u^t u^2\mathrm{d}x$$

$$-\int_{\mathbb{R}^3}\Big((s+t)f(u)u-3F(u)\Big)\mathrm{d}x-\frac{(s+t)2_s^*-3}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x. \tag{1.9}$$

For any $u\in\mathcal{M}$, we say that

$$m:=\inf_{u\in\mathcal{M}}I(u). \tag{1.10}$$

Now, our main result can be stated as follows:

Theorem 1.1 Assume that (f_1)–(f_4) hold and $2s+2t>3$. Then by system (1.1) there exists a nontrivial solution $\tilde{u}$ such that $I(\tilde{u})=m>0$ if one of the following conditions is satisfied:

(C_1) $s>\frac{3}{4}$, $\frac{4s}{3-2s}<p<2_s^*$ and any $\mu>0$;
(C_2) $s>\frac{3}{4}$, $\frac{4s+2t}{s+t}<p\le\frac{4s}{3-2s}$ and $\mu>0$ sufficiently large;
(C_3) $\frac{1}{2}<s\le\frac{3}{4}$, $\frac{4s+2t}{s+t}<p<2_s^*$ and any $\mu>0$.

To prove Theorem 1.1, the main difficulties are threefold. Firstly, in (1.1), the first fractional Schrödinger equation has two nonlocal terms due to the presence of fractional Laplacian operator $(-\Delta)^s$ and the term ϕu, leading to some additional difficulties and making the study interesting. Secondly, if f is not of C^1-class, then $\mathcal{M}$ is not a C^1-manifold. We point out here that the proofs in [28, 35, 36] are based on minimizing the associated functional restricted to a suitable manifold which is C^1. Hence, the arguments used in [28, 35, 36] cannot be applied in this paper and some new tricks will be developed. Thirdly, the unboundedness of the domain $\mathbb{R}^3$ and the critical Sobolev exponent lead to a lack of compactness. Thus more careful analysis is needed.

The outline of this paper is as follows: in Section 2, we provide some preliminary lemmas which will be used later. In Section 3, we prove our main result Theorem 1.1.

2 Preliminaries

In this section, we provide some preliminary lemmas. First, we give the standard notations for the fractional Sobolev spaces. For $\alpha\in(0,1)$, the Hilbert space $H^\alpha(\mathbb{R}^3)$ is defined as

$$H^\alpha(\mathbb{R}^3):=\Big\{u\in L^2(\mathbb{R}^3)\ \big|\ (-\Delta)^{\frac{\alpha}{2}}u\in L^2(\mathbb{R}^3)\Big\}.$$

We endow the space $H^\alpha(\mathbb{R}^3)$ with the inner product, and norm with

$$(u,v):=\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}v\mathrm{d}x+\int_{\mathbb{R}^3}uv\mathrm{d}x$$

and

$$\|u\|_{H^\alpha}:=\Big(\int_{\mathbb{R}^3}\big(|(-\Delta)^{\frac{\alpha}{2}}u|^2+u^2\big)\mathrm{d}x\Big)^{\frac{1}{2}},$$

respectively, where

$$\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u|^2\mathrm{d}x=\frac{C(\alpha)}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}}\mathrm{d}x\mathrm{d}y.$$

It is well known that $H^\alpha(\mathbb{R}^3)$ is continuously embedded into $L^q(\mathbb{R}^3)$ for $2\le q\le 2_\alpha^*$ and is compactly embedded into $L^q_{\mathrm{loc}}(\mathbb{R}^3)$ for $1\le q<2_\alpha^*$. $L^q(\mathbb{R}^3)$ is the usual Lebesgue space with the standard norms

$$\|u\|_{L^q}:=\Big(\int_{\mathbb{R}^3}|u|^q\mathrm{d}x\Big)^{\frac{1}{q}},\ 1\le q<\infty.$$

From [11, 14] we know that $D^{\alpha,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*_\alpha}(\mathbb{R}^3)$ and there exists a best constant $\mathcal{S}_\alpha > 0$ such that

$$\mathcal{S}_\alpha = \inf_{u\in D^{\alpha,2}(\mathbb{R}^3)} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}}u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^3} |u|^{2^*_\alpha} \mathrm{d}x\right)^{\frac{2}{2^*_\alpha}}}, \tag{2.1}$$

where $D^{\alpha,2}(\mathbb{R}^3)$ is defined by

$$D^{\alpha,2}(\mathbb{R}^3) := \left\{u \in L^{2^*_\alpha}(\mathbb{R}^3) \;\middle|\; (-\Delta)^{\frac{\alpha}{2}}u \in L^2(\mathbb{R}^3)\right\}$$

endowed with the norm

$$\|u\|_{D^{\alpha,2}} := \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}}u|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

In what follows, we recall that by the Lax-Milgram theorem, for any $u \in H^s(\mathbb{R}^3)$, there exists a unique $\phi^t_u \in D^{t,2}(\mathbb{R}^3)$ such that

$$(-\Delta)^t \phi^t_u = u^2, \quad x \in \mathbb{R}^3. \tag{2.2}$$

Moreover, ϕ^t_u can be expressed as

$$\phi^t_u(x) = C_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} \mathrm{d}y, \quad x \in \mathbb{R}^3, \tag{2.3}$$

which is called the t-Riesz potential (see [7]), where

$$C_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(\frac{3}{2}-t)}{\Gamma(t)}.$$

Let us now define the operator $\Phi : H^s(\mathbb{R}^3) \to D^{t,2}(\mathbb{R}^3)$ as

$$\Phi(u) = \phi^t_u.$$

The operator Φ has the following properties (see [36]):

Lemma 2.1 Assume that $4s + 2t \geq 3$ for any $u \in H^s(\mathbb{R}^3)$. We have that

(i) Φ is continuous and maps bounded sets into bounded sets;

(ii) $\int_{\mathbb{R}^3} \phi^t_u u^2 \mathrm{d}x \leq \mathcal{S}_t^{-2} \|u\|^4_{L^{\frac{12}{3+2t}}}$;

(iii) if $y \in \mathbb{R}^3$ and $\tilde{u}(x) = u(x+y)$, then $(\Phi(\tilde{u}))(x) = (\Phi(u))(x+y)$ and

$$\int_{\mathbb{R}^3} \phi^t_{\tilde{u}} \tilde{u}^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi^t_u u^2 \mathrm{d}x;$$

(iv) if $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$, then $\Phi(u_n) \rightharpoonup \Phi(u)$ in $D^{t,2}(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \phi^t_u u^2 \mathrm{d}x \leq \liminf_{n\to\infty} \int_{\mathbb{R}^3} \phi^t_{u_n} u_n^2 \mathrm{d}x;$$

(v) if $u_n \to u$ in $H^s(\mathbb{R}^3)$, then $\Phi(u_n) \to \Phi(u)$ in $D^{t,2}(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \phi^t_{u_n} u_n^2 \mathrm{d}x \to \int_{\mathbb{R}^3} \phi^t_u u^2 \mathrm{d}x.$$

Define $\Psi : H^s(\mathbb{R}^3) \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^3} \phi^t_u u^2 \mathrm{d}x.$$

In a fashion similar to the well known Brezis-Lieb lemma, we can establish the Brezis-Lieb splitting property for Ψ and Ψ'; see [36].

Lemma 2.2 Assume that $2s+2t>3$. Letting $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, it holds that

(i) $\Psi(u_n-u)=\Psi(u_n)-\Psi(u)+o_n(1)$;

(ii) $\Psi'(u_n-u)=\Psi'(u_n)-\Psi'(u)+o_n(1)$ in $(H^s(\mathbb{R}^3))^{-1}$,

where $o_n(1)\to 0$ as $n\to\infty$.

The vanishing lemma for the fractional Sobolev space is stated as follows:

Lemma 2.3 ([29, Lemma 2.4]) Assume that $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}^N)$ and that it satisfies

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)} u_n^2\mathrm{d}x=0$$

for some $R>0$. Then $u_n\to 0$ in $L^q(\mathbb{R}^N)$ for every $2<q<2_\alpha^*=\frac{2N}{N-2\alpha}$.

3 Proof of Theorem 1.1

In this section, we will use a constrained minimization on $\mathcal{M}$ to get a nontrivial ground state solution of Nehari-Pohozaev type for system (1.1). Throughout this section, the norm on the $H^s(\mathbb{R}^3)$ is taken as

$$\|u\|:=\Big(\int_{\mathbb{R}^3}\big(|(-\Delta)^{\frac{s}{2}}u|^2+u^2\big)\mathrm{d}x\Big)^{\frac12}.$$

Lemma 3.1 Under the assumption (f_3), we have that

$$\frac{(s+t)(1-\theta^{4s+2t-3})}{4s+2t-3}f(\tau)\tau-\frac{4s+2t-3\theta^{4s+2t-3}}{4s+2t-3}F(\tau)+\theta^{-3}F(\theta^{s+t}\tau)\ge 0,\ \forall\theta\ge 0,\ \tau\in\mathbb{R}.$$

Proof Without loss of generality, we may assume that $\tau\neq 0$, and set

$$h(\theta):=\frac{(s+t)(1-\theta^{4s+2t-3})}{4s+2t-3}f(\tau)\tau-\frac{4s+2t-3\theta^{4s+2t-3}}{4s+2t-3}F(\tau)+\theta^{-3}F(\theta^{s+t}\tau).$$

By a direct computation, we have

$$\begin{aligned}h'(\theta)&=-(s+t)\theta^{4s+2t-4}f(\tau)\tau+3\theta^{4s+2t-4}F(\tau)-3\theta^{-4}F(\theta^{s+t}\tau)+(s+t)\theta^{s+t-4}f(\theta^{s+t}\tau)\tau\\&=\theta^{4s+2t-4}\tau^{\frac{4s+2t}{s+t}}\Big\{\frac{(s+t)f(\theta^{s+t}\tau)\theta^{s+t}\tau-3F(\theta^{s+t}\tau)}{(\theta^{s+t}\tau)^{\frac{4s+2t}{s+t}}}-\frac{(s+t)f(\tau)\tau-3F(\tau)}{\tau^{\frac{4s+2t}{s+t}}}\Big\}.\end{aligned}$$

It follows from (f_3) that $h(\theta)\ge h(1)=0,\ \forall\theta\ge 0$. □

Lemma 3.2 Assume that $(f_1)-(f_3)$ hold and that $4s+2t>3$. Then

$$I(u)\ge I(u_\theta)+\frac{1-\theta^{4s+2t-3}}{4s+2t-3}G(u)+\hat{C}(\theta)\int_{\mathbb{R}^3}u^2\mathrm{d}x,\quad \forall u\in H^s(\mathbb{R}^3),\ \theta\ge 0,$$

where $u_\theta(x):=\theta^{s+t}u(\theta x)$ and $\hat{C}(\theta)\ge 0$.

Proof If $\forall u\in H^s(\mathbb{R}^3),\ \theta\ge 0$, and we set $u_\theta(x):=\theta^{s+t}u(\theta x)$, it follows that

$$\begin{aligned}I(u_\theta)=&\frac{\theta^{4s+2t-3}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x+\frac{\theta^{2s+2t-3}}{2}\int_{\mathbb{R}^3}u^2\mathrm{d}x+\frac{\theta^{4s+2t-3}}{4}\int_{\mathbb{R}^3}\phi_u^t u^2\mathrm{d}x\\&-\theta^{-3}\int_{\mathbb{R}^3}F(\theta^{s+t}u)\mathrm{d}x-\frac{\theta^{(s+t)2_s^*-3}}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x.\end{aligned}\tag{3.1}$$

In view of Lemma 3.1, we have

$I(u)-I(u_\theta)$

$$
\begin{aligned}
&= \frac{1-\theta^{4s+2t-3}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \frac{1-\theta^{2s+2t-3}}{2}\int_{\mathbb{R}^3}u^2\mathrm{d}x + \frac{1-\theta^{4s+2t-3}}{4}\int_{\mathbb{R}^3}\phi_u^t u^2\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3}\Big(F(u)-\theta^{-3}F(\theta^{s+t}u)\Big)\mathrm{d}x - \frac{1-\theta^{(s+t)2_s^*-3}}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x \\
&= \frac{1-\theta^{4s+2t-3}}{4s+2t-3}\Big\{\frac{4s+2t-3}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \frac{2s+2t-3}{2}\int_{\mathbb{R}^3}u^2\mathrm{d}x \\
&\quad + \frac{4s+2t-3}{4}\int_{\mathbb{R}^3}\phi_u^t u^2\mathrm{d}x - \int_{\mathbb{R}^3}\Big((s+t)f(u)u-3F(u)\Big)\mathrm{d}x \\
&\quad - \frac{(s+t)2_s^*-3}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x\Big\} + \hat{C}(\theta)\int_{\mathbb{R}^3}u^2\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}\Big[\frac{(s+t)(1-\theta^{4s+2t-3})}{4s+2t-3}f(u)u - \frac{4s+2t-3\theta^{4s+2t-3}}{4s+2t-3}F(u) + \theta^{-3}F(\theta^{s+t}u)\Big]\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}\Big[\frac{1-\theta^{4s+2t-3}}{2} - \frac{1-\theta^{(s+t)2_s^*-3}}{2_s^*}\Big]|u|^{2_s^*}\mathrm{d}x \\
&\geq \frac{1-\theta^{4s+2t-3}}{4s+2t-3}G(u) + \hat{C}(\theta)\int_{\mathbb{R}^3}u^2\mathrm{d}x,
\end{aligned}
$$

where

$$
\hat{C}(\theta) := \frac{2s-(4s+2t-3)\theta^{2s+2t-3}+(2s+2t-3)\theta^{4s+2t-3}}{2(4s+2t-3)} \geq \hat{C}(1)=0, \quad \forall\theta\geq 0, \tag{3.2}
$$

so this lemma is proved. □

Lemma 3.3 Assume that (f_1)–(f_3) hold and that $(s+t)2_s^* > 4s+2t$. Then for any $u \in H^s(\mathbb{R}^3)\backslash\{0\}$, there is a unique constant $\theta_0 > 0$ such that $u_{\theta_0} \in \mathcal{M}$. Moreover, $I(u_{\theta_0}) = \max\limits_{\theta\geq 0} I(u_\theta)$.

Proof For any $u \in H^s(\mathbb{R}^3)\backslash\{0\}$ and $\theta \geq 0$, we consider $y(\theta) := I(u_\theta)$. It is easy to check that $y(\theta) > 0$ for $\theta > 0$ small and $y(\theta) \to -\infty$ as $\theta \to +\infty$, which gives that $y(\theta)$ has a critical point $\theta_0 > 0$ corresponding to its maximum, i.e., $y(\theta_0) = \max\limits_{\theta\geq 0} y(\theta)$ and $y'(\theta_0) = 0$. Thus

$$
\begin{aligned}
&\frac{4s+2t-3}{2}\theta_0^{4s+2t-4}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \frac{2s+2t-3}{2}\theta_0^{2s+2t-4}\int_{\mathbb{R}^3}u^2\mathrm{d}x \\
&+ \frac{4s+2t-3}{4}\theta_0^{4s+2t-4}\int_{\mathbb{R}^3}\phi_u^t u^2\mathrm{d}x - \theta_0^{-4}\int_{\mathbb{R}^3}\Big((s+t)f(\theta_0^{s+t}u)\theta_0^{s+t}u - 3F(\theta_0^{s+t}u)\Big)\mathrm{d}x \\
&- \frac{(s+t)2_s^*-3}{2_s^*}\theta_0^{(s+t)2_s^*-4}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x = 0,
\end{aligned}
$$

and hence $G(u_{\theta_0}) = 0$, $u_{\theta_0} \in \mathcal{M}$ and $I(u_{\theta_0}) = \max\limits_{\theta\geq 0} I(u_\theta)$.

Moreover, we claim that the critical point of $y(\theta)$ is unique. Indeed, we just suppose that there are two points θ_1, $\theta_2 > 0$ such that $G(u_{\theta_i}) = 0$ for $i = 1, 2$. In a manner similar to the proof of Lemma 3.2, we can deduce that

$$
\begin{aligned}
I(u_{\theta_1}) &\geq I(u_{\theta_2}) + \frac{1-(\frac{\theta_2}{\theta_1})^{4s+2t-3}}{4s+2t-3}G(u_{\theta_1}) + \hat{C}\Big(\frac{\theta_2}{\theta_1}\Big)\theta_1^{2s+2t-3}\int_{\mathbb{R}^3}u^2\mathrm{d}x \\
&= I(u_{\theta_2}) + \hat{C}\Big(\frac{\theta_2}{\theta_1}\Big)\theta_1^{2s+2t-3}\int_{\mathbb{R}^3}u^2\mathrm{d}x
\end{aligned}
$$

and

$$
I(u_{\theta_2}) \geq I(u_{\theta_1}) + \frac{1-(\frac{\theta_1}{\theta_2})^{4s+2t-3}}{4s+2t-3}G(u_{\theta_2}) + \hat{C}\Big(\frac{\theta_1}{\theta_2}\Big)\theta_2^{2s+2t-3}\int_{\mathbb{R}^3}u^2\mathrm{d}x
$$

$$= I(u_{\theta_1}) + \hat{C}\Big(\frac{\theta_1}{\theta_2}\Big)\theta_2^{2s+2t-3}\int_{\mathbb{R}^3} u^2 \mathrm{d}x,$$

where $\hat{C}(\theta)$ is defined in (3.2). This implies that $\theta_1 = \theta_2$, so $\theta_0 > 0$ is the unique critical point of $y(\theta)$. □

Lemma 3.4 Assume that (f_1)–(f_3) hold and that $(s+t)2_s^* > 4s+2t > 3$. It holds that

$$m = \inf_{u\in H^s(\mathbb{R}^3)\setminus\{0\}} \max_{\theta\geq 0} I(u_\theta) > 0.$$

Proof It follows from Lemma 3.3 that

$$m = \inf_{u\in H^s(\mathbb{R}^3)\setminus\{0\}} \max_{\theta\geq 0} I(u_\theta).$$

Next, we claim that $m > 0$. Indeed, it follows from $(f_1)-(f_2)$ that there exists a constant $\bar{C} > 0$ such that

$$|F(u)| \leq \frac{1}{2}u^2 + \bar{C}|u|^{2_s^*}, \quad \forall u \in H^s(\mathbb{R}^3). \tag{3.3}$$

Thus for any $u \in \mathcal{M}$, by (3.3), Lemma 3.2 and Sobolev inequality, we have

$$\begin{aligned}
&I(u) \geq I(u_\theta)\\
\geq{}& \frac{\theta^{4s+2t-3}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \frac{\theta^{2s+2t-3}}{2}\int_{\mathbb{R}^3} u^2\mathrm{d}x - \theta^{-3}\int_{\mathbb{R}^3} F(\theta^{s+t}u)\mathrm{d}x\\
&- \frac{\theta^{(s+t)2_s^*-3}}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*}\mathrm{d}x\\
\geq{}& \frac{\theta^{4s+2t-3}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x - \Big(\bar{C} + \frac{1}{2_s^*}\Big)\theta^{(s+t)2_s^*-3}\int_{\mathbb{R}^3} |u|^{2_s^*}\mathrm{d}x\\
\geq{}& \frac{\theta^{4s+2t-3}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x - \Big(\bar{C} + \frac{1}{2_s^*}\Big)\mathcal{S}_s^{-\frac{2_s^*}{2}}\theta^{(s+t)2_s^*-3}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\Big)^{\frac{2_s^*}{2}}\\
={}& \frac{2s}{3-2s}\tilde{C}^{\frac{3-2s}{2s}}\Big(\frac{1}{2_s^*}\Big)^{\frac{3}{2s}} > 0,
\end{aligned}$$

if we take

$$\theta = \Big[\frac{(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x)^{1-\frac{2_s^*}{2}}}{2_s^*\tilde{C}}\Big]^{\frac{1}{(s+t)2_s^*-(4s+2t)}} \quad \text{and} \quad \tilde{C} = \Big(\bar{C} + \frac{1}{2_s^*}\Big)\mathcal{S}_s^{-\frac{2_s^*}{2}}.$$

As a result, we complete the proof. □

Now, we establish a splitting lemma as follows:

Lemma 3.5 Assume that (f_1)–(f_2) hold and that $2s+2t > 3$. If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then

$$I(u_n) = I(u) + I(u_n - u) + o_n(1), \tag{3.4}$$

$$\langle I'(u_n), u_n\rangle = \langle I'(u), u\rangle + \langle I'(u_n-u), u_n-u\rangle + o_n(1) \tag{3.5}$$

and

$$G(u_n) = G(u) + G(u_n - u) + o_n(1). \tag{3.6}$$

Proof Setting $v_n = u_n - u$, we have $v_n \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$. It follows from (f_1)–(f_2) and the Brezis-Lieb lemma that

$$\|u_n\|^2 = \|u\|^2 + \|v_n\|^2 + o_n(1), \tag{3.7}$$

$$\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^3} F(u)\mathrm{d}x + \int_{\mathbb{R}^3} F(v_n)\mathrm{d}x + o_n(1) \tag{3.8}$$

and

$$\|u_n\|^{2_s^*}_{L^{2_s^*}} = \|u\|^{2_s^*}_{L^{2_s^*}} + \|v_n\|^{2_s^*}_{L^{2_s^*}} + o_n(1). \tag{3.9}$$

Combining Lemma 2.2 (i) and (3.7)–(3.9), we know that (3.4) holds.

Using a similar argument as to that of Lemma 2.7 in [34], we can show that there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\sup_{\varphi\in H^s(\mathbb{R}^3),\|\varphi\|\le 1} \Big| \int_{\mathbb{R}^3} \big(f(u_n) - f(u) - f(v_n)\big)\varphi \mathrm{d}x\Big| = o_n(1). \tag{3.10}$$

Thus, one has

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3} \big(f(u_n) - f(u) - f(v_n)\big)u_n \mathrm{d}x\Big| \\
&\le \|u_n\| \sup_{\varphi\in H^s(\mathbb{R}^3),\|\varphi\|\le 1} \Big| \int_{\mathbb{R}^3} \big(f(u_n) - f(u) - f(v_n)\big)\varphi \mathrm{d}x\Big| = o_n(1).
\end{aligned}$$

From this, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} f(u_n)u_n \mathrm{d}x &= \int_{\mathbb{R}^3} f(u)u\mathrm{d}x + \int_{\mathbb{R}^3} f(v_n)v_n\mathrm{d}x + \int_{\mathbb{R}^3} f(v_n)u\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} f(u)v_n\mathrm{d}x + \int_{\mathbb{R}^3} \big(f(u_n) - f(u) - f(v_n)\big)u_n\mathrm{d}x \\
&= \int_{\mathbb{R}^3} f(u)u\mathrm{d}x + \int_{\mathbb{R}^3} f(v_n)v_n\mathrm{d}x + o_n(1).
\end{aligned} \tag{3.11}$$

The equality (3.5) follows from Lemma 2.2 (ii), (3.7), (3.9) and (3.11).

Finally, we note that

$$\begin{aligned}
G(u) = \frac{4s+2t-3}{2}\langle I'(u), u\rangle - s\int_{\mathbb{R}^3} u^2\mathrm{d}x - \frac{4s+2t-3}{4}\int_{\mathbb{R}^3} \phi_u^t u^2\mathrm{d}x \\
- \int_{\mathbb{R}^3} \Big(\frac{3-2s}{2} f(u)u - 3F(u)\Big)\mathrm{d}x.
\end{aligned}$$

From this, by Lemma 2.2 (i), (3.5), (3.8) and (3.11), we can obtain (3.6). □

In the following, we give an important energy estimate for m:

Lemma 3.6 Assume that (f_1)–(f_4) hold. It holds that $m < \frac{s}{3}\mathcal{S}_s^{\frac{3}{2s}}$ if one of (C_1), (C_2) or (C_3) is satisfied.

Proof Let $\psi \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function such that $\psi(x) = 1$ if $|x| \le r$, and $\psi(x) = 0$ if $|x| \ge 2r$. For $\varepsilon > 0$, we define

$$u_\varepsilon(x) = \psi(x)U_\varepsilon(x), \quad x \in \mathbb{R}^3,$$

where $U_\varepsilon(x) = \varepsilon^{-\frac{3-2s}{2}}u^*(\frac{x}{\varepsilon})$, $u^*(x) = \frac{U\big(x/\mathcal{S}_s^{\frac{1}{2s}}\big)}{\|U\|_{2_s^*}}$ and

$$U(x) = \kappa(\tau^2 + |x - x_0|^2)^{-\frac{3-2s}{2}},$$

with $\kappa \in \mathbb{R}\backslash\{0\}$, $\tau > 0$ and $x_0 \in \mathbb{R}^3$. From [30, 37], we know that

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\mathrm{d}x \le \mathcal{S}_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}),$$

$$\int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*}\mathrm{d}x = \mathcal{S}_s^{\frac{3}{2s}} + O(\varepsilon^3)$$

and

$$\int_{\mathbb{R}^3} |u_\varepsilon|^q \mathrm{d}x = \begin{cases} O(\varepsilon^{3-\frac{3-2s}{2}q}), & \text{for} \quad q > \dfrac{3}{3-2s}, \\ O(\varepsilon^{\frac{3}{2}}|\log\varepsilon|), & \text{for} \quad q = \dfrac{3}{3-2s}, \\ O(\varepsilon^{\frac{3-2s}{2}q}), & \text{for} \quad q < \dfrac{3}{3-2s}. \end{cases}$$

By Lemma 3.3 and Lemma 3.4, there exists a $\theta_\varepsilon > 0$ such that

$$0 < m \leq \max_{\theta \geq 0} I((u_\varepsilon)_\theta) = I((u_\varepsilon)_{\theta_\varepsilon}). \tag{3.12}$$

Next, we claim that there exist two constants $\theta_*,\ \theta^* > 0$ such that $\theta_* \leq \theta_\varepsilon \leq \theta^*$. Indeed, we first prove that θ_ε is bounded from below by a positive constant. Otherwise, we could find a sequence $\varepsilon_n \to 0$ such that $\theta_{\varepsilon_n} \to 0$. By the above estimations, up to a subsequence, we have $(u_{\varepsilon_n})_{\theta_{\varepsilon_n}} \to 0$ in $H^s(\mathbb{R}^3)$. Therefore,

$$0 < m \leq I((u_{\varepsilon_n})_{\theta_{\varepsilon_n}}) \to I(0) = 0,$$

which is a contradiction. On the other hand, it follows from (f_4) that

$$\begin{aligned} 0 \leq I((u_\varepsilon)_{\theta_\varepsilon}) &\leq \frac{\theta_\varepsilon^{4s+2t-3}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \mathrm{d}x + \frac{\theta_\varepsilon^{2s+2t-3}}{2}\int_{\mathbb{R}^3} u_\varepsilon^2 \mathrm{d}x \\ &\quad + \frac{\theta_\varepsilon^{4s+2t-3}}{4}\int_{\mathbb{R}^3} \phi_{u_\varepsilon}^t u_\varepsilon^2 \mathrm{d}x - \frac{\mu\theta_\varepsilon^{(s+t)p-3}}{p}\int_{\mathbb{R}^3} |u_\varepsilon|^p \mathrm{d}x \\ &\leq C\theta_\varepsilon^{4s+2t-3} + C\theta_\varepsilon^{2s+2t-3} - C\theta_\varepsilon^{(s+t)p-3}, \end{aligned}$$

which implies that there exists $\theta^* > 0$ such that $\theta_\varepsilon \leq \theta^*$. Thus, the claim is proved.

Therefore, by using the inequality

$$(a+b)^q \leq a^q + q(a+b)^{q-1}b, \quad \text{for any } a, b > 0,\ q \geq 1,$$

we conclude that

$$\begin{aligned} I((u_\varepsilon)_{\theta_\varepsilon}) &\leq \frac{\theta_\varepsilon^{4s+2t-3}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 \mathrm{d}x + \frac{\theta_\varepsilon^{2s+2t-3}}{2}\int_{\mathbb{R}^3} u_\varepsilon^2 \mathrm{d}x + \frac{\theta_\varepsilon^{4s+2t-3}}{4}\int_{\mathbb{R}^3} \phi_{u_\varepsilon}^t u_\varepsilon^2 \mathrm{d}x \\ &\quad - \frac{\mu\theta_\varepsilon^{(s+t)p-3}}{p}\int_{\mathbb{R}^3} |u_\varepsilon|^p \mathrm{d}x - \frac{\theta_\varepsilon^{(s+t)2_s^*-3}}{2_s^*}\int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} \mathrm{d}x \\ &\leq \left(\frac{\theta_\varepsilon^{4s+2t-3}}{2} - \frac{\theta_\varepsilon^{(s+t)2_s^*-3}}{2_s^*}\right)\mathcal{S}_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) + C\int_{\mathbb{R}^3} u_\varepsilon^2 \mathrm{d}x \\ &\quad + C\left(\int_{\mathbb{R}^3} |u_\varepsilon|^{\frac{12}{3+2t}} \mathrm{d}x\right)^{\frac{3+2t}{3}} - C\mu\int_{\mathbb{R}^3} |u_\varepsilon|^p \mathrm{d}x \\ &\leq \frac{s}{3}\mathcal{S}_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) + C\int_{\mathbb{R}^3} u_\varepsilon^2 \mathrm{d}x + C\left(\int_{\mathbb{R}^3} |u_\varepsilon|^{\frac{12}{3+2t}} \mathrm{d}x\right)^{\frac{3+2t}{3}} - C\mu\int_{\mathbb{R}^3} |u_\varepsilon|^p \mathrm{d}x. \end{aligned}$$

Finally, arguing as in the proof of Lemma 3.3 in [37], we arrive at

$$m \leq I((u_\varepsilon)_{\theta_\varepsilon}) < \frac{s}{3}\mathcal{S}_s^{\frac{3}{2s}},$$

providing that one of (C_1), (C_2) or (C_3) holds. Thus, we complete the proof. $\square$

Lemma 3.7 Assume that (f_1)–(f_4) hold and that $2s+2t>3$. Then $m>0$ is achieved at some $\tilde{u}\in\mathcal{M}$.

Proof Taking $\theta=0$ in Lemma 3.1, we have that

$$f(\tau)\tau-\frac{4s+2t}{s+t}F(\tau)\geq 0,\quad \forall\tau\in\mathbb{R}. \tag{3.13}$$

We set

$$\begin{aligned}\Phi(u):&=I(u)-\frac{1}{4s+2t-3}G(u)\\ &=\frac{s}{4s+2t-3}\int_{\mathbb{R}^3}u^2\mathrm{d}x+\frac{s+t}{4s+2t-3}\int_{\mathbb{R}^3}\Big(f(u)u-\frac{4s+2t}{s+t}F(u)\Big)\mathrm{d}x+\frac{s}{3}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x.\end{aligned} \tag{3.14}$$

Let $\{u_n\}\subset\mathcal{M}$ be a minimizing sequence for m such that

$$I(u_n)\to m.$$

We divide the proof into three steps as follows:

Step 1 The sequence $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.

From (3.13)–(3.14), we know that

$$m+o_n(1)=I(u_n)=\Phi(u_n)\geq\frac{s}{4s+2t-3}\int_{\mathbb{R}^3}u_n^2\mathrm{d}x+\frac{s}{3}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x. \tag{3.15}$$

On the other hand, by $(f_1)-(f_2)$, (3.15) and $G(u_n)=0$, one has that

$$\begin{aligned}\frac{2s+2t-3}{2}\|u_n\|^2&\leq\int_{\mathbb{R}^3}\Big((s+t)f(u_n)u_n-3F(u_n)\Big)\mathrm{d}x+\frac{(s+t)2_s^*-3}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x\\ &\leq C\int_{\mathbb{R}^3}u_n^2\mathrm{d}x+C\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x\leq C.\end{aligned}$$

Thus, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.

Step 2 There exist a sequence $\{y_n\}\subset\mathbb{R}^3$ and constants $R,\ \beta>0$ such that

$$\liminf_{n\to\infty}\int_{B_R(y_n)}u_n^2\mathrm{d}x\geq\beta>0. \tag{3.16}$$

Suppose, by contradiction, that for all $R>0$,

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}u_n^2\mathrm{d}x=0.$$

By Lemma 2.3, we have that

$$u_n\to 0\quad\text{in}\quad L^q(\mathbb{R}^3),\ 2<q<2_s^*. \tag{3.17}$$

It follows from Lemma 2.1 (ii) and (3.17) that

$$\int_{\mathbb{R}^3}\phi_{u_n}^t u_n^2\mathrm{d}x\to 0. \tag{3.18}$$

Since $G(u_n)=0$, by (3.17)–(3.18), we get

$$\frac{4s+2t-3}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x+\frac{2s+2t-3}{2}\int_{\mathbb{R}^3}u_n^2\mathrm{d}x-\frac{(s+t)2_s^*-3}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x=o_n(1). \tag{3.19}$$

From the definition of the constant $\mathcal{S}_s$, we have

$$\frac{4s+2t-3}{2}\mathcal{S}_s\Big(\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x\Big)^{\frac{2}{2_s^*}} \le \frac{(s+t)2_s^*-3}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x+o_n(1). \tag{3.20}$$

Without loss of generality, we may assume that

$$\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x \to l \ge 0.$$

It is easy to check that $l>0$, otherwise $\|u_n\|\to 0$ as $n\to\infty$, which contradicts $m>0$. In view of (3.20), taking the limit as $n\to\infty$, we obtain that $l \ge \mathcal{S}_s^{\frac{3}{2s}}$. On the other hand, since $I(u_n)\to m$, it follows from (3.17)–(3.19) that

$$\begin{aligned}
m &= \frac12\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x+\frac12\int_{\mathbb{R}^3}u_n^2\mathrm{d}x-\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x+o_n(1)\\
&\ge \frac{1}{2(4s+2t-3)}\Big((4s+2t-3)\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x+(2s+2t-3)\int_{\mathbb{R}^3}u_n^2\mathrm{d}x\Big)\\
&\quad -\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x+o_n(1)\\
&= \frac{(s+t)2_s^*-3}{2_s^*(4s+2t-3)}l-\frac{1}{2_s^*}l\\
&= \frac{s}{3}l \ge \frac{s}{3}\mathcal{S}_s^{\frac{3}{2s}},
\end{aligned}$$

which contradicts Lemma 3.6.

Step 3 m is achieved.

If we denote that $\tilde u_n(x)=u_n(x+y_n)$, then $\tilde u_n\in\mathcal{M}$. It is easy to check that $\{\tilde u_n\}$ is still a bounded minimizing sequence for m. Up to a subsequence, we may assume that there is a $\tilde u\in H^s(\mathbb{R}^3)$ such that

$$\begin{cases}\tilde u_n \rightharpoonup \tilde u & \text{in} \quad H^s(\mathbb{R}^3),\\ \tilde u_n\to\tilde u & \text{in} \quad L^q_{\mathrm{loc}}(\mathbb{R}^3),\ 1\le q<2_s^*,\\ \tilde u_n\to\tilde u & \text{a.e. in} \quad \mathbb{R}^3.\end{cases} \tag{3.21}$$

It follows from (3.16) that there exist $R,\ \beta>0$ such that

$$\int_{B_R(0)}\tilde u_n^2\mathrm{d}x \ge \beta>0,$$

which implies that $\tilde u\ne 0$. Set $\tilde v_n=\tilde u_n-\tilde u$. By using Lemma 3.5, (3.14) and (3.21), we deduce that

$$\Phi(\tilde u_n)=\Phi(\tilde u)+\Phi(\tilde v_n)+o_n(1) \tag{3.22}$$

and

$$G(\tilde u_n)=G(\tilde u)+G(\tilde v_n)+o_n(1). \tag{3.23}$$

From this, one has

$$m-\Phi(\tilde u)=\Phi(\tilde v_n)+o_n(1) \quad \text{and} \quad G(\tilde v_n)+o_n(1)=-G(\tilde u). \tag{3.24}$$

Without loss of generality, we may assume that $\tilde v_n\ne 0$. Otherwise, the lemma is trivial. By Lemma 3.3, there exists $\theta_n>0$ such that $(\tilde v_n)_{\theta_n}\in\mathcal{M}$ for any n.

Now we claim that $G(\tilde{u}) \le 0$. Indeed, suppose by contradiction that $G(\tilde{u}) > 0$. From (3.24) we know that $G(\tilde{v}_n) + o_n(1) < 0$. It follows from Lemma 3.2, (3.14) and (3.24) that

$$\begin{aligned} m - \Phi(\tilde{u}) &= \Phi(\tilde{v}_n) + o_n(1) \\ &= I(\tilde{v}_n) - \frac{1}{4s+2t-3} G(\tilde{v}_n) + o_n(1) \\ &\ge I((\tilde{v}_n)_{\theta_n}) - \frac{\theta_n^{4s+2t-3}}{4s+2t-3} G(\tilde{v}_n) + o_n(1) \\ &\ge m + o_n(1), \end{aligned} \tag{3.25}$$

which implies that the claim is true, since $\Phi(\tilde{u}) > 0$.

Lemma 3.3 also implies that there exists $\tilde{\theta} > 0$ such that $\tilde{u}_{\tilde{\theta}} \in \mathcal{M}$. Combining Lemma 3.2, (3.13)–(3.14) and Fatou's lemma, we have that

$$\begin{aligned} m &= \liminf_{n\to\infty} \Big[I(\tilde{u}_n) - \frac{1}{4s+2t-3} G(\tilde{u}_n) \Big] \\ &= \liminf_{n\to\infty} \Big[\frac{s}{4s+2t-3} \int_{\mathbb{R}^3} \tilde{u}_n^2 \mathrm{d}x + \frac{s+t}{4s+2t-3} \int_{\mathbb{R}^3} \Big(f(\tilde{u}_n)\tilde{u}_n - \frac{4s+2t}{s+t} F(\tilde{u}_n) \Big) \mathrm{d}x \\ &\quad + \frac{s}{3} \int_{\mathbb{R}^3} |\tilde{u}_n|^{2_s^*} \mathrm{d}x \Big] \\ &\ge \frac{s}{4s+2t-3} \int_{\mathbb{R}^3} \tilde{u}^2 \mathrm{d}x + \frac{s+t}{4s+2t-3} \int_{\mathbb{R}^3} \Big(f(\tilde{u})\tilde{u} - \frac{4s+2t}{s+t} F(\tilde{u}) \Big) \mathrm{d}x + \frac{s}{3} \int_{\mathbb{R}^3} |\tilde{u}|^{2_s^*} \mathrm{d}x \\ &= I(\tilde{u}) - \frac{1}{4s+2t-3} G(\tilde{u}) \\ &\ge I(\tilde{u}_{\tilde{\theta}}) - \frac{\tilde{\theta}^{4s+2t-3}}{4s+2t-3} G(\tilde{u}) \ge m. \end{aligned}$$

Therefore, we conclude that $I(\tilde{u}) = m$ and $G(\tilde{u}) = 0$. □

Lemma 3.8 Assume that (f_1)–(f_3) hold and that $4s + 2t > 3$. If $I(u) = m$ for $u \in \mathcal{M}$, then u is a critical point of I.

Proof Suppose, by contradiction, that $I'(u) \ne 0$, and there exist $\rho,\ \delta > 0$ such that

$$\|I'(v)\|_{H^{-s}(\mathbb{R}^3)} \ge \rho \quad \text{if } \|u - v\| \le 3\delta,\ \forall v \in H^s(\mathbb{R}^3).$$

We first show that

$$\lim_{\theta\to 1} \|u_\theta - u\| = 0. \tag{3.26}$$

Otherwise, suppose that there exist $\varepsilon_0 > 0$ and a sequence $\{\theta_n\}$ such that

$$\|u_{\theta_n} - u\|^2 \ge \varepsilon_0 \quad \text{as } \theta_n \to 1. \tag{3.27}$$

Notice that there exist two functions U_1 and $U_2 \in C_0(\mathbb{R}^3)$, such that

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u - U_1|^2 \mathrm{d}x < \frac{\varepsilon_0}{20} \quad \text{and} \quad \int_{\mathbb{R}^3} |u - U_2|^2 \mathrm{d}x < \frac{\varepsilon_0}{20}.$$

From this, we get that

$$\begin{aligned} &\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} (u_{\theta_n} - u)|^2 \mathrm{d}x \\ &\le 2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_{\theta_n} - U_1|^2 \mathrm{d}x + 2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u - U_1|^2 \mathrm{d}x \end{aligned}$$

$$\begin{aligned}
&\leq 6\theta_n^{4s+2t-3}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u-U_1|^2\mathrm{d}x+6\theta_n^{4s+2t}\int_{\mathbb{R}^3}|U_1(\theta_n x)-U_1(x)|^2\mathrm{d}x\\
&\quad+6(\theta_n^{2s+t}-1)^2\int_{\mathbb{R}^3}U_1^2\mathrm{d}x+2\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u-U_1|^2\mathrm{d}x\\
&\leq \frac{2\varepsilon_0}{5}+o_n(1)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^3}|u_{\theta_n}-u|^2\mathrm{d}x &\leq 2\int_{\mathbb{R}^3}|u_{\theta_n}-U_2|^2\mathrm{d}x+2\int_{\mathbb{R}^3}|u-U_2|^2\mathrm{d}x\\
&\leq 6\theta_n^{2s+2t-3}\int_{\mathbb{R}^3}|u-U_2|^2\mathrm{d}x+6\theta_n^{2s+2t}\int_{\mathbb{R}^3}|U_2(\theta_n x)-U_2(x)|^2\mathrm{d}x\\
&\quad+6(\theta_n^{s+t}-1)^2\int_{\mathbb{R}^3}U_2^2\mathrm{d}x+2\int_{\mathbb{R}^3}|u-U_2|^2\mathrm{d}x\\
&\leq \frac{2\varepsilon_0}{5}+o_n(1).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|u_{\theta_n}-u\|^2 &= \int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}(u_{\theta_n}-u)|^2\mathrm{d}x+\int_{\mathbb{R}^3}|u_{\theta_n}-u|^2\mathrm{d}x\\
&\leq \frac{4}{5}\varepsilon_0+o_n(1),
\end{aligned}$$

which contradicts (3.27). It follows from (3.26) that for $\delta>0$ given above, there exists $\delta_1>0$ such that

$$\|u_\theta-u\|\leq\delta \quad \text{if } |\theta-1|<\delta_1. \tag{3.28}$$

Lemma 3.2 implies that

$$I(u_\theta)\leq I(u)-\hat{C}(\theta)\int_{\mathbb{R}^3}u^2\mathrm{d}x=m-\hat{C}(\theta)\int_{\mathbb{R}^3}u^2\mathrm{d}x,\quad \forall\theta\geq 0. \tag{3.29}$$

Let $\varepsilon=\min\left\{\frac{1}{3}\min\{\hat{C}(\frac{1}{2}),\hat{C}(\frac{3}{2})\}\int_{\mathbb{R}^3}u^2\mathrm{d}x,1,\frac{\rho\delta}{8}\right\}$ and $S=\{v\in H^s(\mathbb{R}^3)\mid \|u-v\|<\delta\}$. It follows from Lemma 2.3 in [38] that there exists a map $\eta\in C([0,1]\times H^s(\mathbb{R}^3),H^s(\mathbb{R}^3))$ such that

(i) $\eta(1,v)=v$ if $v\notin I^{-1}([m-2\varepsilon,m+2\varepsilon])\cap S_{2\delta}$;

(ii) $\eta(1,I^{m+\varepsilon}\cap S)\subset I^{m-\varepsilon}$;

(iii) $I(\eta(1,v))\leq I(v)$ for all $v\in H^s(\mathbb{R}^3)$;

(iv) $\eta(1,v)$ is a homeomorphism of $H^s(\mathbb{R}^3)$.

Since Lemma 3.2 implies that $I(u_\theta)\leq I(u)=m$ for $\theta\geq 0$, then by (3.28) and (ii), we have

$$I(\eta(1,u_\theta))\leq m-\varepsilon \quad \text{if } |\theta-1|<\delta_1.$$

On the other hand, from (3.29) and (iii), we have

$$I(\eta(1,u_\theta))\leq I(u_\theta)\leq m-\hat{C}(\theta)\int_{\mathbb{R}^3}u^2\mathrm{d}x,\quad \text{if } |\theta-1|\geq\delta_1.$$

In view of (3.2), we can see that $\hat{C}(\theta)>0$ if $|\theta-1|\geq\delta_1$. Thus,

$$\max_{\theta\in[\frac{1}{2},\frac{3}{2}]}I(\eta(1,u_\theta))<m. \tag{3.30}$$

Next, we claim that $\eta(1,u_\theta)\cap\mathcal{M}\neq\emptyset$ for some $\theta\in[\frac{1}{2},\frac{3}{2}]$, which contradicts (3.30). Indeed, we define

$$\Phi_1(\theta)=G(u_\theta)\quad\text{and}\quad\Phi_2(\theta)=G(\eta(1,u_\theta)),\quad\forall\theta\geq 0.$$

From Lemma 3.3 and the definition of the Brouwer degree, we have

$$\deg\Big(\Phi_1,(\frac{1}{2},\frac{3}{2}),0\Big)=1.$$

In view of (3.29) and (i), it is clear that $\eta(1,u_\theta)=u_\theta$ for $\theta=\frac{1}{2}$ and $\theta=\frac{3}{2}$. The homotopy invariance of the Brouwer degree gives that

$$\deg\Big(\Phi_1,(\frac{1}{2},\frac{3}{2}),0\Big)=\deg\Big(\Phi_2,(\frac{1}{2},\frac{3}{2}),0\Big)=1.$$

Thus, there exists $\theta_0\in(\frac{1}{2},\frac{3}{2})$ such that $\Phi_2(\theta_0)=0$, which means that $\eta(1,u_{\theta_0})\in\mathcal{M}$. Therefore, the claim is true and we complete the proof. □

Now, we are ready to prove our main result.

Proof of Theorem 1.1 Combining Lemma 3.7 and Lemma 3.8, one can directly obtain that I has a critical point $\tilde{u}\in\mathcal{M}$ such that $I(\tilde{u})=m>0$.

Next, we want to prove that $\tilde{u}$ is nonnegative. Let us consider the functional

$$I^+(u)=\frac{1}{2}\int_{\mathbb{R}^3}\big(|(-\Delta)^{\frac{s}{2}}u|^2+u^2\big)\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2\mathrm{d}x-\int_{\mathbb{R}^3}F(u)\mathrm{d}x-\frac{1}{2_s^*}\int_{\mathbb{R}^3}(u^+)^{2_s^*}\mathrm{d}x,$$

where $u^+=\max\{u,0\}$ and $u^-=\min\{u,0\}$. Similarly, we can obtain a nontrivial solution u of the equation

$$(-\Delta)^s u+u+\phi_u^t u=f(u)+(u^+)^{2_s^*-1}. \tag{3.31}$$

Multiplying the above equation (3.31) by u^- and integrating over $\mathbb{R}^3$, we find that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{3+2s}}\mathrm{d}xdy\leq 0,$$

but we know that

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\ \geq&\int_{\{y:u<0\}}\int_{\{x:u>0\}}\frac{(u(x)-u(y))(-u^-(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\ &+\int_{\{y:u<0\}}\int_{\{x:u<0\}}\frac{|u^-(x)-u^-(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\ &+\int_{\{y:u>0\}}\int_{\{x:u<0\}}\frac{(u(x)-u(y))u^-(x)}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\ \geq&\ 0.\end{aligned}$$

Thus, $u^-=0$ and $u\geq 0$ is a solution of equation (3.31). Hence, we can assume that $\tilde{u}\geq 0$. □

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