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A LIMIT LAW FOR FUNCTIONALS OF MULTIPLE INDEPENDENT FRACTIONAL BROWNIAN MOTIONS*

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Abstract Let $B=\{B^H(t)\}_{t\geq 0}$ be a d-dimensional fractional Brownian motion with Hurst parameter $H\in(0,1)$. Consider the functionals of k independent d-dimensional fractional Brownian motions

$$\frac{1}{\sqrt{n}}\int_0^{e^{nt_1}}\cdots\int_0^{e^{nt_k}} f(B^{H,1}(s_1)+\cdots+B^{H,k}(s_k))\mathrm{d}s_1\cdots\mathrm{d}s_k,$$

where the Hurst index $H=k/d$. Using the method of moments, we prove the limit law and extending a result by Xu [19] of the case $k=1$. It can also be regarded as a fractional generalization of Biane [3] in the case of Brownian motion.

Key words Limit theorem; fractional Brownian motion; method of moments; chaining argument

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1 Introduction

Let $B=\{B^H(t)\}_{t\geq 0}$ be a fractional Brownian motion (fBm) with Hurst parameter $H\in(0,1)$. A stochastic calculus with respect to it has been intensively developed (see, for example, Biagini et al. [2], Nualart [9]). It is a central Gaussian process with $B^H(0)=0$ and the covariance function

$$\mathrm{E}\left[B^H(t)B^H(s)\right]=\frac{1}{2}\left[t^{2H}+s^{2H}-|t-s|^{2H}\right]$$

for all $t,s\geqslant 0$. This process was first introduced by Kolmogorov and studied by Mandelbrot and Van Ness [8], where a stochastic integral representation in terms of a standard Brownian motion was established:

$$B^H(t)=\frac{\sqrt{2H\Gamma(\frac{3}{2}-H)}}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\int_{\mathbb{R}}\left[(t-s)_+^{H-\frac{1}{2}}-(-s)_+^{H-\frac{1}{2}}\right]\mathrm{d}B(s),$$

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where $u_+ = \max\{u, 0\}$, and $B(s)$ is standard Brownian motion. For $H = \frac{1}{2}$, B^H coincides with the standard Brownian motion B, but B^H is neither a semimartingale nor a Markov process unless $H = \frac{1}{2}$.

On the basis of sufficient study of fBm, the research technology of the fBm is gradually mature, and many results about Brownian motion can be extended to the fBm, especially some classical limit theories. In this article, we will consider the limit law for functionals of d-dimensional fBm. Let $\{B^H(t) = (B^{H_1}(t), \cdots, B^{H_d}(t)), t \geq 0\}$ be a d-dimensional fBm with Hurst index H in $(0,1)$. Let $B^{H,1}$, $B^{H,2}, \cdots, B^{H,k}$ be k independent copies of B^H with $H = k/d$. If $k = 1$, the local time of fBm B^H does not exist. This is called the critical case. In this condition, Xu [19] considered the limit law. That is, for any bounded and integrable function $f: \mathbb{R}^d \to \mathbb{R}$ and $\int_{\mathbb{R}^d} f(x)\mathrm{d}x = 0$,

$$\frac{1}{\sqrt{n}} \int_0^{e^{nt}} f(B^H(s))\mathrm{d}s \xrightarrow{\mathcal{L}} c_{f,d}\sqrt{l(M^{-1}(t))}\eta$$

as n tends to infinity, where $c_{f,d}$ is a constant depend on f and d, $l(t)$ is the local time at 0 of a Brownian motion $B(t)$, $M(t) = \max_{0\leq s\leq t} B(s)$, and η ia a standard normal random variable independent on $l(M^{-1}(t))$. In Biane [3], the limit law with condition $\int_{\mathbb{R}^d} f(x)\mathrm{d}x = 0$ was called the second order limit law, and when $\int_{\mathbb{R}^d} f(x)\mathrm{d}x \neq 0$, which corresponded to the first order limit law. This extended the result of Kallianpur and Robbins [5], Kasahara and Kotani [6] in Brownian motion.

If $k = 2$, the intersection local time of $B^{H,1}$ and $B^{H,2}$ does not exist. For Brownian motion, when $H = \frac{1}{2}$ and $d = 4$, LeGall [7] proved the limit law for functionals of the difference between two independent Brown motions. Bi and Xu [1] generalized the limit theorem to fBms with $d \geq 4$. Recently, Song, Xu, and Yu [16] considered the limit theorems for functionals of two independent Gaussian processes. When the Gaussian process take into fBm, the corresponding convergence in law is as follows:

$$\frac{1}{\sqrt{n}} \int_0^{e^{nt_1}} \int_0^{e^{nt_2}} f(B^{H,1}(u) - B^{H,2}(v))\mathrm{d}u\mathrm{d}v \xrightarrow{\mathcal{L}} \sqrt{D_{f,d}\,(t_1 \wedge t_2)\, N^2}\,\eta$$

as n tends to infinity, where $D_{f,d}$ is a constant depend on f and d, and N and η are two independent real-valued standard normal random variable. This result can be understood as a supplement to the result of Bi and Xu [1] with $d = 3$ and $H = \frac{2}{3}$, and extended the result to second-order limit law with $H = \frac{2}{d}$.

For the condition $k \geq 3$, Biane [3] has given the second-order convergence in law of Brownian motion. On this basis and drawing on the methods of Xu [19] and Song, Xu and Yu [16] at the same time, we deduce that the second-order limit theory of fBm can also be established under $k \geq 3$.

To obtain more precise result, we will prove second-order limit law for fBm in this article, and next theorem is the main result.

Theorem 1.1 Suppose that $Hd = k$ and f is a bounded measurable function on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = 0$ and $\int_{\mathbb{R}^d} |f(x)||x|^\beta\,\mathrm{d}x < \infty$ for some $\beta > 0$. Then, for any $t_1,\ t_2, \cdots, t_k \geq 0$,

$$\frac{1}{\sqrt{n}} \int_0^{e^{nt_1}} \cdots \int_0^{e^{nt_k}} f(B^{H,1}(s_1) + \cdots + B^{H,k}(s_k))\mathrm{d}s_1 \cdots \mathrm{d}s_k$$

$$\xrightarrow{\mathcal{L}} \sqrt{C_{f,d}\,(t_1\wedge\cdots\wedge t_k)\,G_a\big(2^{-(k-1)},2^{-(k-1)}\big)}\,\eta \tag{1.1}$$

as n tends to infinity, where G_a is a random variable of Gamma distribution, η ia a standard normal random variable independent on G_a, and

$$\begin{aligned} C_{f,d}=&\frac{d^{k-1}\Gamma^k(\frac{d+2k}{2k})}{2^{k-1}k^{k-3}\pi^{\frac{d}{2}}}B\Big(\frac{d(k-1)}{2k},\frac{d}{2k}\Big)B\Big(\frac{d(k-2)}{2k},\frac{d}{2k}\Big)\\ &\times\cdots\times B\Big(\frac{d}{2k},\frac{d}{2k}\Big)\left(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{f}(x)|^2|x|^{-d}\,\mathrm{d}x\right), \end{aligned}$$

with $B(\cdot,\cdot)$ being the Beta function and $\Gamma(\cdot)$ being the Gamma function.

The finiteness of $\int_{\mathbb{R}^d}|\widehat{f}(x)|^2|x|^{-d}\,\mathrm{d}x$ given in Theorem 1.1 is explained in the following Remark, which can be find in [16].

Remark 1.2 As the function f is bounded, we can always assume that $\beta\le 1$. Moreover, the assumption on f also implies that $f\in L^p(\mathbb{R}^d)$ for any $p\ge 1$. Note that $\int_{\mathbb{R}^d}f(x)\,\mathrm{d}x=0$. This implies that $|\widehat{f}(x)|\le c_\alpha|x|^\alpha$ for any $\alpha\in[0,\beta]$, where $\widehat{f}(x)$ is the Fourier transform of f. With the help of Plancherel theorem, we can easily obtain the finiteness of $\int_{\mathbb{R}^d}|\widehat{f}(x)|^2|x|^{-d}\,\mathrm{d}x$.

Example 1.3 Let $f(x)=-(2\pi)^{-d/2}xe^{-x^2/2}$ and denoting $p_\varepsilon(x)=(2\pi\varepsilon)^{-d/2}e^{-|x|^2/(2\varepsilon)}$, Theorem 1.1 provides that the exploding rate of derivative of local time for k independent fBms

$$\int_{[0,T]^k}p'_\varepsilon\Big(B^{H,1}(s_1)+\cdots+B^{H,k}(s_k)\Big)\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_k,$$

is $\varepsilon^{-\frac12}\big(\log\frac{1}{\varepsilon}\big)^{\frac12}$, as $\varepsilon\to0$, in the critical case $Hd=k$. Indeed,

$$\begin{aligned} &\varepsilon^{\frac12}\Big(\log\frac1\varepsilon\Big)^{-\frac12}\int_{[0,T]^k}p'_\varepsilon\Big(B^{H,1}(s_1)+\cdots+B^{H,k}(s_k)\Big)\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_k\\ &=(2\pi)^{-d/2}\varepsilon^{\frac{1-d}{2}}\Big(\log\frac1\varepsilon\Big)^{-\frac12}\int_{[0,T]^k}\exp\Big[-\frac{1}{2\varepsilon}\Big(B^{H,1}(s_1)+\cdots+B^{H,k}(s_k)\Big)^2\Big]\\ &\quad\times\Big[-\frac1\varepsilon\Big(B^{H,1}(s_1)+\cdots+B^{H,k}(s_k)\Big)\Big]\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_k\\ &\overset{\text{Law}}{=}\varepsilon^{\frac{k-Hd}{2H}}\Big(\log\frac1\varepsilon\Big)^{-\frac12}\int_{[0,T\varepsilon^{-\frac{1}{2H}}]^k}f\Big(B^{H,1}(s_1)+\cdots+B^{H,k}(s_k)\Big)\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_k\\ &=\frac{1}{\sqrt{2H\log(n)}}\int_{[0,nT]^k}f\Big(B^{H,1}(s_1)+\cdots+B^{H,k}(s_k)\Big)\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_k. \end{aligned}$$

Limit theorems for functionals of multiple independent Brownian motions and their extensions were obtained in Biane [3] and references therein. However, the corresponding results for fBms were not much. As we all know, the general fBm with Hurst index not equal to $\frac12$ is neither a Markov process nor a semimartingale. This means that the methods working for Brownian motions would probably not be used to prove Theorem 1.1 here.

To obtain Theorem 1.1, we would use the methods of moments and some kind of chaining argument, because these methods are becoming more and more mature for fBm. For example, Nualart and Xu [10] first use the chaining argument plus the methods of moments to prove the central limit theorem for an additive functional of the d-dimensional fBm with Hurst index $H\in\big(\frac{1}{d+2},\frac1d\big)$. After that, Nualart and Xu [11] shown the central limit theorem for functionals of two independent fBm with $H\in\big(\frac{2}{d+2},\frac2d\big)$. Later, Bi, and Xu in [1] proved the first-order limit law in the critical case $Hd = 2$ with $H \le 1/2$. Recently, Xu [19] considered the second-order limit laws for additive functionals of d-dimensional fBm with $H = \frac{1}{d}$.

From all above, we can see that the methods of moments and chaining argument are very powerful to prove the limit theorem for functionals of fBm, but for the convergence in law about k independent d-dimensional fBm still has the certain difficulty. The main difficulty is the computational complexity of multiple stochastic integral and the convergence of the corresponding even moments. Multiple stochastic integral with respect to fBm was studied in [4, 18]. Moreover, in this article, the related results can be extended to the cases of other Gaussian processes, for example, the case of sub-fBm, as the stochastic analysis about sub-fBm are very rich (see [14, 15, 17, 20]). At the same time, the corresponding conclusion of non-Gaussian processes (see [12, 13]) will be the direction of future research.

This article is outlined in the following way. After some preliminaries in Section 2, Section 3 is to prove the main result Theorem 1.1, on the basis of the method of moments and the Fourier transform. Throughout this article, if not mentioned otherwise, the letter c, with or without a subscript, denotes a generic positive finite constant whose exact value is independent of n and may change from line to line. Moreover, we use ι to denote $\sqrt{-1}$ and $x \cdot y$ the usual inner product in $\mathbb{R}^d$.

2 Preliminaries

Let $\{B^H(t) = (B^{H,1}(t), \cdots, B^{H,d}(t)), t \ge 0\}$ be a d-dimensional fractional Brownian motion with Hurst index H in $(0,1)$, defined on some probability space $(\Omega, \mathcal{F}, P)$. That is, the components of B^H are independent centered Gaussian processes with covariance function

$$\mathbb{E}\big(B^{H,i}(t)B^{H,i}(s)\big) = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).$$

We shall use the following two properties of fBm B^H to give our proof in next section. The first one is available in [11]. The second one gives estimates to the covariance of increments of fBm on intervals with uncomparable lengths and the key ingredient in the proof of Theorem 1.1, which are available in Lemma 2.4 of [16].

Lemma 2.1 Given $n \ge 1$, there exist two constants κ_H and β_H depending only on n, H, and d, such that for any $0 = s_0 < s_1 < \cdots < s_n$ and $x_i \in \mathbb{R}^d$, $1 \le i \le n$, we have

$$\kappa_H \sum_{i=1}^n |x_i|^2 (s_i - s_{i-1})^{2H} \le \mathrm{Var}\Big(\sum_{i=1}^n x_i \cdot (B^H(s_i) - B^H(s_{i-1}))\Big) \le \beta_H \sum_{i=1}^n |x_i|^2 (s_i - s_{i-1})^{2H}.$$

The first inequality above is called the local nondeterminism property. Moreover, the inequalities in Lemma 2.1 can be rewritten as

$$\kappa_H \sum_{i=1}^n \Big|\sum_{j=i}^n x_j\Big|^2 (s_i - s_{i-1})^{2H} \le \mathrm{Var}\Big(\sum_{i=1}^n x_i \cdot B^H(s_i)\Big) \le \beta_H \sum_{i=1}^n \Big|\sum_{j=i}^n x_j\Big|^2 (s_i - s_{i-1})^{2H}. \quad (2.1)$$

Lemma 2.2 For any $0 < t_1 < t_2 < t_3 < t_4 < \infty$ and $\gamma > 1$, we have the following estimates:

(i) If $\frac{\Delta t_2}{\Delta t_4} \le \frac{1}{\gamma}$ or $\frac{\Delta t_2}{\Delta t_4} \ge \gamma$, there exists a nonnegative decreasing function $\beta_1(\gamma)$ with $\lim\limits_{\gamma\to\infty} \beta_1(\gamma) = 0$, such that

$$\big|\mathbb{E}\big(B^{H,1}(t_4) - B^{H,1}(t_3)\big)\big(B^{H,1}(t_2) - B^{H,1}(t_1)\big)\big| \le \beta_1(\gamma)\,(\Delta t_2)^H (\Delta t_4)^H,$$

where $\Delta t_i = t_i - t_{i-1}$ for $i = 2, 3, 4$.

(ii) If $\frac{\Delta t_2}{\Delta t_3} \le \frac{1}{\gamma}$ and $\frac{\Delta t_4}{\Delta t_3} \le \frac{1}{\gamma}$, there exists a nonnegative decreasing function $\beta_2(\gamma)$ with $\lim\limits_{\gamma\to\infty} \beta_2(\gamma) = 0$, such that

$$\left|\mathbb{E}\big(B^{H,1}(t_4) - B^{H,1}(t_3)\big)\big(B^{H,1}(t_2) - B^{H,1}(t_1)\big)\right| \le \beta_2(\gamma)\,(\Delta t_2)^H (\Delta t_4)^H.$$

3 Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. For simplicity of notation, we use $F_n(t_1, \cdots, t_k)$ to denote the random variables on the left hand side of (1.1), and

$$F_n(t_1, \cdots, t_k) = \frac{1}{\sqrt{n}} \int_0^{e^{nt_1}} \cdots \int_0^{e^{nt_k}} f(B^{H,1}(s_1) + \cdots + B^{H,k}(s_k)) \mathrm{d}s_1 \cdots \mathrm{d}s_k.$$

In order to obtain the limiting distribution of $F_n(t_1, \cdots, t_k)$, we start show that $F_n(t_1, \cdots, t_k)$ have the same limiting distribution as $F_n(t)$ defined in (3.1) with $t = t_1 \wedge \cdots \wedge t_k$ in Lemmas 3.4 and 3.5. Then, we prove that the m-th moment of $F_n(t)$ is asymptotical equal to I_m^n defined in (3.2) by Lemma 3.7. Finally, we obtain the convergence of corresponding moments

$$\lim_{n\to\infty} I_m^n = \mathbb{E}\left[\sqrt{C_{f,d}\,(t_1 \wedge \cdots \wedge t_k)\, G_a\big(2^{-(k-1)}, 2^{-(k-1)}\big)}\,\eta\right]^m$$

in Proposition 3.8, which gives the desired result of Theorem 1.1.

At first, to simplify the proofs of our main results, we give some lemmas as follows.

Lemma 3.1 For fixed $t > 0$, we have

$$\int_{[1,e^{nt}]^k} (s_1^{2H} + \cdots + s_k^{2H})^{-\frac{d}{2}} \mathrm{d}s_1 \cdots \mathrm{d}s_k < c\,n.$$

Proof By the ball coordinate transformation of multiple integral

$$\begin{aligned} s_1^H &= r\cos\varphi_1, \\ s_2^H &= r\sin\varphi_1\cos\varphi_2, \\ &\ \ \vdots \\ s_{k-1}^H &= r\sin\varphi_1\cdots\sin\varphi_{k-2}\cos\varphi_{k-1}, \\ s_k^H &= r\sin\varphi_1\cdots\sin\varphi_{k-2}\sin\varphi_{k-1}, \end{aligned}$$

and for any $a,\ b > 0$,

$$\begin{aligned} \int_0^{\pi/2} (\sin\theta)^{a-1}(\cos\theta)^{b-1}\mathrm{d}\theta &= \int_0^1 x^{a-1}(\sqrt{1-x^2})^{b-1}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x \\ &= \int_0^1 x^{a-1}(1-x^2)^{\frac{b}{2}-1}\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 t^{\frac{a}{2}-1}(1-t)^{\frac{b}{2}-1}\mathrm{d}t \\ &= \frac{1}{2}B\Big(\frac{a}{2}, \frac{b}{2}\Big). \end{aligned}$$

Then,

$$\int_{[1,e^{nt}]^k} (s_1^{2H} + \cdots + s_k^{2H})^{-\frac{d}{2}} \mathrm{d}s_1 \cdots \mathrm{d}s_k$$

$$\leq \int_{\sqrt{k}}^{\sqrt{k}e^{nHt}} r^{-d} r^{\frac{k}{H}-1} \frac{1}{H^k} \int_0^{\pi/2} (\sin\varphi_1)^{\frac{k-1}{H}-1} (\cos\varphi_1)^{\frac{1}{H}-1} \mathrm{d}\varphi_1$$
$$\times \int_0^{\pi/2} (\sin\varphi_2)^{\frac{k-2}{H}-1} (\cos\varphi_2)^{\frac{1}{H}-1} \mathrm{d}\varphi_2 \cdots \int_0^{\pi/2} (\sin\varphi_{k-2})^{\frac{2}{H}-1} (\cos\varphi_{k-2})^{\frac{1}{H}-1} \mathrm{d}\varphi_{k-2}$$
$$\times \int_0^{\pi/2} (\sin\varphi_{k-1})^{\frac{1}{H}-1} (\cos\varphi_{k-1})^{\frac{1}{H}-1} \mathrm{d}\varphi_{k-1}$$
$$= \frac{nt}{(2H)^{k-1}} B\Big(\frac{k-1}{2H}, \frac{1}{2H}\Big) B\Big(\frac{k-2}{2H}, \frac{1}{2H}\Big) \cdots B\Big(\frac{1}{2H}, \frac{1}{2H}\Big) \leq c\, n,$$

where in the equality, we use the fact that $\frac{k}{H} = d$. □

Lemma 3.2 For $\alpha > 0$, we have

$$\int_{\mathbb{R}^d} \int_{[n^{-m}, e^{nt}]^k} |x|^\alpha e^{-|x|^2(s_1^{2H}+\cdots+s_k^{2H})} \mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}x \leq c\, n^{mH\alpha}.$$

Proof Integrating with respect to x gives

$$\int_{\mathbb{R}^d} \int_{[n^{-m}, e^{nt}]^k} |x|^\alpha e^{-|x|^2(s_1^{2H}+\cdots+s_k^{2H})} \mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}x$$
$$= c \int_{[n^{-m}, e^{nt}]^k} (s_1^{2H} + \cdots + s_k^{2H})^{-\frac{d+\alpha}{2}} \mathrm{d}s_1 \cdots \mathrm{d}s_k \leq c\, n^{mH\alpha},$$

where the proof of the last inequality is similar to that of Lemma 3.1. □

Lemma 3.3 Let $\hat{f}$ be the Fourier transform of f, we have

$$\int_{\mathbb{R}^d} \int_{[n^{-m}, e^{nt}]^k} |\widehat{f}(x)|^2 e^{-|x|^2(s_1^{2H}+\cdots+s_k^{2H})} \mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}x < \infty.$$

Proof Using the change of variables $s_i = |x|^{1/H} s_i$ for $i = 1, 2, \cdots, k$,

$$\int_{\mathbb{R}^d} \int_{[n^{-m}, e^{nt}]^k} |\widehat{f}(x)|^2 e^{-|x|^2(s_1^{2H}+\cdots+s_k^{2H})} \mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}x$$
$$\leq \int_{\mathbb{R}^d} |\widehat{f}(x)|^2 |x|^{-\frac{2}{H}} \mathrm{d}x \int_0^\infty e^{-s_1^{2H}} \mathrm{d}u \cdots \int_0^\infty e^{-s_k^{2H}} \mathrm{d}s_k$$
$$\leq c \int_{\mathbb{R}^d} |\widehat{f}(x)|^2 |x|^{-d} \mathrm{d}x,$$

where the last integral is finite by Remark 1.2. □

The following result shows that the limiting distribution of $F_n(t_1, \cdots, t_k)$ depends on $t_1 \wedge \cdots \wedge t_k$.

Lemma 3.4 For fixed $t_i \to 0, i = 1, \cdots, k$, we have

$$\lim_{n\to\infty} \mathbb{E}\Big[|F_n(t_1, \cdots, t_k) - F_n(t_1 \wedge \cdots \wedge t_k, \cdots, t_1 \wedge \cdots \wedge t_k)|\Big] = 0.$$

Proof Without loss of generality, we assume that $t_1 = \min\{t_1, \cdots, t_k\}$, then, we have

$$\mathbb{E}\Big[|F_n(t_1, \cdots, t_k) - F_n(t_1 \wedge \cdots \wedge t_k, \cdots, t_1 \wedge \cdots \wedge t_k)|\Big]$$
$$\leq \frac{1}{\sqrt{n}} \mathbb{E}\Big[\int_0^{e^{nt_1}} \int_{e^{nt_1}}^\infty \int_0^\infty \cdots \int_0^\infty |f(B^{H,1}(s_1) + \cdots + B^{H,k}(s_k))| \mathrm{d}s_1 \cdots \mathrm{d}s_k\Big]$$
$$\leq \frac{1}{\sqrt{n}} \int_0^{e^{nt_1}} \int_{e^{nt_1}}^\infty \int_0^\infty \cdots \int_0^\infty \int_{\mathbb{R}^d} |f(x)| (s_1^{2H} + \cdots + s_k^{2H})^{-\frac{d}{2}} \mathrm{d}x\, \mathrm{d}s_1 \cdots \mathrm{d}s_k$$

$$\leq \frac{1}{\sqrt{n}}\Big(\int_{\mathbb{R}^d}|f(x)|\,\mathrm{d}x\Big)\int_0^1\int_1^\infty\int_0^\infty\cdots\int_0^\infty(s_1^{2H}+\cdots+s_k^{2H})^{-\frac{d}{2}}e^{(k-Hd)nt_1}\mathrm{d}s_1\cdots\mathrm{d}s_k$$
$$=\frac{1}{\sqrt{n}}\int_{\mathbb{R}^d}|f(x)|\,\mathrm{d}x\to 0,\quad n\to\infty,$$

where we use the fact that the probability density function of $B^{H,1}(s_1)+\cdots+B^{H,k}(s_k)$ is less than $(2\pi)^{-\frac{d}{2}}(s_1^{2H}+\cdots+s_k^{2H})^{-\frac{d}{2}}$ in the second inequality and make the change of variables $s_j=e^{-nt_1}s_j$, $j=1,2,\cdots,k$ in the third inequality. □

Lemma 3.5 Let

$$J_n(t)=\frac{1}{\sqrt{n}}\int_{[0,e^{nt}]^k-[1,e^{nt}]^k}f(B^{H,1}(s_1)+\cdots+B^{H,k}(s_k))\mathrm{d}s_1\cdots\mathrm{d}s_k.$$

Then, we have

$$\lim_{n\to\infty}\mathbb{E}[|J_n(t)|]=0.$$

Proof It is easy to find that

$$[0,e^{nt}]^k-[1,e^{nt}]^k=[0,1]^k+[0,1]^{k-1}\times[1,e^{nt}]+\cdots+[0,1]\times[1,e^{nt}]^{k-1},$$

then, by the boundedness and integrability of function f, we can obtain

$$\begin{aligned}\mathbb{E}[|J_n(t)|]\leq&\frac{1}{\sqrt{n}}\|f\|_\infty\int_{[0,1]^k}\mathrm{d}s_1\cdots\mathrm{d}s_k\\&+\frac{1}{\sqrt{n}}\Big(\int_{\mathbb{R}^d}|f(x)|\,\mathrm{d}x\Big)\int_{[0,1]^{k-1}\times[1,e^{nt}]}(s_1^{2H}+\cdots+s_k^{2H})^{-\frac{d}{2}}\mathrm{d}s_1\cdots\mathrm{d}s_k\\&+\cdots+\frac{1}{\sqrt{n}}\Big(\int_{\mathbb{R}^d}|f(x)|\,\mathrm{d}x\Big)\int_{[0,1]\times[1,e^{nt}]^{k-1}}(s_1^{2H}+\cdots+s_k^{2H})^{-\frac{d}{2}}\mathrm{d}s_1\cdots\mathrm{d}s_k\\\leq&\frac{c}{\sqrt{n}}\bigg(\|f\|_\infty+\int_{\mathbb{R}^d}|f(x)|\,\mathrm{d}x\bigg).\end{aligned}$$

This completes the proof. □

Combining Lemma 3.4 with Lemma 3.5, we only need to consider the convergence of random variables

$$F_n(t)=\frac{1}{\sqrt{n}}\int_{[1,e^{nt}]^k}f(B^{H,1}(s_1)+\cdots+B^{H,k}(s_k))\mathrm{d}s_1\cdots\mathrm{d}s_k. \tag{3.1}$$

Let $\mathscr{P}$ be the set of all permutations of $\{1,2,\cdots,m\}$ and define

$$I_m^n=\frac{1}{n^{\frac{m}{2}}}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\mathbb{E}\Big[\int_{[D_m^n]^k}\prod_{i=1}^m f(B^{H,1}_{\sigma_1(i)}(s_1)+\cdots+B^{H,k}_{\sigma_k(i)}(s_k))\mathrm{d}s_1\cdots\mathrm{d}s_k\Big], \tag{3.2}$$

where $B^{H,u}_{\sigma_u(i)}(s_u)$ denotes $B^{H,u}((s_u)_i)$ with the permutation for time variable $(s_u)_i$, for $u=1,\cdots,k$, $i=1,\cdots,m$,

$$D_m^n=\big\{u\in[1,e^{nt}]^m:u_1<\cdots<u_m,\ u_{i+1}-u_i\geq n^{-m},\ i=1,2,\cdots,m-1\big\}.$$

In the following content, we will use the local nondeterminisim and the chaining argument properly to obtain the main estimates in Lemma 3.6, which is helpful in proving Lemma 3.7 and play a very important role in computing $\lim_{n\to\infty}I_m^n$ in Proposition 3.8.

Let

$$I_{nt}(x)=\int_{D_m^n}\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B_i^{H,1}(s_1)\big)\Big)\mathrm{d}s_1,$$

where $B_i^{H,1}(s_1)$ denotes $B^{H,1}((s_1)_i)$ with the time variable $(s_1)_i$, for $i = 1, \cdots, m$.

Applying the similar techniques as [16] and the local nondeterminism property in Lemma 2.1, we have

$$
\begin{aligned}
|I_m^n| &\le \frac{1}{(2\pi)^d \sqrt{n})^m} \int_{\mathbb{R}^{md}} \prod_{i=1}^m \big|\widehat{f}(x_i)\big| (I_{nt}(x))^k \mathrm{d}x \\
&\le cn^{-\frac{m}{2}} \int_{\mathbb{R}^{md}} \int_{[D_m^n]^k} \prod_{i=1}^m \big|\widehat{f}(y_i - y_{i+1})\big| \\
&\quad \times \exp\Big(-\frac{1}{2} \sum_{j=1}^k \mathrm{Var}\Big(\sum_{i=1}^m y_i (B_i^{H,j}(s_j) - B_{i-1}^{H,j}(s_j)) \Big)\Big) \mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}y \\
&\le cn^{-\frac{m}{2}} \int_{\mathbb{R}^{md}} \int_{[D_m^n]^k} \prod_{i=1}^m \big|\widehat{f}(y_i - y_{i+1})\big| \\
&\quad \times \exp\Big(-\frac{\kappa_H}{2} \sum_{i=1}^m |y_i|^2 \big((s_1(i) - s_1(i-1)^{2H} \\
&\quad + \cdots + (s_k(i) - s_k(i-1)^{2H}) \big) \Big) \mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}y \\
&\le cn^{-\frac{m}{2}} \int_{\mathbb{R}^{md}} \int_{[n^{-m}, e^{nt}]^{km}} \prod_{i=1}^m \big|\widehat{f}(x_i - x_{i+1})\big| \\
&\quad \times \exp\Big(-\frac{\kappa_H}{2} \sum_{i=1}^m |x_i| \big(s_1^{2H}(i) + \cdots + s_k^{2H}(i) \big) \Big) \mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}x
\end{aligned}
\tag{3.3}
$$

where $y_i = \sum\limits_{j=i}^m x_j$ with the convention $y_{m+1} = 0$.

Using the chaining argument introduced in [10], we have

$$
\prod_{i=1}^m \big|\widehat{f}(x_i - x_{i+1})\big| \le \sum_{l=1}^m I_l,
$$

where

$$
I_l = \Big(\prod_{j=1}^{l-1} \big|\widehat{f}(x_{2\lfloor \frac{j+1}{2} \rfloor})\big| \Big) \big|\widehat{f}(x_l - x_{l+1}) - \widehat{f}((-1)^l x_{2\lfloor \frac{l+1}{2} \rfloor})\big| \prod_{j=l+1}^m \big|\widehat{f}(x_j - x_{j+1})\big|
$$

for $l = 1, 2, \cdots, m-1$, $\lfloor \frac{i+1}{2} \rfloor$ denotes the integer part of $\frac{i+1}{2}$, and

$$
I_m = \Big(\prod_{j=1}^{m-1} \big|\widehat{f}(x_{2\lfloor \frac{j+1}{2} \rfloor})\big| \Big) \big|\widehat{f}(x_m)\big|.
$$

In this way, we obtain the decomposition

$$
\big|I_m^n\big| \le \frac{(m!)^k}{(2\pi)^{md}} \sum_{k=1}^m A_{l,m}, \tag{3.4}
$$

where

$$
A_{l,m} = n^{-\frac{m}{2}} \int_{\mathbb{R}^{md}} \int_{[n^{-m}, e^{nt}]^{km}} I_l \exp\Big(-\frac{\kappa_H}{2} \sum_{i=1}^m |x_i|^2 (s_1^{2H}(i) + \cdots + s_k^{2H}(i)) \Big) \mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}x.
$$

The estimation of each term $A_{l,m}$ is given as follows.

Lemma 3.6 For any fixed positive constant $\lambda < 1/2$, there exists a positive constant c such that

(i) $A_{l,m} \le c\,n^{-\lambda}$ for $l = 1, 2, \cdots, m-1$,

(ii) $A_{m,m} \le c\,n^{-\frac{1}{2}}$ if m is odd and $A_{m,m} \le c$ if m is even.

Proof To prove part (i), we first consider the case when k is odd. By the assumption on f, we can obtain $|\widehat{f}(x)| \le c_\alpha(|x|^\alpha \wedge 1)$ for any $\alpha \in [0, \beta]$. Then, $A_{l,m}$ is less than a constant multiple of

$$n^{-\frac{m}{2}} \int_{\mathbb{R}^{md}} \int_{[n^{-m}, e^{nt}]^{km}} |x_l|^\alpha \prod_{j=\frac{l+1}{2}}^{\lfloor \frac{m}{2} \rfloor} (|x_{2j}|^\alpha + |x_{2j+1}|^\alpha) \prod_{j=1}^{\frac{l-1}{2}} |\widehat{f}(x_{2j})|^2$$
$$\times \exp\Big(- \frac{\kappa_H}{2} \sum_{i=1}^{m} |x_i|^2 (s_1^{2H}(i) + \cdots + s_k^{2H}(i))\Big) \mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}x.$$

Integrating with respect to the x_is and $s_j(i)$ (for $j = 1, 2, \cdots, k$) with $i \le l-1$ gives

$$A_{l,m} \le c_1\, n^{-\frac{m-(l-1)}{2}} \int_{\mathbb{R}^{(m-l+1)d}} \int_{[n^{-m}, e^{nt}]^{k(m-l+1)}} |x_l|^\alpha \prod_{j=\frac{l+1}{2}}^{\lfloor \frac{m}{2} \rfloor} (|x_{2j}|^\alpha + |x_{2j+1}|^\alpha)$$
$$\times \exp\Big(- \frac{\kappa_H}{2} \sum_{i=l}^{m} |x_i|^2 \, (s_1^{2H}(i) + \cdots + s_k^{2H}(i))\Big) \, \mathrm{d}\overline{s_1} \cdots \mathrm{d}\overline{s_k} \, \mathrm{d}\overline{x},$$

where $\mathrm{d}\overline{s_j} = \mathrm{d}s_j(l) \cdots \mathrm{d}s_j(m)$, for $j = 1, 2, \cdots, k$ and $\mathrm{d}\overline{x} = \mathrm{d}x_k \cdots \mathrm{d}x_m$.

By Lemmas 3.1 and 3.2,

$$\begin{aligned} A_{l,m} &\le c\, n^{-\frac{m-l+1}{2} + (\lfloor \frac{m-l+1}{2} \rfloor + 1)(mH\alpha) + (m-l-\lfloor \frac{m-l+1}{2} \rfloor)} \\ &= c\, n^{\frac{m}{2} - \lfloor \frac{m}{2} \rfloor - 1 + (\lfloor \frac{m-l+1}{2} \rfloor + 1)(mH\alpha)}. \end{aligned}$$

Choosing α small enough such that

$$\frac{m}{2} - \lfloor \frac{m}{2} \rfloor - 1 + (\lfloor \frac{m-l+1}{2} \rfloor + 1)(mH\alpha) = -\lambda$$

gives

$$A_{l,m} \le c\, n^{-\lambda}. \tag{3.5}$$

By the same way, we obtain the case when l is even.

$$\begin{aligned} A_{l,m} &\le c\, n^{-\frac{m}{2}} \int_{\mathbb{R}^{md}} \int_{[n^{-m}, e^{nt}]^{km}} |x_l|^\alpha |x_{l+1}|^\alpha \prod_{j=\frac{l+2}{2}}^{\lfloor \frac{m}{2} \rfloor} (|x_{2j}|^\alpha + |x_{2j+1}|^\alpha) \\ &\quad \times \Big(\prod_{j=1}^{\frac{l-2}{2}} |\widehat{f}(x_{2j})|^2 \Big) \exp\Big(- \frac{\kappa_H}{2} \sum_{i=1}^{m} |x_i|^2 \, (s_1^{2H}(i) + \cdots + s_k^{2H}(i))\Big) \, \mathrm{d}\overline{s_1} \cdots \mathrm{d}\overline{s_k} \, \mathrm{d}\overline{x} \\ &\le c\, n^{-\frac{m-(l-2)}{2} + (\lfloor \frac{m-l}{2} \rfloor + 2)(mH\alpha) + (m-l-\lfloor \frac{m-l}{2} \rfloor)} \\ &= c\, n^{\frac{m}{2} - \lfloor \frac{m}{2} \rfloor - 1 + (\lfloor \frac{m-l}{2} \rfloor + 2)(mH\alpha)}. \end{aligned}$$

Choosing α small enough such that

$$\frac{m}{2} - \lfloor \frac{m}{2} \rfloor - 1 + (\lfloor \frac{m-k}{2} \rfloor + 2)(mH\alpha) = -\lambda$$

gives

$$A_{l,m} \leq c\, n^{-\lambda}. \tag{3.6}$$

Combining (3.5) and (3.6) gives the desired estimates in part (i).

Finally, we study part (ii). If m is odd, then,

$$\begin{aligned} A_{m,m} &= n^{-\frac{m}{2}} \int_{\mathbb{R}^{md}} \int_{[n^{-m}, e^{nt}]^{km}} |\widehat{f}(x_m)| \prod_{j=1}^{\frac{m-1}{2}} |\widehat{f}(x_{2j})|^2 \\ &\quad \times \exp\Big(-\frac{\kappa_H}{2} \sum_{i=1}^m |x_i|^2 \,(s_1^{2H}(i) + \cdots + s_k^{2H}(i))\Big) \mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}x \\ &\leq c\, n^{-\frac{1}{2}} \int_{\mathbb{R}^d} \int_{[n^{-m}, e^{nt}]^k} |\widehat{f}(x_m)| \\ &\quad \times \exp\Big(-\frac{\kappa_H}{2} |x_m|^2 \,(s_1^{2H}(m) + \cdots + s_k^{2H}(m))\Big) \mathrm{d}s_1(m) \cdots \mathrm{d}s_k(m) \mathrm{d}x \\ &\leq c\, n^{-\frac{1}{2}} \int_{\mathbb{R}^d} |\widehat{f}(x_m)| |x_m|^{-d}\, \mathrm{d}x_m \\ &\leq c\, n^{-\frac{1}{2}}, \end{aligned}$$

where the last second inequality follows from Lemma 3.3 and the last inequality follows easily from Remark 1.2.

If m is even, then, by Lemma 3.3,

$$\begin{aligned} A_{m,m} &= n^{-\frac{m}{2}} \int_{\mathbb{R}^{md}} \int_{[n^{-m}, e^{nt}]^{km}} \prod_{j=1}^{\frac{m}{2}} |\widehat{f}(x_{2j})|^2 \\ &\quad \times \exp\Big(-\frac{\kappa_H}{2} \sum_{i=1}^m |x_i|^2 \,(s_1^{2H}(i) + \cdots + s_k^{2H}(i))\Big) \mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}x \leq c. \end{aligned}$$

This completes the proof. □

Next, we will show that $\mathbb{E}[F_n^m(t)]$ is asymptotical equal to I_m^n.

Lemma 3.7

$$\lim_{n\to\infty} |\mathbb{E}[F_n^m(t)] - I_m^n| = 0.$$

Proof Notice that

$$\begin{aligned} &|\mathbb{E}[F_n^m(t)] - I_m^n| \\ &\leq \frac{1}{n^{\frac{m}{2}}} \Big| \sum_{\sigma_u \in \mathscr{P}, u=1,\cdots,k} \mathbb{E}\Big[\int_{[\widetilde{D}_m^n]^k - [\overline{D}_m^n]^k} \prod_{i=1}^m f(B^{H,1}_{\sigma_1(i)}(s_1) + \cdots + B^{H,k}_{\sigma_k(i)}(s_k)) \mathrm{d}s_1 \cdots \mathrm{d}s_k \Big] \Big| \\ &\quad + \frac{1}{n^{\frac{m}{2}}} \Big| \sum_{\sigma_u \in \mathscr{P}, u=1,\cdots,k} \mathbb{E}\Big[\int_{[\overline{D}_m^n]^k - [D_m^n]^k} \prod_{i=1}^m f(B^{H,1}_{\sigma_1(i)}(s_1) + \cdots + B^{H,k}_{\sigma_k(i)}(s_k)) \mathrm{d}s_1 \cdots \mathrm{d}s_k \Big] \Big| \\ &:= \frac{1}{n^{\frac{m}{2}}} (I_1 + I_2), \end{aligned} \tag{3.7}$$

where $\widetilde{D}_m^n = \big\{ u \in [1, e^{nt}]^m : u_1 < \cdots < u_m \big\}$ and

$$\overline{D}_m^n = \widetilde{D}_m^n \cap \big\{ u_{i+1} - u_i \geq e^{-2mnt},\ i = 1, 2, \cdots, m-1 \big\}.$$

It is easy to find that $\frac{1}{n^{\frac{m}{2}}}I_1 \to 0$ as $n\to\infty$, because f is bounded. For I_2, using Fourier transform, we could obtain

$$\begin{aligned}
I_2 \le{}& \frac{1}{(2\pi)^{md}}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\int_{\mathbb{R}^{md}}\int_{[\overline{D}^n_m]^k-[D^n_m]^k}\prod_{i=1}^m|\widehat{f}(x_i)|\,\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}_{\sigma_1(i)}(s_1)\big)\Big)\\
&\times\cdots\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}_{\sigma_k(i)}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x\\
&=\frac{1}{(2\pi)^{md}}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\int_{\mathbb{R}^{md}}\int_{[\overline{D}^n_m-D^n_m]^k}\prod_{i=1}^m|\widehat{f}(x_i)|\,\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}_{\sigma_1(i)}(s_1)\big)\Big)\\
&\times\cdots\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}_{\sigma_k(i)}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x\\
&+\frac{1}{(2\pi)^{md}}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\int_{\mathbb{R}^{md}}\int_{[\overline{D}^n_m-D^n_m]^{k-1}\times[D^n_m]}\prod_{i=1}^m|\widehat{f}(x_i)|\\
&\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}_{\sigma_1(i)}(s_1)\big)\Big)\\
&\times\cdots\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}_{\sigma_k(i)}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x\\
&+\cdots+\frac{1}{(2\pi)^{md}}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\int_{\mathbb{R}^{md}}\int_{[\overline{D}^n_m-D^n_m]\times[D^n_m]^{k-1}}\prod_{i=1}^m|\widehat{f}(x_i)|\\
&\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}_{\sigma_1(i)}(s_1)\big)\Big)\\
&\times\cdots\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}_{\sigma_k(i)}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x\\
={}& I_{2,1}+I_{2,2}+\cdots+I_{2,k}.
\end{aligned}$$

Applying Hölder inequality to the above integral on the term $I_{2,2}$, we have

$$\begin{aligned}
I_{2,2}\le{}&\frac{c}{(2\pi)^{md}}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\Big[\int_{\mathbb{R}^{md}}\int_{[\overline{D}^n_m-D^n_m]^k}\prod_{i=1}^m|\widehat{f}(x_i)|\,\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}(s_1)\big)\Big)\\
&\times\cdots\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x\Big]^{\frac{k-1}{k}}\\
&\times\Big[\int_{\mathbb{R}^{md}}\int_{[D^n_m]^k}\prod_{i=1}^m|\widehat{f}(x_i)|\,\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}(s_1)\big)\Big)\\
&\times\cdots\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x\Big]^{\frac{1}{k}}.
\end{aligned}$$

Using the similar techniques, we obtain

$$I_{2,3}\le\frac{c}{(2\pi)^{md}}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\Big[\int_{\mathbb{R}^{md}}\int_{[\overline{D}^n_m-D^n_m]^k}\prod_{i=1}^m|\widehat{f}(x_i)|\,\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}(s_1)\big)\Big)$$

$$\times \cdots \times \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x\Big]^{\frac{k-2}{k}}$$
$$\times\Big[\int_{\mathbb{R}^{md}}\int_{[D_m^n]^k}\prod_{i=1}^m|\widehat{f}(x_i)|\,\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}(s_1)\big)\Big)$$
$$\times \cdots \times \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x\Big]^{\frac{2}{k}}$$

and

$$I_{2,k}\le\frac{c}{(2\pi)^{md}}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\Big[\int_{\mathbb{R}^{md}}\int_{[\overline{D}_m^n-D_m^n]^k}\prod_{i=1}^m|\widehat{f}(x_i)|\,\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}(s_1)\big)\Big)$$
$$\times \cdots \times \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x\Big]^{\frac{1}{k}}$$
$$\times\Big[\int_{\mathbb{R}^{md}}\int_{[D_m^n]^k}\prod_{i=1}^m|\widehat{f}(x_i)|\,\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}(s_1)\big)\Big)$$
$$\times \cdots \times \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x\Big]^{\frac{k-1}{k}}.$$

Thus, we only need to consider the convergence of

$$\int_{\mathbb{R}^{md}}\int_{[\overline{D}_m^n-D_m^n]^k}\prod_{i=1}^m|\widehat{f}(x_i)|\,\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}(s_1)\big)\Big)$$
$$\times \cdots \times \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x$$

and

$$\int_{\mathbb{R}^{md}}\int_{[D_m^n]^k}\prod_{i=1}^m|\widehat{f}(x_i)|\,\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}(s_1)\big)\Big)$$
$$\times \cdots \times \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x.$$

From (3.2), (3.4), and Lemma 3.6, we have

$$\int_{\mathbb{R}^{md}}\int_{[D_m^n]^k}\prod_{i=1}^m|\widehat{f}(x_i)|\,\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}(s_1)\big)\Big)$$
$$\times \cdots \times \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x\le n^{m/2}|I_m^n|\le c\,n^{m/2}. \tag{3.8}$$

Using Fourier inverse transform,

$$\int_{\mathbb{R}^{md}}\int_{[\overline{D}_m^n-D_m^n]^k}\prod_{i=1}^m\big|\widehat{f}(x_i)\big|\,\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,1}_{\sigma_1(i)}(s_1)\big)$$
$$-\cdots-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot B^{H,k}_{\sigma_k(i)}(s_k)\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x$$
$$=(2\pi)^{md}\,\mathbb{E}\Big[\int_{[\overline{D}_m^n-D_m^n]^k}\prod_{i=1}^m U_f(B^{H,1}_{\sigma_1(i)}(s_1)+\cdots+B^{H,k}_{\sigma_k(i)}(s_k))\mathrm{d}s_1\cdots\mathrm{d}s_k\Big]$$

$$
\begin{aligned}
&\le c\,n^{-mk}\mathbb{E}\Big[\int_{[\overline{D}_m^n-D_m^n]^k}\prod_{i=1}^{m-1}U_f(B^{H,1}_{\sigma_1(i)}(s_1)+\cdots+B^{H,k}_{\sigma_k(i)}(s_k))\mathrm{d}s_1\cdots\mathrm{d}s_k\Big]\\
&\le c\,n^{-mk}\mathbb{E}\Big[\int_{[\overline{D}_m^n-D_m^n]^k}\int_{\mathbb{R}^{(m-1)d}}\exp\Big(-\frac{1}{2}\mathrm{Var}(\sum_{i=1}^{m-1}x_i(B^{H,1}_{\sigma_1(i)}(s_1)\\
&\quad+\cdots+B^{H,k}_{\sigma_k(i)}(s_k)))\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\Big]\\
&\le c\,n^{-(k-1)m-1},
\end{aligned}
$$

where U_f is the Fourier inverse of $|\widehat{f}|$ and we use Lemmas 2.1 and 3.1 in the last inequality. From all above, we can find

$$
\begin{aligned}
|\mathbb{E}[F_n^m(t)]-I_m^n|\le c\,n^{-\frac{m}{2}}\Big[&n^{-(k-1)m-1}+n^{\frac{k-1}{k}(-(k-1)m-1)}\times n^{\frac{m}{2}}\\
&+\cdots+n^{\frac{1}{k}(-(k-1)m-1)}\times n^{\frac{k-1}{k}\frac{m}{2}}\Big]\to 0,\quad n\to\infty.
\end{aligned}
$$

Hence, this completes the proof. □

From (3.2), applying Fourier transform, we can rewrite

$$
\begin{aligned}
I_m^n=&\frac{1}{[(2\pi)^d\sqrt{n}]^m}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\int_{[D_m^n]^k}\int_{\mathbb{R}^{md}}\prod_{i=1}^m\widehat{f}(x_i)\\
&\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m(B^{H,1}_{\sigma_1(i)}(s_1)+\cdots+B^{H,k}_{\sigma_k(i)}(s_k))\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k.
\end{aligned}
$$

Proposition 3.8 If m is odd, then, $\lim_{n\to\infty}I_m^n=0$. If m is even, then,

$$
\lim_{n\to\infty}I_m^n=(C_{f,d}\,t)^{m/2}\Big[\prod_{i=1}^{m/2}(1+2^{k-1}(i-1))\Big]^2.
$$

Proof The convergence of odd moments follows easily from Lemma 3.6. So, we only need to show that the convergence of even moments.

Define

$$
\widetilde{O}_m=D_m^n\cap\big\{n^2<\Delta u_{2i-1}<e^{nt}/m,\ n^{-1}<\Delta u_{2i}<n,\ i=1,2,\cdots m/2\big\},
$$

where $\Delta u_k=u_k-u_{k-1}$ for $k=1,2,\cdots,m$ with the convention $u_0=0$.

Let

$$
\begin{aligned}
\widetilde{I}_m^n=&\frac{1}{((2\pi)^d\sqrt{n})^m}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\int_{[\widetilde{O}_m]^k}\int_{\mathbb{R}^{md}}\prod_{i=1}^m\widehat{f}(x_i)\\
&\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot(B^{H,1}_{\sigma_1(i)}(s_1)+\cdots+B^{H,k}_{\sigma_k(i)}(s_k))\big)\Big)\,\mathrm{d}x\mathrm{d}s_1\cdots\mathrm{d}s_k.
\end{aligned}
$$

Then, by (3.3) and Lemma 3.6, we have

$$
\begin{aligned}
\limsup_{n\to\infty}|I_m^n-\widetilde{I}_m^n|\le c\limsup_{n\to\infty}n^{-\frac{m}{2}}&\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\int_{\mathbb{R}^{md}}\int_{[D_m^n]^k-[\widetilde{O}_m]^k}\prod_{i=1}^m\widehat{f}(x_i)\\
&\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot(B^{H,1}_{\sigma_1(i)}(s_1)+\cdots+B^{H,k}_{\sigma_k(i)}(s_k))\big)\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
&\le c\limsup_{n\to\infty} n^{-\frac{m}{2}}\int_{\mathbb{R}^{md}}\int_{[D_m^n]^k-[\widetilde{O}_m]^k}\prod_{i=1}^m|\widehat{f}(y_i-y_{i+1})|\\
&\quad\times\exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^m|y_i|^2\big[(\Delta s_1(i))^{2H}+\cdots+(\Delta s_k(i))^{2H}\big]\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\mathrm{d}y\\
&\le c\limsup_{n\to\infty} n^{-\frac{m}{2}}\int_{[D_m^n]^k-[\widetilde{O}_m]^k}\int_{\mathbb{R}^{md}}\prod_{l=1}^{m/2}|\widehat{f}(y_{2l})|^2\\
&\quad\times\exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^m|y_i|^2\big[(\Delta s_1(i))^{2H}+\cdots+(\Delta s_k(i))^{2H}\big]\Big)\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k\\
&\le c\limsup_{n\to\infty} n^{-\frac{m}{2}}\Big(\int_{[D_m^n-\widetilde{O}_m]^k}+\int_{[D_m^n-\widetilde{O}_m]^{k-1}\times[\widetilde{O}_m]}\\
&\quad+\cdots+\int_{[D_m^n-\widetilde{O}_m]\times[\widetilde{O}_m]^{k-1}}\Big)\int_{\mathbb{R}^{md}}\prod_{l=1}^{m/2}|\widehat{f}(y_{2l})|^2\\
&\quad\times\exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^m|y_i|^2\big[(\Delta s_1(i))^{2H}+\cdots+(\Delta s_k(i))^{2H}\big]\Big)\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k\\
&:= II_1+II_2+\cdots+II_k.
\end{aligned}$$

For the convergence of II_j, $(j=1,2,\cdots,k)$, applying the same way as the convergence of I_2 in the proof Lemma 3.7 and using Hölder inequality, we only need to consider the convergence of

$$\begin{aligned}
&\limsup_{n\to\infty} n^{-\frac{m}{2}}\int_{[D_m^n-\widetilde{O}_m]^k}\int_{\mathbb{R}^{md}}\prod_{l=1}^{m/2}|\widehat{f}(y_{2l})|^2\\
&\times\exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^m|y_i|^2\big[(\Delta s_1(i))^{2H}+\cdots+(\Delta s_k(i))^{2H}\big]\Big)\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
&\limsup_{n\to\infty} n^{-\frac{m}{2}}\int_{[\widetilde{O}_m]^k}\int_{\mathbb{R}^{md}}\prod_{l=1}^{m/2}|\widehat{f}(y_{2l})|^2\\
&\times\exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^m|y_i|^2\big[(\Delta s_1(i))^{2H}+\cdots+(\Delta s_k(i))^{2H}\big]\Big)\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k.
\end{aligned}\tag{3.10}$$

It easy to find that, for $l=1,2,\cdots,m$,

$$D_m^n-\widetilde{O}_m=\begin{cases}D_m^n\cap\{n^{-m}\le\Delta u_l\le n^2\text{ or }e^{nt}/m\le\Delta u_l\le e^{nt}\} & \text{if } l \text{ is odd;}\\ D_m^n\cap\{n^{-m}\le\Delta u_l\le n^{-1}\text{ or }n\le\Delta u_l\le e^{nt}\} & \text{otherwise.}\end{cases}$$

If l is odd, (3.9) is less than

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{c}{n}\Big(\int_{[n^{-m},n^2]^2}+\int_{[e^{nt}/m,e^{nt}]^2}\Big)\int_{\mathbb{R}^d}\\
&\times\exp\Big(-\frac{\kappa_H}{2}|y_l|^2\big[(\Delta s_1(l))^{2H}+\cdots+(\Delta s_k(l))^{2H}\big]\Big)\mathrm{d}y\mathrm{d}\Delta s_1\cdots\mathrm{d}\Delta s_k\\
\le\ &\limsup_{n\to\infty}\frac{c}{n}\Big(\int_{[n^{-m},n^2]^2}+\int_{[e^{nt}/m,e^{nt}]^2}\Big)\\
&\times\big[(\Delta s_1(l))^{2H}+\cdots+(\Delta s_k(l))^{2H}\big]^{-\frac{d}{2}}\mathrm{d}\Delta s_1\cdots\mathrm{d}\Delta s_k=0.
\end{aligned}$$

where in the last equality, we use similar arguments as in the proof of Lemma 3.1.

If l is even, (3.9) is less than

$$\begin{aligned}
&c\limsup_{n\to\infty}\Big(\int_{[n^{-m},n^{-1}]^2}+\int_{[n,e^{nt}]^2}\Big)\int_{\mathbb{R}^d}\big|\widehat{f}(y_l)\big|^2\\
&\quad\times\exp\Big(-\frac{\kappa_H}{2}|y_l|^2\big[(\Delta s_1(l))^{2H}+\cdots+(\Delta s_k(l))^{2H}\big]\Big)\mathrm{d}y\mathrm{d}\Delta s_1\cdots\mathrm{d}\Delta s_k\\
&\le c\limsup_{n\to\infty}\int_{[e^{nt}/m,e^{nt}]^2}\int_{\mathbb{R}^d}|y_l|^{2\alpha}\big[(\Delta s_1(l))^{2H}+\cdots+(\Delta s_k(l))^{2H}\big]^{-\frac{d}{2}}\mathrm{d}\Delta s_1\cdots\mathrm{d}\Delta s_k\\
&\le c\limsup_{n\to\infty}\int_{[e^{nt}/m,e^{nt}]^2}\big[(\Delta s_1(l))^{2H}+\cdots+(\Delta s_k(l))^{2H}\big]^{-\frac{d}{2}-\alpha}\mathrm{d}\Delta s_1\cdots\mathrm{d}\Delta s_k\\
&\le c\limsup_{n\to\infty}(e^{-2\alpha nHt}-n^{-2\alpha H})=0,
\end{aligned}$$

where we use Remark 1.2 in the first inequality and the proof of Lemma 3.2 in the second inequality. From (3.8), we can see (3.10) is less than a constant c. Then,

$$\limsup_{n\to\infty}|I_m^n-\widetilde{I}_m^n|=0. \tag{3.11}$$

For any $\gamma>1$, we define

$$\begin{aligned}
\widetilde{I}^n_{m,\gamma}=&\frac{m!}{((2\pi)^d\sqrt{n})^m}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\int_{[\widetilde{O}_m^\gamma]^k}\int_{\mathbb{R}^{md}}\prod_{i=1}^m\widehat{f}(x_i)\\
&\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot(B^{H,1}_{\sigma_1(i)}(s_1)+\cdots+B^{H,k}_{\sigma_k(i)}(s_k))\big)\Big)\mathrm{d}x\mathrm{d}s_1\cdots\mathrm{d}s_k,
\end{aligned}\tag{3.12}$$

where

$$\widetilde{O}_m^\gamma=\widetilde{O}_m\cap\Big\{\frac{\Delta u_{2i-1}}{\Delta u_{2j-1}}>\gamma\ \text{or}\ \frac{\Delta u_{2j-1}}{\Delta u_{2i-1}}>\gamma\ \text{for all}\ i,j\in\{1,2,\cdots,m/2\}\ \text{with}\ i\neq j\Big\}.$$

Then, by (3.3) and Lemma 3.6, using similar arguments as the proof of $\limsup\limits_{n\to\infty}|I_m^n-\widetilde{I}_m^n|=0$, we have

$$\begin{aligned}
|\widetilde{I}_m^n-\widetilde{I}_{m,\gamma}^n|\le\,&c\frac{1}{((2\pi)^d\sqrt{n})^m}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\int_{[\widetilde{O}_m]^k-[\widetilde{O}_m^\gamma]^k}\int_{\mathbb{R}^{md}}\prod_{i=1}^m|\widehat{f}(x_i)|\\
&\times\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m x_i\cdot(B^{H,1}_{\sigma_1(i)}(s_1)+\cdots+B^{H,k}_{\sigma_k(i)}(s_k))\big)\Big)\mathrm{d}x\mathrm{d}s_1\cdots\mathrm{d}s_k\\
\le\,&c\frac{1}{((2\pi)^d\sqrt{n})^m}\int_{[\widetilde{O}_m]^k-[\widetilde{O}_m^\gamma]^k}\int_{\mathbb{R}^{md}}\prod_{i=1}^m|\widehat{f}(y_i-y_{i+1})|\\
&\times\exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^m|y_i|^2\big[(\Delta s_1(i))^{2H}+\cdots+(\Delta s_k(i))^{2H}\big]\Big)\mathrm{d}x\mathrm{d}s_1\cdots\mathrm{d}s_k\\
\le\,&c\,n^{-\lambda}+c\,n^{-\frac{m}{2}}\int_{[\widetilde{O}_m]^k-[\widetilde{O}_m^\gamma]^k}\int_{\mathbb{R}^{md}}\prod_{j=1}^{m/2}|\widehat{f}(y_{2j})|^2\\
&\times\exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^m|y_i|^2\big[(\Delta s_1(i))^{2H}+\cdots+(\Delta s_k(i))^{2H}\big]\Big)\mathrm{d}x\mathrm{d}s_1\cdots\mathrm{d}s_k\\
\le\,&c\,n^{-\lambda}+c\,n^{-\frac{m}{2}}\Big(\int_{[\widetilde{O}_m-\widetilde{O}_m^\gamma]^k}+\int_{[\widetilde{O}_m-\widetilde{O}_m^\gamma]^{k-1}\times[\widetilde{O}_m^\gamma]}+\cdots+\int_{[\widetilde{O}_m-\widetilde{O}_m^\gamma]\times[\widetilde{O}_m^\gamma]^{k-1}}\Big)
\end{aligned}$$

$$\times \int_{\mathbb{R}^{md}} \prod_{j=1}^{m/2} |\widehat{f}(y_{2j})|^2 \exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^{m} |y_i|^2 \big[(\Delta s_1(i))^{2H}$$
$$+\cdots+(\Delta s_k(i))^{2H}\big]\Big)\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k$$
$$:= c\,n^{-\lambda} + III_1 + III_2 + \cdots + III_k.$$

Applying the same way as the convergence of $II_j,\ (j=1,2,\cdots,k)$, using Hölder inequality, we only need to consider the convergence of

$$\limsup_{n\to\infty} n^{-\frac{m}{2}} \int_{[\widetilde{O}_m-\widetilde{O}_m^{\gamma}]^k} \int_{\mathbb{R}^{md}} \prod_{j=1}^{m/2} |\widehat{f}(y_{2j})|^2$$
$$\times \exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^{m} |y_i|^2 \big[(\Delta s_1(i))^{2H}+\cdots+(\Delta s_k(i))^{2H}\big]\Big)\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k \tag{3.13}$$

and

$$\limsup_{n\to\infty} n^{-\frac{m}{2}} \int_{[\widetilde{O}_m^{\gamma}]^k} \int_{\mathbb{R}^{md}} \prod_{j=1}^{m/2} |\widehat{f}(y_{2j})|^2$$
$$\times \exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^{m} |y_i|^2 \big[(\Delta s_1(i))^{2H}+\cdots+(\Delta s_k(i))^{2H}\big]\Big)\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k. \tag{3.14}$$

We can see (3.14) is less than (3.10), which is less than a constant c. By some calculation, we can see (3.13) is less than

$$\limsup_{n\to\infty} c\,n^{-2} \int_{[n^{-m},e^{nt}]^{2k}} \big[(\Delta s_1(l_1))^{2H}+\cdots+(\Delta s_k(l_1))^{2H}\big]^{-\frac{d}{2}}$$
$$\times \big[(\Delta s_1(l_2))^{2H}+\cdots+(\Delta s_k(l_2))^{2H}\big]^{-\frac{d}{2}} 1_{\{1/\gamma\le \Delta s_1(l_1)/\Delta s_1(l_2)\le\gamma\}}$$
$$\times\cdots\times 1_{\{1/\gamma\le \Delta s_k(l_1)/\Delta s_k(l_2)\le\gamma\}}\Big)\mathrm{d}s_1(l_1)\cdots\mathrm{d}s_k(l_1)\mathrm{d}s_1(l_2)\cdots\mathrm{d}s_k(l_2)$$
$$\le \limsup_{n\to\infty} c\,\frac{\log\gamma}{n} = 0.$$

Therefore,

$$\limsup_{n\to\infty} |\widetilde{I}_m^n - \widetilde{I}_{m,\gamma}^n| = 0. \tag{3.15}$$

Making the change of variables $y_i = \sum\limits_{j=i}^{m} x_j$ for $i=1,2,\cdots,m$ (with the convention $y_{m+1}=0$), we can rewrite

$$\widetilde{I}_{m,\gamma}^n = \frac{1}{((2\pi)^d\sqrt{n})^m} \sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k} \int_{[\widetilde{O}_m^{\gamma}]^k} \int_{\mathbb{R}^{md}} \prod_{i=1}^{m} \widehat{f}(y_i - y_{i+1})$$
$$\times \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^{m} y_{\sigma_1(i)} \cdot (B^H_{\sigma_1(i)}(s_1) - B^H_{\sigma_1(i-1)}(s_1))\big)\Big)$$
$$\times \prod_{l=2}^{k} \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^{m}\sum_{j=i}^{m} (y_{\sigma_l(j)} - y_{\sigma_l(j)+1})$$
$$\times (B^H_{\sigma_l(i)}(s_l) - B^H_{\sigma_l(i-1)}(s_l))\big)\Big)\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k.$$

For any $\varepsilon \in (0,1)$, we define

$$\begin{aligned}\widetilde{I}^{n,\varepsilon}_{m,\gamma} = \frac{1}{((2\pi)^d\sqrt{n})^m} \sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k} \int_{[\widetilde{O}^{\gamma}_m]^k} \int_{T^{\sigma}_{\varepsilon}} \prod_{i=1}^m \widehat{f}(y_i-y_{i+1}) \\ \times \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m y_{\sigma_1(i)}\cdot(B^H_{\sigma_1(i)}(s_1)-B^H_{\sigma_1(i-1)}(s_1))\big)\Big) \\ \times \prod_{l=2}^k \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m\sum_{j=i}^m (y_{\sigma_l(j)}-y_{\sigma_l(j)+1}) \\ \times (B^H_{\sigma_l(i)}(s_l)-B^H_{\sigma_l(i-1)}(s_l))\big)\Big)\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k, \qquad (3.16)\end{aligned}$$

where

$$T^{\sigma}_{\varepsilon} = \mathbb{R}^{md}\cap\Big\{|y_{2i-1}|<\varepsilon, \Big|\sum_{j=2i-1}^m (y_{\sigma_l(j)}-y_{\sigma_l(j)+1})\Big|<\varepsilon,\ i=1,2,\cdots,m/2,\ l=1,2,\cdots,k\Big\}.$$

Let

$$T_{\sigma,\varepsilon} = \mathbb{R}^{md}\cap\Big\{\big|\sum_{j=2i-1}^m (y_{\sigma_l(j)}-y_{\sigma_l(j)+1})\big|<\varepsilon,\ i=1,2,\cdots,m/2,\ l=1,2,\cdots,k\Big\}$$

and

$$T_{\varepsilon} = \mathbb{R}^{md}\cap\big\{|y_{2i-1}|<\varepsilon,\ i=1,2,\cdots,m/2\big\}.$$

Then, $T^{\sigma}_{\varepsilon} = T_{\varepsilon}\cap T_{\sigma,\varepsilon}$. This implies that

$$\mathbb{R}^{md}-T^{\sigma}_{\varepsilon} = \big(\mathbb{R}^{md}-T_{\varepsilon}\big)\cup\big(\mathbb{R}^{md}-T_{\sigma,\varepsilon}\big)$$

and the value of $|\widetilde{I}^n_{m,\gamma}-\widetilde{I}^{n,\varepsilon}_{m,\gamma}|$ in the interval $\mathbb{R}^{md}-T_{\sigma,\varepsilon}$ is zero as $\varepsilon\to 0$. Thus,

$$\begin{aligned}|\widetilde{I}^n_{m,\gamma}-\widetilde{I}^{n,\varepsilon}_{m,\gamma}| &\le c\,n^{-\frac{m}{2}} \sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k} \int_{[\widetilde{O}^{\gamma}_m]^k} \int_{\mathbb{R}^{md}-T_{\varepsilon}} \prod_{i=1}^m \widehat{f}(y_i-y_{i+1}) \\ &\quad\times \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m y_{\sigma_1(i)}\cdot(B^H_{\sigma_1(i)}(s_1)-B^H_{\sigma_1(i-1)}(s_1))\big)\Big) \\ &\quad\times \prod_{l=2}^k \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m\sum_{j=i}^m (y_{\sigma_l(j)}-y_{\sigma_l(j)+1}) \\ &\quad\times (B^H_{\sigma_l(i)}(s_l)-B^H_{\sigma_l(i-1)}(s_l))\big)\Big)\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k \\ &\le c\,n^{-\frac{m}{2}} \int_{[\widetilde{O}^{\gamma}_m]^k} \int_{\mathbb{R}^{md}-T_{\varepsilon}} \prod_{i=1}^m |\widehat{f}(y_i-y_{i+1})| \\ &\quad\times \prod_{l=1}^k \exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^m y_i\cdot(B^H_i(s_l)-B^H_{i-1}(s_l))\big)\Big)\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k \\ &\le c\,n^{-\lambda} + c\,n^{-\frac{m}{2}} \int_{[\widetilde{O}^{\gamma}_m]^k} \int_{\mathbb{R}^{md}-T_{\varepsilon}} \prod_{j=1}^{m/2} |\widehat{f}(y_{2j})|^2 \\ &\quad\times \exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^m |y_i|^2\big[(\Delta s_1(i))^{2H}+\cdots+(\Delta s_k(i))^{2H}\big]\Big)\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k\end{aligned}$$

$$
\begin{aligned}
&\le c n^{-\lambda}+c n^{-1}\int_{[n^2,e^{nt}/m]^{2k}}\int_{|x|\ge\varepsilon}\\
&\quad\times \exp\Big(-\frac{\kappa_H}{2}|x|^2(s_1^{2H}+\cdots+s_k^{2H})\Big)\,\mathrm{d}x\mathrm{d}s_1\cdots\mathrm{d}s_k\\
&\le c n^{-\lambda}+c\, e^{-\frac{k}{4}\kappa_H\varepsilon^2 n^{4H}},
\end{aligned}
$$

where we use the same argument as in (3.3) in the second inequality, and use Lemmas 2.1 and 3.6 in the third inequality. This gives

$$
\limsup_{n\to\infty}|\widetilde{I}^{n}_{m,\gamma}-\widetilde{I}^{n,\varepsilon}_{m,\gamma}|=0. \tag{3.17}
$$

Making the change of variables $y_i=\sum\limits_{j=i}^{m}x_j$ for $i=1,2,\cdots,m$ again, we can rewrite

$$
\begin{aligned}
\widetilde{I}^{n,\varepsilon}_{m,\gamma}=&\frac{1}{((2\pi)^d\sqrt{n})^m}\sum_{\sigma_u\in\mathscr{P},u=1,\cdots,k}\int_{[\widetilde{O}^{\gamma}_m]^k}\int_{\overline{T}^{\sigma}_{\varepsilon}}\prod_{i=1}^{m}\widehat{f}(x_i)\\
&\times\prod_{l=1}^{k}\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^{m}\sum_{j=i}^{m}x_{\sigma_l(j)}\cdot(B^H_{\sigma_l(i)}(s_l)-B^H_{\sigma_l(i-1)}(s_l))\big)\Big)\,\mathrm{d}x\mathrm{d}s_1\cdots\mathrm{d}s_k,
\end{aligned}
$$

where

$$
\overline{T}^{\sigma}_{\varepsilon}=\mathbb{R}^{md}\cap\Big\{\Big|\sum_{j=2l-1}^{m}x_{\sigma_i(j)}\Big|<\varepsilon,\ l=1,2,\cdots,m/2,\ i=1,2,\cdots,k\Big\}. \tag{3.18}
$$

For any $\sigma_i\in\mathscr{P}$ $(i=1,2,\cdots,k)$, define

$$
\mathscr{P}_1=\{\sigma_i\in\mathscr{P}:\ \#A(\sigma_1,\cdots,\sigma_k)=m/2, i=1,2,\cdots,k\} \tag{3.19}
$$

and $\mathscr{P}_0=\mathscr{P}-\mathscr{P}_1$, where

$$
\begin{aligned}
A(\sigma_1,\cdots,\sigma_k)=&\Big\{\{\sigma_1(2l),\sigma_1(2l-1)\},\ l=1,2,\cdots,m/2\Big\}\\
&\cap\cdots\cap\Big\{\{\sigma_k(2l),\sigma_k(2l-1)\},\ l=1,2,\cdots,m/2\Big\}.
\end{aligned}
$$

We can see that $\#\mathscr{P}_1=k^{\frac{m}{2}}(m/2)!$. For any $\sigma_i\in\mathscr{P}$ $(i=1,2,\cdots,k)$, let

$$
\begin{aligned}
\widetilde{I}^{n,\varepsilon,\sigma}_{m,\gamma}=&\frac{1}{((2\pi)^d\sqrt{n})^m}\int_{[\widetilde{O}^{\gamma}_m]^k}\int_{\overline{T}^{\sigma}_{\varepsilon}}\prod_{i=1}^{m}\widehat{f}(x_i)\\
&\times\prod_{l=1}^{k}\exp\Big(-\frac{1}{2}\mathrm{Var}\big(\sum_{i=1}^{m}\sum_{j=i}^{m}x_{\sigma_l(j)}\cdot(B^H_{\sigma_l(i)}(s_l)-B^H_{\sigma_l(i-1)}(s_l))\big)\Big)\,\mathrm{d}x\mathrm{d}s_1\cdots\mathrm{d}s_k.
\end{aligned}
$$

When $\sigma\in\mathscr{P}_0$, by Lemmas 3.1 and 3.3, using the same argument as in the proof of Proposition 4.5 in [16], as $\varepsilon\to0$, we have

$$
\limsup_{n\to\infty}|\widetilde{I}^{n,\varepsilon,\sigma}_{m,\gamma}|\le c_{25}\int_{\mathbb{R}^{2d}}|\widehat{f}(x)|^2|\widehat{f}(y)|^2|x|^{-d}|y|^{-d}\,1_{\{|x-y|\le m\varepsilon\}}\,\mathrm{d}x\mathrm{d}y\to0,
$$

When $\sigma\in\mathscr{P}_1$, note that $\mathscr{P}$ is the set of all permutations of $\{1,2,\cdots,m\}$. For any $(\tau,\sigma)\in\mathscr{P}\times\mathscr{P}$, define

$$
\phi_{\tau,\sigma}(j)=\sup\{\tau(i):i\in A^{\sigma}_j\},
$$

where

$$
A^{\sigma}_j=\{\sigma(1),\sigma(2),\cdots,\sigma(j)\}\triangle\{\sigma(1)+1,\sigma(2)+1,\cdots,\sigma(j)+1\}.
$$

Let Ω_m be the set of all $(\tau, \sigma_1, \sigma_2, \cdots, \sigma_{k-1})$ such that $\phi_{\tau,\sigma_1}, \phi_{\tau,\sigma_2}, \cdots, \phi_{\tau,\sigma_{k-1}}$ is bijective. Then, from Biane [3], the number of elements in Ω_m is

$$\prod_{i=1}^{m}(1+2^{k-1}(i-1)).$$

Define the set

$$T_{\varepsilon,\gamma}^{\sigma} = \{T_{\varepsilon}^{\sigma} \cap \{|y_{2i}| > \gamma\varepsilon : i=1,2,\cdots,m/2\}\} - \bigcup_{i\neq j\in\{2k-1:k=1,2,\cdots,m/2\}} \{|y_j|/\gamma < |y_i| < \gamma|y_j|\}.$$

For γ large enough, on $T_{\varepsilon,\gamma}^{\sigma}$, we have the fact (details can be find in [16]),

$$\Big|\sum_{j=i}^{m}(y_{\sigma(j)} - y_{\sigma(j)+1})\Big| \geq (1-\frac{m}{\gamma})\sup_{j\in A_i^{\sigma}}|y_j|.$$

Thus, by Lemma 2.2, using the same argument as the proof Proposition 4.5 in [16], we have

$$\begin{aligned}
&\limsup_{n\to\infty} \widetilde{I}_{m,\gamma}^{n,\varepsilon,\sigma} \\
&\leq \limsup_{n\to\infty} \frac{1}{((2\pi)^d\sqrt{n})^m} \int_{[\widetilde{O}_m^{\gamma}]^k} \int_{T_{\varepsilon}^{\sigma}} \prod_{j=1}^{m/2} |\widehat{f}(y_{2j})|^2 \\
&\quad \times \exp\Big(-\frac{1}{2}(1-\frac{c}{\gamma^H}-\frac{c}{n^{\theta}})\sum_{i=1}^{m}|y_i|^2(\Delta s_1(i))^{2H}\Big) \\
&\quad \times \prod_{l=2}^{k} \exp\Big(-\frac{1}{2}(1-\frac{c}{\gamma^H}-\frac{c}{n^{\theta}})\sum_{i=1}^{m}|\sum_{j=i}^{m}(y_{\sigma_l(j)}-y_{\sigma_l(j)+1})|^2(\Delta s_l(i))^{2H}\Big)\,\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_k \\
&\leq c\int_{|y|\leq\lambda\varepsilon} |\widehat{f}(y)|^2|y|^{-d}\,\mathrm{d}y \\
&\quad + \frac{k^m[\prod_{i=1}^{m/2}(1+2^{k-1}(i-1))][(m-1)!!]}{(2\pi)^{md/4}} \Big(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{f}(z)|^2|z|^{-d}\,\mathrm{d}z\Big)^{m/2} \\
&\quad \times \Big(\int_{[0,\infty)^k} \exp\Big(-\frac{1}{2}(1-\frac{c}{\gamma^H})(1-\frac{m}{\gamma})(s_1)^{2H}+\cdots+(s_k)^{2H}\Big)\mathrm{d}s_1\cdots\mathrm{d}s_k\Big)^{m/2} \\
&\quad \times \Big(\limsup_{n\to\infty}\frac{1}{n}\int_{[n^{-m},e^{nt}]^k}(1-\frac{c}{\gamma^H})^{-\frac{d}{2}})(s_1)^{2H}+\cdots+(s_k)^{2H})^{-\frac{d}{2}}\mathrm{d}s_1\cdots\mathrm{d}s_k\Big)^{m/2} \\
&\to \frac{k^m[\prod_{i=1}^{m/2}(1+2^{k-1}(i-1))][(m-1)!!]}{(2\pi)^{md/4}} \Big(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{f}(x)|^2|x|^{-d}\,\mathrm{d}x\Big)^{m/2} \\
&\quad \times \Big(\int_0^{+\infty} e^{-\frac{1}{2}r^{2H}}\,\mathrm{d}r\Big)^{\frac{km}{2}} \Big(\lim_{n\to\infty}\frac{1}{n}\int_{[1,e^{nt}]^k}(s_1^{2H}+\cdots+s_k^{2H})^{-\frac{d}{2}}\,\mathrm{d}s_1\cdots\mathrm{d}s_k\Big)^{\frac{m}{2}},
\end{aligned}$$

as $\varepsilon\to 0$ first, and then, $\gamma\to\infty$, where θ is $(1-2H)\wedge H$ if $H\leq 1/2$ and $2-2H$ otherwise.

On the other hand, using the same way, we have

$$\liminf_{n\to\infty} \widetilde{I}_{m,\gamma}^{n,\varepsilon,\sigma} \geq \frac{k^m[\prod_{i=1}^{m/2}(1+2^{k-1}(i-1))][(m-1)!!]}{(2\pi)^{md/4}} \Big(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{f}(x)|^2|x|^{-d}\,\mathrm{d}x\Big)^{m/2}$$

$$\times \left(\int_0^{+\infty} e^{-\frac{1}{2}r^{2H}}\,\mathrm{d}r\right)^{\frac{km}{2}} \left(\lim_{n\to\infty}\frac{1}{n}\int_{[1,e^{nt}]^k}(s_1^{2H}+\cdots+s_k^{2H})^{-\frac{d}{2}}\,\mathrm{d}s_1\cdots\mathrm{d}s_k\right)^{\frac{m}{2}}.$$

Therefore,

$$\lim_{n\to\infty}\widetilde{I}_{m,\gamma}^{n,\varepsilon} = \frac{k^m[\prod_{i=1}^{m/2}(1+2^{k-1}(i-1))][(m-1)!!]}{(2\pi)^{md/4}}\left(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{f}(x)|^2|x|^{-d}\,\mathrm{d}x\right)^{m/2}$$
$$\times \left(\int_0^{+\infty} e^{-\frac{1}{2}r^{2H}}\,\mathrm{d}r\right)^{\frac{km}{2}} \left(\lim_{n\to\infty}\frac{1}{n}\int_{[1,e^{nt}]^k}(s_1^{2H}+\cdots+s_k^{2H})^{-\frac{d}{2}}\,\mathrm{d}s_1\cdots\mathrm{d}s_k\right)^{\frac{m}{2}}. \tag{3.20}$$

It is easy to find that

$$\left(\int_0^{\infty} e^{-\frac{1}{2}r^{2H}}\mathrm{d}r\right)^k = 2^{\frac{d}{2}}\Gamma^k\Big(\frac{d}{2k}+1\Big),$$

where $\Gamma(\cdot)$ is a Gamma function, and by Lemma 3.1, we find that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\int_{[1,e^{nt}]^k}(s_1^{2H}+\cdots+s_k^{2H})^{-\frac{d}{2}}\,\mathrm{d}s_1\cdots\mathrm{d}s_k\\
&= \lim_{n\to\infty}\frac{1}{n}\int_{\sqrt{k}}^{\sqrt{k}e^{nHt}} r^{-d}r^{\frac{k}{H}-1}\\
&\quad\times\frac{1}{H^k}\int_0^{\pi/2}(\sin\varphi_1)^{\frac{k-1}{H}-1}(\cos\varphi_1)^{\frac{1}{H}-1}\mathrm{d}\varphi_1\int_0^{\pi/2}(\sin\varphi_2)^{\frac{k-2}{H}-1}(\cos\varphi_2)^{\frac{1}{H}-1}\mathrm{d}\varphi_2\\
&\quad\times\cdots\times\int_0^{\pi/2}(\sin\varphi_{k-2})^{\frac{2}{H}-1}(\cos\varphi_{k-2})^{\frac{1}{H}-1}\mathrm{d}\varphi_{k-2}\\
&\quad\times\int_0^{\pi/2}(\sin\varphi_{k-1})^{\frac{1}{H}-1}(\cos\varphi_{k-1})^{\frac{1}{H}-1}\mathrm{d}\varphi_{k-1}\\
&= \frac{t}{(2H)^{k-1}}B\Big(\frac{k-1}{2H},\frac{1}{2H}\Big)B\Big(\frac{k-2}{2H},\frac{1}{2H}\Big)\cdots B\Big(\frac{1}{2H},\frac{1}{2H}\Big).
\end{aligned}$$

Define

$$\begin{aligned}
C_{f,d} = &\frac{d^{k-1}\Gamma^k(\frac{d+2k}{2k})}{2^{k-1}k^{k-3}\pi^{\frac{d}{2}}}B\Big(\frac{d(k-1)}{2k},\frac{d}{2k}\Big)B\Big(\frac{d(k-2)}{2k},\frac{d}{2k}\Big)\\
&\times\cdots\times B\Big(\frac{d}{2k},\frac{d}{2k}\Big)\left(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{f}(x)|^2|x|^{-d}\,\mathrm{d}x\right).
\end{aligned}$$

Combining (3.10), (3.15), (3.17) with(3.20), we have

$$\lim_{n\to\infty}I_m^n = (C_{f,d}\,t)^{m/2}\left[\prod_{i=1}^{m/2}(1+2^{k-1}(i-1))\right][(m-1)!!].$$

□

Proof of Theorem 1.1 This follows from Lemmas 3.4–3.6 and Proposition 3.8. □

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