Acta Mathematica Scientia, 2020, **40B**(2): 442–456
https://doi.org/10.1007/s10473-020-0210-x

Mathematica Acta Scientia
数学物理学报
http://actams.wipm.ac.cn

LOCAL WELL-POSEDNESS OF STRONG SOLUTIONS FOR THE NONHOMOGENEOUS MHD EQUATIONS WITH A SLIP BOUNDARY CONDITIONS*

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Abstract This article is concerned with the 3D nonhomogeneous incompressible magnetohydrodynamics equations with a slip boundary conditions in bounded domain. We obtain weighted estimates of the velocity and magnetic field, and address the issue of local existence and uniqueness of strong solutions with the weaker initial data which contains vacuum states.

Key words Nonhomogeneous MHD equations; local existence and uniqueness; vacuum; t-weighted H^2 estimate; Galerkin approximation

2010 MR Subject Classification 35Q35; 76D03

1 Introduction

In this article, we consider the following nonhomogeneous incompressible MHD system

$$\partial_t\rho + u\cdot\nabla\rho = 0, \tag{1.1}$$

$$\rho\partial_t u - \Delta u + \rho(u\cdot\nabla)u + \nabla\pi = (b\cdot\nabla)b, \tag{1.2}$$

$$\partial_t b - \Delta b + (u\cdot\nabla)b - (b\cdot\nabla)u = 0, \tag{1.3}$$

$$\nabla\cdot u = 0, \quad \nabla\cdot b = 0, \tag{1.4}$$

in $\Omega\times(0,T)$, where $\Omega\subset\mathbb{R}^3$ is a bounded open set with smooth boundary, and $T>0$. The unknowns $\rho, u=(u_1,u_2,u_3), b=(b_1,b_2,b_3)$, and π denote the density, the velocity field, the magnetic field, and the pressure of the fluid, respectively.

Concerning the nonhomogeneous fluid equations, there exists a considerable number of papers devoted to their mathematical analysis [1, 2, 4, 9–11, 19, 27, 28]. It is worth noting that density may play different roles. For compressible fluid, the pressure is commonly expressed as

*Received September 29, 2018; revised April 8, 2019. This work was supported by Natural Science Foundation of China (11871412).

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exponential function about the density. Navier-Stokes equations with density-dependent viscosity is considered [17]. In [29], Wu studied incompressible magnetohydrodynamic equations with the coefficients depending on the density and temperature. In general, the initial vacuum is taken into consideration; for instance, see [7, 11, 23, 24]. Li [22] considered 3D nonhomogeneous incompressible magnetohydrodynamic equations with density-dependent viscosity and resistivity coefficients, and the vacuum of initial density was also allowed.

Because of its mathematical challenges and physical complexity, rich phenomena, MHD has been the important subject of many studies by mathematicians and physicists; see [5, 6, 8, 12, 14, 21, 26, 30, 32] and the references cited therein. Global well-posedness of three-dimensional incompressible magneto-hydrodynamical system with small and smooth initial data in whole space was established by Xu and Zhang in [33]. The authors in [34] studied the 3D incompressible MHD equations with density-dependent viscosity, and obtained global strong solutions under the assumption that the initial energy was suitably small. Unique global strong solution of the 2D MHD problem with boundary conditions

$$u = 0, \quad b \cdot n = 0, \quad \nabla \times b = 0, \quad \text{on } \partial\Omega \times [0, T),$$

is studied by Huang, as the viscosity $\mu = \mu(\rho)$ is a function in $C^1(0, \infty)$, seeing [16]. In [7], the authors obtained the strong solution of incompressible magnetohydrodynamic equations (1.1)–(1.4), with boundary conditions

$$u = 0, \quad b \cdot n = 0, \quad \nabla \times b \times n = 0, \quad \text{on } \partial\Omega \times [0, T),$$

and the initial data satisfy that

$$0 \le \rho_0 \in L^{\frac{3}{2}} \cap L^\infty \cap H^1, u_0 \in H_0^1 \cap H^2, b_0 \in H^2,$$

and the compatibility conditions

$$\begin{aligned} &\Delta u_0 + \nabla \times b_0 \times b_0 - \nabla p_0 = \sqrt{\rho_0} g, \\ &\nabla \cdot u_0 = \nabla \cdot b_0 = 0, \end{aligned}$$

for some $(p_0, g) \in H^1 \times L^2$.

In bounded domain, various boundary conditions were proposed. Generally, n and τ denote the outward normal vector and the unit tangential vector on $\partial\Omega$, respectively. The classical no-slip boundary condition, $u = 0$ on $\partial\Omega$, which gives rise to the phenomenon of strong boundary layers. Later, Xiao and Xin [31] obtained a unique strong solution of the 3D incompressible Navier-Stokes equations with a slip boundary condition, $u \cdot n = \nabla \times u \cdot \tau = 0$ on $\partial\Omega$, and proved strong convergence for the vanishing viscosity limit in flat case. Guo and Wang studied the similar problem for MHD system with generalized Navier slip boundary conditions, $u \cdot n = 0$ and $n \times (\nabla \times u) = [\beta u]_\tau$ on $\partial\Omega$, where β is a given smooth symmetric tensor on the boundary [13]. For nonhomogeneous fluid, under slip boundary condition, this is quite difficult to get a strong solution because of variable density. Thus, we supplement the system with initial and boundary conditions:

$$(\rho, u, b)(x, 0) = (\rho_0, u_0, b_0)(x), \qquad \text{in } \Omega, \tag{1.5}$$

$$u \cdot n = 0, \ b \cdot n = 0, \ n \times \nabla \times u = 0, \ n \times \nabla \times b = 0, \qquad \text{on } \partial\Omega \times (0, T), \tag{1.6}$$

It is remarkable that Jiu, Wang, and Xin [18] established a unique global classical solution which may contain vacuums to compressible Navier-Stokes equations in weighted spaces. We investigate the existence and uniqueness of strong solutions for system (1.1)–(1.6) in 3D smooth bounded domains. The motivation of the t-weighted estimate originate from the study of several incompressible models, to weaken the regularity of the initial data; see [25]. Moreover, it is worth noting that we do not need the compatibility of initial data. We only assume that the initial data satisfy

$$0 \le \rho_0 \le \bar{\rho}, \quad \rho_0 \in W^{1,p} \cap L^\infty, \quad u_0, b_0 \in H^1_\sigma, \quad p > 1, \bar{\rho} > 0. \tag{1.7}$$

The article is arranged as follows: In Section 2, we briefly recall the notations, some lemmas, and main results for the system. Next, in Section 3, we will obtain some prior estimates. Finally, in Section 4, we show Theorem 2.3.

2 Preliminaries and Main Results

In this section, we introduce some notations, which are used throughout this article, definition, main result, and preliminary facts. We use standard notations for Lebesgue and Sobolev spaces. We denote by $\|\cdot\|_p$ the norm in $L^p(\Omega)$, and for simplicity, use $\|\cdot\|$ to denote $\|\cdot\|_2$. Moreover, $\|(\rho,u,b)\|^p := \|\rho\|^p + \|u\|^p + \|b\|^p$, also, use H^m to denote $W^{m,2}$. As usual, we will use the same notation for vector valued and scalar valued spaces without danger of confusion. Spaces L^2_σ and H^1_σ are the closures of the space $C^\infty_\sigma := \{f \in C^\infty(\Omega) : \nabla\cdot f = 0\}$ in L^2 and H^1, respectively.

Strong solutions to system (1.1)–(1.6) are defined as follows.

Definition 2.1 Given a positive time $T \in (0,\infty)$, and the initial data (ρ_0, u_0, b_0) satisfying (1.7), then (ρ, u, b, π) is called a strong solution to system (1.1)–(1.6), on $\Omega \times (0,T)$, if it has the regularities

$$\begin{aligned}
&\rho \in L^\infty(0,T;W^{1,p}\cap L^\infty) \cap C([0,T];L^p), p>1,\\
&u,b \in L^\infty(0,T;H^1_\sigma) \cap L^2(0,T;H^2), \pi \in L^2(0,T;H^1), \rho u \in C([0,T];L^2),\\
&\sqrt{t}u, \sqrt{t}b \in L^\infty(0,T;H^2) \cap L^2(0,T;W^{2,6}), \sqrt{t}\partial_t u, \sqrt{t}\partial_t b \in L^2(0,T;H^1),
\end{aligned}$$

satisfying system (1.1)–(1.6) a.e. in $\Omega \times (0,T)$.

Remark 2.2 According to the regularities of the strong solutions stated in Definition 2.1, and by (1.1)–(1.3), we can infer that the strong solutions have the additional regularities $\partial_t\rho \in L^4(0,T;L^p)$, $\sqrt{\rho}\partial_t u, \partial_t b \in L^2(0,T;L^2)$, and $\sqrt{t\rho}\partial_t u \in L^\infty(0,T;L^2)$.

Our aim is to prove the following theorem:

Theorem 2.3 Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that the initial data (ρ_0, u_0, b_0) satisfies (1.7). Then, there is a positive time T_0, depending only on $\bar{\rho}, \Omega$, and $\|(u_0, b_0)\|$, such that system (1.1)–(1.6) admits a strong solution (ρ, u, b) on $\Omega \times (0, T_0)$. Moreover, if $p \in [2,\infty)$, then the strong solution is unique.

Remark 2.4 In fact, the positive time T_0 also depends on the viscosity-resistivity coefficient, so we just assume that the viscosity-resistivity coefficients are both 1.

Now, we introduce the following lemmas to be needed later on.

Lemma 2.5 ([20]) Let $v \in L^1(0,T;\text{Lip})$ a vector field, such that $\nabla \cdot v = 0$, and $v \cdot n = 0$ on $\partial\Omega$. Let $\rho_0 \in W^{1,q}$, with $q \in [1,\infty]$.

Then, the following system

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho v) = 0, & \text{in } \ \Omega \times (0,T), \\ \rho(0) = \rho_0, & \text{in } \ \Omega, \end{cases}$$

has a unique solution $\rho \in L^\infty(0,T;W^{1,\infty}) \cap C([0,T]; \bigcap\limits_{1\le p<\infty} W^{1,p})$ if $q=\infty$; and $\rho \in C([0,T]; W^{1,q})$ if $1 \le q < \infty$.

Moveover, the following estimate holds:

$$\|\rho\|_{W^{1,q}} \le \|\rho_0\|_{W^{1,q}} \exp\left\{ \int_0^t \|\nabla v(\tau)\|_\infty \mathrm{d}\tau \right\}$$

for any $t \in [0,T]$.

Lemma 2.6 ([3]) Let Ω satisfy the assumptions made in Section 1. Then, the following inequality holds true for all $1 < p < \infty$, with a constant C depending only on p and Ω:

$$\|f\|_{W^{1,p}} \le C\|\nabla \times f\|_p,$$

for each divergence-free vector field $f \in W^{1,p}(\Omega)$ such that either the boundary condition $f \cdot n|_{\partial\Omega} = 0$ or $f \times n|_{\partial\Omega} = 0$ is satisfied.

Lemma 2.7 ([23]) Given a positive time T, let f, g, G be nonnegative functions on $[0,T]$, with f and g being absolutely continuous on $[0,T]$. Suppose that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} f(t) \le A\sqrt{G(t)}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} g(t) + G(t) \le \alpha(t) g(t) + \beta(t) f^2(t), \\ f(0) = 0, \end{cases}$$

a.e. on $(0,T)$, where A is a positive constant, and α and β are two nonnegative functions, satisfying

$$\alpha(t) \in L^1((0,T)) \quad \text{and} \quad t\beta(t) \in L^1((0,T)).$$

Then, the following estimates hold:

$$f(t) \le A\sqrt{g(0)}\sqrt{t} \exp\left\{ \frac{1}{2} \int_0^t (\alpha(s) + A^2 s\beta(s)) \mathrm{d}s \right\},$$
$$g(t) + \int_0^t G(s)\mathrm{d}s \le g(0) \exp\left\{ \int_0^t (\alpha(s) + A^2 s\beta(s)) \mathrm{d}s \right\},$$

for $t \in [0,T]$, which, in particular, imply that $f \equiv 0, g \equiv 0$, and $G \equiv 0$, provided $g(0) = 0$.

3 Galerkin Approximation

In this section, we construct approximate solutions via a Galerkin scheme to system (1.1)–(1.4). We first introduce the approximation scheme, then show the existence of the approximate solutions, and finally derive uniform bounds to be used in next section.

3.1 The scheme

Suppose that Ω is simply connected, as in [32], X^m is defined by

$$X^m = \text{span}\{w_1, \cdots, w_m\} \quad (m = 1, 2, \cdots),$$

which is finite-dimensional subspaces of basic function space

$$X = \{w \in H^1_\sigma(\Omega) \cap H^2(\Omega) | w \cdot n = 0 \text{ on } \partial\Omega\}.$$

Then, consider the functions $w_j \in X$ satisfying

$$(\nabla\times)^2 w_j = \lambda_j w_j,$$
$$\nabla \times w_j \times n|_{\partial\Omega} = 0.$$

These functions w_j make up a suitable orthogonal system of smooth functions in the sense of the following scalar product on X, and are dense in X:

$$a(w_1, w_2) = \int_\Omega \nabla \times w_1 \cdot \nabla \times w_2 \text{d}x.$$

We will solve the following system:

$$\partial_t \rho^m + u^m \cdot \nabla \rho^m = 0, \tag{3.1}$$
$$\big(\rho^m \partial_t u^m, w\big) + \big(\nabla \times u^m, \nabla \times w\big) + \big(\rho^m (u^m \cdot \nabla) u^m, w\big) = \big((b^m \cdot \nabla) b^m, w\big), \tag{3.2}$$
$$\big(\partial_t b^m, \psi\big) + \big(\nabla \times b^m, \nabla \times \psi\big) + \big((u^m \cdot \nabla) b^m, \psi\big) - \big((b^m \cdot \nabla) u^m, \psi\big) = 0, \tag{3.3}$$
$$\rho^m(0) = \rho^m_0, \quad u^m(0) = u^m_0, \quad b^m(0) = b^m_0, \tag{3.4}$$

for all $w, \psi \in X^m$, where $u^m(0), b^m(0)$ are given by

$$u^m_0 = \sum_{j=1}^m (u_0, w_j) w_j, \quad b^m_0 = \sum_{j=1}^m (b_0, w_j) w_j, \quad j = 1, \cdots, m, \tag{3.5}$$

and $\{\rho^m_0\}^\infty_{m=1}$ is a sequence of functions from $C^2(\bar{\Omega})$, satisfying

$$0 < \underline{\rho} \le \rho^m_0 \le \bar{\rho}, \quad \rho^m_0 \to \rho_0 \text{ in } W^{1,q}(\Omega), \quad q \in (3, \infty). \tag{3.6}$$

Applying the results in [23], for any $v \in C([0,T]; X^m)$, the following problem possesses a unique solution $\rho^m \in C^2(\bar{\Omega} \times [0,T])$:

$$\begin{cases} \partial_t \rho^m + v \cdot \nabla \rho^m = 0, \\ \rho^m(0) = \rho^m_0. \end{cases} \tag{3.7}$$

We define the mapping $R : C([0,T]; X^m) \to C^2(\bar{\Omega} \times [0,T])$ as $\rho^m = R[v]$, then $\rho^m = R[v]$ is continuous with respect to $v \in C([0,T]; X^m)$.

Next, for any $v \in C([0,T]; X^m)$, we consider following system

$$\begin{cases} \partial_t b^m - \Delta b^m + (v \cdot \nabla) b^m - (b^m \cdot \nabla) v = 0, \\ b^m(0) = b^m_0, \end{cases} \tag{3.8}$$

which is a linear parabolic-type problem, hence the existence of a solution can be obtained by the standard Galerkin methods. From the results in [15], we obtain the unique solution $b^m \in Y$ of problem (3.8), here $Y = C([0,T]; X^m) \cap L^2([0,T]; H^1_\sigma(\Omega)) \cap L^\infty([0,T]; L^2(\Omega))$, and the solution operator $S : C([0,T]; X^m) \to Y$ as $b^m = S[v]$ is continuous with respect to $v \in C([0,T]; X^m)$.

In order to prove the solvability of system (3.1)–(3.4), it suffices to find a solution $u^m \in C([0,T];X^m)$ to the following system

$$\begin{aligned}\big(R[u^m](\partial_t u^m + (u^m\cdot\nabla)u^m), w\big) + \big(\nabla\times u^m, \nabla\times w\big) &= \big((S[u^m]\cdot\nabla)S[u^m], w\big),\\ u^m(0) &= u_0^m,\end{aligned}$$

where $R[u^m], S[u^m]$, as defined before, are the unique solutions to systems (3.7) and (3.8), respectively, with v replaced by u^m. For this purpose, we consider the following linearized system

$$\begin{cases}\big(R[v](\partial_t u^m + (v\cdot\nabla)u^m), w\big) + \big(\nabla\times u^m, \nabla\times w\big) = \big((S[v]\cdot\nabla)S[v], w\big),\\ \qquad\qquad\qquad\qquad\qquad\qquad u^m(0) = u_0^m,\end{cases} \tag{3.9}$$

where $v \in C([0,T];X^m)$ is given. We define 3-th solution mapping $Q : C([0,T];X^m) \to C([0,T];X^m)$ as $u^m = Q[v]$, then u^m is the unique solution to (3.9). Later, one will shown that the mapping Q is well-defined. Therefore, to prove the solvability of system (3.1)–(3.4), it only require to find a fixed point of the mapping Q in $C([0,T];X^m)$.

3.2 Solvability of approximated system

We first study the solvability of system (3.9). Given $v \in C([0,T];X^m)$, and $\rho^m = R[v]$ and $b^m = S[v]$, as defined before, then $\rho^m \in C^2(\Omega\times[0,T])$ and $\underline{\rho} \le \rho^m \le \bar\rho$, and (3.9) is equivalent to

$$\begin{aligned}&\sum_{j=1}^m a_{ij}^1 \partial_t c_j^m(t) + \sum_{j=1}^m a_{ij}^2 c_j^m(t) + \lambda_i c_i^m + a_i^3 = 0 \qquad (3.10)\\ &c_j^m(0) = (u_0, w_j), \quad j = 1,2,\cdots,m,\end{aligned}$$

where the coefficients a_{ij}^1 , a_{ij}^2, and a_i^3 are given by

$$a_{ij}^1(t) = (\rho^m w_j, w_i), \quad a_{ij}^2 = (\rho^m (v\cdot\nabla)w_j, w_i) \ \text{ and } \ a_i^3 = -((b^m\cdot\nabla)b^m, w_i).$$

Because $\rho^m \ge \underline{\rho}$, it holds that

$$\sum_{i,j=1}^m a_{ij}^1 \xi_i\xi_j = \int_\Omega \rho^m(x,t)\Big(\sum_{i=1}^m \xi_i w_i(x)\Big)^2 \mathrm{d}x \ge \underline{\rho}\sum_{i=1}^m |\xi_i|^2, \quad \forall\ \xi\in\mathbb{R}^m.$$

It follows that $A = \{a_{ij}^1\}_{ij}$ is a symmetric and positive definite matrix, and in particular A is invertible. From the classical theory of ordinary differential equations, system (3.10) has a unique solution $\{c_j^m\}_{j=1}^\infty \in (C^1([0,T]))^m$, thus, this guarantee the solvability of the system (3.9).

Next, we discuss some properties of operator Q defined as before. Taking $w = u^m$ in (3.9), and noticing that the L^∞ norm is equivalent to the norm $\|\cdot\|_{X^m}$, as X^m is a finite dimensional Banach space, then it follows from integration by parts and using equation (3.7) that

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho^m}u^m\|^2 + \|\nabla\times u^m\|^2 &= \int_\Omega (b^m\cdot\nabla)b^m\cdot u^m\mathrm{d}x\\ &\le \|b^m\|_\infty\|\nabla b^m\|\|u^m\|\\ &\le C_m\sqrt{\underline{\rho}^{-1}}\|b^m\|_{X^m}\|\nabla b^m\|\|\sqrt{\rho^m}u^m\|\\ &\le C_m(\|b^m\|_{X^m}^2 + \|\nabla b^m\|^2)\|\sqrt{\rho^m}u^m\|.\end{aligned}$$

Using a generalized Gronwall lemma [27], we can obtain

$$\|\sqrt{\rho^m}u^m\| \le \|\sqrt{\rho_0^m}u_0^m\| + C_m\int_0^T(\|b^m\|_{X^m}^2 + \|\nabla b^m\|^2)\mathrm{d}t.$$

Clearly, u^m is bounded in $C([0,T];L^2(\Omega))$ from $b^m\in Y$. Thus, because of $u^m=\sum\limits_{j=1}^m c_j^m(t)w_j(x)$ and $\sum\limits_{j=1}^m |c_j^m(t)| = \|u^m\|$, we infer that $\{c_j\}$ is bounded in $C([0,T])$. This implies that u^m is bounded (independent of v) in $C([0,T];X^m)$. Moreover, if v is bounded in $C([0,T];X^m)$, from (3.10) and the symmetry of $A=\{a_{ij}^1\}$, the set $\{\partial_t c_j^m : 1\le j\le m\}$ is bounded in $C([0,T])$ which implies that $\partial_t u^m$ is bounded in $C([0,T];X^m)$. Thus, we conclude that there are $K_1,K_2>0$ such that $\|u^m\|_{C^1([0,T];X^m)}\le K_2$ if $\|v\|_{C([0,T];X^m)}\le K_1$. Denote by B_1 the closed ball in $C([0,T];X)$ of radio K_1, and B_2 the closed ball in $C^1([0,T];X^m)$ of radio K_2. Then, the mapping Q is continuous from B_1 to B_2. The Arzelà-Ascoli theorem implies that B_2 is compact in $C([0,T];X^m)$. And therefore Q is continuous and compact mapping from B_1 into B_2, and by the Schauder fixed point theorem, there is a fixed point u^m for a given $T>0$. Taking ρ^m, b^m as the corresponding solution of (3.7) and (3.8), respectively, we obtain a unique solutions (ρ^m,u^m,b^m) of (3.1)–(3.4).

3.3 Uniform estimates

In this subsection, we will derive some a priori estimates, which are uniform in m, in a short time, to the solution (ρ^m,u^m,b^m) as above. Now, we study the H^1 estimate, that is the following proposition:

Proposition 3.1 Let (ρ^m,u^m,b^m) be the solution of system (3.1)–(3.4). Then, there is a positive time T_0 depending only on $\bar\rho,\Omega$, and $\|(u_0,b_0)\|$, such that

$$\sup_{0\le t\le T_0}\|(\nabla\times u^m,\nabla\times b^m)\|^2 + \int_0^{T_0}(\|(\sqrt{\rho^m}\partial_t u^m,\partial_t b^m)\|^2 + \|(\nabla^2u^m,\nabla^2b^m)\|^2)\mathrm{d}t \le C,$$

for a positive constant C depending only on $\bar\rho$, Ω, and $\|(\nabla u_0^m,\nabla b_0^m)\|$.

Proof Taking $w=\partial_t u^m$ in (3.2) and $\psi=\partial_t b^m$ in (3.3), then it follows from integration by parts and the Young's inequality that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\times u^m\|^2 + \|\sqrt{\rho^m}\partial_t u^m\|^2 &= \int_\Omega (b^m\cdot\nabla)b^m\cdot\partial_t u^m - \rho^m(u^m\cdot\nabla)u^m\cdot\partial_t u^m\mathrm{d}x\\ &\le \frac12\|\sqrt{\rho^m}\partial_t u^m\|^2 + C\int_\Omega |b^m|^2|\nabla b^m|^2 + \rho^m|u^m|^2|\nabla u^m|^2\mathrm{d}x,\\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\times b^m\|^2 + \|\partial_t b^m\|^2 &= \int_\Omega (b^m\cdot\nabla)u^m\cdot\partial_t b^m - (u^m\cdot\nabla)b^m\cdot\partial_t b^m\mathrm{d}x\\ &\le \frac12\|\partial_t b^m\|^2 + C\int_\Omega |b^m|^2|\nabla u^m|^2 + |u^m|^2|\nabla b^m|^2\mathrm{d}x,\end{aligned}$$

and thus

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla\times u^m,\nabla\times b^m)\|^2 + \|(\sqrt{\rho^m}\partial_t u^m,\partial_t b^m)\|^2\\ &\le C\int_\Omega |b^m|^2|\nabla b^m|^2 + \rho^m|u^m|^2|\nabla u^m|^2 + |b^m|^2|\nabla u^m|^2 + |u^m|^2|\nabla b^m|^2\mathrm{d}x. \qquad (3.11)\end{aligned}$$

Applying the H^2 estimate to (3.2) and (3.3), we deduce

$$\begin{aligned}\|\nabla^2 u^m\|^2 &\le C\|(b^m\cdot\nabla)b^m-\rho^m\partial_t u^m-\rho^m(u^m\cdot\nabla)u^m\|^2\\ &\le M\|\sqrt{\rho^m}\partial_t u^m\|^2+C\int_\Omega |b^m|^2|\nabla b^m|^2+\rho^m|u^m|^2|\nabla u^m|^2\mathrm{d}x,\end{aligned}\tag{3.12}$$

$$\begin{aligned}\|\nabla^2 b^m\|^2 &\le C\|(b^m\cdot\nabla)u^m-(u^m\cdot\nabla)b^m-\partial_t b^m\|^2\\ &\le M\|\partial_t b^m\|^2+C\int_\Omega |b^m|^2|\nabla u^m|^2+|u^m|^2|\nabla b^m|^2\mathrm{d}x,\end{aligned}\tag{3.13}$$

where M and C are positive constants depending only on $\bar\rho$ and Ω. Multiplying (3.11) by $2M$, and summing the resultants with (3.12) and (3.13), we obtain

$$\begin{aligned}&2M\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla\times u^m,\nabla\times b^m)\|^2+M\|(\sqrt{\rho^m}\partial_t u^m,\partial_t b^m)\|^2+\|(\nabla^2 u^m,\nabla^2 b^m)\|^2\\ \le\ & C\int_\Omega |b^m|^2|\nabla b^m|^2+\rho^m|u^m|^2|\nabla u^m|^2+|b^m|^2|\nabla u^m|^2+|u^m|^2|\nabla b^m|^2\mathrm{d}x\\ =\ &\sum_{i=1}^4 J_i,\end{aligned}\tag{3.14}$$

for a positive constant C depending only on $\bar\rho$ and Ω. Noticing that

$$J_1\le C\|b^m\|_6^2\|\nabla b^m\|\|\nabla b^m\|_6\le C\|\nabla b^m\|^3\|\nabla^2 b^m\|\le\frac14\|\nabla^2 b^m\|^2+C\|\nabla b^m\|^6,$$

similarly,

$$\begin{aligned}J_2&\le\frac14\|\nabla^2 u^m\|^2+C\|\nabla u^m\|^6,\\ J_3&\le\frac14\|\nabla^2 u^m\|^2+C\|\nabla b^m\|^4\|\nabla u^m\|^2,\\ J_4&\le\frac14\|\nabla^2 b^m\|^2+C\|\nabla u^m\|^4\|\nabla b^m\|^2,\end{aligned}$$

which, substituted into (3.14), gives

$$\begin{aligned}&2M\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla\times u^m,\nabla\times b^m)\|^2+M\|(\sqrt{\rho^m}\partial_t u^m,\partial_t b^m)\|^2+\frac12\|(\nabla^2 u^m,\nabla^2 b^m)\|^2\\ \le\ & C\|(\nabla\times u^m,\nabla\times b^m)\|^6\end{aligned}\tag{3.15}$$

for a positive constant C depending only on $\bar\rho$ and Ω.

Set

$$F(t)=2M\|(\nabla\times u^m,\nabla\times b^m)\|^2(t)+\int_0^t M\|(\sqrt{\rho^m}\partial_t u^m,\partial_t b^m)\|^2+\frac12\|(\nabla^2 u^m,\nabla^2 b^m)\|^2\mathrm{d}s.$$

Then, it follows from (3.15) that

$$F'(t)\le C_1F^3(t),\quad t\in[0,T],$$

where C_1 is a positive constant depending only on $\bar\rho$ and Ω. By ordinary differential inequality, we find that there is time $T_0>0$, such that

$$F(t)\le C,\quad t\in[0,T_0].$$

This completes the proof of Proposition 3.1. □

Next, we can work on the t-weighted H^2 estimate, which is stated in the following proposition.

Proposition 3.2 Let (ρ^m, u^m, b^m) be the solution of system (3.1)–(3.4) and T_0 the number in Proposition 3.1. Then, the following estimate holds

$$\sup_{0\le t\le T_0} t(\|(\sqrt{\rho^m}\partial_t u^m, \partial_t b^m)\|^2 + \|(\nabla^2 u^m, \nabla^2 b^m)\|^2) + \int_0^{T_0} t\|(\nabla\times\partial_t u^m, \nabla\times\partial_t b^m)\|^2 \mathrm{d}t \le C,$$

for a positive constant C depending only on $\bar\rho, T_0, \Omega$, and $\|(\nabla u_0, \nabla b_0)\|$.

Proof Differentiating (3.2) and (3.3) with respect to t yield

$$\begin{aligned}&(\rho^m\partial_t^2 u^m + \rho^m(u^m\cdot\nabla)\partial_t u^m, w) + (\nabla\times\partial_t u^m), \nabla\times w)\\ =\ &(-\partial_t\rho^m(\partial_t u^m + (u^m\cdot\nabla)u^m), w) - (\rho^m(\partial_t u^m\cdot\nabla)u^m, w)\\ &+((\partial_t b^m\cdot\nabla)b^m + (b^m\cdot\nabla)\partial_t b^m, w),\end{aligned}$$

$$\begin{aligned}&(\partial_t^2 b^m, \psi) + (\nabla\times\partial_t b^m, \nabla\times\psi)\\ =\ &((\partial_t b^m\cdot\nabla)u^m + (b^m\cdot\nabla)\partial_t u^m, \psi) - ((\partial_t u^m\cdot\nabla)b^m + (u^m\cdot\nabla)\partial_t b^m, \psi),\end{aligned}$$

for all $w, \psi\in X^m$. Taking $w=\partial_t u^m$ and $\psi = \partial_t b^m$ in the above equalities respectively, then it follows from integration by parts and using (3.1) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\sqrt{\rho^m}\partial_t u^m, \partial_t b^m)\|^2 + \|(\nabla\times\partial_t u^m, \nabla\times\partial_t b^m)\|^2\\ =\ &(\nabla\cdot(\rho^m u^m)(\partial_t u^m + (u^m\cdot\nabla)u^m), \partial_t u^m) - (\rho^m(\partial_t u^m\cdot\nabla)u^m, \partial_t u^m)\\ &+((\partial_t b^m\cdot\nabla)b^m + (b^m\cdot\nabla)\partial_t b^m, \partial_t u^m) + ((\partial_t b^m\cdot\nabla)u^m + (b^m\cdot\nabla)\partial_t u^m, \partial_t b^m)\\ &-((\partial_t u^m\cdot\nabla)b^m + (u^m\cdot\nabla)\partial_t b^m, \partial_t b^m)\\ \le\ &\int_\Omega 2\rho^m|u^m||\partial_t u^m||\nabla\partial_t u^m| + \rho^m|u^m||\partial_t u^m||\nabla u^m|^2 + \rho^m|u^m|^2|\partial_t u^m||\nabla^2 u^m|\\ &+\rho^m|u^m|^2|\nabla\partial_t u^m||\nabla u^m| + \rho^m|\partial_t u^m|^2|\nabla u^m| + 2|\partial_t b^m||\nabla b^m||\partial_t u^m| + |\partial_t b^m|^2|\nabla u^m|\mathrm{d}x\\ =\ &\sum_{i=1}^7 I_i,\end{aligned}\tag{3.16}$$

where we use $((u^m\cdot\nabla)\partial_t b^m, \partial_t b^m) = 0, ((b^m\cdot\nabla)\partial_t b^m, \partial_t u^m) + ((b^m\cdot\nabla)\partial_t u^m, \partial_t b^m) = 0$. We deuce

$$I_1 \le 2\|\sqrt{\rho^m}\|_\infty\|u^m\|_\infty\|\sqrt{\rho^m}\partial_t u^m\|\|\nabla\partial_t u^m\| \le C\|\nabla u^m\|^{\frac12}\|\nabla^2 u^m\|^{\frac12}\|\sqrt{\rho^m}\partial_t u^m\|\|\nabla\partial_t u^m\|,$$

$$I_2 \le \|\rho^m\|_\infty\|u^m\|_6\|\partial_t u^m\|_6\|\nabla u^m\|\|\nabla u^m\|_6 \le C\|\nabla u^m\|^2\|\nabla^2 u^m\|\|\nabla\partial_t u^m\|,$$

$$I_3 \le \|\rho^m\|_\infty\|u^m\|_6^2\|\partial_t u^m\|_6\|\nabla^2 u^m\| \le C\|\nabla u^m\|^2\|\nabla^2 u^m\|\|\nabla\partial_t u^m\|,$$

$$I_4 \le \|\rho^m\|_\infty\|u^m\|_\infty^2\|\nabla u^m\|\|\nabla\partial_t u^m\| \le C\|\nabla u^m\|^2\|\nabla^2 u^m\|\|\nabla\partial_t u^m\|,$$

$$I_5 \le \|\sqrt{\rho^m}\|_\infty\|\nabla u^m\|_3\|\sqrt{\rho^m}\partial_t u^m\|_2\|\partial_t u^m\|_6 \le C\|\nabla u^m\|^{\frac12}\|\nabla^2 u^m\|^{\frac12}\|\sqrt{\rho^m}\partial_t u^m\|\|\nabla\partial_t u^m\|,$$

$$I_6 \le 2\|\partial_t b^m\|_3\|\nabla b^m\|\|\partial_t u^m\|_6 \le C\|\nabla\partial_t u^m\|\|\nabla b^m\|\|\nabla\partial_t b^m\|^{\frac12}\|\partial_t b^m\|^{\frac12},$$

$$I_7 \le \|\nabla u^m\|\|\partial_t b^m\|_6\|\partial_t b^m\|_3 \le C\|\nabla u^m\|\|\nabla\partial_t b^m\|^{\frac32}\|\partial_t b^m\|^{\frac12}.$$

Substituting the estimates on $I_i,\ i=1,2,\cdots,7$, into (3.16), and using the Young inequality, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\sqrt{\rho^m}\partial_t u^m, \partial_t b^m)\|^2 + \|(\nabla\times\partial_t u^m, \nabla\times\partial_t b^m)\|^2$$

$$
\begin{aligned}
&\leq C\|\nabla u^m\|^{\frac{1}{2}}\|\nabla^2 u^m\|^{\frac{1}{2}}\|\sqrt{\rho^m}\partial_t u^m\|\|\nabla\partial_t u^m\| + C\|\nabla u^m\|^2\|\nabla^2 u^m\|\|\nabla\partial_t u^m\| \\
&\quad + C\|\nabla\partial_t u^m\|\|\nabla b^m\|\|\nabla\partial_t b^m\|^{\frac{1}{2}}\|\partial_t b^m\|^{\frac{1}{2}} + C\|\nabla u^m\|\|\nabla\partial_t b^m\|^{\frac{3}{2}}\|\partial_t b^m\|^{\frac{1}{2}} \\
&\leq \frac{1}{2}\|(\nabla\times\partial_t u^m, \nabla\times\partial_t b^m)\|^2 + C\|\nabla u^m\|\|\nabla^2 u^m\|\|\sqrt{\rho^m}\partial_t u^m\|^2 \\
&\quad + C\|\nabla u^m\|^4\|\nabla^2 u^m\|^2 + C\|\nabla b^m\|^4\|\partial_t b^m\|^2 + C\|\nabla u^m\|^4\|\partial_t b^m\|^2,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|(\sqrt{\rho^m}\partial_t u^m, \partial_t b^m)\|^2 + \|(\nabla\times\partial_t u^m, \nabla\times\partial_t b^m)\|^2 \\
&\leq C(\|(\nabla u^m, \nabla b^m)\|^4 + \|\nabla u^m\|\|\nabla^2 u^m\|)\|(\sqrt{\rho^m}\partial_t u^m, \partial_t b^m)\|^2 + C\|\nabla u^m\|^4\|\nabla^2 u^m\|^2.
\end{aligned}
$$

Multiplying the above inequality by t yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(t\|(\sqrt{\rho^m}\partial_t u^m, \partial_t b^m)\|^2) + t\|(\nabla\times\partial_t u^m, \nabla\times\partial_t b^m)\|^2 \\
&\leq C(\|(\nabla u^m, \nabla b^m)\|^4 + \|\nabla u^m\|\|\nabla^2 u^m\|)t\|(\sqrt{\rho^m}\partial_t u^m, \partial_t b^m)\|^2 \\
&\quad + C(t\|\nabla u^m\|^4\|\nabla^2 u^m\|^2 + \|(\sqrt{\rho^m}\partial_t u^m, \partial_t b^m)\|^2).
\end{aligned}
$$

By the Gronwall's inequality, and using Proposition 3.1, we obtain

$$
\sup_{0\leq t\leq T_0} t(\|(\sqrt{\rho^m}\partial_t u^m, \partial_t b^m)\|^2 + \int_0^{T_0} t\|(\nabla\times\partial_t u^m, \nabla\times\partial_t b^m\|^2\mathrm{d}t \leq C, \tag{3.17}
$$

for a positive constant C depending only on $\bar{\rho}, T_0, \Omega$, and $\|(\nabla u_0, \nabla b_0)\|$.

Substituting $J_i's$ into (3.12) and (3.13), one obtains

$$
\|(\nabla^2 u^m, \nabla^2 b^m)\|^2 \leq C(\|(\sqrt{\rho^m}\partial_t u^m, \partial_t b^m)\|^2 + \|(\nabla u^m, \nabla b^m)\|^6),
$$

for a positive constant C depending only on $\bar{\rho}, T_0$, and Ω, which along with (3.17) completes the proof of Proposition 3.2. □

4 Existence and Uniqueness of Solution

This section is devoted to proving Theorem 2.3, We start with the following result.

Proposition 4.1 Suppose that the initial data $(\rho_0, u_0, b_0) \in (W^{1,p}\cap L^\infty)\times H^1_{0,\sigma}\times H^1_\sigma$ for some $p\in[1,\infty)$, and that $0<\underline{\rho}\leq\rho_0\leq\bar{\rho}$, and T_0 stated in Proposition 3.1. Then, there is a strong solution (ρ, u, b) to system (1.1)–(1.4), subject to (1.5)–(1.6), in $\Omega\times(0, T_0)$, such that $\underline{\rho}\leq\rho\leq\bar{\rho}$ and

$$
\begin{aligned}
&\sup_{0\leq t\leq T_0}[\|(\nabla u, \nabla b)\|^2 + \|\nabla\rho\|_p^p + t(\|(\sqrt{\rho}\partial_t u, \partial_t b)\|^2 + \|(\nabla^2 u, \nabla^2 b)\|^2)] \\
&+ \int_0^{T_0}[t(\|(\nabla\partial_t u, \nabla\partial_t b)\|^2 + \|(\nabla^2 u, \nabla^2 b)\|_6^2) + \|\partial_t\rho\|_p^4 \\
&+ \|(\sqrt{\rho}\partial_t u, \partial_t b)\|^2 + \|(\nabla^2 u, \nabla^2 b)\|^2 + \|(\nabla u, \nabla b)\|_\infty]\mathrm{d}t \leq C
\end{aligned}
$$

for a positive constant C depending only on $\bar{\rho}, \Omega, \|\nabla\rho_0\|_p$, and $\|(\nabla u_0, \nabla b_0)\|$.

Proof Choose a sequence of $\rho_0^m\in C^2(\bar{\Omega})$, such that

$$
\underline{\rho}\leq\rho_0^m\leq\bar{\rho}, \quad \rho_0^m\to\rho_0 \ \text{ in } W^{1,p}(\Omega), \quad \|\nabla\rho_0^m\|_p\leq\|\nabla\rho_0\|_p.
$$

Recall (3.5), then, $u_0^m\to u_0, b_0^m\to b_0$ in $H^1(\Omega)$.

By Propositions 3.1–3.2, for any positive integer m, there is a solution (ρ^m, u^m, b^m) to system (3.1)–(3.4), such that $\underline{\rho} \le \rho^m \le \bar{\rho}$ and

$$\sup_{0\le t\le T_0}[\|(\nabla u^m,\nabla b^m)\|^2+t(\|(\sqrt{\rho^m}\partial_t u^m,\partial_t b^m)\|^2+\|(\nabla^2u^m,\nabla^2b^m)\|^2)] \\ +\int_0^{T_0}(t\|(\nabla\partial_t u^m,\nabla\partial_t b^m)\|^2+\|(\sqrt{\rho}\partial_t u^m,\partial_t b^m)\|^2+\|(\nabla^2u^m,\nabla^2b^m)\|^2)\mathrm{d}t\le C,$$

where T_0 is the positive time stated in Proposition 3.1, and C is a positive constant depending only on $\bar{\rho}, T_0, \Omega$, and $\|(\nabla u_0, \nabla b_0)\|$.

Because of the above estimate, there exists a subsequence of $\{(\rho^m, u^m, b^m)\}_{m=1}^\infty$, still denoted by $\{(\rho^m, u^m, b^m)\}_{m=1}^\infty$, and a triple (ρ, u, b), with $\underline{\rho} \le \rho \le \bar{\rho}$, and

$$\sup_{0\le t\le T_0}[\|(\nabla u,\nabla b)\|^2+t(\|(\sqrt{\rho}\partial_t u,\partial_t b)\|^2+\|(\nabla^2u,\nabla^2b)\|^2)] \\ +\int_0^{T_0}(t\|(\nabla\partial_t u,\nabla\partial_t b)\|^2+\|(\sqrt{\rho}\partial_t u,\partial_t b)\|^2+\|(\nabla^2u,\nabla^2b)\|^2)\mathrm{d}t\le C, \tag{4.1}$$

for a positive constant C depending only on $\bar{\rho}, T_0, \Omega$, and $\|(\nabla u_0, \nabla b_0)\|$, such that

$$\begin{aligned}
&\rho^m\to\rho,\quad \text{in } C([0,T_0];L^q),\quad q\in[1,\infty),\\
&u^m\overset{*}{\rightharpoonup}u, b^m\overset{*}{\rightharpoonup}b\ \text{ in } L^\infty(0,T_0;H^1)\cap L^\infty(\tau,T_0;H^2),\\
&u^m\rightharpoonup u, b^m\rightharpoonup b\ \text{ in } L^2(0,T_0;H^2),\\
&\partial_t u^m\overset{*}{\rightharpoonup}\partial_t u,\partial_t b^m\overset{*}{\rightharpoonup}\partial_t b\ \text{ in } L^\infty(\tau,T_0;L^2),\\
&\partial_t u^m\rightharpoonup\partial_t u,\partial_t b^m\rightharpoonup\partial_t b\ \text{ in } L^2(0,T_0;L^2)\cap L^2(\tau,T_0;H^1),
\end{aligned}$$

for any $\tau = \frac{T_0}{k}, k = 2, 3, \cdots$, and thus for any $\tau \in (0, T_0)$. Noticing that $H^2 \hookrightarrow\hookrightarrow H^1 \hookrightarrow L^2$, by the Aubin-Lions compactness lemma, we have $u^m \to u, b^m \to b$ in $C([0, T_0]; L^2) \cap L^2(0, T_0; H^1)$. Therefore, we have $(\rho, u, b)|_{t=0} = (\rho_0, u_0, b_0)$.

By the previous convergences, it is clear that (ρ, u, b) satisfies (1.1)–(1.4), in the sense of distribution, and moreover, because $\rho \in L^\infty(0, T_0; W^{1,p})$ and $\partial_t\rho \in L^4(0, T_0; L^p)$, which will be proven in the below, (ρ, u, b) satisfies equations (1.1)–(1.4) a.e. in $\Omega \times (0, T_0)$. The previous convergences also imply that

$$\begin{aligned}
&\rho^m\partial_t u^m\rightharpoonup\rho\partial_t u,\ \rho^m(u^m\cdot\nabla)u^m\rightharpoonup\rho(u\cdot\nabla)u,\ (b^m\cdot\nabla)b^m\rightharpoonup(b\cdot\nabla)b,\\
&(u^m\cdot\nabla)b^m\rightharpoonup(u\cdot\nabla)b,\ (b^m\cdot\nabla)u^m\rightharpoonup(b\cdot\nabla)u,\quad \text{in } L^2(0,T_0;L^2).
\end{aligned}$$

Consequently, as $m \to \infty$ in (3.2), by integration by parts, we deduce that

$$(\rho\partial_t u+\rho(u\cdot\nabla)u-\Delta u-(b\cdot\nabla)b,w_i)=0.$$

Noticing that

$$\rho\partial_t u+\rho(u\cdot\nabla)u-\Delta u-(b\cdot\nabla)b\in L^2(0,T_0;L^2),$$

because $u^m = \sum_{j=1}^m (c_j, w_j)w_j$ and $b^m = \sum_{j=1}^m (d_j, v_j)v_j$, with coefficients c_j and d_j continuous in $[0, T]$, are dense in $L^2(0, T_0; H^1)$, then, there is a function $\pi \in L^2(0, T_0; H^1)$, such that

$$\rho\partial_t u+\rho(u\cdot\nabla)u-\Delta u-(b\cdot\nabla)b=-\nabla\pi.$$

Next, one still need to verify

$$\sup_{0\le t\le T_0}\|\nabla\rho\|_p^p+\int_0^{T_0}(t\|(\nabla^2u,\nabla^2b)\|_6^2+\|(\nabla u,\nabla b)\|_\infty+\|\partial_t\rho\|_p^4)\mathrm{d}t\le C.$$

By the elliptic estimate and using the Sobolev inequality, we deduce

$$\begin{aligned}\|\nabla^2u\|_6^2&\le C(\|\rho\partial_tu+\rho(u\cdot\nabla)u-(b\cdot\nabla)b\|_6^2)\\&\le C(\|\partial_tu\|_6^2+\|u\|_\infty^2\|\nabla u\|_6^2+\|b\|_\infty^2\|\nabla b\|_6^2)\\&\le C(\|\nabla\partial_tu\|^2+\|\nabla u\|\|\nabla^2u\|^3+\|\nabla b\|\|\nabla^2b\|^3),\end{aligned}$$

similarly,

$$\|\nabla^2b\|_6^2\le C(\|\nabla\partial_tb\|^2+\|\nabla u\|\|\nabla^2u\|\|\nabla^2b\|^2+\|\nabla b\|\|\nabla^2b\|\|\nabla^2u\|^2),$$

and thus, recalling (4.1), we obtain

$$\int_0^{T_0}t\|(\nabla^2u,\nabla^2b)\|_6^2\mathrm{d}t\le C\int_0^{T_0}t(\|(\nabla\partial_tu,\nabla\partial_tb)\|^2+\|(\nabla u,\nabla b)\|\|(\nabla^2u,\nabla^2b)\|^3)\mathrm{d}t\le C,$$

for a positive constant C depending only on $\bar\rho,T_0,\Omega$, and $\|(\nabla u_0,\nabla b_0)\|$.

Noticing that $\|\nabla u\|_\infty\le C\|\nabla^2u\|^{\frac12}\|\nabla^2u\|_6^{\frac12}$, it follows from the Hölder inequality and (4.1) that

$$\begin{aligned}\int_0^{T_0}\|(\nabla u,\nabla b)\|_\infty\mathrm{d}t&\le C\int_0^{T_0}\|(\nabla^2u,\nabla^2b)\|^{\frac12}(t\|(\nabla^2u,\nabla^2b)\|_6^2)^{\frac14}t^{-\frac14}\mathrm{d}t\\&\le C\Big(\int_0^{T_0}\|(\nabla^2u,\nabla^2b)\|^2\mathrm{d}t\Big)^{\frac14}\Big(\int_0^{T_0}t\|(\nabla^2u,\nabla^2b)\|_6^2\mathrm{d}t\Big)^{\frac14}\Big(\int_0^{T_0}t^{-\frac12}\mathrm{d}t\Big)^{\frac12}\\&\le C,\end{aligned}$$

for a positive constant C as above. Applying Lemma 2.5, it follows that

$$\sup_{0\le t\le T_0}\|\nabla\rho\|_p^p\le C,$$

for a positive constant C depending only on $\bar\rho,T_0,\Omega,\|\nabla\rho_0\|_p$, and $\|(\nabla u_0,\nabla b_0)\|$. It follows from the continuity equation (1.1) that

$$\int_0^{T_0}\|\partial_t\rho\|_p^4\mathrm{d}t\le\int_0^{T_0}\|u\|_\infty^4\|\nabla\rho\|_p^4\mathrm{d}t\le C\int_0^{T_0}\|\nabla u\|^2\|\nabla^2u\|^2\|\nabla\rho\|_p^4\mathrm{d}t\le C,$$

for a positive constant C as above. This completes the proof of Proposition 4.1. □

Finally, we are ready to prove Theorem 2.3.

Proof Take a sequence $\{\rho_{0n}\}_{n=1}^\infty$, such that

$$\frac1n\le\rho_{0n}\le\bar\rho+1,\quad \rho_{0n}\to\rho_0\ \text{ in } W^{1,p},\quad \|\nabla\rho_{0n}\|_p\to\|\nabla\rho_0\|_p+1.$$

By Proposition 4.1, there is a positive constant T_0 depending only on $\bar\rho,\Omega$, and $\|(\nabla u_0,\nabla b_0)\|$, such that for each n, there is a strong solution (ρ_n,u_n,b_n) to system (1.1)–(1.4), subject to (1.5)–(1.6), with initial data $(\rho_{0n},u_{0n},b_{0n})$, in $\Omega\times(0,T_0)$, satisfying $\frac1n\le\rho_{0n}\le\bar\rho+1$ and

$$\begin{aligned}&\sup_{0\le t\le T_0}[\|(\nabla u_n,\nabla b_n)\|^2+\|\nabla\rho_n\|_p^p+t(\|(\sqrt{\rho_n}\partial_tu_n,\partial_tb_n)\|^2+\|(\nabla^2u_n,\nabla^2b_n)\|^2)]\\&+\int_0^{T_0}[t(\|(\nabla\partial_tu_n,\nabla\partial_tb_n)\|^2+\|(\nabla^2u_n,\nabla^2b_n)\|_6^2)+\|(\sqrt{\rho_n}\partial_tu_n,\partial_tb_n)\|^2\end{aligned}$$

$$+\|(\nabla^2 u_n,\nabla^2 b_n)\|^2+\|(\nabla u_n,\nabla b_n)\|_\infty+\|\partial_t\rho_n\|_p^4]\mathrm{d}t\le C,$$

for a positive constant C depending only on $\bar\rho, T_0, \Omega, \|\nabla\rho_0\|_p$, and $\|(\nabla u_0,\nabla b_0)\|$.

Because of the above estimates, by the Cantor diagonal argument, there is a subsequence of $\{(\rho_n,u_n,b_n)\}_{n=1}^\infty$, still denoted by $\{(\rho_n,u_n,b_n)\}$, and a triple $\{(\rho,u,b)\}_{n=1}^\infty$, such that

$$\begin{aligned}
&\rho_n \overset{*}{\rightharpoonup}\rho,\quad \text{in } L^\infty(0,T_0;W^{1,p}\cap L^\infty),\partial_t\rho_n\rightharpoonup\partial_t\rho\ \ \text{in } L^4(0,T_0;L^p),\\
&u_n\overset{*}{\rightharpoonup}u, b_n\overset{*}{\rightharpoonup}b\ \ \text{in } L^\infty(0,T_0;H^1)\cap L^\infty(\tau,T_0;H^2),\\
&u_n\rightharpoonup u,b_n\rightharpoonup b\ \ \text{in } L^2(0,T_0;H^2)\cap L^2(\tau,T_0;W^{2,6}),\\
&\partial_t u_n\rightharpoonup\partial_t u,\partial_t b_n\rightharpoonup\partial_t b\ \ \text{in } L^2(0,T_0;L^2)\cap L^2(\tau,T_0;H^1),
\end{aligned}$$

for any $\tau=\frac{T_0}{k}, k=2,3,\cdots$, and thus for any $\tau\in(0,T_0)$. Hence, by the Aubin-Lions compactness lemma, the following strong convergences hold:

$$\begin{aligned}
&\rho_n\to\rho,\quad \text{in } C([0,T_0];L^q),\quad \text{for any } q\in[1,\infty),\\
&u_n\to u, b_n\to b\ \ \text{in } C([\tau,T_0];H^1\cap L^6)\cap L^2(\tau,T_0;C(\bar\Omega)).
\end{aligned}$$

Similar to the results in [23], we conclude that (ρ,u,b) has the regularities stated in Theorem 2.3 and satisfies system (1.1)–(1.4) in $\Omega\times(0,T_0)$. Thus, the existence part of Theorem 2.3 is proved.

We now prove the uniqueness part of Theorem 2.3. Let (ρ_1,u_1,b_1) and (ρ_2,u_2,b_2) be two local strong solutions to system (1.1)–(1.4), subject to (1.5)–(1.6), on $\Omega\times(0,T)$, for a positive time T, with the same initial data (ρ_0,u_0,b_0). Then, we have following regularities:

$$\rho_1,\rho_2\in L^\infty(0,T;H^1), u_1,u_2,b_1,b_2\in L^2(0,T;H^2), \tag{4.2}$$

$$\sqrt{t}u_1,\sqrt{t}u_2,\sqrt{t}b_1,\sqrt{t}b_2\in L^\infty(0,T;H^2),\sqrt{t}\partial_t u_1,\sqrt{t}\partial_t u_2,\sqrt{t}\partial_t b_1,\sqrt{t}\partial_t b_2\in L^2(0,T;H^1) \tag{4.3}$$

Let $\rho=\rho_1-\rho_2, u=u_1-u_2$, and $b=b_1-b_2$. Thus, ρ, u, and b satisfy

$$\partial_t\rho+u_1\cdot\nabla\rho=-u\cdot\nabla\rho_2 \tag{4.4}$$

$$\begin{aligned}
&\rho_1\partial_t u+\rho_1(u_1\cdot\nabla)u-\Delta u+\nabla\pi\\
=&-\rho_1(u\cdot\nabla)u_2-\rho(\partial_t u_2+(u_2\cdot\nabla)u_2)+b_1\cdot\nabla b+b\cdot\nabla b_2
\end{aligned}\tag{4.5}$$

$$\partial_t b-\Delta b=-(u_1\cdot\nabla)b-(u\cdot\nabla)b_2+(b_1\cdot\nabla)u+(b\cdot\nabla)u_2 \tag{4.6}$$

a.e. in $\Omega\times(0,T)$.

Multiplying equations (4.5) by u, and (4.6) by b and using equation (4.4), it follows from integration by parts, the Hölder, Sobolev, Poincaré and Young's inequalities that

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|(\sqrt{\rho_1}u,b)\|^2+\|(\nabla\times u,\nabla\times b)\|^2\\
=&-\int_\Omega[\rho_1(u\cdot\nabla)u_2+\rho(\partial_t u_2+(u_2\cdot\nabla)u_2)]\cdot u\mathrm{d}x\\
&-\int_\Omega[(b\cdot\nabla)b_2\cdot u-(u\cdot\nabla)b_2\cdot b+(b\cdot\nabla)u_2\cdot b]\mathrm{d}x\\
\le&\|\sqrt{\rho_1}\|_\infty\|\sqrt{\rho_1}u\|\|u\|_6\|\nabla u_2\|_3+\|\rho\|_{\frac32}(\|\partial_t u_2\|_6\|u\|_6+\|u_2\|_\infty\|\nabla u_2\|_6\|u\|_6)\\
&+C\|b\|\|\nabla b_2\|_3\|u\|_6+C\|u\|_6\|\nabla b_2\|_3\|b\|+C\|b\|\|\nabla u_2\|_3\|b\|_6\\
\le&C\|\sqrt{\rho_1}u\|\|\nabla u\|\|\nabla u_2\|_{H^1}+C\|\rho\|_{\frac32}(\|\nabla\partial_t u_2\|\|\nabla u\|+\|\nabla u_2\|^{\frac12}\|\nabla u_2\|_{H^1}^{\frac32}\|\nabla u\|)
\end{aligned}$$

$$
\begin{aligned}
&+C\|b\|\|\nabla b_2\|_{H^1}\|\nabla u\|+C\|b\|\|\nabla u_2\|_{H^1}\|\nabla b\| \\
\le\ & \frac{1}{2}\|(\nabla\times u,\nabla\times b)\|^2+C\|(\nabla u_2,\nabla b_2)\|_{H^1}^2\|(\sqrt{\rho_1}u,b)\|^2 \\
&+C\|\rho\|_{\frac{3}{2}}^2(\|\nabla\partial_t u_2\|^2+\|\nabla u_2\|^2\|\nabla u_2\|_{H^1}^3)
\end{aligned}
$$

namely,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|(\sqrt{\rho_1}u,b)\|^2+\|(\nabla\times u,\nabla\times b)\|^2\le\alpha(t)\|(\sqrt{\rho_1}u,b)\|^2+\beta(t)\|\rho\|_{\frac{3}{2}}^2 \tag{4.7}
$$

where

$$
\alpha(t)=C\|(\nabla u_2,\nabla b_2))\|_{H^1}^2(t),\quad \beta(t)=C(\|\nabla\partial_t u_2\|^2+\|\nabla u_2\|^2\|\nabla u_2\|_{H^1}^3)
$$

Recalling (4.3), it is clear that $\alpha\in L^1((0,T))$ and $t\beta(t)\in L^1((0,T))$. From [23], we also get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{\frac{3}{2}}\le K\|(\nabla\times u,\nabla\times b)\|, \tag{4.8}
$$

for a positive constant K. As a result, combining (4.7) and (4.8), and applying Lemma 2.7, one obtains $\|\rho\|_{\frac{3}{2}}\equiv\|\sqrt{\rho_1}u\|\equiv\|b\|\equiv 0$. Thus, $\rho\equiv u\equiv b\equiv 0$; the uniqueness part of Theorem 2.3 is proved. □

References

[1] Agapito R, Schonbek M. Non-uniform decay of MHD equations with and without magnetic diffusion. Commu Part Diff Eq, 2007, **32**(11): 1791–1821

[2] Beirão da Veiga H, Valli A. On the Euler equations for nonhomogeneous fluids II. J Math Anal Appl, 1980, **73**(2): 338–350

[3] Berselli L, Spirito S. On the vanishing viscosity limit of 3D Navier-Stokes equations under slip boundary conditions in general domains. Commun Math Phys, 2012, **316**(1): 171–198

[4] Biskamp D. Nonlinear magnetohydrodynamics. Cambridge University Press, 1997

[5] Chen G Q, Wang D H. Existence and continuous dependence of large solutions for the magnetohydrodynamic equations. Z Angew Math Phys, 2003, **54**(4): 608–632

[6] Chen P F, Xiao Y L, Zhang H. Vanishing viscosity limit for the 3D nonhomogeneous incompressible Navier-Stokes equation with special slip boundary conditions. Electron J Differ Eq, 2017, **2017**(169): 1–13

[7] Chen Q, Tan Z, Wang Y J. Strong solutions to the incompressible magnetohydrodynamic equations. Math Method Appl Sci, 2011, **34**(1): 94–107

[8] Ducomet B, Feireisl E. The equations of Magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars. Commun Math Phys, 2006, **266**(3): 595–629

[9] Fan J S, Jiang S, Nakamura G. Vanishing shear viscosity limit in the magnetohydrodynamic equations. Commun Math Phys, 2007, **270**(3): 691–708

[10] Feireisl E, Novotný A, Petzeltová H. On the existence of globally defined weak solutions to the Navier-Stokes equations. J Math Fluid Mech, 2001, **3**(4): 358–392

[11] Ferreira L C F, Planas G, Vllamizar-Roa E J. On the Nonhomogeneous Navier-Stokes System with Navier Friction Boundary Conditions. SIAM J Math Anal, 2013, **45**(4): 2576–2595

[12] Freidberg J. Ideal Magnetohydrodynamics. New York London Plenum Press, 1987

[13] Guo B L, Wang G W. Vanishing viscosity limit for the 3D magnetohydrodynamic system with generalized Navier slip boundary conditions. Math Meth Appl Sci, 2016, **39**(15): 4526–4534

[14] Hoff D, Tsyganov E. Uniqueness and continuous dependence of weak solutions in compressible magnetohydrodynamics. Z Angew Math Phys, 2005, **56**(5): 791–804

[15] Hu X P, Wang D H. Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows. Arch Rational Mech Anal, 2010, **197**(1): 203–238

[16] Huang X D, Wang Y. Global strong solution to the 2D nonhomogeneous incompressible MHD system. J Differ Equations, 2013, **254**(2): 511–527

[17] Huang X D, Wang Y. Global strong solution of 3D inhomogeneous Navier-Stokes equations with density-dependent viscosity. J Differ Equations, 2015, **259**(4): 1606–1627

[18] Jiu Q S, Wang Y, Xin Z P. Global well-posedness of the Cauchy problem of two-dimensional compressible Navier-Stokes equations in weighted spaces. J Differ Equations, **255**(3): 351–404

[19] Ladyzhenskaya O A. The Mathematical Theory of Viscous Incompressible Flow. second English edition, Mathematics and Its Applications. Vol. 2. New York, London, Paris: Gordon and Breach, Science Publishers, 1969

[20] Ladyzhenskaya O A, Solonnikov V A. Unique solvability of an initial and boundary value problem for viscous incompressible non-homogeneous fluids. J Sov Math, 1978, **9**(5): 697–749

[21] Li F C, Zhang Z P. Zero viscosity-resistivity limit for the 3D incompressible magnetohydrodynamic equations in Gevrey class. Discrete and continuous dynamical systems, 2018, **38**(9): 4279–4304

[22] Li H Y. Global strong solution to the three dimensional nonhomogeneous incompressible magnetohydrodynamic equations with density-dependent viscosity and resistivity. Math Methods Appl Sci, 2018, **41**(8): 3062–3092

[23] Li J. Local existence and uniqueness of strong solutions to the Navier-Stokes equations with nonnegative density. J Differ Equations, 2017, **263**(10): 6512–6536

[24] Lv B Q, Xu Z H, Zhong X. Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent magnetohydrodynamic equations with vacuum. J Math Pures Appl, 2017, **108**(1): 41–62

[25] Paicu M, Zhang P, Zhang Z F. Global unique solvability of homogeneous Navier-Stokes equations with bounded density. Commun Part Diff Eq, 2013, **38**(7): 1208–1234

[26] Sart R. Existence of finite energy weak solutions for the equations MHD of compressible fluids. Appl Anal, 2009, **88**(3): 357–379

[27] Simon J. Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure. SIAM J Math Anal, 1990, **21**(5): 1093–1117

[28] Temam R. Navier-Stokes equations and nonlinear functional analysis. CBMS-NSF Regional Conference Series in Applied Mathematics. Philadelphia: SIAM, 1983

[29] Wu H W. Strong solutions to the incompressible magnetohydrodynamic equations with vacuum. Comput Math Appl, 2011, **61**(9): 2742–2753

[30] Wu J H. Generalized MHD equations. J Differ Equations, 2003, **195**(2): 284–312

[31] Xiao Y L, Xin Z P. On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition. Commun Pur Appl Math, 2007, **60**(7): 1027–1055

[32] Xiao Y L, Xin Z P, Wu J H. Vanishing viscosity limit for the 3D magnetohydrodynamic system with a slip boundary condition. J Funct Anal, 2009, **257**(11): 3375–3394

[33] Xu L, Zhang P. Global small solutions to three-dimensional incompressible magnetohydrodynamical system. SIAM J Math Anal, 2015, **47**(1): 26–65

[34] Yu H B, Zhang P X, Shi X J. Global strong solutions to the 3D incompressible MHD equations with density-dependent viscosity. Comput Math Appl, 2018, **75**(8): 2825–2834