Acta Mathematica Scientia, 2019, **39B**(5): 1240–1254
https://doi.org/10.1007/s10473-019-0504-z

Mathematica Acta Scientia
数学物理学报
http://actams.wipm.ac.cn

SHARP HÖLDER CONTINUITY OF THE INTEGRATED DENSITY OF STATES FOR EXTENDED HARPER'S MODEL WITH A LIOUVILLE FREQUENCY*

Wenwen JIAN (菅雯雯)
School of Mathematical Sciences, Fudan University, Shanghai 200433, China
E-mail: *wwjian16@fudan.edu.cn*

Yunfeng SHI (石云峰)†
School of Mathematical Sciences, Peking University, Beijing 100871, China
E-mail: *yunfengshi18@gmail.com*

Abstract In this article, the non-self dual extended Harper's model with a Liouville frequency is considered. It is shown that the corresponding integrated density of states is $\frac{1}{2}$-Hölder continuous. As an application, the homogeneity of the spectrum is proven.

Key words extended Harper's model; Liouville frequency; $\frac{1}{2}$-Hölder continuity; homogeneity

2010 MR Subject Classification 47B36; 37C55; 82B26

1 Introduction and Main Results

Let us consider the extended Harper's model (EHM)

$$(H_{\lambda,\alpha,x}u)_n = c(x+n\alpha)u_{n+1} + \overline{c}(x+(n-1)\alpha)u_{n-1} + 2\cos 2\pi(x+n\alpha)u_n, \tag{1.1}$$

where $u=\{u_n\}\in \ell^2(\mathbb{Z})$ and

$$c(x) = c_\lambda(x) = \lambda_1 \mathrm{e}^{-2\pi\mathrm{i}(x+\frac{\alpha}{2})} + \lambda_2 + \lambda_3 \mathrm{e}^{2\pi\mathrm{i}(x+\frac{\alpha}{2})},$$
$$\overline{c}(x) = \overline{c}_\lambda(x) = \lambda_1 \mathrm{e}^{2\pi\mathrm{i}(x+\frac{\alpha}{2})} + \lambda_2 + \lambda_3 \mathrm{e}^{-2\pi\mathrm{i}(x+\frac{\alpha}{2})}.$$

We call $\lambda=(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}_+^3$ the coupling, $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ the frequency, and $x\in\mathbb{R}$ the phase. If $\lambda_1=\lambda_3=0$, the EHM reduces to the famous almost Mathieu operator (AMO). Physically, the EHM describes the influence of a transversal magnetic field of flux α on a single tight-binding electron in a 2-dimensional crystal layer (see [27]).

For irrational frequency α, the spectrum of $H_{\lambda,\alpha,x}$ does not depend on x, and is denoted by $\Sigma_{\lambda,\alpha}$.

*Received June 28, 2018; revised October 27, 2018. The second author was supported by China Postdoctoral Science Foundation (2018M641050).

† Corresponding author: Yunfeng SHI.

In general, we can split the coupling region into three parts (see Figure 1):

$$\mathrm{I} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_+^3 :\ 0 < \max\{\lambda_1 + \lambda_3, \lambda_2\} < 1\},$$
$$\mathrm{II} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_+^3 :\ 0 < \max\{\lambda_1 + \lambda_3, 1\} < \lambda_2\},$$
$$\mathrm{III} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_+^3 :\ 0 < \max\{\lambda_2, 1\} < \lambda_1 + \lambda_3\}.$$

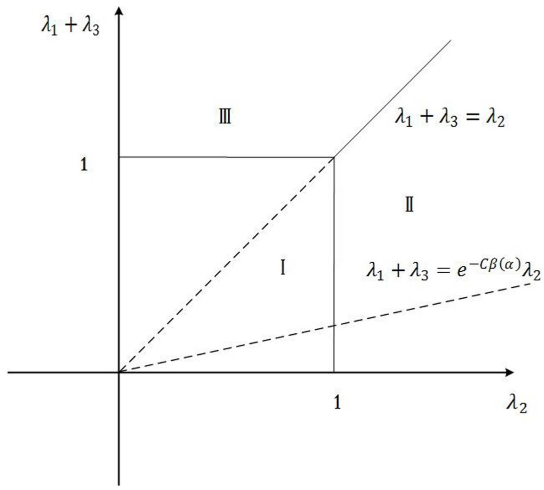

Figure 1

Considering $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we call α a Liouville frequency if $\beta(\alpha) > 0$, where

$$\beta(\alpha) = \limsup_{k \to \infty} \frac{-\ln \|k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|}, \ \|x\|_{\mathbb{R}/\mathbb{Z}} = \min_{k \in \mathbb{Z}} |x - k|. \tag{1.2}$$

On the contrary, α is called a weak Diophantine frequency for $\beta(\alpha) = 0$.

In the present article, we mainly focus on the regularity of the integrated density of states (IDS) $\mathcal{N}_{\lambda,\alpha}(\cdot)$ (see subsection 2.2 for details) for EHM.

It is well-known that the IDS for a general analytic quasi-periodic Schrödinger operator is continuous, which does not imply the Hölder continuity. In fact, many research efforts have focused on the regularity of the IDS for both quasi-periodic Schrödinger operators and quasi-periodic Jacobi operators.

On the one hand, we consider in the regime of positive Lyapunov exponent. In this case, Bourgain-Goldstein-Schlag developed a series of powerful techniques, such as the large deviation theorem, avalanche principle and semi-algebtraic sets theory to study the Hölder continuity of the IDS. Along this line, the breakthrough was made by Goldstein-Schlag [11]. They proved Hölder continuity of the IDS for a quasi-periodic Schrödinger operator with large analytic potential and a Diophantine frequency (we call $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ a Diophantine frequency if there exist $\gamma > 1$ and $\mu > 0$ such that $\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\mu}{|k|^\gamma}$ for $\forall\ k \in \mathbb{Z} \setminus \{0\}$). Later, Bourgain [5] showed that the IDS for AMO is $(\frac{1}{2} - \epsilon)$-Hölder continuous (for any small $\epsilon > 0$) if the coupling λ_2 is small and the frequency is Diophantine. Recently, Tao-Voda [26] dealt with quasi-periodic Jacobi operators and obtained especially the $(\frac{1}{2} - \epsilon)$-Hölder continuous of the IDS for EHM if the Lyapunov exponent is positive and the frequency is strong Diophantine (we call $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ a strong Diophantine frequency if there exist $\mu > 0$ and $\gamma > 1$ such that $\forall\ k \in \mathbb{Z} \setminus \{0\}$, $\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\mu}{|k|(\ln|k|)^\gamma}$).

On the other hand, in the subcritical regime[1], by using KAM techniques, Amor [13] proved that the IDS for Schrödinger operator with a small (in perturbative sense) analytic potential and a Diophantine frequency is $\frac{1}{2}$-Hölder continuous. After that, Avila-Jitomirskaya developed the quantitative almost reducibility methods [2], which makes them show the $\frac{1}{2}$-Hölder continuity of the IDS for AMO with $\lambda_2 \neq \pm 1$ and a Diophantine frequency. They also obtained $\frac{1}{2}$-Hölder continuity of the IDS for Schrödinger operators with small (in non-perturbative sense) analytic potentials and Diophantine frequencies. Subsequently, Avila-Jitomirskaya [3] established $\frac{1}{2}$-Hölder continuity of the spectral measures for Schrödinger operator with a small analytic potential and a Diophantine frequency. Leguil-You-Zhao-Zhou [21] proved the $\frac{1}{2}$-Hölder continuity of IDS for Schrödinger operators in the sub-critical regime with a weak Diophantine frequency. In a recent work of Cai-Chavaudret-You-Zhou [6], they proved $\frac{1}{2}$-Hölder continuity of the IDS for Schrödinger operators with small finitely differentiable potentials and Diophantine frequencies.

Note that all results mentioned above are in Diophantine frequencies case. You-Zhang [28] extended Goldstein-Schlag's results to weak Liouville frequencies case (that is $0 < \beta(\alpha) \ll 1$). In [23], Liu-Yuan improved Avila-Jitomirskaya's results to quasi-periodic Schrödinger operators with Liouville frequencies. Hölder continuity of the IDS for quasi-periodic Schrödinger operators with any Liouville frequencies was recently studied by Han-Zhang [18], with generalization to the Jacobi case by Tao [25].

We also refer to [14, 16, 17, 19] for the study of the EHM.

The present article is the first one to investigate the sharp Hölder continuity of the IDS for the quasi-periodic Jacobi operators with any Liouville frequencies. The main result of this article is the following theorem.

Theorem 1.1 Suppose that $0 < \beta(\alpha) < \infty$ and $\lambda \in \mathrm{I} \cup \mathrm{II}$. Then there is an absolute constant $C > 0$ such that if $\max\{\mathcal{L}_{\overline{\lambda}}, \mathcal{L}_\lambda\} > C\beta(\alpha)$, we have for $E_1, E_2 \in \mathbb{R}$,

$$|\mathcal{N}_{\lambda,\alpha}(E_1) - \mathcal{N}_{\lambda,\alpha}(E_2)| \le C_\star |E_1 - E_2|^{\frac{1}{2}},$$

where

$$\begin{aligned} \mathcal{L}_{\overline{\lambda}} &= \ln \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}}{\max\{\lambda_1+\lambda_3, 1\} + \sqrt{\max\{\lambda_1+\lambda_3,1\}^2 - 4\lambda_1\lambda_3}}, \\ \mathcal{L}_\lambda &= \ln \frac{1 + \sqrt{1 - 4\lambda_1\lambda_3}}{\max\{\lambda_1+\lambda_3, \lambda_2\} + \sqrt{\max\{\lambda_1+\lambda_3,\lambda_2\}^2 - 4\lambda_1\lambda_3}}, \end{aligned} \tag{1.3}$$

and $C_\star > 0$ is a constant depending on α, λ.

Remark 1.2 In EHM case, the involved $A_{\lambda,E}(x) \notin \mathrm{SL}(2,\mathbb{R})$ (see (2.2)), which makes the proof more technical (see section 3 for details).

Remark 1.3 Almost reducibility results were first obtained by Han [15] for EHM for weak Diophantine frequencies, later generalized to the Liouville frequencies case by Shi-Yuan in [24]. The proof of Theorem 1.1 relies on some almost reducibility results of [24] for EHM with Liouville frequencies, but more quantitative.

[1]Subcritical regime means the corresponding transfer matrices $A_n(z)$ (see section 2) are uniformly subexponentially bounded through some strip $\{z \in \mathbb{C} : \ |\Im z| \le h, \ h > 0\}$.

Combining Theorem 1.1 and the estimates on specrtal gaps in [24], we can obtain the homogeneity of the spectrum.

Theorem 1.4 Assume that the conditions of Theorem 1.1 are satisfied. Then for any $\epsilon > 0$, there exists $\sigma_\star = \sigma_\star(\lambda, \alpha, \epsilon) > 0$ such that for all $E \in \Sigma_{\lambda,\alpha}$ and $\sigma \in (0, \sigma_\star)$, we have

$$\mathrm{Leb}\left((E-\sigma, E+\sigma) \cap \Sigma_{\lambda,\alpha}\right) \geq (1-\epsilon)\sigma,$$

where $\mathrm{Leb}(\cdot)$ is the Lebesgue measure.

Actually, there are also many results on the homogeneity of the spectrum for quasi-periodic operators. In continuous Schrödinger operators case, Damanik-Goldstein-Lukic [7] set up Carleson homogeneity of the spectrum for small analytic potentials and Diophantine frequencies. Later, in the regime of positive Lyapunov exponent, Goldsein-Damanik-Schlag-Voda [10] proved Carleson homogeneity of the spectrum for (discrete) Schrödinger operators with Diophantine frequencies. In [12], Goldstein-Schlag-Voda got the Carleson homogeneity of the spectrum for Diophantine multi-frequency Schrödinger operators. Recently, Leguil-You-Zhao-Zhou [21] considered the non-critical AMO with a weak Diophantine frequency and showed that the spectrum is homogenous. They also proved the homogeneity of the spectrum for Schrödinger operators with (measure-theoretically) typical quasi-periodic analytic potentials and fixed strong Diophantine frequency. Actually, we remark that all these results are restricted to (weak) Diophantine frequencies. Liu-Shi [22] extended some results of [21] to Liouville frequencies case. In [9], Fillman-Lukic established Carleson homogeneity of the spectrum for limit-periodic Schrödinger operators.

The present article is organized as follows. In Section 2, we give some basic concepts and notations. In Section 3, we prove a "remormalization" lemma. In Section 4, we prove $\frac{1}{2}$-Hölder continuity of the IDS by establishing some quantitative almost reducibility results. The proof of Theorem 1.4 is included in the Appendix.

2 Some Basic Concepts and Notations

2.1 Cocycle, Transfer Matrix and Lyapunov Exponent

By a cocycle, we mean a pair $(\alpha, A) \in (\mathbb{R}\setminus\mathbb{Q}) \times C^0(\mathbb{R}/\mathbb{Z}, M_2(\mathbb{C}))$ satisfying

$$\int_{\mathbb{R}/\mathbb{Z}} \ln_+ ||A(x)|| \mathrm{d}x < +\infty.$$

We can regard the cocycle (α, A) as a dynamical system on $(\mathbb{R}/\mathbb{Z}) \times \mathbb{C}^2$ with

$$(\alpha, A) : (x, v) \longmapsto (x+\alpha, A(x)v),\ (x, v) \in (\mathbb{R}/\mathbb{Z}) \times \mathbb{C}^2.$$

For $k > 0$, we define

$$A_k(x) = \prod_{l=k}^{1} A(x+(l-1)\alpha).$$

The Lyapunov exponent for (α, A) is defined by

$$\mathcal{L}(\alpha, A) = \lim_{k\to+\infty} \frac{1}{k}\int_{\mathbb{R}/\mathbb{Z}} \ln\|A_k(x)\|\mathrm{d}x = \inf_{k>0}\frac{1}{k}\int_{\mathbb{R}/\mathbb{Z}} \ln\|A_k(x)\|\mathrm{d}x.$$

2.2 Spectral Measure and the IDS

Let H be a bounded self-adjoint operator on $\ell^2(\mathbb{Z})$. Then $(H-z)^{-1}$ is analytic in $\mathbb{C}\setminus\Sigma(H)$, where $\Sigma(H)$ is the spectrum of H, and we have for $f\in\ell^2(\mathbb{Z})$,

$$\Im\langle (H-z)^{-1}f,f\rangle=\Im z\cdot\|(H-z)^{-1}f\|^2_{\ell^2(\mathbb{Z})},$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product in $\ell^2(\mathbb{Z})$. Thus $\phi_f(z)=\langle (H-z)^{-1}f,f\rangle$ is an analytic function in the upper half plane with $\Im\phi_f\geq 0$ (ϕ_f is the so-called Herglotz function). Therefore, one has a representation

$$\phi_f(z)=\int_{\mathbb{R}}\frac{1}{x-z}\mathrm{d}\mu^f(x),$$

where μ^f is the spectral measure associated to the vector f. Alternatively, for any Borel set $\Omega\subseteq\mathbb{R}$,

$$\mu^f(\Omega)=\langle\mathbb{E}(\Omega)f,f\rangle,$$

where $\mathbb{E}$ is the corresponding spectral projection of H.

Denote by $\mu^f_{\lambda,\alpha,x}$ the spectral measure of the operator $H_{\lambda,\alpha,x}$ and the vector f as above with $||f||_{\ell^2(\mathbb{Z})}=1$. The IDS $\mathcal{N}_{\lambda,\alpha}:\mathbb{R}\to[0,1]$ is obtained by averaging the spectral measure $\mu^f_{\lambda,\alpha,x}$ with respect to x, that is,

$$\mathcal{N}_{\lambda,\alpha}(E)=\int_{\mathbb{R}/\mathbb{Z}}\mu^f_{\lambda,\alpha,x}(-\infty,E]\mathrm{d}x.$$

It is a continuous, non-decreasing surjective function and the definition is independent of the choices of f.

2.3 Gap Labelling and the IDS

Each connected component of $[E_{\min},E_{\max}]\setminus\Sigma_{\lambda,\alpha}$ is called a spectral gap, where $E_{\min}=\min\{E:E\in\Sigma_{\lambda,\alpha}\}$ and $E_{\max}=\max\{E:E\in\Sigma_{\lambda,\alpha}\}$. By the well-known gap labelling theorem [8, 20], for every spectral gap G there is a unique nonzero integer m such that $\mathcal{N}_{\lambda,\alpha}|_G=m\alpha \mod \mathbb{Z}$ and

$$[E^-_m,E^+_m]=\{E_{\min}\leq E\leq E_{\max}:\ \mathcal{N}_{\lambda,\alpha}(E)=m\alpha \mod \mathbb{Z}\}. \tag{2.1}$$

2.4 Dynamical Relation

Recalling (1.1), for $c(x)\neq 0$ the equation

$$H_{\lambda,\alpha,x}u=Eu$$

is equivalent to

$$\begin{pmatrix}u_{k+1}\\ u_k\end{pmatrix}=A_{\lambda,E}(x+k\alpha)\begin{pmatrix}u_k\\ u_{k-1}\end{pmatrix},$$

where

$$A_{\lambda,E}(x)=\frac{1}{c(x)}\begin{bmatrix}E-2\cos 2\pi x & -\overline{c}(x-\alpha)\\ c(x) & 0\end{bmatrix}. \tag{2.2}$$

2.5 Aubry Duality

The map $\sigma : \lambda = (\lambda_1, \lambda_2, \lambda_3) \to \overline{\lambda} = (\frac{\lambda_3}{\lambda_2}, \frac{1}{\lambda_2}, \frac{\lambda_1}{\lambda_2})$ induces the duality between region I and region II. We call $H_{\overline{\lambda},\alpha,x}$ the Aubry duality of $H_{\lambda,\alpha,x}$. We have $\Sigma_{\lambda,\alpha} = \lambda_2 \Sigma_{\overline{\lambda},\alpha}$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Aubry duality expresses an algebraic relation between the families of operators $\{H_{\overline{\lambda},\alpha,x}\}_{x\in\mathbb{R}}$ and $\{H_{\lambda,\alpha,x}\}_{x\in\mathbb{R}}$ by Bloch waves, that is, if $u : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is an L^2 function whose Fourier coefficients $\hat{u}$ satisfy $H_{\overline{\lambda},\alpha,\theta}\hat{u} = \frac{E}{\lambda_2}\hat{u}$, then there exist $\theta \in \mathbb{R}$, such that $U(x) = \begin{pmatrix} e^{2\pi i\theta}u(x) \\ u(x-\alpha) \end{pmatrix}$ satisfies

$$A_{\lambda,E}(x) \cdot U(x) = e^{2\pi i\theta} U(x+\alpha). \tag{2.3}$$

2.6 Some Notations

We briefly comment on the constants and norms in the following proofs. Let $C(\alpha)$ be a large constant depending on α, and let $C_\star$ (or $c_\star$) be a large (or small) constant depending on λ and α. Define the strip $\Delta_s = \{z \in \mathbb{C}/\mathbb{Z} : |\Im z| < s\}$ and let $\|v\|_s = \sup_{z\in\Delta_s} \|v(z)\|$, where v is a mapping from Δ_s to some Banach space $(\mathcal{B}, \|\cdot\|)$. In this article, $\mathcal{B}$ may be $\mathbb{C}$, $\mathbb{C}^2$ or $\mathrm{SL}(2,\mathbb{C})$.

Remark 2.1 Recall that

$$\mathcal{N}_{\lambda,\alpha}(E) = \mathcal{N}_{\overline{\lambda},\alpha}(\lambda_2^{-1}E),\ \Sigma_{\lambda,\alpha} = \lambda_2\Sigma_{\overline{\lambda},\alpha}.$$

It suffices to prove our main results in case $\lambda \in$ II.

3 Renormalization of the Jacobi Cocycle with Liouville Frequencies

In EHM case, $A_{\lambda,E}(x) \notin \mathrm{SL}(2,\mathbb{R})$ (see (2.2)), which is difficult to deal with directly. Thus it needs make some "renormalization". The "renormalization" from EHM cocycle to $\mathrm{SL}(2,\mathbb{R})$ cocycle was first introduced by Avila-Jitomirskaya-Marx [4], and then improved by Han [15]. The following lemma extends Han's results to Liouville frequencies case and is one of the key ingredients of this article.

Lemma 3.1 Let $0 < \beta(\alpha) < \infty$ and $\lambda \in$ II. If $\mathcal{L}_{\overline{\lambda}} \geq 5\beta(\alpha)$, then there are some analytic mapping Q_λ from $\Delta_{\frac{\mathcal{L}_{\overline{\lambda}}}{4\pi}}$ to $M_2(\mathbb{C})$ and its inverse Q_λ^{-1} which is analytic in the same region, such that for all $x \in \Delta_{\frac{\mathcal{L}_{\overline{\lambda}}}{4\pi}}$,

$$Q_\lambda^{-1}(x+\alpha)A_{\lambda,E}(x)Q_\lambda(x) = \overline{A}_{\lambda,E}(x),$$

where

$$\overline{A}_{\lambda,E}(x) = \frac{1}{\sqrt{|c|(x)|c|(x-\alpha)}} \begin{bmatrix} E - 2\cos 2\pi x & -|c|(x-\alpha) \\ |c|(x) & 0 \end{bmatrix}$$

with $|c|(x) = \sqrt{c(x)\overline{c}(x)}$. [2]

Proof Let

$$\epsilon_\star = \min\left\{ \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}}{2\lambda_1},\ \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}}{2\lambda_3} \right\}.$$

[2] $\overline{c}(x)$ is the complex conjugate of $c(x)$ for $x \in \mathbb{R}$ and its analytic extension for $x \notin \mathbb{R}$.

As $\lambda_2 > \lambda_1 + \lambda_3$, we have for any $\epsilon \in \mathbb{R}$, $|\epsilon| < \epsilon_\star$,

$$\lambda_2 - (\lambda_1 \mathrm{e}^{2\pi\epsilon} + \lambda_3 \mathrm{e}^{-2\pi\epsilon}) > 0, \tag{3.1}$$

$$\lambda_2 - (\lambda_1 \mathrm{e}^{-2\pi\epsilon} + \lambda_3 \mathrm{e}^{2\pi\epsilon}) > 0. \tag{3.2}$$

Thus $\Re c(x + \mathrm{i}\epsilon) > 0, \Re \overline{c}(x + \mathrm{i}\epsilon) > 0$ for any $x \in \mathbb{R}/\mathbb{Z}$. We have showed that $c(x)$, $\overline{c}(x)$ have no zeros on $\Delta_{\frac{\mathcal{L}_{\overline{\lambda}}}{2\pi}}$. Recalling (3.1) and (3.2) again, the rotation numbers of $c(x)$, $\overline{c}(x)$ on $\Delta_{\frac{\mathcal{L}_{\overline{\lambda}}}{2\pi}}$ are identically vanishing. Consequently, there are single-valued analytic functions $g_1(x) = \log|c(x)| + \mathrm{i} \arg c(x)$ and $g_2(x) = \log|\overline{c}(x)| + \mathrm{i} \arg \overline{c}(x)$ on $\Delta_{\frac{\mathcal{L}_{\overline{\lambda}}}{2\pi}}$ such that $c(x) = \mathrm{e}^{g_1(x)}$, $\overline{c}(x) = \mathrm{e}^{g_2(x)}$.

In view of $x \in \mathbb{R}/\mathbb{Z}$,

$$\Re c(1 - \alpha - x) = \Re c(x), \Im c(1 - \alpha - x) = -\Im c(x),$$

then we have

$$\begin{aligned} \int_{\mathbb{R}/\mathbb{Z}} \arg c(x) \mathrm{d}x &= \int_{-\frac{\alpha}{2}}^{\frac{1}{2}-\frac{\alpha}{2}} \arg\, c(x) \mathrm{d}x + \int_{\frac{1}{2}-\frac{\alpha}{2}}^{1-\frac{\alpha}{2}} \arg\, c(x) \mathrm{d}x \\ &= -\int_{-\frac{\alpha}{2}}^{\frac{1}{2}-\frac{\alpha}{2}} \arg\, c(1 - \alpha - x) \mathrm{d}x + \int_{\frac{1}{2}-\frac{\alpha}{2}}^{1-\frac{\alpha}{2}} \arg\, c(x) \mathrm{d}x \\ &= 0. \end{aligned}$$

Similarly, $\int_{\mathbb{R}/\mathbb{Z}} \arg \overline{c}(x) \mathrm{d}x = 0$. Hence $\widehat{g_1 - g_2}(0) = \int_{\mathbb{R}/\mathbb{Z}} (g_1(x) - g_2(x)) \,\mathrm{d}x = 0$ and the function $f(x) = \sum\limits_{k \in \mathbb{Z}} \hat{f}_k \mathrm{e}^{2\pi k \mathrm{i} x}$ will solve the equation

$$2f(x + \alpha) - 2f(x) = g_1(x) - g_2(x),$$

where $\hat{f}_0 = 0$ and $\hat{f}_k = \frac{\widehat{g_1 - g_2}(k)}{2(\mathrm{e}^{2\pi k \mathrm{i}\alpha} - 1)}, k \neq 0$. From the definition of $\beta(\alpha)$ in (1.2), we have the small divisor estimate

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq C(\alpha) \mathrm{e}^{-\frac{3}{2}\beta(\alpha)|k|} \text{ for } k \neq 0. \tag{3.3}$$

Because $\mathcal{L}_{\overline{\lambda}} \geq 5\beta(\alpha)$, $f(x)$ is analytic on $\Delta_{\frac{\mathcal{L}_{\overline{\lambda}}}{4\pi}}$. Thus $c(x) = |c|(x) \mathrm{e}^{f(x+\alpha) - f(x)}$, $\overline{c}(x) = |c|(x) \mathrm{e}^{-f(x+\alpha) + f(x)}$ for all $x \in \Delta_{\frac{\mathcal{L}_{\overline{\lambda}}}{4\pi}}$.

Let

$$Q_\lambda(x) = \mathrm{e}^{f(x)} \sqrt{|c|(x - \alpha)} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\dfrac{\overline{c}(x - \alpha)}{c(x - \alpha)}} \end{bmatrix}.$$

Then the proof is complete (the detailed calculation is similar to that of [15]). □

We denote by $\mathcal{L}_\lambda(E) = \mathcal{L}(\alpha, \overline{A}_{\lambda,E})$ the Lyapunov exponent of $(\alpha, \overline{A}_{\lambda,E})$.

The Thouless formula relates the Lyapunov exponent to the IDS,

$$\mathcal{L}_\lambda(E) = -\int_{\mathbb{R}/\mathbb{Z}} \ln|c_\lambda(x)| \mathrm{d}x + \int_{\mathbb{R}} \ln|E' - E| \mathrm{d}\mathcal{N}_{\lambda,\alpha}(E'). \tag{3.4}$$

4 $\frac{1}{2}$-Hölder Continuity of the IDS

In this section, we will prove the $\frac{1}{2}$-Hölder continuity of the IDS for EHM. To this end, one needs to establish quantitative almost reducibility results for the extended Harper's cocycle. Let us begin with some useful definitions and lemmas.

Definition 4.1 Fix $\theta \in \mathbb{R}, \epsilon_0 > 0$. We call $n \in \mathbb{Z}$ an ϵ_0-resonance of θ if

$$\min_{|k|\le|n|} \|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} = \|2\theta - n\alpha\|_{\mathbb{R}/\mathbb{Z}} \le \mathrm{e}^{-\epsilon_0|n|}.$$

Given $\theta \in \mathbb{R}$, we order all the ϵ_0-resonances of θ as $0 < |n_1| \le |n_2| < \cdots$. We say θ is ϵ_0-resonant if the set of all ϵ_0-resonances of θ is infinite and ϵ_0-non-resonant for otherwise. Supposing $\{0, n_1, \cdots, n_j\}$ is the set of all ϵ_0-resonances of θ, we let $n_{j+1} = \infty$.

Lemma 4.2 (Theorem 3.3 of [2]) Let $E \in \Sigma_{\lambda,\alpha}$. Then there exist $\theta = \theta(E) \in \mathbb{R}$ and solution u of $H_{\overline{\lambda},\alpha,\theta}u = \frac{E}{\lambda_2}u$ with $u_0 = 1, |u_k| \le 1$.

Throughout this section we fix $E, \theta = \theta(E)$ and u which are given by Lemma 4.2.

In the following, we let C_2, C_1 ($C_2 \gg C_1$) be large absolute constants which are bigger than any positive absolute constant C. Moreover, we assume that

$$h = \frac{\mathcal{L}_{\overline{\lambda}}}{200\pi},\ \mathcal{L}_{\overline{\lambda}} > C_2\beta(\alpha).$$

From Theorem 3.3 in [24], we have

$$|u_k| \le C_\star \mathrm{e}^{-2\pi h|k|} \quad \text{for } 3|n_j| < |k| < \frac{|n_{j+1}|}{3}, \tag{4.1}$$

where $\{n_j\}$ is the set of all $C_1^2\beta(\alpha)$-resonances of $\theta = \theta(E)$.

Lemma 4.3 (Lemma 6.6 of [24]) We have

$$\sup_{0\le k\le \mathrm{e}^{\frac{hn}{20}}} \|\overline{A}_k\|_{\frac{h}{20}} \le C_\star \mathrm{e}^{C\beta(\alpha)n}, \tag{4.2}$$

where $C > 0$ is some absolute constant.

Lemma 4.4 (Theorem 2.6 of [1]) Given $\eta > 0$, we let $U : \mathbb{C}/\mathbb{Z} \to \mathbb{C}^2$ be analytic in Δ_η and satisfy $\delta_1 \le \|U(x)\| \le \delta_2^{-1}$ *for* $\forall\ x \in \Delta_\eta$. Then there exists $B(x) : \mathbb{C}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$ being analytic in Δ_η with the first column being $U(x)$ and $\|B\|_\eta \le C\delta_1^{-2}\delta_2^{-1}(1 - \ln(\delta_1\delta_2))$, where $C > 0$ is some absolute constant.

For simplicity, we write $n = |n_j| < \infty$ and $N = |n_{j+1}|$.

Define $I_2 = \left[-\lfloor\frac{N}{9}\rfloor, \lfloor\frac{N}{9}\rfloor\right]$ and

$$U^{I_2}(x) = \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}\theta} \displaystyle\sum_{k\in I_2} u_k \mathrm{e}^{2\pi\mathrm{i}kx} \\ \displaystyle\sum_{k\in I_2} u_k \mathrm{e}^{2\pi\mathrm{i}k(x-\alpha)} \end{pmatrix},$$

where $\lfloor x\rfloor$ denotes the integer part of some $x \in \mathbb{R}$. Suppose that $U_\star^{I_2}(x) = Q_\lambda(x)\cdot U^{I_2}(x)$. Recalling (2.3) and (4.1), we have

$$\overline{A}_{\lambda,E}(x)U_\star^{I_2}(x) = \mathrm{e}^{2\pi\mathrm{i}\theta}U_\star^{I_2}(x+\alpha) + G_\star(x), \tag{4.3}$$

where

$$\|G_\star\|_{\frac{h}{3}} \le C_\star \mathrm{e}^{-\frac{h}{10}N}. \tag{4.4}$$

We have the following useful estimate.

Lemma 4.5 (Lemma 6.6 in [24]) For $n > n(\lambda, \alpha)$,

$$\inf_{x\in\Delta_{\frac{h}{3}}} \|U_\star^{I_2}(x)\| \geq \mathrm{e}^{-C\beta(\alpha)n}, \tag{4.5}$$

where $C > 0$ is some absolute constant.

We now turn to the upper bound. From (4.1) and the definition of u in Lemma 4.2, one has

$$\begin{aligned}\|U_\star^{I_2}(x)\|_{C_1\beta(\alpha)} &\leq C_\star \sum_{|k|\leq 3n} |u_k|\mathrm{e}^{2\pi C_1\beta(\alpha)|k|} + C_\star \sum_{3n<|k|\leq \frac{N}{9}} |u_k|\mathrm{e}^{2\pi C_1\beta(\alpha)|k|} \\ &\leq C_\star \mathrm{e}^{CC_1\beta(\alpha)n}.\end{aligned} \tag{4.6}$$

The purpose of the following is to construct quantitative almost reducibility (in $\mathrm{SL}(2,\mathbb{C})$) results. Suppose that $B(x)$ is as in Lemma 4.4 with $U(x) = U_\star^{I_2}(x)$ and $\eta = C_1\beta(\alpha)$. Then from (4.5), (4.6) and Lemma 4.4, we obtain

$$\|B\|_{C_1\beta(\alpha)}, \|B^{-1}\|_{C_1\beta(\alpha)} \leq C_\star \mathrm{e}^{CC_1\beta(\alpha)n}. \tag{4.7}$$

More precisely, by letting $B(x) = (U_\star^{I_2}(x), V(x))$ and recalling (4.3), we have

$$\begin{aligned}\overline{A}_{\lambda,E}(x)B(x) &= \left[\mathrm{e}^{2\pi \mathrm{i}\theta}U_\star^{I_2}(x+\alpha) + G_\star(x), \overline{A}_{\lambda,E}(x)V(x)\right] \\ &= B(x+\alpha)\begin{bmatrix}\mathrm{e}^{2\pi\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}\theta}\end{bmatrix} + \left[G_\star(x), \overline{A}_{\lambda,E}(x)V(x) - \mathrm{e}^{-2\pi\mathrm{i}\theta}V(x+\alpha)\right].\end{aligned}$$

In other words,

$$B^{-1}(x+\alpha)\overline{A}_{\lambda,E}(x)B(x) = \begin{bmatrix}\mathrm{e}^{2\pi\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}\theta}\end{bmatrix} + \begin{bmatrix}\beta_1(x) & b(x) \\ \beta_2(x) & \beta_3(x)\end{bmatrix}. \tag{4.8}$$

From (4.4) and (4.7), we obtain

$$\|\beta_1\|_{C_1\beta(\alpha)}, \|\beta_2\|_{C_1\beta(\alpha)} \leq C_\star \mathrm{e}^{-\frac{h}{20}N} \tag{4.9}$$

and

$$\|b\|_{C_1\beta(\alpha)} \leq C_\star \mathrm{e}^{CC_1\beta(\alpha)n}. \tag{4.10}$$

By taking determinant on (4.8) and in view of $\overline{A}_{\lambda,E}, B \in \mathrm{SL}(2,\mathbb{C})$, one has

$$\|\beta_3\|_{C_1\beta(\alpha)} \leq \|b\|_{C_1\beta(\alpha)}\|\beta_2\|_{C_1\beta(\alpha)} + \|\beta_1\|_{C_1\beta(\alpha)} \leq C_\star \mathrm{e}^{-\frac{h}{30}N}. \tag{4.11}$$

Actually, one can obtain the following refinement.

Theorem 4.6 Under the previous assumptions, there exists $\Phi(x) : \mathbb{C}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$ being analytic in $\Delta_{\frac{1}{2}C_1\beta(\alpha)}$ with $\|\Phi\|_{\frac{1}{2}C_1\beta(\alpha)} \leq C_\star \mathrm{e}^{CC_1\beta(\alpha)n}$ such that

$$\Phi^{-1}(x+\alpha)\overline{A}_{\lambda,E}(x)\Phi(x) = \begin{bmatrix}\mathrm{e}^{2\pi\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}\theta}\end{bmatrix} + \begin{bmatrix}\beta_1'(x) & b'(x) \\ \beta_2'(x) & \beta_3'(x)\end{bmatrix}, \tag{4.12}$$

where

$$\|\beta_1'\|_{\frac{1}{2}C_1\beta(\alpha)}, \|\beta_2'\|_{\frac{1}{2}C_1\beta(\alpha)}, \|\beta_3'\|_{\frac{1}{2}C_1\beta(\alpha)} \leq C_\star \mathrm{e}^{-\frac{h}{50}N}, \tag{4.13}$$

and

$$\|b'\|_{\frac{1}{2}C_1\beta(\alpha)} \leq C_\star \mathrm{e}^{-\frac{1}{20}C_1^2\beta(\alpha)n}. \tag{4.14}$$

Proof It suffices to assume that $n > n(\lambda,\alpha)$. Recalling (4.8), we can write $b(x) = b^r(x) + b^l(x) + b^h(x)$, where $b^l(x) = \sum\limits_{|k|\le C_1 n,\ k\ne n_j} \hat{b}_k \mathrm{e}^{2\pi \mathrm{i} kx}$, $b^r(x) = \hat{b}_{n_j}\mathrm{e}^{2\pi\mathrm{i} n_j x}$ and $b^h(x) = \sum\limits_{|k|>C_1 n} \hat{b}_k \mathrm{e}^{2\pi\mathrm{i} kx}$. Then by (4.10),

$$\|b^h\|_{\frac{1}{2}C_1\beta(\alpha)} \le \sum_{|k|>C_1 n} \|b\|_{C_1\beta(\alpha)} \mathrm{e}^{-\pi C_1\beta(\alpha)|k|} \le C_\star \mathrm{e}^{-2C_1^2\beta(\alpha)n}. \tag{4.15}$$

We then eliminate the term $b^l(x)$ by solving some homological equation. From (3.3) and the definition of ϵ_0-resonance, one has for $|k|\le C_1 n$ and $k\ne n_j$,

$$\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge \|(n_j-k)\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge c_\star \mathrm{e}^{-3C_1\beta(\alpha)n}. \tag{4.16}$$

Let $\hat{w}_k = -\hat{b}_k \frac{\mathrm{e}^{-2\pi\mathrm{i}\theta}}{1-\mathrm{e}^{-2\pi\mathrm{i}(2\theta-k\alpha)}}$ for $|k|\le C_1 n$ and $k\ne n_j$, and $\hat{w}_k = 0$ for $|k| > C_1 n$ or $k = n_j$. Consequently, the function $w(x) = \sum\limits_{k\in\mathbb{Z}} \hat{w}_k \mathrm{e}^{2\pi\mathrm{i}kx}$ will satisfy $\|w\|_{\frac{1}{2}C_1\beta(\alpha)} \le C_\star \mathrm{e}^{CC_1\beta(\alpha)n}$ from (4.10) and (4.16). If define

$$W(x) = \begin{bmatrix} 1 & w(x) \\ 0 & 1 \end{bmatrix},$$

then

$$W^{-1}(x+\alpha)\begin{bmatrix} \mathrm{e}^{2\pi\mathrm{i}\theta} & b^l(x) \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}\theta} \end{bmatrix} W(x) = \begin{bmatrix} \mathrm{e}^{2\pi\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}\theta} \end{bmatrix},$$

and

$$\|W\|_{\frac{1}{2}C_1\beta(\alpha)} \le C_\star \mathrm{e}^{CC_1\beta(\alpha)n}. \tag{4.17}$$

We now set $\Phi(x) = B(x)W(x)$. Then $\|\Phi\|_{\frac{1}{2}C_1\beta(\alpha)} \le C_\star \mathrm{e}^{CC_1\beta(\alpha)n}$. By direct computation, we have

$$\Phi^{-1}(x+\alpha)\overline{A}_{\lambda,E}(x)\Phi(x) = Z(x) + \Psi(x),$$

where

$$Z(x) = \begin{bmatrix} \mathrm{e}^{2\pi\mathrm{i}\theta} & b^r(x) \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}\theta} \end{bmatrix}$$

and

$$\Psi(x) = \begin{bmatrix} \beta_1'(x) & b^h(x) \\ \beta_2'(x) & \beta_3'(x) \end{bmatrix} = W^{-1}(x+\alpha)\begin{bmatrix} \beta_1(x) & b^h(x) \\ \beta_2(x) & \beta_3(x) \end{bmatrix} W(x).$$

Hence, we can obtain (4.13) and

$$\|\Psi\|_{\frac{1}{2}C_1\beta(\alpha)} \le C_\star \mathrm{e}^{-C_1^2\beta(\alpha)n} \tag{4.18}$$

from (4.9), (4.11), (4.15) and (4.17).

Thus what remains is to estimate the term $b^r(x)$. For $s\in\mathbb{N}$, we set

$$Z_s(x) = \prod_{k=s-1}^{0} Z(x+k\alpha) = \begin{bmatrix} \mathrm{e}^{2\pi\mathrm{i}s\theta} & b_s^r(x) \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}s\theta} \end{bmatrix},$$

where

$$b_s^r(x) = \hat{b}_{n_j}\mathrm{e}^{2\pi\mathrm{i}((s-1)\theta+n_j x)} \sum_{k=0}^{s-1} \mathrm{e}^{-2\pi\mathrm{i}k(2\theta-n_j\alpha)}.$$

Therefore,

$$\|b_s^r\|_0 = \left|\hat{b}_{n_j}\frac{\sin\pi s(2\theta - n_j\alpha)}{\sin\pi(2\theta - n_j\alpha)}\right|$$

if $\sin\pi(2\theta - n_j\alpha) \neq 0$, and $\|b_s^r\|_0 = s|\hat{b}_{n_j}|$ otherwise. As

$$2\|x\|_{\mathbb{R}/\mathbb{Z}} \leq \sin(\pi\|x\|_{\mathbb{R}/\mathbb{Z}}) \leq \pi\|x\|_{\mathbb{R}/\mathbb{Z}},$$

we have for $0 \leq s \leq \frac{1}{2}\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}}^{-1}$,

$$\frac{2s}{\pi}|\hat{b}_{n_j}| \leq \|b_s^r\|_0 \leq s|\hat{b}_{n_j}|.$$

Therefore, for $0 \leq s \leq \frac{1}{2}\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}}^{-1}$,

$$\frac{2s}{\pi}|\hat{b}_{n_j}| \leq \|Z_s\|_0 \leq 1 + s|\hat{b}_{n_j}| \leq C_\star(1+s)\mathrm{e}^{CC_1\beta(\alpha)n}. \tag{4.19}$$

Because of

$$\Phi^{-1}(x+s\alpha)\overline{A}_s(x)\Phi(x) = Z_s(x) + \sum_{k=1}^{s}\sum_{s-1\geq j_1>j_2>\cdots>j_k\geq 0}\Psi(x+j_1\alpha)\cdots\Psi(x+j_k\alpha)$$
$$\times Z_{s-1-j_1}(x+(j_1+1)\alpha)Z_{j_1-j_2-1}(x+(j_2+1)\alpha)\cdots Z_{j_k}(x)$$

and combining with (4.18) and (4.19), we have for $s \sim \mathrm{e}^{\frac{1}{10}C_1^2\beta(\alpha)n} < \frac{1}{2}\|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}}^{-1}$,

$$\begin{aligned}\|\overline{A}_s\|_0 &\geq \|\Phi\|_0^{-2}\left(\|Z_s\|_0 - \sum_{k=1}^{s}\binom{s}{k}\|\Psi\|_0^k(\max_{0\leq j\leq s-1}\|Z_j\|_0)^{1+k}\right)\\ &\geq \|\Phi\|_0^{-2}\left(\|Z_s\|_0 - C_\star\mathrm{e}^{\frac{1}{10}C_1^2\beta(\alpha)n}\sum_{k=1}^{s}\binom{s}{k}2^k\mathrm{e}^{-\frac{1}{2}C_1^2\beta(\alpha)nk}\right)\\ &\geq \|\Phi\|_0^{-2}\left(\|Z_s\|_0 - C_\star\mathrm{e}^{\frac{1}{10}C_1^2\beta(\alpha)n}((1+2\mathrm{e}^{-\frac{1}{2}C_1^2\beta(\alpha)n})^s - 1)\right)\\ &\geq c_\star\mathrm{e}^{-CC_1\beta(\alpha)n}(\|Z_s\|_0 - C_\star\mathrm{e}^{-\frac{3}{10}C_1^2\beta(\alpha)n}).\end{aligned}$$

Thus, from Lemma 4.3 and (4.19), we have for $s \sim \mathrm{e}^{\frac{1}{10}C_1^2\beta(\alpha)n}$,

$$|\hat{b}_{n_j}| \leq C_\star\mathrm{e}^{-\frac{1}{15}C_1^2\beta(\alpha)n}.$$

Hence,

$$\|b'\|_{\frac{1}{2}C_1\beta(\alpha)} \leq C_\star\mathrm{e}^{-\frac{1}{20}C_1^2\beta(\alpha)n}.$$

The proof is completed. □

Now we give the proof of Theorem 1.1.

Proof If the energy E is in the resolvent set, then $\mathcal{N}_{\lambda,\alpha}(E)$ is clearly Lipschitz continuous. Thus it suffices to consider the case $E \in \Sigma_{\lambda,\alpha}$. Given $\epsilon > 0$, we define $D = \begin{bmatrix} d^{-1} & 0 \\ 0 & d\end{bmatrix}$, where $d = \|\Phi\|_{\frac{1}{2}C_1\beta(\alpha)}\epsilon^{\frac{1}{4}}$ and Φ is given by Theorem 4.6. Let $\Phi'(x) = \Phi(x)D$. If $\epsilon \leq c_\star\mathrm{e}^{-C_1^2\beta(\alpha)n}$, we have

$$\|\Phi'\|_{\frac{1}{2}C_1\beta(\alpha)} \leq C_\star\epsilon^{-\frac{1}{4}}. \tag{4.20}$$

Setting $B'(x) = \Phi'^{-1}(x+\alpha)\overline{A}_{\lambda,E}(x)\Phi'(x)$, then

$$B'(x) = \begin{bmatrix}\mathrm{e}^{2\pi\mathrm{i}\theta} & 0\\ 0 & \mathrm{e}^{-2\pi\mathrm{i}\theta}\end{bmatrix} + \begin{bmatrix}\beta_1'(x) & d^2b'(x)\\ d^{-2}\beta_2'(x) & \beta_3'(x)\end{bmatrix}$$

with

$$\|\beta_1'\|_{\frac{1}{2}C_1\beta(\alpha)}, \|\beta_3'\|_{\frac{1}{2}C_1\beta(\alpha)} \le C_\star \mathrm{e}^{-\frac{1}{50}hN},$$

$$\left\|d^2 b'\right\|_{\frac{1}{2}C_1\beta(\alpha)} \le C_\star \mathrm{e}^{-\frac{1}{50}C_1^2\beta(\alpha)n}\epsilon^{\frac{1}{2}},$$

and

$$\left\|d^{-2}\beta_2'\right\|_{\frac{1}{2}C_1\beta(\alpha)} \le C_\star \mathrm{e}^{-\frac{1}{100}hN}\epsilon^{-\frac{1}{2}}.$$

If $\epsilon \ge C_\star \mathrm{e}^{-\frac{1}{100}hN}$, then

$$\left\|d^{-2}\beta_2'\right\|_{\frac{1}{2}C_1\beta(\alpha)} \le C_\star \epsilon^{\frac{1}{2}},$$

and

$$\|B'\|_{\frac{1}{2}C_1\beta(\alpha)} \le 1 + C_\star \epsilon^{\frac{1}{2}}. \tag{4.21}$$

As a result, for $C_\star \mathrm{e}^{-\frac{1}{100}hN} \le \epsilon \le c_\star \mathrm{e}^{-C_1^2\beta(\alpha)n}$,

$$\mathcal{L}_\lambda(E) = \mathcal{L}(\alpha, B') \le \ln \|B'\|_{\frac{1}{2}C_1\beta(\alpha)} \le \ln\left(1 + C_\star\epsilon^{\frac{1}{2}}\right) \le C_\star\epsilon^{\frac{1}{2}}.$$

Define

$$I_j := \{\epsilon \in \mathbb{R}: \ C_\star \mathrm{e}^{-\frac{1}{100}h|n_{j+1}|} \le \epsilon \le c_\star \mathrm{e}^{-C_1^2\beta(\alpha)|n_j|}\}.$$

Then for any small $\epsilon_0 > 0$, there exists $j_0 \in \mathbb{Z}^+$ such that $[0, \epsilon_0] \subset \bigcup\limits_{j \ge j_0} I_j$. Let $\epsilon = |E - E'| \in [0, \epsilon_0]$ with $E' \in \mathbb{C}$. Then by (4.20) and (4.21), one has

$$\begin{aligned} \mathcal{L}_\lambda(E') = \ & \mathcal{L}\left(\alpha, \Phi'^{-1}(x+\alpha)\overline{A}_{\lambda,E'}(x)\Phi'(x)\right) \\ \le \ & \ln\left\|B' + \Phi'^{-1}(x+\alpha)\left(\overline{A}_{\lambda,E'}(x) - \overline{A}_{\lambda,E}(x)\right)\Phi'(x)\right\|_{\frac{1}{2}C_1\beta(\alpha)} \\ \le \ & \ln\left(1 + C_\star\epsilon^{\frac{1}{2}}\right) \le C_\star\epsilon^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$|\mathcal{L}_\lambda(E') - \mathcal{L}_\lambda(E)| \le C_\star |E' - E|^{\frac{1}{2}}. \tag{4.22}$$

From the Thouless formula (3.4), we have

$$\begin{aligned} \left|\mathcal{L}\left(\alpha, \overline{A}_{\lambda,E+\mathrm{i}\epsilon}\right) - \mathcal{L}_\lambda(E)\right| &= \frac{1}{2}\int \ln\left(1 + \frac{\epsilon^2}{(E-E')^2}\right)\mathrm{d}\mathcal{N}_{\lambda,\alpha}(E') \\ &\ge \frac{1}{2}\ln 2\left(\mathcal{N}_{\lambda,\alpha}(E+\epsilon) - \mathcal{N}_{\lambda,\alpha}(E-\epsilon)\right). \end{aligned}$$

Thus, recalling (4.22), we obtain

$$\mathcal{N}_{\lambda,\alpha}(E+\epsilon) - \mathcal{N}_{\lambda,\alpha}(E-\epsilon) \le C_\star\epsilon^{\frac{1}{2}},$$

which means precisely that $\mathcal{N}_{\lambda,\alpha}(E)$ is $\frac{1}{2}$-Hölder continuous.

This completes the proof of Theorem 1.1. □

The proof Theorem 1.4 is included in Appendix.

Acknowledgements The authors would like to thank Prof. Xiaoping Yuan for his helpful suggestions.

References

[1] Avila A. The absolutely continuous spectrum of the almost Mathieu operator. arXiv, 2008, 0810.2965

[2] Avila A, Jitomirskaya S. Almost localization and almost reducibility. J Eur Math Soc, 2010, 12: 93–131

[3] Avila A, Jitomirskaya S. Hölder continuity of absolutely continuous spectral measures for one-frequency Schrödinger operators. Comm Math Phys, 2011, **301**(2): 563–581

[4] Avila A, Jitomirskaya S, Marx C A. Spectral theory of extended Harper's model and a question by Erdos and Szekeres. Invent Math, 2017, **210**(1): 283–339

[5] Bourgain J. Hölder regularity of integrated density of states for the almost Mathieu operator in a perturbative regime. Lett Math Phys, 2000, **51**(2): 83–118

[6] Cai A, Chavaudret C, You J G, Zhou Q. Sharp Hölder continuity of the Lyapunov exponent of finitely differentiable quasi-periodic cocycles. Math Z, 2019, **29**(3/4): 931–958

[7] Damanik D, Goldstein M, Lukic M. The spectrum of a Schrödinger operator with small quasi-periodic potential is homogeneous. J Spectral Theory, 2016, **6**(2): 415–427

[8] Delyon F, Souillard B. The rotation number for finite difference operators and its properties. Comm Math Phys, 1983, **89**(3): 415–426

[9] Fillman J, Lukic M. Spectral homogeneity of limit-periodic Schrödinger operators. J Spectr Theory, 2017, **7**(2): 387–406

[10] Damanik D, Goldstein M, Schlag W, Voda M. Homogeneity of the spectrum for quasi-perioidic Schrödinger operators. J Eur Math Soc, 2018, **20**(12): 3073–3111

[11] Goldstein M, Schlag W. Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions. Ann Math, 2001, **154**(1): 155–203

[12] Goldstein M, Schlag W, Voda M. On localization and the spectrum of multi-frequency quasi-periodic operators. arXiv, 2016, 1610.00380

[13] Hadj Amor S. Hölder continuity of the rotation number for quasi-periodic co-cycles in $SL(2,\mathbb{R})$. Comm Math Phys, 2009, **287**(2): 565–588

[14] Han R. Absence of point spectrum for the self-dual extended Harper's model. Int Math Res Not, 2018, (9): 2801–2809

[15] Han R. Dry Ten Martini problem for the non-self-dual extended Harper's model. Trans Amer Math Soc, 2018, **370**(1): 197–217

[16] Han R, Jitomirskaya S. Full measure reducibility and localization for quasiperiodic Jacobi operators: a topological criterion. Adv Math, 2017, **319**: 224–250

[17] Han R, Yang F, Zhang S W. Spectral dimension for β-almost periodic singular Jacobi operators and the extended Harper's model. arXiv, 2018, 1804.04322

[18] Han R, Zhang S W. Optimal large deviation estimates and Hölder regularity of the Lyapunov exponents for quasi-periodic Schrödinger cocycles. arXiv, 2018, 1803.02035

[19] Jitomirskaya S, Koslover D A, Schulteis M S. Localization for a family of one-dimensional quasiperiodic operators of magnetic origin. Ann Henri Poincaré, 2005, **6**(1): 103–124

[20] Johnson R, Moser J. The rotation number for almost periodic potentials. Comm Math Phys, 1983, **90**(2): 317–318

[21] Leguil M, You J G, Zhao Z Y, Zhou Q. Asymptotics of spectral gaps of quasi-periodic Schrödinger operators. arXiv, 2017, 1712.04700

[22] Liu W C, Shi Y F. Upper bounds on the spectral gaps of quasi-periodic Schrödinger operators with Liouville frequencies. To appear in J Spectr Theory

[23] Liu W C, Yuan X P. Hölder continuity of the spectral measures for one-dimensional Schrödinger operator in exponential regime. J Math Phys, 2015, **56**(1): 012701–21

[24] Shi Y F, Yuan X P. Exponential decay of the lengths of the spectral gaps for the extended Harper's model with a Liouvillean frequency. To appear in J Dynam Differential Equations

[25] Tao K. Strong Birkhoff ergodic theorem for subharmonic functions with irrational shift and its application to analytic quasi-periodic cocycles. arXiv, 2018, 1805.00431

[26] Tao K, Voda M. Hölder continuity of the integrated density of states for quasi-periodic Jacobi operators. J Spectr Theory, 2017, **7**(2): 361–386

[27] Thouless D J. Bandwidths for a quasiperiodic tight-binding model. Phys Rev B, 1983, **28**(8): 4272–4276

[28] You J G, Zhang S W. Hölder continuity of the Lyapunov exponent for analytic quasiperiodic Schrödinger cocycle with weak Liouville frequency. Ergodic Theory Dynam Systems, 2014, **34**(4): 1395–1408

Appendix

In this section, we will complete the proof of Theorem 1.4. The proof is similar to that of [22]. For convenience, we include the details in the following.

Lemma 0.7 (Theorem 1.1 of [24]) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $0 \le \beta(\alpha) < \infty$ and E_m^-, E_m^+ be given by (2.1). Then there exists absolute constant $C > 1$ such that, if $\lambda \in \mathrm{II}$ and $\mathcal{L}_{\overline{\lambda}} > C\beta(\alpha)$, one has for $|m| \ge m_\star$,

$$E_m^+ - E_m^- \le \mathrm{e}^{-C^{-1}\mathcal{L}_{\overline{\lambda}}|m|},$$

where $m_\star$ is a positive constant only depending on λ and α, and $\mathcal{L}_{\overline{\lambda}}$ is given by (1.3).

Lemma 0.8 Let $G_m = (E_m^-, E_m^+)$ for $m \in \mathbb{Z} \setminus \{0\}$ and $G_0 = (-\infty, E_{\min}) \bigcup (E_{\max}, +\infty)$. Then, for $m' \ne m \in \mathbb{Z} \setminus \{0\}$ with $|m'| \ge |m|$, we have

$$\mathrm{dist}(G_m, G_{m'}) = \inf_{x \in G_m, x' \in G_{m'}} |x - x'| \ge c_\star \mathrm{e}^{-6\beta(\alpha)|m'|}, \tag{A.1}$$

and for $m \in \mathbb{Z} \setminus \{0\}$,

$$\mathrm{dist}(G_m, G_0) \ge c_\star \mathrm{e}^{-6\beta(\alpha)|m|}. \tag{A.2}$$

Proof From the small divisor condition (3.3), one has

$$\|(m - m')\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge C(\alpha)\mathrm{e}^{-3\beta(\alpha)|m'|} \tag{A.3}$$

for $|m'| \ge |m|$.

Without loss of generality, we assume that $E_m^+ \le E_{m'}^-$. By Theorem 1.1, (2.1) and (A.3), we have

$$\begin{aligned} \mathrm{dist}(G_m, G_{m'}) &= |E_{m'}^- - E_m^+| \\ &\ge \left(\frac{1}{C_\star} \left|\mathcal{N}_{\lambda,\alpha}(E_{m'}^-) - \mathcal{N}_{\lambda,\alpha}(E_m^+)\right|\right)^2 \\ &\ge c_\star \|(m - m')\alpha\|_{\mathbb{R}/\mathbb{Z}}^2 \\ &\ge c_\star \mathrm{e}^{-6\beta(\alpha)|m'|}, \end{aligned}$$

which completes the proof of (A.1). The proof of (A.2) is similar. □

Now we can give the proof of Theorem 1.4.

Proof Assume that $0 < \sigma \le \sigma_\star(\lambda, \alpha, \epsilon)$. For $E \in \Sigma_{\lambda,\alpha}$ and σ, let

$$\mathcal{R}(E, \sigma) = \{m \in \mathbb{Z} \setminus \{0\} : (E - \sigma, E + \sigma) \cap G_m \ne \emptyset\}.$$

Define $m_0 \in \mathbb{Z} \setminus \{0\}$ with $|m_0| = \min\limits_{m \in \mathcal{R}(E,\sigma)} |m|$. For any $m \in \mathcal{R}(E, \sigma)$, one has

$$\mathrm{dist}(G_m, G_{m_0}) \le 2\sigma.$$

We first assume that $(E-\sigma, E+\sigma) \cap G_0 = \emptyset$. Recalling (A.1), we have for any $m \in \mathcal{R}(E, \sigma)$ and $m \ne m_0$,

$$2\sigma \ge c_\star \mathrm{e}^{-6\beta(\alpha)|m|},$$

that is,

$$|m| \ge \frac{-\ln(C_\star \sigma)}{6\beta(\alpha)}. \tag{A.4}$$

Then, by Lemma 0.7, we obtain

$$\begin{aligned}\sum_{m\in\mathcal{R}(E,\sigma),m\neq m_0}\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_m) &\leq \sum_{m\in\mathcal{R}(E,\sigma),m\neq m_0}(E_m^+-E_m^-)\\ &\leq \sum_{|m|\geq\frac{-\ln(C_\star\sigma)}{6\beta(\alpha)}} C_\star \mathrm{e}^{-C^{-1}\mathcal{L}_{\overline{\lambda}}|m|}\\ &\leq \epsilon\sigma. \end{aligned}\tag{A.5}$$

Moreover, $E\in\Sigma_{\lambda,\alpha}$ implies $E\notin G_{m_0}$. Thus, we have

$$\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_{m_0})\leq\sigma. \tag{A.6}$$

In this case, (A.5) and (A.6) implies

$$\begin{aligned}&\mathrm{Leb}((E-\sigma,E+\sigma)\cap\Sigma_{\lambda,\alpha})\\ \geq{}& 2\sigma-\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_{m_0})-\sum_{m\in\mathcal{R}(E,\sigma),m\neq m_0}\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_m)\\ \geq{}& 2\sigma-\sigma-\epsilon\sigma\geq(1-\epsilon)\sigma.\end{aligned}$$

In the case $(E-\sigma,E+\sigma)\cap G_0\neq\emptyset$, we have

$$0<E_m^--E_{\min}\leq2\sigma$$

for any $m\in\mathcal{R}(E,\sigma)$. Thus, (A.4) also holds for any $m\in\mathcal{R}(E,\sigma)$ by (A.2). From the proof of (A.5), we have

$$\sum_{m\in\mathcal{R}(E,\sigma)}\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_m)\leq\epsilon\sigma. \tag{A.7}$$

Due to $E\in\Sigma_{\lambda,\alpha}$ and $E\notin G_0$, one has

$$\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_0)\leq\sigma. \tag{A.8}$$

By (A.7) and (A.8), we obtain

$$\begin{aligned}&\mathrm{Leb}((E-\sigma,E+\sigma)\cap\Sigma_{\lambda,\alpha})\\ \geq{}& 2\sigma-\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_0)-\sum_{m\in\mathcal{R}(E,\sigma)}\mathrm{Leb}((E-\sigma,E+\sigma)\cap G_m)\\ \geq{}& 2\sigma-\sigma-\epsilon\sigma\geq(1-\epsilon)\sigma.\end{aligned}$$

This completes the proof of Theorem 1.4. □