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ON SHRINKING GRADIENT RICCI SOLITONS WITH POSITIVE RICCI CURVATURE AND SMALL WEYL TENSOR*

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Abstract We show that closed shrinking gradient Ricci solitons with positive Ricci curvature and sufficiently pinched Weyl tensor are Einstein. When Weyl tensor vanishes, this has been proved before but our proof here is much simpler.

Key words shrinking gradient Ricci solitons; positive Ricci curvature; pinched Weyl tensor

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1 Introduction

A Riemannian manifold (M, g), coupled with a smooth potential function f, is called a gradient Ricci soliton, if there exists a constant λ, such that

$$R_{ij} + f_{ij} = \lambda g_{ij}.$$

It is called shrinking, steady, or expanding, if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Note that, when the potential function f is a constant, a gradient Ricci soliton is an Einstein manifold. Gradient Ricci solitons also correspond to self-similar solutions of the Ricci flow, and play a fundamental role in the singularity study of the flow. They have been widely studied (see the survey of Cao [1, 2] for nice overviews) and, in particular, compact gradient steady and expanding Ricci solitons are shown to be Einstein by Hamilton [7] and Ivey [9], and Perelman [13] showed that general compact Ricci solitons are necessarily gradient. For the shrinking case, Ivey [9] proved that the only compact three-dimensional shrinking Ricci solitons are, up to quotients, isometric to the round sphere S^3. Dimension four is then the lowest dimension allowing "nontrivial" examples of compact shrinking Ricci solitons and such examples do exist (see the survey [1] and references therein).

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So we need more conditions to obtain the triviality of compact shrinking gradient Ricci solitons. In this decade, many results were built on conditions of Weyl tensor, for instance [4–6, 11, 12, 14]. In particular, a full classification is achieved in [15] for locally conformally flat shrinking gradient Ricci solitons. However, since the Ricci flow does not preserve the vanishing of Weyl tensor, i.e., such property is not dynamically stable, generic singular models are probably not locally conformally flat. This shows the necessity of investigating solitons with nonvanishing Weyl tensor.

Many phenomena shows that, for a solution of the Ricci flow, the Weyl part in the curvature decomposition cannot be too small. There have been some results which proposed formulations about this suspicion, see for example [3, 10]. In this article, we show that closed shrinking solitons with positive Ricci curvature and small Weyl tensor must be Einstein.

Theorem 1.1 Given an n-dimensional closed shrinking Ricci soliton with $\mathrm{Ric} \geq \epsilon R > 0$. If $|W| \leq \frac{\epsilon^2}{2(n-2)} R$, then it must be Einstein. Furthermore, it should be the spherical space form.

Our proof is purely algebraic. In particular, when Weyl vanishes, the proof becomes even simpler and the following proposition follows.

Proposition 1.2 Let (M, g, f) be a locally conformally flat compact gradient shrinking soliton with positive Ricci curvature. Then it is necessary an Einstein metric. Furthermore, it must be the spherical space form.

Note that above theorem is just a part of a full classification theorem of the first author [15]. We will prove this special case in the next section and demonstrate the general case in Section 3.

2 Locally Conformally Flat Solitons

We first recall some basic identities of gradient Ricci soliton (M, g, f). By using the notion of f-Laplacian which acts on a symmetric tensor T as

$$\Delta_f T = \Delta T - \nabla f \cdot \nabla T,$$

the identities are the following.

Lemma 2.1 For a gradient Ricci soliton,

$$\begin{cases} \Delta_f R_{ij} = 2\lambda R_{ij} - 2R_{ikjl}R_{kl}, \\ \Delta_f R \quad = 2\lambda R - 2|R_{ij}|^2, \end{cases}$$

where $\{R_{ij}\}$ is the Ricci curvature tensor, and R is the scalar curvature.

For the computation of the matrix $Q_{ij} = R_{ikjl}R_{kl}$, we can choose orthonormal basis $\{e_i\}_{i=1}^n$, such that the Ricci tensor are diagonalized with eigenvalue $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then we have the following simple facts.

Lemma 2.2 If Weyl tensor vanishes, we have

$$\begin{cases} |R_{ij}|^2 = \lambda_1^2 + (n-1)\bar{\lambda}^2 + \sum\limits_{k\geq 2} \delta_k^2, \\ Q_{11} = \bar{\lambda}\lambda_1 + \dfrac{1}{n-2} \sum\limits_{k\geq 2} \delta_k^2, \end{cases}$$

where $\bar\lambda = \frac{1}{n-1}(\lambda_2 + \cdots + \lambda_n)$, and $\delta_k = \lambda_k - \bar\lambda$, $k = 2, \cdots, n$.

Proof The fact $\sum\limits_{k\geq 2}\delta_k = 0$ implies that

$$|R_{ij}|^2 = \sum_{i=1}^n \lambda_i^2 = \lambda_1^2 + \sum_{k\geq 2}(\bar\lambda + \delta_k)^2 = \lambda_1^2 + (n-1)\bar\lambda^2 + \sum_{k\geq 2}\delta_k^2.$$

Thus the first equation holds.

Now since Weyl tensor vanishes, we have

$$R_{1k1k} = \frac{1}{n-2}(\lambda_1 + \lambda_k) - \frac{1}{(n-1)(n-2)}R = \frac{\lambda_1}{n-1} + \frac{\delta_k}{n-2}.$$

Thus one can see that

$$Q_{11} = \sum_{k\geq 2}(\bar\lambda + \delta_k)\cdot(\frac{\lambda_1}{n-1} + \frac{\delta_k}{n-2}) = \bar\lambda\lambda_1 + \frac{1}{n-2}\sum_{k\geq 2}\delta_k^2.$$

□

Now we can use the maximum principle to show Proposition 1.2.

Proof of Proposition 1.2 Since the manifold (M, g) is compact, we have $R_{ij} \geq \epsilon R g_{ij} > 0$ for some positive constant ϵ. Furthermore, we can choose an optimal ϵ, such that there exist a point $p \in M$ and unit vector $e_1 \in T_pM$, such that the least eigenvalue of the Ricci tensor $\lambda_1(p) = \mathrm{Ric}(e_1, e_1) = \epsilon R(p)$.

Since e_1 realizes the minimum of $R_{ij} - \epsilon R g_{ij}$, we have

$$\Delta_f(R_{ij} - \epsilon R g_{ij})(e_1, e_1) \geq 0.$$

Then by Lemma 2.1,

$$Q_{11} - \epsilon|R_{ij}|^2 \leq 0.$$

Thus we have

$$\begin{aligned}
0 &\geq \frac{Q_{11}}{\lambda_1} - \frac{|R_{ij}|^2}{R}\\
&= \bar\lambda + \frac{1}{(n-2)\lambda_1}\sum_{k\geq 2}\delta_k^2 - \frac{1}{R}\left[\lambda_1^2 + (n-1)\bar\lambda^2 + \sum_{k\geq 2}\delta_k^2\right]\\
&= \bar\lambda - \frac{1}{R}\left[\lambda_1\bar\lambda + (n-1)\bar\lambda^2 - \lambda_1(\bar\lambda - \lambda_1)\right] + \left[\frac{1}{(n-2)\lambda_1} - \frac{1}{R}\right]\cdot\sum_{k\geq 2}\delta_k^2\\
&= \frac{\bar\lambda}{R}\left[R - \lambda_1 - (n-1)\bar\lambda\right] + \frac{1}{R}\cdot\lambda_1(\bar\lambda - \lambda_1) + \left[\frac{1}{(n-2)\lambda_1} - \frac{1}{R}\right]\cdot\sum_{k\geq 2}\delta_k^2\\
&= \frac{1}{R}\cdot\lambda_1(\bar\lambda - \lambda_1) + \left[\frac{1}{(n-2)\lambda_1} - \frac{1}{R}\right]\cdot\sum_{k\geq 2}\delta_k^2\\
&\geq \frac{1}{R}\cdot\lambda_1(\bar\lambda - \lambda_1).
\end{aligned}$$

So $\bar\lambda(p) = \lambda_1(p)$, which implies that $\lambda_1(p) = \cdots = \lambda_n(p)$, and then $\epsilon = \frac{1}{n}$. Hence $R_{ij} \geq \frac{1}{n}Rg_{ij}$ holds everywhere, and (M, g) is an Einstein manifold. So we complete the proof of Proposition 1.2. □

3 Solitons with Small Weyl Tensor

In this section, we turn to consider the gradient shrinking solitons with pinched Weyl tensor $\{W_{ijkl}\}$. In this case, the sectional curvature

$$R_{1k1k} = W_{1k1k} + \frac{1}{n-2}(\lambda_1 + \lambda_k) - \frac{1}{(n-1)(n-2)}R = W_{1k1k} + \frac{\lambda_1}{n-1} + \frac{\delta_k}{n-2}.$$

Then by a similar argument as Lemma 2.2, we obtain the following lemma.

Lemma 3.1 We have

$$\begin{cases} |R_{ij}|^2 = \lambda_1^2 + (n-1)\bar{\lambda}^2 + \sum\limits_{k\geq 2}\delta_k^2, \\ Q_{11} = \bar{\lambda}\lambda_1 + \dfrac{1}{n-2}\sum\limits_{k\geq 2}\delta_k^2 + \sum\limits_{k\geq 2}\delta_k W_{1k1k} \end{cases}$$

where $\bar{\lambda} = \frac{1}{n-1}(\lambda_2 + \cdots + \lambda_n)$, and $\delta_k = \lambda_k - \bar{\lambda}$, $k = 2, \cdots, n$.

Proof of Theorem 1.1 Suppose $R_{ij} \geq \epsilon_0 R g_{ij} > 0$ for an optimal positive constant $\epsilon_0 \geq \epsilon$. Then there exist a point $p \in M$ and unit vector $e_1 \in T_pM$, such that the least eigenvalue of the Ricci tensor $\lambda_1(p) = \mathrm{Ric}(e_1, e_1) = \epsilon_0 R(p)$.

Similar to the proof in Section 2, we have $\Delta_f(R_{ij} - \epsilon R g_{ij})(e_1, e_1) \geq 0$, which implies that

$$\frac{Q_{11}}{\lambda_1} - \frac{|R_{ij}|^2}{R} \leq 0.$$

When $|W| \leq \eta R$ for some $\eta > 0$ to be determined later, one can divide the Weyl term in Lemma 3.1 by λ_1 and derives

$$\frac{1}{\lambda_1}\sum_{k\geq 2}\delta_k W_{1k1k} \geq -\frac{|W|}{\lambda_1}\sum_{k\geq 2}|\delta_k| \geq -\frac{\eta}{\epsilon_0}\sum_{k\geq 2}|\delta_k|.$$

Recalling the last equality in the proof of Proposition 1.2, we have

$$\begin{aligned} 0 &\geq \frac{Q_{11}}{\lambda_1} - \frac{|R_{ij}|^2}{R} \\ &\geq \frac{1}{R}\cdot\lambda_1(\bar{\lambda} - \lambda_1) - \frac{\eta}{\epsilon_0}\sum_{k\geq 2}|\delta_k| + \left[\frac{1}{(n-2)\lambda_1} - \frac{1}{R}\right]\cdot\sum_{k\geq 2}\delta_k^2 \\ &= \epsilon_0(\bar{\lambda} - \lambda_1) - \frac{\eta}{\epsilon_0}\sum_{k\geq 2}|\delta_k| + \left[\frac{1}{(n-2)\lambda_1} - \frac{1}{R}\right]\cdot\sum_{k\geq 2}\delta_k^2. \end{aligned}$$

Observing that the term $\sum\limits_{k\geq 2}|\delta_k|$ can be estimated by

$$\sum_{k\geq 2}|\delta_k| \leq 2(n-2)(\bar{\lambda} - \lambda_2),$$

we immediately get that

$$0 \geq \epsilon_0(\bar{\lambda} - \lambda_1) - \frac{2(n-2)\eta}{\epsilon_0}\cdot(\bar{\lambda} - \lambda_2) + \left[\frac{1}{(n-2)\lambda_1} - \frac{1}{R}\right]\cdot\sum_{k\geq 2}\delta_k^2.$$

By the choice of $\eta = \frac{\epsilon^2}{2(n-2)} \leq \frac{\epsilon_0^2}{2(n-2)}$, we have $\lambda_2 = \lambda_1$ and $\sum\limits_{k\geq 2}\delta_k^2 = 0$. So $\lambda_1(p) = \cdots = \lambda_n(p)$, which implies that $\epsilon = \frac{1}{n}$. Hence $R_{ij} \geq \frac{1}{n}Rg_{ij}$ everywhere and (M, g) is an Einstein manifold.

Now the traceless Ricci curvature $R_{ij} - \frac{R}{n} g_{ij}$ vanishes, and the Weyl tensor satisfies $|W| \le \frac{\epsilon^2}{2(n-2)} R \le \frac{1}{2n^2(n-2)} R$.

(1) If $n = 4$. Then $|W|^2 \le \left(\frac{1}{64}\right)^2 R^2 < \frac{1}{5} \cdot \frac{2}{4 \cdot 3} R^2 = \frac{1}{5} \cdot \frac{2}{n(n-1)} R^2$.

(2) If $n = 5$. Then $|W|^2 \le \left(\frac{1}{150}\right)^2 R^2 < \frac{1}{10} \cdot \frac{2}{5 \cdot 4} R^2 = \frac{1}{10} \cdot \frac{2}{n(n-1)} R^2$.

(3) If $n \ge 6$. Then $|W|^2 \le \left[\frac{1}{2n^2(n-2)}\right]^2 R^2 < \frac{2}{(n-2)(n+1)} \cdot \frac{2}{n(n-1)} R^2$.

So by the result of Huisken [8], the soliton should be the spherical space form. And we complete the proof. $\square$

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