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PRECISE MOMENT ASYMPTOTICS FOR THE STOCHASTIC HEAT EQUATION OF A TIME-DERIVATIVE GAUSSIAN NOISE*

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Abstract This article establishes the precise asymptotics

$$\mathbb{E}u^m(t,x) \quad (t\to\infty \quad \text{or} \quad m\to\infty)$$

for the stochastic heat equation

$$\frac{\partial u}{\partial t}(t,x)=\frac{1}{2}\Delta u(t,x)+u(t,x)\frac{\partial W}{\partial t}(t,x)$$

with the time-derivative Gaussian noise $\frac{\partial W}{\partial t}(t,x)$ that is fractional in time and homogeneous in space.

Key words Stochastic heat equation; time-derivative Gaussian noise; Brownian motion; Feynman-Kac representation; Schilder's large deviation

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1 Introduction

Moment asymptotics for solutions to stochastic partial differential equations are known as the problem of intermittency that has been studied extensively in the past two decades [1, 8]. In this work, we investigate the asymptotics problem

$$\mathbb{E}u^m(t,x) \qquad (t\to\infty \;\; \text{or} \;\; m\to\infty)$$

for the stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x)=\dfrac{1}{2}\Delta u(t,x)+u(t,x)\dfrac{\partial W}{\partial t}(t,x) \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d \\ u(0,x)=u_0(x) \end{cases} \tag{1.1}$$

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with the Gaussian noise $\frac{\partial}{\partial t}W(t,x)$ that is formally given as the time derivative of the mean-zero Gaussian field $\{W(t,x);\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\}$ with the covariance function

$$\mathrm{Cov}\Big(W(t,x),W(s,y)\Big)=\frac{1}{2}(t^{2H_0}+s^{2H_0}-|t-s|^{2H_0})\Gamma(x,y)\qquad (t,x),(s,y)\in\mathbb{R}^+\times\mathbb{R}^d,$$

where the time Hurst parameter $H_0\in(0,1)$ and we assume that the space covariance function $\Gamma(x,y)$ is locally bounded and has the homogeneity in the sense that

$$\begin{cases}\Gamma(Cx,Cx)=|C|^{2H}\Gamma(x,x)\\ \Gamma(x,x)+\Gamma(y,y)-2\Gamma(x,y)=\Gamma(x-y,x-y)\end{cases}\tag{1.2}$$

for any $x,y\in\mathbb{R}^d$ and $C\in\mathbb{R}$, where the constant $H\in(0,1)$. Assumption (1.2) can be restated as

$$\begin{cases}\big\{W(c_0t,cx);\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\big\}\overset{d}{=}\big\{c_0^{H_0}c^HW(t,x);\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\big\}\\ W(t,x)-W(s,y)\overset{d}{=}W(t-s,x-y)\qquad (c_0,c>0,\ (t,x),(s,y)\in\mathbb{R}^+\times\mathbb{R}^d).\end{cases}\tag{1.3}$$

For simplicity, we assume that bounded initial condition is as follows:

$$0<\inf_{x\in\mathbb{R}^d}u_0(x)\le\sup_{x\in\mathbb{R}^d}u_0(x)<\infty.\tag{1.4}$$

Mathematically, $\frac{\partial}{\partial t}W(t,x)$ is defined as a generalized centered Gaussian field with

$$\mathrm{Cov}\Big(\frac{\partial}{\partial t}W(t,x),\frac{\partial}{\partial s}W(s,y)\Big)=\gamma_0(t-s)\Gamma(x,y)\qquad (t,x),(s,y)\in\mathbb{R}^+\times\mathbb{R}^d.\tag{1.5}$$

Here, the time-covariance $\gamma_0(t-s)$ is morally considered as the derivative

$$\frac{\partial^2}{\partial t\partial s}\Big\{\frac{1}{2}(t^{2H_0}+s^{2H_0}-|t-s|^{2H_0})\Big\}.$$

In particular,

$$\gamma_0(t-s)=\begin{cases}H_0(2H_0-1)|t-s|^{-(2-2H_0)} & H_0>1/2\\ \delta_0(t-s) & H_0=1/2.\end{cases}\tag{1.6}$$

The function $\gamma_0(\cdot)$ defined in (1.6) is qualified as covariance function as it is non-negative definite. Indeed, it can be shown that

$$\gamma_0(u)=\frac{\Gamma(2H_0+1)\sin(\pi H_0)}{2\pi}\int_{\mathbb{R}}e^{\mathrm{i}\lambda u}|\lambda|^{1-2H_0}\mathrm{d}\lambda\qquad u\in\mathbb{R}.\tag{1.7}$$

When $H_0<1/2$, the function $|\cdot|^{-(2-2H_0)}$ is no longer non-negative definite and is not qualified for being a covariance function. As consequence, the covariance function $\gamma_0(\cdot)$ can not be legally defined by (1.6) when $H_0<1/2$. As $H_0<1/2$, the function $\gamma_0(\cdot)$ is defined as a generalized function given in (1.7). It should be emphasized that $\gamma_0(\cdot)$ is not defined point-wise when $H_0<1/2$.

Under suitable conditions (such as the one assumed in our main theorem), the solution to (1.1) yields the following Feynman-Kac formula:

$$u(t,x)=\mathbb{E}_x\bigg[u_0(B_t)\exp\bigg\{\int_0^tW(\mathrm{d}s,B_{t-s})\bigg\}\bigg]\qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\tag{1.8}$$

where $\{B_t;\ t\ge0\}$ is a d-dimensional Brownian motion independent of $\{W(t,x);\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^d\}$ with $B_0=x$, and "$\mathbb{E}_x$" stands for the Brownian expectation. We point to [5] and [6] for existing literature.

Recently, Chen, Hu, Kalbasi, and Nualart ((1.3), [1]) establish the bounds

$$C_1 \exp\left\{C_1 m^{\frac{2-H}{1-H}} t^{\frac{2H_0+H}{1-H}}\right\} \le \mathbb{E}u^m(t,x) \le C_2 \exp\left\{C_2 m^{\frac{2-H}{1-H}} t^{\frac{2H_0+H}{1-H}}\right\} \tag{1.9}$$

for the case $H_0 < 1/2$, where the constant $C_1, C_2 > 0$ are independent of $t > 0$ and $m = 1, 2, \cdots$.

In connection to (1.9), our goal is to obtain the precise asymptotics as $t \to \infty$ or as $m \to \infty$.

Let $C_0\{[0,1], \mathbb{R}^d\}$ be the space of continuous functions $x(s)$: $[0,1] \longrightarrow \mathbb{R}^d$ with $x(0) = 0$ and $\mathcal{H}_d$ be the Cameron-Martin space given as

$$\mathcal{H}_d = \Big\{x(\cdot) \in C_0\big([0,1], \mathbb{R}^d\big);\quad x(s) \text{ is absolutely continuous and } \int_0^1 |\dot{x}(s)|^2 \mathrm{d}s < \infty\Big\}.$$

Set

$$\mathcal{E}(H_0) = \sup_{x\in\mathcal{H}_d}\left\{\frac{C_{H_0}}{2}\int_0^1\int_0^1 \frac{\Gamma(x(s), x(r))}{|s-r|^{(2-2H_0)}}\mathrm{d}s\mathrm{d}r - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2\mathrm{d}s\right\}\ (1/2 < H_0 < 1), \tag{1.10}$$

$$\mathcal{E}\Big(\frac{1}{2}\Big) = \sup_{x\in\mathcal{H}_d}\left\{\frac{1}{2}\int_0^1 \Gamma(x(s), x(s))\mathrm{d}s - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2\mathrm{d}s\right\}\ (H_0 = 1/2), \tag{1.11}$$

$$\begin{aligned}\mathcal{E}(H_0) = \sup_{x\in\mathcal{H}_d}\bigg\{&\frac{|C_{H_0}|}{4}\int_0^1\int_0^1 \frac{\Gamma(x(s)-x(r), x(s)-x(r))}{|s-r|^{(2-2H_0)}}\mathrm{d}s\mathrm{d}r\\ &+\frac{H_0}{2}\int_0^1\Big\{s^{-(1-2H_0)} + (1-s)^{-(1-2H_0)}\Big\}\Gamma(x(s), x(s))\mathrm{d}s\\ &-\frac{1}{2}\int_0^1 |\dot{x}(s)|^2\mathrm{d}s\bigg\}\quad (0 < H_0 < 1/2),\end{aligned} \tag{1.12}$$

where $C_{H_0} = H_0(2H_0 - 1)$. In connection to the Feynman-Kac representation (1.8) and in view of the variance identities given in (2.26), (2.27), and (2.28) below, all variations can be unified into the following form:

$$\mathcal{E}(H_0) = \sup_{x\in\mathcal{H}_d}\left\{\frac{1}{2}\mathrm{Var}\left(\int_0^1 W\big(\mathrm{d}s, x(s)\big)\right) - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2\mathrm{d}s\right\}\quad (0 < H_0 < 1). \tag{1.13}$$

Clearly, $\mathcal{E}(H_0) > 0$ for every $0 < H_0 < 1$. We now show that $\mathcal{E}(H_0) < \infty$ whenever $1 - 2H_0 < H < 1$. For similarity, we only consider the case $H_0 < 1/2$. By the local boundedness and homogeneity of $\Gamma(\cdot,\cdot)$, there exists a constant $C_0 > 0$ such that

$$\Gamma(x, x) \le C_0|x|^{2H} \qquad x \in \mathbb{R}^d. \tag{1.14}$$

In particular,

$$\Gamma(x(s)-x(r), x(s)-x(r)) \le C_0|x(s)-x(r)|^{2H} \quad \text{and} \quad \Gamma(x(s), x(s)) \le C_0|x(s)|^{2H}$$

for any $x \in \mathcal{H}_d$. Notice that

$$|x(s) - x(r)| = \left|\int_r^s \dot{x}(t)\mathrm{d}t\right| \le |s-r|^{1/2}\left(\int_0^1 |\dot{x}(s)|^2\mathrm{d}s\right)^{1/2}.$$

By the fact that $x(0) = 0$, we have

$$|x(s)| = \left|\int_0^s \dot{x}(t)\mathrm{d}t\right| \le |s|^{1/2}\left(\int_0^1 |\dot{x}(s)|^2\mathrm{d}s\right)^{1/2}.$$

Hence,

$$\begin{aligned}&\int_0^1\int_0^1\frac{\Gamma(x(s)-x(r),x(s)-x(r))}{|s-r|^{(2-2H_0)}}\mathrm{d}s\mathrm{d}r\\&\le C_0\Big(\int_0^1\int_0^1|s-r|^{-(2-H-2H_0)}\mathrm{d}s\mathrm{d}r\Big)\Big(\int_0^1|\dot{x}(s)|^2\mathrm{d}s\Big)^H\\&\le C\Big(\int_0^1|\dot{x}(s)|^2\mathrm{d}s\Big)^H,\end{aligned}$$

where the last step follows from the fact that $2-H-2H_0<1$. Similarly,

$$\int_0^1\Big\{s^{-(1-2H_0)}+(1-s)^{-(1-2H_0)}\Big\}\Gamma(x(s),x(s))\mathrm{d}s\le C\Big(\int_0^1|\dot{x}(s)|^2\mathrm{d}s\Big)^H.$$

Hence,

$$\mathcal{E}(H_0)\le\sup_{\lambda>0}\Big\{C\Big(\frac{|C_{H_0}|}{4}+\frac{H_0}{2}\Big)\lambda^H-\frac{1}{2}\lambda\Big\}<\infty$$

by the assumption that $H<1$.

Our concern is the asymptotic behavior of $\mathbb{E}u^m(t,x)$ when at least one of t and m goes to infinity. This limit pattern is described as $t\vee m\to\infty$.

Theorem 1.1 Assume that $0<H_0,H<1$ and $1-2H_0<H<1$. For every $x\in\mathbb{R}^d$, we have

$$\lim_{t\vee m\to\infty}t^{-\frac{2H_0+H}{1-H}}m^{-\frac{2-H}{1-H}}\log\mathbb{E}u^m(t,x)=\mathcal{E}(H_0). \tag{1.15}$$

The interesting models covered by Theorem 1.1 are the case when $W(t,x)$ is a fractional Brownian sheet with the Hurst parameter $(H_0,H_1,\cdots,H_d)$ and the case when $W(t,x)$ is a spatial radial fractional Brownian sheet with Hurst parameter (H_0,H).

When $W(t,x)$ is a fractional Brownian sheet,

$$\Gamma(x,y)=\prod_{j=1}^d R_{H_j}(x_j,y_j)\qquad x=(x_1,\cdots,x_d),\ y=(y_1,\cdots,y_d)\in\mathbb{R}^d, \tag{1.16}$$

where

$$R_{H_j}(x_j,y_j)=\frac{1}{2}\big\{|x_j|^{2H_j}+|y_j|^{2H_j}-|x_j-y_j|^{2H_j}\big\}\qquad j=1,\cdots,d.$$

One can verify assumption (1.2) with $H=H_1+\cdots+H_d$.

As for spatial radial fractional Brownian sheet,

$$\Gamma(x,y)=\frac{1}{2}\big\{|x|^{2H}+|y|^{2H}-|x-y|^{2H}\big\}\qquad x,y\in\mathbb{R}^d. \tag{1.17}$$

The variations appearing in Theorem 1.1 can be evaluated in some cases. Here, we report an interesting link to the historic article [7] by Strassen who prove that ((5), [7])

$$\begin{aligned}&\sup\Big\{\int_0^1|x(s)|^a\mathrm{d}s;\quad x\in\mathcal{H}_1\text{ and }\int_0^1|\dot{x}(s)|^2\mathrm{d}s=1\Big\}\\&=2(a+2)^{\frac{a-2}{2}}a^{-a/2}\Big(\int_0^1\frac{\mathrm{d}s}{\sqrt{1-s^a}}\Big)^{-a}=2(a+2)^{\frac{a-2}{2}}a^{a/2}B\Big(\frac{1}{2},\frac{1}{a}\Big)^{-a}\end{aligned} \tag{1.18}$$

for every $a\ge1$, where $B(\cdot,\cdot)$ is the beta function. We point out that the computation (by Lagrange multiplier) in [7] allows (1.18) for every $a>0$.

The identity leads to the evaluation of the variation $\mathcal{E}(H_0)$ in the case when $d=1$, $H_0=1/2$, and the the space covariance $\Gamma(\cdot,\cdot)$ is given by (1.16) or (1.17). More precisely,

$$\begin{aligned}\mathcal{E}(1/2) &= \sup_{x\in\mathcal{H}_1}\left\{\frac{1}{2}\int_0^1 |x(s)|^{2H}\mathrm{d}s-\frac{1}{2}\int_0^1 |\dot{x}(s)|^2\mathrm{d}s\right\}\\ &= 2^{\frac{1+H}{1-H}}(1-H^2)H^{\frac{2H}{1-H}}B\Big(\frac{1}{2},\frac{1}{2H}\Big)^{-\frac{2H}{1-H}}.\end{aligned} \tag{1.19}$$

Indeed, let

$$\Lambda=\sup\left\{\int_0^1 |x(s)|^{2H}\mathrm{d}s;\quad x\in\mathcal{H}_1 \text{ and } \int_0^1 |\dot{x}(s)|^2\mathrm{d}s=1\right\}.$$

One can easily see that

$$\begin{aligned}&\sup_{x\in\mathcal{H}_1}\left\{\frac{1}{2}\int_0^1 |x(s)|^{2H}\mathrm{d}s-\frac{1}{2}\int_0^1 |\dot{x}(s)|^2\mathrm{d}s\right\}\\ &=\frac{1}{2}\sup_{x\in\mathcal{H}_1}\left\{\Lambda\Big(\int_0^1 |\dot{x}(s)|^2\mathrm{d}s\Big)^H-\int_0^1 |\dot{x}(s)|^2\mathrm{d}s\right\}\\ &=\frac{1}{2}\sup_{\theta>0}\left\{\Lambda\theta^H-\theta\right\}=\frac{1}{2}(1-H)H^{\frac{H}{1-H}}\Lambda^{\frac{1}{1-H}}.\end{aligned}$$

Taking $a=2H$ in (1.18), then

$$\Lambda=2^{2H}(H+1)^{H-1}H^HB\Big(\frac{1}{2},\frac{1}{2H}\Big)^{-2H}.$$

So, we have (1.19).

Theorem 1.1 is proved in Section 2. Large deviation theory (Schilder theorem, more precisely) is essential to our approach. Unlike the setting of the time-space derivative noise

$$\frac{\partial^2 W}{\partial t\partial x}(t,x)$$

where ([2, 3]) the Brownian motion B_t appearing in the Feynman-Kac formula (1.8) plays a role as Markov process, the Brownian motion in this work plays role as Gaussian process.

2 Proof of Theorem 1.1

By (1.8), the solution $u(t,x)$ is monotonic in the initial state $u_0(x)$. By the bounded initial condition (1.4), therefore, we may assume $u_0(x)=1$.

Our proof relies on the Feynman-Kac representation (1.8), where time integral is defined by the approximation

$$\int_0^t W(\mathrm{d}s,B_{t-s})\overset{\text{def}}{=}\lim_{\epsilon\to 0^+}\int_0^t\frac{\partial W_\epsilon}{\partial s}(s,B_{t-s})\mathrm{d}s \quad \text{in } \mathcal{L}^2(\Omega,\mathcal{A},\mathbb{P}),$$

where $W_\epsilon(s,x)$ is a properly smoothed version of $W(s,x)$. As a side remark, we point that the existence of the limit and the existence of the solution are secured by the finite expectation of the Feynman-Kac representation (1.8). First notice that whenever the expectation exists, the time integral defined by the limit is Gaussian conditioning on the Brownian motion B_t. Taking expectation with respect to W in (1.8) and by Fubini theorem, we obtain

$$\mathbb{E}u(t,x)=\mathbb{E}_x\exp\left\{\frac{1}{2}\mathrm{Var}\left(\int_0^t W(\mathrm{d}s,B_{t-s})\Big|B\right)\right\},$$

where $\mathrm{Var}\,(\cdot|B)$ is the variance conditioning on the Brownian motion B_t. It is not hard to see that the Feynman-Kac representation (1.8) is well established and in $\mathcal{L}(\Omega, \mathcal{A}, \mathbb{P})$ if the conditional variance

$$\mathrm{Var}\left(\int_0^t W(\mathrm{d}s, B_{t-s})\Big|B\right) \tag{2.1}$$

is exponentially integrable. Such exponential integrability holds under the condition of Theorem (1.1). As a matter of fact, the proof given below largely depends on our investigation of the asymptotics for the exponential moment of the conditioning variance given in (2.1).

2.1 Asymptotics for $\mathbb{E}u^m(t,0)$

In this sub-section, we prove (1.15) in the case when $x = 0$, that is,

$$\lim_{t\vee m\to\infty} t^{-\frac{2H_0+H}{1-H}} m^{-\frac{2-H}{1-H}} \log \mathbb{E}u^m(t,0) = \mathcal{E}(H_0). \tag{2.2}$$

First, we point out that the conditioning variance in (2.1) can be explicitly written. When $H_0 > 1/2$, by the direct computation, we have

$$\begin{aligned}\mathrm{Var}\left(\int_0^t W(\mathrm{d}s, B_{t-s})\Big|B\right) &= \int_0^t\int_0^t \mathrm{Cov}\left(\frac{\partial}{\partial t}W(s, B_{t-s}), \frac{\partial}{\partial s}W(r, B_{t-r})\Big|B\right)\mathrm{d}s\mathrm{d}r\\ &= C_{H_0}\int_0^t\int_0^t |s-r|^{-(2-2H_0)}\Gamma(B_{t-s}, B_{t-r})\mathrm{d}s\mathrm{d}r\\ &= C_{H_0}\int_0^t\int_0^t |s-r|^{-(2-2H_0)}\Gamma(B_s, B_r)\mathrm{d}s\mathrm{d}r,\end{aligned} \tag{2.3}$$

where the second step follows from (1.5) with $\gamma_0(\cdot) = C_{H_0}|\cdot|^{-(2-2H_0)}$ in (1.6) and the last step follows from time-reversal substitution.

When $H_0 = 1/2$, a similar treatment leads to

$$\mathrm{Var}\left(\int_0^t W(\mathrm{d}s, B_{t-s})\Big|B\right) = \int_0^t \Gamma(B_s, B_s)\mathrm{d}s. \tag{2.4}$$

The case $H_0 < 1/2$ gets a little tricky as $\gamma_0(\cdot)$ is not pointwise defined. By approximation, Chen, Hu, Kalbasi, and Nualart (Theorem 2.2 and Remark 2.3, [1]) prove that

$$\begin{aligned}&\mathrm{Var}\left(\int_0^t W(\mathrm{d}s, B_{t-s})\Big|B\right)\\ &= H_0\int_0^t s^{-(1-2H_0)}\big\{\Gamma(B_s, B_s) + \Gamma(B_{t-s}, B_{t-s})\big\}\mathrm{d}s\\ &\quad + \frac{|C_{H_0}|}{2}\int_0^t\int_0^t |s-r|^{-(2-2H_0)}\big\{\Gamma(B_s, B_s) + \Gamma(B_r, B_r) - \Gamma(B_s, B_r) - \Gamma(B_r, B_s)\big\}\mathrm{d}s\mathrm{d}r\\ &= H_0\int_0^t s^{-(1-2H_0)}\big\{\Gamma(B_s, B_s) + \Gamma(B_{t-s}, B_{t-s})\big\}\mathrm{d}s\\ &\quad + \frac{|C_{H_0}|}{2}\int_0^t\int_0^t \frac{\Gamma(B_s - B_r, B_s - B_r)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r,\end{aligned} \tag{2.5}$$

where the second equality follows from the second equation in (1.2).

Set

$$t_m = m^{\frac{1}{2(1-H)}} t^{\frac{2H_0+H}{2(1-H)}}.$$

By the first equation in (1.2) and the Brownian scaling, all conditional variances in (2.3), (2.4), and (2.5) satisfy the identity in law:

$$\mathrm{Var}\left(\int_0^t W(\mathrm{d}s, B_{t-s})\Big|B\right) \overset{d}{=} m^{-1}t_m^2\,\mathrm{Var}\left(\int_0^1 W(\mathrm{d}s, t_m^{-1}B_{1-s})\Big|B\right). \tag{2.6}$$

By (1.8) (with $u_0(\cdot)=1$ and $x=0$) and (conditional) Gaussian property,

$$u(t,0) = \mathbb{E}_0 \exp\left\{m^{-1/2}t_m \int_0^1 W(\mathrm{d}s, t_m^{-1}B_{1-s})\right\}. \tag{2.7}$$

Let $B_t^1, \cdots, B_t^m$ be independent d-dimensional Brownian motions with $B_0^j = 0$ ($j = 1, \cdots, m$). From (2.7), we have

$$u^m(t,0) = \mathbb{E}_0 \exp\left\{m^{-1/2}t_m \sum_{j=1}^m \int_0^1 W(\mathrm{d}s, t_m^{-1}B_{1-s}^j)\right\}.$$

Here, we extend our notation "$\mathbb{E}_0$" naturally for the expectation with respect to the Brownian motions $B_t^1, \cdots, B_t^m$.

By the conditional Gaussian property,

$$\mathbb{E}u^m(t,0) = \mathbb{E}_0 \exp\left\{\frac{1}{2}m^{-1}t_m^2\,\mathrm{Var}\left(\sum_{j=1}^m \int_0^1 W(\mathrm{d}s, t_m^{-1}B_{1-s}^j)\Big|B\right)\right\} \tag{2.8}$$

where $\mathrm{Var}\,(\cdot|B)$ is the variance conditioning on $B_t^1, \cdots, B_t^m$. By Jensen's inequality,

$$m^{-1}\mathrm{Var}\left(\sum_{j=1}^m \int_0^1 W(\mathrm{d}s, t_m^{-1}B_{1-s}^j)\Big|B\right) \le \sum_{j=1}^m \mathrm{Var}\left(\int_0^1 W(\mathrm{d}s, t_m^{-1}B_{1-s}^j)\Big|B\right).$$

By independence, therefore,

$$\mathbb{E}u^m(t,0) \le \left(\mathbb{E}_0 \exp\left\{\frac{1}{2}t_m^2\,\mathrm{Var}\left(\int_0^1 W(\mathrm{d}s, t_m^{-1}B_{1-s})\Big|B\right)\right\}\right)^m. \tag{2.9}$$

Taking $t=1$ and replacing B by $t_m^{-1}B$ in (2.3), (2.4), and (2.5), respectively, we have

$$\begin{aligned}&\mathbb{E}_0 \exp\left\{\frac{1}{2}t_m^2\,\mathrm{Var}\left(\int_0^1 W(\mathrm{d}s, t_m^{-1}B_{1-s})\Big|B\right)\right\}\\ &= \mathbb{E}_0 \exp\left\{\frac{C_{H_0}}{2}t_m^2 \int_0^1\int_0^1 |s-r|^{-(2-2H_0)}\Gamma\big(t_m^{-1}B_s, t_m^{-1}B_r\big)\mathrm{d}s\mathrm{d}r\right\}\end{aligned} \tag{2.10}$$

for $H_0 > 1/2$,

$$\mathbb{E}_0 \exp\left\{\frac{1}{2}t_m^2\,\mathrm{Var}\left(\int_0^1 W(\mathrm{d}s, t_m^{-1}B_{1-s})\Big|B\right)\right\} = \mathbb{E}_0 \exp\left\{\frac{1}{2}t_m^2 \int_0^1 \Gamma\big(t_m^{-1}B_s, t_m^{-1}B_s\big)\mathrm{d}s\right\} \tag{2.11}$$

for $H_0 = 1/2$, and

$$\begin{aligned}&\mathbb{E}_0 \exp\left\{\frac{1}{2}t_m^2\,\mathrm{Var}\left(\int_0^1 W(\mathrm{d}s, t_m^{-1}B_{1-s})\Big|B\right)\right\}\\ &= \mathbb{E}_0 \exp\left\{\frac{H_0}{2}t_m^2 \int_0^1 s^{-(1-2H_0)}\big\{\Gamma(t_m^{-1}B_s, t_m^{-1}B_s) + \Gamma(t_m^{-1}B_{1-s}, t_m^{-1}B_{1-s})\big\}\mathrm{d}s\right.\\ &\quad\left. + \frac{|C_{H_0}|}{4}t_m^2 \int_0^1\int_0^1 \frac{\Gamma\big(t_m^{-1}(B_s-B_r), t_m^{-1}(B_s-B_r)\big)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r\right\}\end{aligned} \tag{2.12}$$

for $H_0 < 1/2$.

Recall the Schilder's large deviation (Theorem 5.2.3, p.153, [4]) for the Brownian motion $B = \{B_s;\ s \in [0,1]\}$ which is viewed as a Gaussian random variable taking values in $C_0\Big\{[0,1], \mathbb{R}^d\Big\}$. Let the rate function $I(x)$ on $C_0\Big\{[0,1], \mathbb{R}^d\Big\}$ be defined as

$$I(x) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 |\dot{x}(s)|^2 \mathrm{d}s, & x \in \mathcal{H}_d; \\ \infty, & \text{elsewhere.} \end{cases}$$

Schilder's large deviation states that

$$\limsup_{\epsilon\to 0^+} \epsilon^2 \log \mathbb{P}_0\{\epsilon B \in F\} \le -\inf_{x\in F} I(x) \qquad \text{for any close set } F \text{ in } C_0\Big\{[0,1], \mathbb{R}^d\Big\}$$

and

$$\liminf_{\epsilon\to 0^+} \epsilon^2 \log \mathbb{P}_0\{\epsilon B \in G\} \ge -\inf_{x\in G} I(x) \qquad \text{for any open set } G \text{ in } C_0\Big\{[0,1], \mathbb{R}^d\Big\}.$$

Consequently, by Varadhan's integral lemma (Theorem 4.3.1, p.137, [4]), we obtain

$$\lim_{\epsilon\to 0^+} \epsilon^2 \log \mathbb{E}_0 \exp\Big\{\epsilon^{-2}\Psi(\epsilon B)\Big\} = \sup_{x\in\mathcal{H}_d}\Big\{\Psi(x) - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 \mathrm{d}s\Big\} \tag{2.13}$$

for every continuous function $\Psi(x)$ on $C_0\Big\{[0,1], \mathbb{R}^d\Big\}$ satisfying

$$\limsup_{\epsilon\to 0^+} \epsilon^2 \log \mathbb{E}_0 \exp\Big\{\theta\epsilon^{-2}\Psi(\epsilon B)\Big\} < \infty \tag{2.14}$$

for some $\theta > 1$.

In connection to (2.8), the function

$$\Psi(x) = \frac{C_{H_0}}{2}\int_0^1\int_0^1 |s-r|^{-(2-2H_0)}\Gamma\big(x(s), x(r)\big)\mathrm{d}s\mathrm{d}r \qquad x \in C_0\Big\{[0,1], \mathbb{R}^d\Big\}$$

is clearly continuous on $C_0\Big\{[0,1], \mathbb{R}^d\Big\}$ under the uniform topology. By (1.14), we have

$$\int_0^1\int_0^1 |s-r|^{-(2-2H_0)}\Gamma\big(t_m^{-1}B_s, t_m^{-1}B_r\big)\mathrm{d}s\mathrm{d}r \le Ct_m^{-2H}\max_{0\le s\le 1}|B_s|^{2H}$$

therefore, Gaussian tail

$$\lim_{t\vee m\to\infty} t_m^{-2}\log \mathbb{E}\exp\Big\{\frac{1}{2}\theta t_m^2\int_0^1\int_0^1 |s-r|^{-(2-2H_0)}\Gamma\big(t_m^{-1}B_s, t_m^{-1}B_r\big)\mathrm{d}s\mathrm{d}r\Big\} < \infty$$

for every $\theta > 1$.

Applying (2.13) to (2.10), then by (2.9), we have

$$\limsup_{t\vee m\to\infty} m^{-1}t_m^{-2}\log \mathbb{E}u^m(t,0) \le \mathcal{E}(H_0), \tag{2.15}$$

which is the desired upper bound for (2.2) in the case $H_0 > 1/2$.

Similarly, playing (2.13) with

$$\Psi(x) = \frac{1}{2}\int_0^1 \Gamma\big(x(s), x(s)\big)\mathrm{d}s \quad \text{and} \qquad x \in C_0\Big\{[0,1], \mathbb{R}^d\Big\}$$

proves the upper bound (2.15) in the setting of $H_0 = 1/2$.

Some additional care is needed when it comes to the case $H_0 < 1/2$ as the second part of the function

$$\Psi(x) = \frac{H_0}{2}\int_0^1 s^{-(1-2H_0)}\big\{\Gamma\big(x(s),x(s)\big)+\Gamma\big(x(1-s),x(1-s)\big)\big\}\mathrm{d}s + \frac{|C_{H_0}|}{4}\int_0^1\int_0^1 \frac{\Gamma\big(x(s)-x(r),x(s)-x(r)\big)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r$$

is not continuous on $C_0\Big\{[0,1],\mathbb{R}^d\Big\}$ as $H_0 < 1/2$.

Given a small number $0<\delta<1$, set

$$D_\delta = \{(s,r)\in[0,1]^2;\ |s-r|\le\delta\} \quad\text{and}\quad \hat{D}_\delta = [0,1]^2\setminus D_\delta,$$

$$\Psi_1(x) = \frac{H_0}{2}\int_0^1 s^{-(1-2H_0)}\big\{\Gamma\big(x(s),x(s)\big)+\Gamma\big(x(1-s),x(1-s)\big)\big\}\mathrm{d}s + \frac{|C_{H_0}|}{4}\iint_{\hat{D}_\delta}\frac{\Gamma\big(x(s)-x(r),x(s)-x(r)\big)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r,$$

$$\Psi_2(x) = \frac{|C_{H_0}|}{4}\iint_{D_\delta}\frac{\Gamma\big(x(s)-x(r),x(s)-x(r)\big)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r.$$

Let $p,q>1$ be two conjugate numbers with p being close to 1. By Hölder's inequality,

$$\mathbb{E}_0\exp\Big\{t_m^2\Psi(t_m^{-1}B)\Big\} \le \bigg(\mathbb{E}_0\exp\Big\{t_m^2 p\Psi_1(t_m^{-1}B)\Big\}\bigg)^{1/p}\bigg(\mathbb{E}_0\exp\Big\{t_m^2 q\Psi_2(t_m^{-1}B)\Big\}\bigg)^{1/q}. \tag{2.16}$$

Applying (2.13) to the continuous function $p\Psi_1(\cdot)$ gives

$$\begin{aligned}&\lim_{t\vee m\to\infty} t_m^{-2}\log\mathbb{E}_0\exp\Big\{t_m^2 p\Psi_1(t_m^{-1}B)\Big\}\\
&=\sup_{x\in\mathcal{H}_d}\bigg\{p\Psi_1(x)-\frac12\int_0^1|\dot{x}(s)|^2\mathrm{d}s\bigg\} \le \sup_{x\in\mathcal{H}_d}\bigg\{p\Psi(x)-\frac12\int_0^1|\dot{x}(s)|^2\mathrm{d}s\bigg\}\\
&= p^{\frac{1}{1-H}}\sup_{x\in\mathcal{H}_d}\bigg\{\Psi(x)-\frac12\int_0^1|\dot{x}(s)|^2\mathrm{d}s\bigg\} = p^{\frac{1}{1-H}}\mathcal{E}(H_0),\end{aligned}\tag{2.17}$$

where the third step follows from the space homogeneity assumed in (1.2).

By the assumption $1-2H_0<H$, one can fix β with $\frac{1-2H_0}{2H}<\beta<1/2$. By (1.14),

$$\begin{aligned}\Psi_2(t_m^{-1}B) &\le C_0 t_m^{-2H}\iint_{D_\delta}\frac{|B_s-B_r|^{2H}}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r\\
&\le C_0 t_m^{-2H}\bigg(\sup_{u,v\in[0,1]}\frac{|B_u-B_v|}{|u-v|^\beta}\bigg)^{2H}\iint_{D_\delta}|s-r|^{-(2-2H_0-2H\beta)}\mathrm{d}s\mathrm{d}r\\
&= C_0\frac{2}{2H_0+2H\beta-1}\delta^{2H_0+2H\beta-1}t_m^{-2H}\bigg(\sup_{\substack{u\neq v\\u,v\in[0,1]}}\frac{|B_u-B_v|}{|u-v|^\beta}\bigg)^{2H}\\
&= C\delta^\alpha t_m^{-2H}\bigg(\sup_{\substack{u\neq v\\u,v\in[0,1]}}\frac{|B_u-B_v|}{|u-v|^\beta}\bigg)^{2H}.\end{aligned}$$

We remark that $\alpha = 2H_0+2H\beta-1>0$ and the constant C is independent of $\delta>0$, and recall the well-known fact that the Brownian motion is β-Hölder continuous and the random variable

$$\sup_{\substack{u\neq v\\u,v\in[0,1]}}\frac{|B_u-B_v|}{|u-v|^\beta}$$

has a Gaussian tail. With the bound we derive and with the fact that $H < 1$, therefore,

$$\limsup_{t\vee m\to\infty} t_m^{-2}\log \mathbb{E}_0 \exp\Big\{t_m^2 q\Psi_2(t_m^{-1}B)\Big\} \le C_q \delta^{\frac{2\alpha}{2-2H}} \tag{2.18}$$

for a constant $C_q > 0$ independent of δ.

Together, (2.16), (2.17) and (2.18) imply that

$$\limsup_{t\vee m\to\infty} t_m^{-2}\log \mathbb{E}_0 \exp\Big\{t_m^2 \Psi(t_m^{-1}B)\Big\} \le p^{\frac{H}{1-H}}\mathcal{E}(H_0) + q^{-1}C_q\delta^{\frac{2\alpha}{2-2H}}.$$

On the right hand side, let $\delta \to 0^+$ and then $p \to 1^+$. By (2.12),

$$\limsup_{t\vee m\to\infty} t_m^{-2}\log \mathbb{E}_0 \exp\bigg\{\frac{1}{2}t_m^2 \mathrm{Var}\bigg(\int_0^1 W(\mathrm{d}s, t_m^{-1}B_{1-s})\Big|B\bigg)\bigg\} \le \mathcal{E}(H_0).$$

Therefore, the desired upper bound (2.15) follows from (2.9) in the setting of $H_0 < 1/2$.

To complete the proof of (2.2), it remains to prove its lower bound

$$\liminf_{t\vee m\to\infty} m^{-1}t_m^{-2}\log \mathbb{E}u^m(t,0) \ge \mathcal{E}(H_0). \tag{2.19}$$

For any $x \in C_0\Big\{[0,1];\mathbb{R}^d\Big\}$, define the W-measurable random variable

$$\eta(x) = \int_0^1 W\big(\mathrm{d}s, x(1-s)\big)$$

whenever the stochastic integral makes sense. Let $y \in \mathcal{H}_d$ be fixed but arbitrary. In connection to (2.8), we have

$$\begin{aligned}\mathrm{Var}\Big(\frac{1}{m}\sum_{j=1}^m \eta(t_m^{-1}B^j)\Big|B\Big) &\ge -\mathrm{Var}\big(\eta(y)\big) + 2\mathrm{Cov}\Big(\eta(y), \frac{1}{m}\sum_{j=1}^m \eta(t_m^{-1}B^j)\Big|B\Big)\\ &= -\mathrm{Var}\big(\eta(y)\big) + \frac{2}{m}\sum_{j=1}^m \mathrm{Cov}\big(\eta(y), \eta(t_m^{-1}B^j)\big|B\big).\end{aligned}$$

By (2.8) and by independence,

$$\mathbb{E}u^m(t,0) \ge \exp\Big\{-\frac{1}{2}mt_m^2\mathrm{Var}\big(\eta(y)\big)\Big\}\bigg(\mathbb{E}_0\exp\Big\{t_m^2\mathrm{Cov}\big(\eta(y),\eta(t_m^{-1}B)\big|B\big)\Big\}\bigg)^m. \tag{2.20}$$

By a computation similar to the one in (2.3), we have

$$\mathrm{Cov}\big(\eta(y),\eta(t_m^{-1}B)\big|B\big) = C_{H_0}\int_0^1\int_0^1 |s-r|^{-(2-2H_0)}\Gamma\big(y(s), t_m^{-1}B_r\big)\mathrm{d}s\mathrm{d}r \ (H_0 > 1/2) \tag{2.21}$$

and

$$\mathrm{Cov}\big(\eta(y),\eta(t_m^{-1}B)\big|B\big) = \int_0^1 \Gamma\big(y(s), t_m^{-1}B_s\big)\mathrm{d}s \qquad (H_0 = 1/2). \tag{2.22}$$

When $H_0 < 1/2$, (by Theorem 2.2 and Remark 2.3 in [1])

$$\begin{aligned}\mathrm{Cov}\big(\eta(y),\eta(t_m^{-1}B)\big|B\big) &= H_0\int_0^1 s^{-(1-2H_0)}\big\{\Gamma(y(s),t_m^{-1}B_s) + \Gamma(y(1-s),t_m^{-1}B_{1-s})\big\}\mathrm{d}s\\ &\quad + \frac{|C_{H_0}|}{2}\int_0^1\int_0^1 \frac{\Gamma\big(y(s)-y(r), t_m^{-1}(B_s-B_r)\big)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r.\end{aligned} \tag{2.23}$$

We now claim that in all three cases, we have

$$\liminf_{t\vee m\to\infty} t_m^{-2}\log \mathbb{E}_0\exp\Big\{t_m^2\mathrm{Cov}\big(\eta(y),\eta(t_m^{-1}B)\big|B\big)\Big\}$$

$$\geq \sup_{x\in\mathcal{H}_d}\left\{\mathrm{Cov}\big(\eta(y),\eta(x)\big)-\frac{1}{2}\int_0^1|\dot{x}(s)|^2\mathrm{d}s\right\}. \tag{2.24}$$

Given the covariance representations (2.21) and (2.22), the verification of (2.24), with the liminf and the inequality being strengthened into limit and equality, respectively, appears to be a straightforward application of the Schilder's large deviation and (2.13) in the setting $H_0\geq 1/2$. We now prove (2.24) in the case $H_0<1/2$. Consider the function

$$\psi(x)=H_0\int_0^1 s^{-(1-2H_0)}\big\{\Gamma\big(y(s),x(s)\big)+\Gamma\big(y(1-s),x(s)\big)\big\}\mathrm{d}s + \frac{|C_{H_0}|}{2}\int_0^1\int_0^1\frac{\Gamma\big(y(s)-y(r),x(s)-x(r)\big)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r.$$

Under our set-up, $\psi(x)$ is not continuous on $C_0\Big\{[0,1];\mathbb{R}^d\Big\}$. In the following, we approximate $\psi(x)$ by continuous function. For any $0<\delta<1$, recall that

$$D_\delta=\{(s,r)\in[0,1]^2;\ |s-r|\leq\delta\}\quad\text{and}\quad \hat{D}_\delta=\{(s,r)\in[0,1]^2;\ |s-r|>\delta\}.$$

Define

$$\psi_1(x)=H_0\int_0^1 s^{-(1-2H_0)}\big\{\Gamma\big(y(s),x(s)\big)+\Gamma\big(y(1-s),x(s)\big)\big\}\mathrm{d}s + \frac{|C_{H_0}|}{2}\iint_{\hat{D}_\delta}\frac{\Gamma\big(y(s)-y(r),x(s)-x(r)\big)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r,$$

$$\psi_2(x)=\frac{|C_{H_0}|}{2}\iint_{D_\delta}\frac{\Gamma\big(y(s)-y(r),x(s)-x(r)\big)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r.$$

Letting $\epsilon>0$ be small but fixed, we have

$$\mathbb{E}_0\exp\Big\{t_m^2\psi(t_m^{-1}B)\Big\}\geq e^{-\epsilon t_m^2}\mathbb{E}_0\Big[\exp\Big\{t_m^2\psi_1(t_m^{-1}B)\Big\};\ |\psi_2(t_m^{-1}B)|\leq\epsilon\Big]$$
$$=e^{-\epsilon t_m^2}\bigg\{\mathbb{E}_0\exp\Big\{t_m^2\psi_1(t_m^{-1}B)\Big\}-\mathbb{E}_0\Big[\exp\Big\{t_m^2\psi_1(t_m^{-1}B)\Big\};\ |\psi_2(t_m^{-1}B)|>\epsilon\Big]\bigg\}.$$

By Cauchy-Schwarz inequality,

$$\mathbb{E}_0\Big[\exp\Big\{t_m^2\psi_1(t_m^{-1}B)\Big\};\ |\psi_2(t_m^{-1}B)|>\epsilon\Big]$$
$$\leq\bigg(\mathbb{E}_0\exp\Big\{2t_m^2\psi_1(t_m^{-1}B)\Big\}\bigg)^{1/2}\Big(\mathbb{P}_0\{|\psi_2(t_m^{-1}B)|>\epsilon\}\Big)^{1/2}.$$

So, we have

$$\mathbb{E}_0\exp\Big\{t_m^2\psi(t_m^{-1}B)\Big\}+e^{-\epsilon t_m^2}\Big(\mathbb{P}_0\{|\psi_2(t_m^{-1}B)|>\epsilon\}\Big)^{1/2}\bigg(\mathbb{E}_0\exp\Big\{2t_m^2\psi_1(t_m^{-1}B)\Big\}\bigg)^{1/2}$$
$$\geq e^{-\epsilon t_m^2}\mathbb{E}_0\exp\Big\{t_m^2\psi_1(t_m^{-1}B)\Big\}.$$

Applying the Schilder's large deviation and (2.13) to the continuous function $\psi_1(x)$ and $2\psi_1(x)$, respectively, we obtain

$$\lim_{t\vee m\to\infty}t_m^{-2}\log\mathbb{E}_0\exp\Big\{t_m^2\psi_1(t_m^{-1}B)\Big\}=\sup_{x\in\mathcal{H}_d}\left\{\psi_1(x)-\frac{1}{2}\int_0^1|\dot{x}(s)|^2\mathrm{d}s\right\}$$

and

$$\lim_{t\vee m\to\infty} t_m^{-2}\log\mathbb{E}_0\exp\Big\{2t_m^2\psi_1(t_m^{-1}B)\Big\}=\sup_{x\in\mathcal{H}_d}\Big\{2\psi_1(x)-\frac{1}{2}\int_0^1|\dot{x}(s)|^2\mathrm{d}s\Big\}.$$

Therefore,

$$\begin{aligned}&\max\bigg\{\liminf_{t\vee m\to\infty} t_m^{-2}\log\mathbb{E}_0\exp\Big\{t_m^2\psi(t_m^{-1}B)\Big\}\\&+\frac{1}{2}\bigg[\sup_{x\in\mathcal{H}_d}\Big\{2\psi_1(x)-\frac{1}{2}\int_0^1|\dot{x}(s)|^2\mathrm{d}s\Big\}+\limsup_{t\vee m\to\infty}t_m^{-2}\log\mathbb{P}_0\{|\psi_2(t_m^{-1}B)|>\epsilon\}\bigg]\bigg\}\\ \geq&\ -\epsilon+\sup_{x\in\mathcal{H}_d}\Big\{\psi_1(x)-\frac{1}{2}\int_0^1|\dot{x}(s)|^2\mathrm{d}s\Big\}.\end{aligned}\tag{2.25}$$

By (1.14), on the other hand, we have

$$\begin{aligned}&\big|\Gamma\big(y(s)-y(r),t_m^{-1}(B_s-B_r)\big)\big|\\ \leq&\Big\{\Gamma\big(y(s)-y(r),y(s)-y(r)\big)\Big\}^{1/2}\Big\{\Gamma\big(t_m^{-1}(B_s-B_r),t_m^{-1}(B_s-B_r)\big)\Big\}^{1/2}\\ \leq&\ C_0t_m^{-H}|y(s)-y(r)|^H|B_s-B_r|^H\leq Ct_m^{-H}|s-r|^{H/2}|B_s-B_r|^H,\qquad s,r\in[0,1],\end{aligned}$$

where the constant $C>0$ is independent of s,r, and the last step follows from the fact that "$y\in\mathcal{H}_d$" implies the (1/2)-Hölder continuity of $y(s)$. Hence,

$$\begin{aligned}|\psi_2(t_m^{-1}B)|&\leq C\frac{|C_{H_0}|}{2}t_m^{-H}\iint_{D_\delta}|s-r|^{-(2-2H_0-2^{-1}H)}|B_s-B_r|^H\mathrm{d}s\mathrm{d}r\\&\leq C\frac{|C_{H_0}|}{2}t_m^{-H}\bigg(\sup_{\substack{u\neq v\\u,v\in[0,1]}}\frac{|B_u-B_v|}{|u-v|^\alpha}\bigg)^H\iint_{D_\delta}|s-r|^{-(2-2H_0-(2^{-1}+\alpha)H)}\mathrm{d}s\mathrm{d}r\\&=C\frac{|C_{H_0}|}{2}\frac{2}{2H_0+(2^{-1}+\alpha)H-1}\delta^{2H_0+(2^{-1}+\alpha)H-1}t_m^{-H}\bigg(\sup_{\substack{u\neq v\\u,v\in[0,1]}}\frac{|B_u-B_v|}{|u-v|^\alpha}\bigg)^H.\end{aligned}$$

Here, the constant α satisfies

$$\frac{1-2H_0-2^{-1}H}{H}<\alpha<\frac{1}{2},\quad\text{therefore}\quad\theta\equiv2H_0+(2^{-1}+\alpha)H-1>0.$$

By the fact that the random variable

$$\sup_{\substack{u\neq v\\u,v\in[0,1]}}\frac{|B_u-B_v|}{|u-v|^\alpha}$$

has a Gaussian tail, we obtain

$$\limsup_{t\vee m\to\infty}t_m^{-2}\log\mathbb{P}_0\{|\psi_2(t_m^{-1}B)|>\epsilon\}\leq-\lambda\delta^{-2\theta/H}\epsilon^{2/H}$$

for a constant $\lambda>0$ independent of ϵ,δ. In view of (2.25),

$$\begin{aligned}&\max\bigg\{\liminf_{t\vee m\to\infty}t_m^{-2}\log\mathbb{E}_0\exp\Big\{t_m^2\psi(t_m^{-1}B)\Big\}\\&+\frac{1}{2}\bigg[\sup_{x\in\mathcal{H}_d}\Big\{2\psi_1(x)-\frac{1}{2}\int_0^1|\dot{x}(s)|^2\mathrm{d}s\Big\}-\lambda\delta^{-2\theta/H}\epsilon^{2/H}\bigg]\bigg\}\end{aligned}$$

$$\geq -\epsilon + \sup_{x\in\mathcal{H}_d}\left\{\psi_1(x) - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 \mathrm{d}s\right\}.$$

We now let $\delta \to 0^+$ in the above inequality. Noticing that

$$\lim_{\delta\to 0^+}\sup_{x\in\mathcal{H}_d}\left\{2\psi_1(x) - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 \mathrm{d}s\right\} = \sup_{x\in\mathcal{H}_d}\left\{2\psi(x) - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 \mathrm{d}s\right\}$$

and

$$\lim_{\delta\to 0^+}\sup_{x\in\mathcal{H}_d}\left\{\psi_1(x) - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 \mathrm{d}s\right\} = \sup_{x\in\mathcal{H}_d}\left\{\psi(x) - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 \mathrm{d}s\right\},$$

we obtain

$$\liminf_{t\vee m\to\infty} t_m^{-2}\log\mathbb{E}_0\exp\left\{t_m^2\psi(t_m^{-1}B)\right\} \geq -\epsilon + \sup_{x\in\mathcal{H}_d}\left\{\psi(x) - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 \mathrm{d}s\right\}.$$

Notice that $\psi(x) = \mathrm{Cov}\big(\eta(y), \eta(x)\big)$. Letting $\epsilon \to 0^+$ leads to (2.24).

Picking $x = y$ in the variation on the right hand side of (2.24),

$$\liminf_{t\vee m\to\infty} t_m^{-2}\log\mathbb{E}_0\exp\left\{t_m^2\psi(t_m^{-1}B)\right\} \geq \mathrm{Var}\big(\eta(y)\big) - \frac{1}{2}\int_0^1 |\dot{y}(s)|^2 \mathrm{d}s.$$

Bringing this to (2.20), we have

$$\liminf_{t\vee m\to\infty} m^{-1}t_m^{-2}\log\mathbb{E}u^m(t,0) \geq \frac{1}{2}\mathrm{Var}\big(\eta(y)\big) - \frac{1}{2}\int_0^1 |\dot{y}(s)|^2 \mathrm{d}s.$$

Because $y \in \mathcal{H}_d$ can be arbitrary, taking supremum over y leads to

$$\liminf_{t\vee m\to\infty} m^{-1}t_m^{-2}\log\mathbb{E}u^m(t,0) \geq \sup_{x\in\mathcal{H}_d}\left\{\frac{1}{2}\mathrm{Var}\big(\eta(x)\big) - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 \mathrm{d}s\right\}.$$

Finally, the desired lower bound follows from the variance representation

$$\mathrm{Var}\big(\eta(x)\big) = C_{H_0}\int_0^1\int_0^1 |s-r|^{-(2-2H_0)}\Gamma\big(x(s), x(r)\big)\mathrm{d}s\mathrm{d}r \qquad (H_0 > 1/2), \tag{2.26}$$

$$\mathrm{Var}\big(\eta(x)\big) = \int_0^1 \Gamma\big(x(s), x(s)\big)\mathrm{d}s \qquad (H_0 = 1/2), \tag{2.27}$$

and (in [1])

$$\begin{aligned}\mathrm{Var}\big(\eta(x)\big) = {} & H_0\int_0^1 s^{-(1-2H_0)}\big\{\Gamma(x(s), x(s)) + \Gamma(x(1-s), x(1-s))\big\}\mathrm{d}s \\ & + \frac{|C_{H_0}|}{2}\int_0^1\int_0^1 \frac{\Gamma\big(x(s)-x(r), x(s)-x(r)\big)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r \qquad (H_0 < 1/2). \end{aligned}\tag{2.28}$$

□

2.2 Asymptotics for $\mathbb{E}u^m(t,x)$

In this section, we establish (1.15) for any $x \in \mathbb{R}^d$. Let $x \in \mathbb{R}^d$ be fixed and keep in mind that in the Feynman-Kac formula (1.18), the notation "$\mathbb{E}_x$" stands for the expectation with respect to the Brownian motion B_t starting at x. Hence, (1.8) can be rewritten as (for $u_0(x) = 1$)

$$u(t,x) = \mathbb{E}_0\exp\left\{\int_0^t W(\mathrm{d}s, x + B_{t-s})\right\}.$$

Let $p, q > 1$ be a conjugate pair. By the Hölder's inequality,

$$u(t,x) \le \left(\mathbb{E}_0 \exp\left\{p\int_0^t W(\mathrm{d}s, B_{t-s})\right\}\right)^{1/p}$$
$$\times \left(\mathbb{E}_0 \exp\left\{q\Big(\int_0^t W(\mathrm{d}s, x+B_{t-s}) - \int_0^t W(\mathrm{d}s, B_{t-s})\Big)\right\}\right)^{1/q}.$$

By the Hölder's inequality again, we have

$$\mathbb{E}u(t,x)^m \le \left\{\mathbb{E}\left(\mathbb{E}_0 \exp\left\{p\int_0^t W(\mathrm{d}s, B_{t-s})\right\}\right)^m\right\}^{1/p}$$
$$\times \left\{\mathbb{E}\left(\mathbb{E}_0 \exp\left\{q\Big(\int_0^t W(\mathrm{d}s, x+B_{t-s}) - \int_0^t W(\mathrm{d}s, B_{t-s})\Big)\right\}\right)^m\right\}^{1/q}. \quad (2.29)$$

Given $\theta > 0$, write $u_\theta(t,x)$ for the solution to (1.1) with the constant 1 as its initial value and with $W(t,x)$ being replaced by $\theta W(t,x)$. Clearly, the corespondent space covariance is $\theta^2\Gamma(\cdot,\cdot)$. In view of (2.8), (2.29) can be rewritten as

$$\mathbb{E}u(t,x)^m \le \Big(\mathbb{E}u_p^m(t,0)\Big)^{1/p}$$
$$\times \left\{\mathbb{E}\left(\mathbb{E}_0 \exp\left\{q\Big(\int_0^t W(\mathrm{d}s, x+B_{t-s}) - \int_0^t W(\mathrm{d}s, B_{t-s})\Big)\right\}\right)^m\right\}^{1/q}.$$

Similar to (2.8), we have

$$\mathbb{E}\left(\mathbb{E}_0 \exp\left\{q\Big(\int_0^t W(\mathrm{d}s, x+B_{t-s}) - \int_0^t W(\mathrm{d}s, B_{t-s})\Big)\right\}\right)^m$$
$$= \mathbb{E}_0 \exp\left\{\frac{q^2}{2}\mathrm{Var}\left(\sum_{j=1}^m \Big(\int_0^t W(\mathrm{d}s, x+B^j_{t-s}) - \int_0^t W(\mathrm{d}s, B^j_{t-s})\Big)\Big| B\right)\right\}.$$

By Jensen's inequality, we obtain

$$\mathrm{Var}\left(\sum_{j=1}^m \Big(\int_0^t W(\mathrm{d}s, x+B^j_{t-s}) - \int_0^t W(\mathrm{d}s, B^j_{t-s})\Big)\Big| B\right)$$
$$\le m\sum_{j=1}^m \mathrm{Var}\left(\int_0^t W(\mathrm{d}s, x+B^j_{t-s}) - \int_0^t W(\mathrm{d}s, B^j_{t-s})\Big| B\right).$$

Considering independence among the Brownian motions,

$$\mathbb{E}_0 \exp\left\{\frac{q^2}{2}\mathrm{Var}\left(\sum_{j=1}^m \Big(\int_0^t W(\mathrm{d}s, x+B^j_{t-s}) - \int_0^t W(\mathrm{d}s, B^j_{t-s})\Big)\Big| B\right)\right\}$$
$$\le \left(\mathbb{E}_0 \exp\left\{\frac{q^2}{2}m\mathrm{Var}\left(\int_0^t W(\mathrm{d}s, x+B_{t-s}) - \int_0^t W(\mathrm{d}s, B_{t-s})\Big| B\right)\right\}\right)^m.$$

Summarizing our computation, we conclude that

$$\mathbb{E}u(t,x)^m \le \Big(\mathbb{E}u_p^m(t,0)\Big)^{1/p}$$
$$\times \left(\mathbb{E}_0 \exp\left\{\frac{q^2}{2}m\mathrm{Var}\left(\int_0^t W(\mathrm{d}s, x+B_{t-s}) - \int_0^t W(\mathrm{d}s, B_{t-s})\Big| B\right)\right\}\right)^{m/q}. \quad (2.30)$$

An obvious modification of the above procedure also leads to

$$\begin{aligned}&\mathbb{E}u_{1/p}(t,0)^m \le \Big(\mathbb{E}u^m(t,x)\Big)^{1/p}\\&\times\left(\mathbb{E}_0\exp\left\{\frac{q^2}{2p}m\mathrm{Var}\left(\int_0^t W(\mathrm{d}s,x+B_{t-s})-\int_0^t W(\mathrm{d}s,B_{t-s})\Big|B\right)\right\}\right)^{m/q}\end{aligned}$$

or,

$$\begin{aligned}&\mathbb{E}u(t,x)^m \ge \Big(\mathbb{E}u^m_{1/p}(t,0)\Big)^{p}\\&\times\left(\mathbb{E}_0\exp\left\{\frac{q^2}{2p}m\mathrm{Var}\left(\int_0^t W(\mathrm{d}s,x+B_{t-s})-\int_0^t W(\mathrm{d}s,B_{t-s})\Big|B\right)\right\}\right)^{-\frac{pm}{q}}. \qquad (2.31)\end{aligned}$$

We claim that

$$\mathrm{Var}\left(\int_0^t W(\mathrm{d}s,x+B_{t-s})-\int_0^t W(\mathrm{d}s,B_{t-s})\Big|B\right)=\Gamma(x,x)t^{2H_0}. \qquad (2.32)$$

Because of similarity, we only consider the case $H_0<1/2$. Write

$$\begin{aligned}&\mathrm{Var}\left(\int_0^t W(\mathrm{d}s,x+B_{t-s})-\int_0^t W(\mathrm{d}s,B_{t-s})\Big|B\right)\\&=\mathrm{Var}\left(\int_0^t W(\mathrm{d}s,x+B_{t-s})\Big|B\right)+\mathrm{Var}\left(\int_0^t W(\mathrm{d}s,B_{t-s})\Big|B\right)\\&\quad-2\mathrm{Cov}\left(\int_0^t W(\mathrm{d}s,x+B_{t-s}),\ \int_0^t W(\mathrm{d}s,B_{t-s})\Big|B\right).\end{aligned}$$

By Theorem 2.2 in [1],

$$\begin{aligned}\mathrm{Var}\left(\int_0^t W(\mathrm{d}s,B_{t-s})\Big|B\right)&=H_0\int_0^t s^{-(1-2H_0)}\big\{\Gamma(B_s,B_s)+\Gamma(B_{t-s},B_{t-s})\big\}\mathrm{d}s\\&\quad+\frac{|C_{H_0}|}{2}\int_0^t\int_0^t\frac{\Gamma(B_s-B_r,B_s-B_r)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r,\end{aligned}$$

$$\begin{aligned}&\mathrm{Var}\left(\int_0^t W(\mathrm{d}s,x+B_{t-s})\Big|B\right)\\&=H_0\int_0^t s^{-(1-2H_0)}\big\{\Gamma(x+B_s,x+B_s)+\Gamma(x+B_{t-s},x+B_{t-s})\big\}\mathrm{d}s\\&\quad+\frac{|C_{H_0}|}{2}\int_0^t\int_0^t\frac{\Gamma(B_s-B_r,B_s-B_r)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r,\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Cov}\left(\int_0^t W(\mathrm{d}s,x+B_{t-s}),\ \int_0^t W(\mathrm{d}s,B_{t-s})\Big|B\right)\\&=H_0\int_0^t s^{-(1-2H_0)}\big\{\Gamma(x+B_s,B_s)+\Gamma(x+B_{t-s},B_{t-s})\big\}\mathrm{d}s\\&\quad+\frac{|C_{H_0}|}{2}\int_0^t\int_0^t\frac{\Gamma(B_s-B_r,B_s-B_r)}{|s-r|^{2-2H_0}}\mathrm{d}s\mathrm{d}r.\end{aligned}$$

Hence, (2.32) is established as

$$\mathrm{Var}\left(\int_0^t W(\mathrm{d}s,x+B_{t-s})-\int_0^t W(\mathrm{d}s,B_{t-s})\Big|B\right)$$

$$\begin{aligned}&= H_0\int_0^t s^{-(1-2H_0)}\Big\{\big(\Gamma(B_s,B_s)+\Gamma(x+B_s,x+B_s)-2\Gamma(x+B_s,B_s)\big)\\&\quad+\big(\Gamma(B_{t-s},B_{t-s})+\Gamma(x+B_{t-s},x+B_{t-s})-2\Gamma(x+B_{t-s},B_{t-s})\big)\Big\}\mathrm{d}s\\&=2H_0\Gamma(x,x)\int_0^t s^{-(1-2H_0)}\mathrm{d}s=\Gamma(x,x)t^{2H_0},\end{aligned}$$

where the second equality follows from assumption (1.2).

Together, (2.30), (2.31) and (2.32) imply that

$$\begin{aligned}&\exp\Big\{-\frac{q}{2p}\Gamma(x,x)m^2t^{2H_0}\Big\}\Big(\mathbb{E}u^m_{1/p}(t,0)\Big)^p\le \mathbb{E}u^m(t,x)\\&\le \exp\Big\{\frac{q}{2}\Gamma(x,x)m^2t^{2H_0}\Big\}\Big\{\mathbb{E}u^m_p(t,0)\Big\}^{1/p}.\end{aligned}\tag{2.33}$$

Replacing $u(t,0)$ by $u_{1/p}(t,0)$ and $u_p(t,0)$ in (2.2), respectively, we have

$$\lim_{t\vee m\to\infty} t^{-\frac{2H_0+H}{1-H}}m^{-\frac{2-H}{1-H}}\log\mathbb{E}u^m_{1/p}(t,0)=\mathcal{E}_{1/p}(H_0)$$

and

$$\lim_{t\vee m\to\infty} t^{-\frac{2H_0+H}{1-H}}m^{-\frac{2-H}{1-H}}\log\mathbb{E}u^m_{p}(t,0)=\mathcal{E}_{p}(H_0),$$

where $\mathcal{E}_p(H_0)$ and $\mathcal{E}_{1/p}(H_0)$ are the variations given in (1.10) ($H_0>1/2$), (1.11) ($H_0=1/2$), and (1.12) ($H_0<1/2$). By the space homogeneity given in the first identity in (1.2), we have

$$\mathcal{E}_p(H_0)=p^{\frac{2}{1-H}}\mathcal{E}(H_0)\quad\text{and}\quad \mathcal{E}_{1/p}(H_0)=p^{-\frac{2}{1-H}}\mathcal{E}(H_0).$$

By (2.33), therefore,

$$\begin{aligned}p^{-\frac{1+H}{1-H}}\mathcal{E}(H_0)&\le\liminf_{t\vee m\to\infty}t^{-\frac{2H_0+H}{1-H}}m^{-\frac{2-H}{1-H}}\log\mathbb{E}u^m(t,x)\\&\le\limsup_{t\vee m\to\infty}t^{-\frac{2H_0+H}{1-H}}m^{-\frac{2-H}{1-H}}\log\mathbb{E}u^m(t,x)\le p^{\frac{1+H}{1-H}}\mathcal{E}(H_0).\end{aligned}$$

Letting $p\to1^+$ in the both ends, we complete the proof of Theorem 1.1. □

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