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A FOUR-WEIGHT WEAK TYPE MAXIMAL INEQUALITY FOR MARTINGALES*

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Abstract In this article, some necessary and sufficient conditions are shown in order that weighted inequality of the form

$$\int_{\{f^*>\lambda\}} \Phi_1(\lambda w_1)w_2 \mathrm{d}\mathbb{P} \le C \int_{\Omega} \Phi_2(C \mid f_\infty \mid w_3)w_4 \mathrm{d}\mathbb{P}$$

holds a.e. for uniformly integrable martingales $f=(f_n)_{n\ge 0}$ with some constant $C>0$, where Φ_1, Φ_2 are Young functions, w_i $(i=1,2,3,4)$ are weights, $f^*=\sup\limits_{n\ge 0} \mid f_n \mid$ and $f_\infty = \lim\limits_{n\to\infty} f_n$ a.e. As an application, two-weight weak type maximal inequalities of martingales are considered, and particularly a new equivalence condition is presented.

Key words weight; weak type inequality; martingale maximal operator; Young function

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1 Introduction

Muckenhoupt [1] proved that two-weight weak type of (p,p) inequality holds for Hardy-Littlewood maximal function if and only if the couple of weights satisfies the A_p condition$(1<p<\infty)$. From then on, much attention was attracted to weightd theory for Hardy-Littlewood maximal function, and a series of important results were obtained successively, see for example in [2–8]. In martingale setting, the countpart to Hardy-Littlewood maximal function is Doob maximal operator, and it is well-known that there are a lot of similarities between them. Along with that of Hardy-Littlewood maximal function, the research on the weighted theory of martingale maximal operator was carried out. The earlier work on this aspect can be found in [16], the more recent work one can see in [9–15].

The purpose of this article is to consider weighted inequality of the form

$$\int_{\{f^*>\lambda\}} \Phi_1(\lambda w_1)w_2 \mathrm{d}\mathbb{P} \le C \int_{\Omega} \Phi_2(C \mid f_\infty \mid w_3)w_4 \mathrm{d}\mathbb{P},$$

where Φ_1, Φ_2 are Young functions, w_i $(i=1,2,3,4)$ are weights, $f^*=\sup\limits_{n\ge 0} \mid f_n \mid$ is martingale maximal operator. The weighted inequality mentioned above for Hardy-Littlewood maximal

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function was studied in [7], where $\Phi_1 = \Phi_2$ are quasi-convex functions. In this article, we make the best of properties of conditional expectation to consider the weighted inequality, which is different from [7]. Some necessary and sufficient conditions for the inequality to hold are obtained, and our main theorem generalizes some known results. As an application, two-weight weak type maximal inequalities of martingales are considered, and particularly a new equivalence condition is presented.

The organization of this article is divided into two further sections. Some basic knowledge, which we will use, is collected in the next section. A four-weight weak type maximal inequality for martingales and its application to the case of two-weight are considered in Section 3.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\{\mathcal{F}_n\}_{n\geq 0}$ a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. The expectation operator is denoted by $\mathbb{E}$. A weight is a measurable function that is positive and finite almost everywhere. Let u be a weight, we denote by $\mathbb{P}_u$ the weighted measure $ud\mathbb{P}$, and by $\mathbb{E}_u$ the expectation relative to $\mathbb{P}_u$.

A Young function is a convex function given by $\Phi(t) = \int_0^t \varphi(s)\mathrm{d}s$, where φ is a nonnegative, nondecreasing and right-continuous function on $(0, \infty)$. We call Φ an N-function if φ satisfies the following three conditions:

(i) $\varphi(0) = \lim\limits_{s\to 0+} \varphi(s) = 0$;

(ii) $0 < s < \infty \Leftrightarrow 0 < \varphi(s) < \infty$;

(iii) $\lim\limits_{s\to\infty} \varphi(s) = \infty$.

Every N-function is strictly increasing and thus it has the inverse function. Notice that for an N-function Φ: $\Phi(at) \leq a\Phi(t)$ when $0 \leq a \leq 1$, and $a\Phi(t) \leq \Phi(at)$ when $a \geq 1$, $t \in (0, \infty)$. The right-continuous inverse function of φ is given by

$$\psi(t) = \inf\{s \in (0, \infty) : \varphi(s) \geq t\}, \quad t \in (0, \infty).$$

The Young function given by

$$\Psi(t) = \int_0^t \psi(s)\mathrm{d}s, \quad t \in (0, \infty)$$

is called the complementary function of Φ. Note that Ψ is an N-function if and only if so is Φ. Let us recall the Young inequality $st \leq \Phi(s) + \Psi(t)$.

Let (Φ, Ψ) be a pair of complementary N-functions, denote

$$R_\Phi(t) = \frac{\Phi(t)}{t}, S_\Phi(t) = \frac{\Psi(t)}{t}, t > 0, R_\Phi(0) = S_\Phi(0) = 0.$$

If Φ is a Young function, then R_Φ and S_Φ are continuous and increasing functions which map $[0, \infty)$ into itself and satisfy

$$\Phi(S_\Phi(t)) \leq \Psi(t), \Psi(R_\Phi(t)) \leq \Phi(t) \tag{2.1}$$

for every $t \geq 0$ (see [8]).

Let $f = (f_n)_{n\geq 0}$ be a martingale relative to $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_n)_{n\geq 0})$, denote its maximal function by $f^* = \sup\limits_{n\geq 0} \mid f_n \mid$. We denote by $\mathcal{M}$ the collection of all uniformly integrable martingales with respect to $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_n)_{n\geq 0})$, and by $\mathcal{T}$ the collection of all stopping times with respect to $(\mathcal{F}_n)_{n\geq 0}$. For more information about martingale theory see [16, 17].

Throughout this article, we denote the set of non-negative integers and the set of integers by $\mathbb{N}$ and $\mathbf{Z}$, respectively. We use C and C_1 to denote constants and may denote different constants at different occurrences.

3 Main Results and Proofs

In this section, we devote to study a four-weight weak type maximal inequality for martingales. Some necessary and sufficient conditions for it are shown. First, we give a useful lemma.

Lemma 3.1 Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{F}$, (Φ_1, Ψ_1) and (Φ_2, Ψ_2) are two pairs of complementary N-functions, w_i $(i = 1, 2, 3, 4)$ are weights. Then the following statements are equivalent.

(i) There is a constant $C > 0$ such that

$$\mathbb{E}(\Phi_1(\mathbb{E}(x \mid \mathcal{A})w_1)w_2 \mid \mathcal{A}) \leq C\mathbb{E}(\Phi_2(Cxw_3)w_4 \mid \mathcal{A}) \tag{3.1}$$

holds a.e. for any positive random variable x.

(ii) There are constants $\varepsilon_1 > 0$ and $C_1 > 0$ such that

$$\mathbb{E}(\Psi_2(\varepsilon_1 \frac{\mathbb{E}(\Phi_1(\lambda w_1)w_2 \mid \mathcal{A})}{\lambda w_3 w_4})w_4 \mid \mathcal{A}) \leq C_1\mathbb{E}(\Phi_1(\lambda w_1)w_2 \mid \mathcal{A}) \tag{3.2}$$

holds a.e. for any positive $\mathcal{A}$-measurable random variable λ.

(iii) There are constants $\varepsilon_2 > 0$ and $C_2 > 0$ such that

$$\mathbb{E}(\Phi_1(\varepsilon_2\mathbb{E}(\Psi_2(\frac{1}{\lambda w_3 w_4})\lambda w_4 \mid \mathcal{A})w_1)w_2 \mid \mathcal{A}) \leq C_2\mathbb{E}(\Psi_2(\frac{1}{\lambda w_3 w_4})w_4 \mid \mathcal{A}) \tag{3.3}$$

holds a.e. for any positive $\mathcal{A}$-measurable random variable λ.

Proof We complete the proof by showing that (i)$\Leftrightarrow$(ii) and (i)$\Leftrightarrow$(iii).

(i)$\Rightarrow$(ii) Assume that (3.1) holds a.e.. Choose k sufficient large such that $\mathbb{P}(B) > 0$, where $B = \{x : \frac{1}{k} \leq w_3(x)w_4(x), w_1(x) \leq k, w_2(x) \leq k, w_4(x) \leq k\}$, and let

$$g = (\frac{\mathbb{E}(\Phi_1(\lambda w_1)w_2 \mid \mathcal{A})}{\lambda w_3 w_4})^{-1}\Psi_2(\varepsilon_1 \frac{\mathbb{E}(\Phi_1(\lambda w_1)w_2 \mid \mathcal{A})}{\lambda w_3 w_4})\chi(B)$$

with ε_1 will be specified later. Then by (3.1), we have

$$\begin{aligned}
&\mathbb{E}(\Psi_2(\varepsilon_1 \frac{\mathbb{E}(\Phi_1(\lambda w_1)w_2 \mid \mathcal{A})}{\lambda w_3 w_4})w_4\chi(B) \mid \mathcal{A}) \\
&= \mathbb{E}(\frac{g}{\lambda w_3} \mid \mathcal{A})\mathbb{E}(\Phi_1(\lambda w_1)w_2 \mid \mathcal{A}) \\
&= \mathbb{E}(\frac{g}{\lambda w_3} \mid \mathcal{A})\mathbb{E}(\Phi_1(\lambda w_1)w_2 \mid \mathcal{A})\chi(\{\mathbb{E}(\frac{g}{w_3} \mid \mathcal{A}) \leq \lambda\}) \\
&\quad +\mathbb{E}(\frac{g}{\lambda w_3} \mid \mathcal{A})\mathbb{E}(\Phi_1(\lambda w_1)w_2 \mid \mathcal{A})\chi(\{\mathbb{E}(\frac{g}{w_3} \mid \mathcal{A}) > \lambda\}) \\
&\leq \mathbb{E}(\Phi_1(\lambda w_1)w_2 \mid \mathcal{A}) + \mathbb{E}(\Phi_1(\mathbb{E}(\frac{g}{w_3} \mid \mathcal{A})w_1)w_2 \mid \mathcal{A}) \\
&\leq \mathbb{E}(\Phi_1(\lambda w_1)w_2 \mid \mathcal{A}) + C\mathbb{E}(\Phi_2(Cg)w_4 \mid \mathcal{A}).
\end{aligned}$$

Now let ε_1 small enough such that $\varepsilon_1 C<1$ and $\varepsilon_1 C^2<1$, then by (2.1), we obtain

$$\begin{aligned}&\mathbb{E}(\Psi_2(\varepsilon_1\frac{\mathbb{E}(\Phi_1(\lambda w_1)w_2\mid\mathcal{A})}{\lambda w_3w_4})w_4\chi(B)\mid\mathcal{A})\\ \le\ &\mathbb{E}(\Phi_1(\lambda w_1)w_2\mid\mathcal{A})+C\mathbb{E}(\Phi_2(Cg)w_4\mid\mathcal{A})\\ \le\ &\mathbb{E}(\Phi_1(\lambda w_1)w_2\mid\mathcal{A})+\varepsilon_1C^2\mathbb{E}(\Psi_2(\varepsilon_1\frac{\mathbb{E}(\Phi_1(\lambda w_1)w_2\mid\mathcal{A})}{\lambda w_3w_4})w_4\chi(B)\mid\mathcal{A}).\end{aligned}$$

Note that $\mathbb{E}(\Psi_2(\varepsilon_1\frac{\mathbb{E}(\Phi_1(\lambda w_1)w_2|\mathcal{A})}{\lambda w_3w_4})w_4\chi(B)\mid\mathcal{A})<\infty$, so we have

$$\mathbb{E}(\Psi_2(\varepsilon_1\frac{\mathbb{E}(\Phi_1(\lambda w_1)w_2\mid\mathcal{A})}{\lambda w_3w_4})w_4\chi(B)\mid\mathcal{A})\le\frac{1}{1-\varepsilon_1C^2}\mathbb{E}(\Phi_1(\lambda w_1)w_2\mid\mathcal{A}).$$

Now let $k\to+\infty$, we obtian (3.2).

(ii)⇒(i) We can directly assume that $\mathbb{E}(\Phi_1(\mathbb{E}(x\mid\mathcal{A})\mid w_1)w_2\mid\mathcal{A})<\infty$ a.e.. Otherwise, we can similarly take a set B as above. Then by Young inequality and (2.1) we have

$$\begin{aligned}&\mathbb{E}(\Phi_1(\mathbb{E}(x\mid\mathcal{A})\mid w_1)w_2\mid\mathcal{A})\\ =\ &\frac{1}{\varepsilon_1C}\mathbb{E}(\varepsilon_1\frac{\mathbb{E}(\Phi_1(\mathbb{E}(x\mid\mathcal{A})\mid w_1)w_2\mid\mathcal{A})}{\mathbb{E}(x\mid\mathcal{A})w_3w_4}\cdot Cxw_3w_4\mid\mathcal{A})\\ \le\ &\frac{1}{\varepsilon_1C}(\mathbb{E}(\Psi_2(\varepsilon_1\frac{\mathbb{E}(\Phi_1(\mathbb{E}(x\mid\mathcal{A}w_1)w_2\mid\mathcal{A})}{\mathbb{E}(x\mid\mathcal{A}w_3w_4})w_4\mid\mathcal{A})+\mathbb{E}(\Phi_2(Cxw_3)w_4\mid\mathcal{A})\\ \le\ &\frac{1}{\varepsilon_1C}(\mathbb{E}(\Phi_1(\mathbb{E}(x\mid\mathcal{A})\mid w_1)w_2\mid\mathcal{A})+\mathbb{E}(\Phi_2(Cxw_3)w_4\mid\mathcal{A}).\end{aligned}$$

Choose C large enough such that $\frac{1}{\varepsilon_1C}<1$, we then obtain (3.1).

(i)⇒(iii) Set $x=\varepsilon_2\Psi_2(\frac{1}{\lambda w_3w_4})\lambda w_4$, choose ε_2 such that $\varepsilon_2C<1$, then by (2.1) and (3.1) we obtain (3.3).

(iii)⇒(i) By Young inequality, we have

$$\begin{aligned}\mathbb{E}(x\mid\mathcal{A})&=\frac{1}{C}\mathbb{E}(Cxw_3\cdot\frac{1}{\lambda w_3w_4}\cdot\lambda w_4\mid\mathcal{A})\\ &\le\frac{\lambda}{C}\mathbb{E}(\Phi_2(Cxw_3)w_4\mid\mathcal{A})+\frac{1}{C}\mathbb{E}(\Psi_2(\frac{1}{\lambda w_3w_4})\lambda w_4\mid\mathcal{A}).\end{aligned}$$

Set $\lambda=\frac{C\mathbb{E}(x|\mathcal{A})}{2\mathbb{E}(\Phi_2(Cxw_3)w_4|\mathcal{A})}$, then

$$\mathbb{E}(x\mid\mathcal{A})\le\frac{2}{C}\mathbb{E}(\Psi_2(\frac{1}{\lambda w_3w_4})\lambda w_4\mid\mathcal{A}).$$

Since $\frac{\Phi(t)}{t}$ is increasing, choose C large enough such that $\frac{2}{C\varepsilon_2}<1$, we have

$$\frac{\Phi_1(\mathbb{E}(x\mid\mathcal{A})w_1)}{\mathbb{E}(x\mid\mathcal{A})}\le\frac{\Phi_1(\varepsilon_2\mathbb{E}(\Psi_2(\frac{1}{\lambda w_3w_4})\lambda w_4\mid\mathcal{A})w_1)}{\varepsilon_2\mathbb{E}(\Psi_2(\frac{1}{\lambda w_3w_4})\lambda w_4\mid\mathcal{A})},$$

that is

$$\Phi_1(\mathbb{E}(x\mid\mathcal{A})w_1)\le\Phi_1(\varepsilon_2\mathbb{E}(\Psi_2(\frac{1}{\lambda w_3w_4})\lambda w_4\mid\mathcal{A})w_1)\cdot\frac{\mathbb{E}(x\mid\mathcal{A})}{\varepsilon_2\mathbb{E}(\Psi_2(\frac{1}{\lambda w_3w_4})\lambda w_4\mid\mathcal{A})}.$$

It follows from (3.3) and the definition of λ that

$$\mathbb{E}(\Phi_1(\mathbb{E}(x\mid\mathcal{A})w_1)w_2\mid\mathcal{A})\le\frac{2C_2}{C\varepsilon_2}\mathbb{E}(\Phi_2(Cxw_3)w_4\mid\mathcal{A}).$$

The proof is completed. □

Remark 3.2 If x is a random variable that is not equal to zero a.e., then each of the inequality (3.2) and (3.3) is equivalent with the following inequality

$$\mathbb{E}(\Phi_1(\mid \mathbb{E}(x \mid \mathcal{A}) \mid w_1)w_2 \mid \mathcal{A}) \leq C\mathbb{E}(\Phi_2(C \mid x \mid w_3)w_4 \mid \mathcal{A})$$

In fact, we only use the inequality $\mid \mathbb{E}(x \mid \mathcal{A}) \mid \leq \mathbb{E}(\mid x \mid\mid \mathcal{A})$ in the proof of Lemma 3.1.

Now let us state our main result below.

Theorem 3.3 Let (Φ_1, Ψ_1), (Φ_2, Ψ_2) be two pairs of complementary N-functions and $w_i (i = 1, 2, 3, 4)$ be weights. Then the following statements are equivalent:

(i) There is a constant $C_1 > 0$, independent of $f = (f_n)_{n\geq 0} \in \mathcal{M}$, such that

$$\int_{\{f^*>\lambda\}} \Phi_1(\lambda w_1)w_2 \mathrm{d}\mathbb{P} \leq C_1 \int_\Omega \Phi_2(C_1 \mid f_\infty \mid w_3)w_4 \mathrm{d}\mathbb{P} \quad \text{a.e..}$$

(ii) There is a constant $C_2 > 0$, independent of $f = (f_n)_{n\geq 0} \in \mathcal{M}$ and $n \in \mathbb{N}$, such that

$$\mathbb{E}(\Phi_1(\mid f_n \mid w_1)w_2 \mid \mathcal{F}_n) \leq C_2\mathbb{E}(\Phi_2(C_2 \mid f_\infty \mid w_3)w_4 \mid \mathcal{F}_n) \quad \text{a.e..}$$

(iii) There is a constant $C_3 > 0$, independent of $f = (f_n)_{n\geq 0} \in \mathcal{M}$ and $\tau \in \mathcal{T}$, such that

$$\mathbb{E}(\Phi_1(\mid f_\tau \mid w_1)w_2 \mid \mathcal{F}_\tau) \leq C_3\mathbb{E}(\Phi_2(C_3 \mid f_\infty \mid w_3)w_4 \mid \mathcal{F}_\tau) \quad \text{a.e..}$$

(iv) There are constants $\varepsilon > 0$ and $C_4 > 0$, such that

$$\mathbb{E}(\Psi_2(\varepsilon \frac{\mathbb{E}(\Phi_1(\lambda w_1)w_2 \mid \mathcal{F}_n)}{\lambda w_3 w_4})w_4 \mid \mathcal{F}_n) \leq C_4\mathbb{E}(\Phi_1(\lambda w_1)w_2 \mid \mathcal{F}_n), \quad \forall n \in \mathbb{N}$$

holds a.e. for any positive $\mathcal{F}_n$-measurable random variable λ.

(v) There are constants $\varepsilon_1 > 0$ and $C_5 > 0$ such that

$$\mathbb{E}(\Phi_1(\varepsilon_1\mathbb{E}(\Psi_2(\frac{1}{\lambda w_3 w_4})\lambda w_4 \mid \mathcal{F}_n)w_1)w_2 \mid \mathcal{F}_n) \leq C_5\mathbb{E}(\Psi_2(\frac{1}{\lambda w_3 w_4})w_4 \mid \mathcal{F}_n), \quad \forall n \in \mathbb{N}$$

holds a.e. for any positive $\mathcal{F}_n$-measurable random variable λ.

Proof By the corollary of Lemma 3.1, we obtain (ii)⇔(iv) and (ii)⇔(v). Now we complete the proof by showing that (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii) Suppose that (i) holds, and let $(f_n)_{n\geq 0} \in \mathcal{M}$. For any $A \in \mathcal{F}_n$ and $\lambda \in (0, \infty)$, we have

$$\begin{aligned}\mathbb{E}[\Phi_1(\lambda w_1)w_2\chi(\{\mid f_n \mid > \lambda\} \bigcap A)] &\leq \mathbb{E}[\Phi_1(\lambda w_1)w_2\chi(\sup_n \mathbb{E}(\mid f_\infty \mid \chi(A) \mid \mathcal{F}_n) > \lambda)] \\ &\leq C_1\mathbb{E}[\Phi_2(C_1 \mid f_\infty \mid \chi(A)w_3)w_4],\end{aligned}$$

where $\chi(A)$ denotes the characteristic function of set A. Hence

$$\mathbb{E}(\Phi_1(\lambda w_1)w_2\chi(\{\mid f_n \mid > \lambda\}) \mid \mathcal{F}_n) \leq C_1\mathbb{E}(\Phi_2(C_1 \mid f_\infty \mid w_3)w_4 \mid \mathcal{F}_n) \quad \text{a.e..}$$

For all $k \in \mathbf{Z}$, set $B_k = \{2^k <\mid f_n \mid\leq 2^{k+1}\} \subseteq \{2^k <\mid f_n \mid\}$, then for any $B \in \mathcal{F}_n$,

$$\begin{aligned}\int_B \mathbb{E}(\Phi_1(\mid f_n \mid w_1)w_2 \mid \mathcal{F}_n)\mathrm{d}\mathbb{P} &= \sum_{k\in\mathbf{Z}} \int_{B\bigcap B_k} \mathbb{E}(\Phi_1(\mid f_n \mid w_1)w_2 \mid \mathcal{F}_n)\mathrm{d}\mathbb{P} \\ &\leq \sum_{k\in\mathbf{Z}} \int_{B\bigcap B_k} \mathbb{E}(\Phi_1(2^{k+1}w_1)w_2\chi(\{2 \mid f_n \mid > 2^{k+1}\}) \mid \mathcal{F}_n)\mathrm{d}\mathbb{P} \\ &\leq C_1 \sum_{k\in\mathbf{Z}} \int_{B\bigcap B_k} \mathbb{E}(\Phi_2(2C_1 \mid f_\infty \mid w_3)w_4 \mid \mathcal{F}_n)\mathrm{d}\mathbb{P}\end{aligned}$$

$$= C_1 \int_B \mathbb{E}(\Phi_2(2C_1 \mid f_\infty \mid w_3) w_4 \mid \mathcal{F}_n) \mathrm{d}\mathbb{P},$$

from which we obtain

$$\mathbb{E}(\Phi_1(\mid f_n \mid w_1) w_2 \mid \mathcal{F}_n) \le C_2 \mathbb{E}(\Phi_2(C_2 \mid f_\infty \mid w_3) w_4 \mid \mathcal{F}_n) \quad \text{a.e..}$$

(ii)⇒(iii) is obvious.

(iii)⇒(i) Let $\lambda \in (0,\infty)$, and define

$$\tau = \inf\{n \in \mathbb{N} : \mid f_n \mid > \lambda\} \in \mathcal{T}, \quad \inf \emptyset = \infty.$$

Then $\{\tau < \infty\} = \{f^* > \lambda\}$ and $\mid f_\tau \mid > \lambda$ on $\{\tau < \infty\}$. Using (iii) we obtain

$$\begin{aligned} \int_{\{f^*>\lambda\}} \Phi_1(\lambda w_1) w_2 \mathrm{d}\mathbb{P} &\le \mathbb{E}(\Phi_1(\mid f_\tau \mid w_1) w_2 \chi(\tau < \infty)) \\ &= \mathbb{E}(\mathbb{E}(\Phi_1(\mid f_\tau \mid w_1) w_2 \mid \mathcal{F}_\tau) \chi(\tau < \infty)) \\ &\le C_3 \mathbb{E}(\Phi_2(C_3 \mid f_\infty \mid w_3) w_4 \chi(\tau < \infty)) \\ &\le C_3 \int_\Omega \Phi_2(C_3 \mid f_\infty \mid w_3) w_4 \mathrm{d}\mathbb{P}. \end{aligned}$$

The proof is completed. □

According to Theorem 3.3, we can easily show the following two corollaries.

Corollary 3.4 Let (Φ_1, Ψ_1), (Φ_2, Ψ_2) be two pairs of complementary N-functions and (u, v) a pair of weights. Then the following statements are equivalent:

(i) There is a constant $C_1 > 0$, independent of $f = (f_n)_{n\ge 0} \in \mathcal{M}$, such that

$$\int_{\{f^*>\lambda\}} \Phi_1(\frac{\lambda}{u}) u \mathrm{d}\mathbb{P} \le C_1 \int_\Omega \Phi_2(C_1 \frac{\mid f_\infty \mid}{v}) v \mathrm{d}\mathbb{P} \quad \text{a.e..}$$

(ii) There are positive constants ε and K, such that

$$\lambda S_{\Phi_2}(\mathbb{E}(\varepsilon R_{\Phi_1}(\frac{1}{\lambda u}) \mid \mathcal{F}_n)) \mathbb{E}(v \mid \mathcal{F}_n) \le K$$

holds a.e. for any positive $\mathcal{F}_n$-measurable random variable λ.

Corollary 3.5 Let (Φ_1, Ψ_1), (Φ_2, Ψ_2) be two pairs of complementary N-functions and (u, v) a pair of weights. Then the following statements are equivalent:

(i) There is a constant $C > 0$, independent of $f = (f_n)_{n\ge 0} \in \mathcal{M}$, such that

$$\int_{\{f^*>\lambda\}} \Psi_2(\frac{\lambda}{v}) v \mathrm{d}\mathbb{P} \le C \int_\Omega \Psi_1(C \frac{\mid f_\infty \mid}{u}) u \mathrm{d}\mathbb{P} \quad \text{a.e..}$$

(ii) There is a constant $C > 0$, independent of $f = (f_n)_{n\ge 0} \in \mathcal{M}$, such that

$$\Phi_1(\lambda) \mathbb{P}_u(f^* > \lambda) \le C \mathbb{E}_\nu(\Phi_2(C \mid f_\infty \mid)) \quad \text{a.e..}$$

(iii) There is a constant $C > 0$, independent of $f = (f_n)_{n\ge 0} \in \mathcal{M}$ and $n \in \mathbb{N}$, such that

$$\Phi_1(\mid f_n \mid) \mathbb{E}(u \mid \mathcal{F}_n) \le C \mathbb{E}(\Phi_2(C \mid f_\infty \mid) v \mid \mathcal{F}_n) \quad \text{a.e..}$$

(iv) There is a constant $C > 0$, independent of $f = (f_n)_{n\ge 0} \in \mathcal{M}$ and $\tau \in \mathcal{T}$, such that

$$\Phi_1(\mid f_\tau \mid) \mathbb{E}(u \mid \mathcal{F}_\tau) \le C \mathbb{E}(\Phi_2(C \mid f_\infty \mid) v \mid \mathcal{F}_\tau) \quad \text{a.e..}$$

(v) There are positive constants ε and K, such that

$$\lambda R_{\Phi_1}(\mathbb{E}(\varepsilon S_{\Phi_2}(\frac{1}{\lambda v}) \mid \mathcal{F}_n)) \mathbb{E}(u \mid \mathcal{F}_n) \le K$$

holds a.e. for any positive $\mathcal{F}_n$-measurable random variable λ.

(vi) There are constants $\varepsilon > 0$ and $C > 0$ such that

$$\mathbb{E}(\Psi_2(\varepsilon\frac{\Phi_1(\lambda)\mathbb{E}(u \mid \mathcal{F}_n)}{\lambda v})v \mid \mathcal{F}_n) \le C\Phi_1(\lambda)\mathbb{E}(u \mid \mathcal{F}_n), \quad \forall n \in \mathbb{N}$$

holds a.e. for any positive $\mathcal{F}_n$-measurable random variable λ.

Remark 3.6 Corollary 3.5 gives some necessary and sufficient conditions for two-weight weak type maximal inequality. It was studied in [10] and [11], respectively. Here we use Theorem 3.3 to recondisider it, and a new equivalent condition ((i) in Corollary 3.5) is presented.

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