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TIME-PERIODIC ISENTROPIC SUPERSONIC EULER FLOWS IN ONE-DIMENSIONAL DUCTS DRIVING BY PERIODIC BOUNDARY CONDITIONS*

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Abstract We show existence of time-periodic supersonic solutions in a finite interval, after certain start-up time depending on the length of the interval, to the one space-dimensional isentropic compressible Euler equations, subjected to periodic boundary conditions. Both classical solutions and weak entropy solutions, as well as high-frequency limiting behavior are considered. The proofs depend on the theory of Cauchy problems of genuinely nonlinear hyperbolic systems of conservation laws.

Key words supersonic flow; isentropic; compressible Euler equations; duct; time-periodic solution; initial-boundary-value problem

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1 Introduction and Main Results

We study the one-space-dimensional isentropic compressible Euler equations (cf. [1, p. 31] or [2, p. 63])

$$\begin{cases}\rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0,\end{cases} \tag{1.1}$$

here $t > 0$ represents time, and for a given constant $L > 0$, the space variable x lies in the interval $[0, L]$, which represents a one-dimensional rectilinear duct with constant cross-section and occupied by gas. The unknowns are ρ, p and u, which are respectively the density of mass, scalar pressure, and velocity of the gas flow. We consider isentropic polytropic gas, namely,

$$p = a\rho^\gamma \tag{1.2}$$

with $a > 0$ and $\gamma > 1$ being constants. The local sonic speed is given by

$$c = \sqrt{a\gamma}\rho^{\frac{\gamma-1}{2}}. \tag{1.3}$$

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In this paper, we focus on supersonic flows in the duct; that is, $u(t,x) > c(t,x)$ for $t > 0$ and $x \in [0,L]$. Suppose the gas flow is uniform initially

$$\rho(0,x) = \underline{\rho}, \quad u(0,x) = \underline{u}, \tag{1.4}$$

where $\underline{\rho} > 0, \underline{u} > 0$ are constants and $\underline{u} > \underline{c} = \sqrt{a\gamma}\underline{\rho}^{\frac{\gamma-1}{2}}$. To control the flow, we propose the boundary conditions

$$\rho(t,0) = \rho_l(t), \quad u(t,0) = u_l(t), \qquad t \geq 0 \tag{1.5}$$

at the inflow boundary $\{x = 0\}$. The problem is, if ρ_l, u_l are periodic functions (i.e., there is a constant $P > 0$ so that $\rho_l(t+P) = \rho_l(t)$, $u_l(t+P) = u_l(t)$ for all $t > 0$) and $u_l > c_l = \sqrt{a\gamma}\rho_l^{\frac{\gamma-1}{2}}$, can we obtain a periodic solution of the initial-boundary-value problem (1.1)(1.4)(1.5)?

We will prove the following three theorems.

Theorem 1.1 (Classical solutions) Suppose that $\rho_l(t)$ and $u_l(t)$ are periodic functions with a period $P > 0$ and $\rho_l(0) = \underline{\rho}, u_l(0) = \underline{u}$. There are positive constants ε_0, T_0 and C_0 such that if

$$\|\rho_l - \underline{\rho}\|_{C^1([0,P])} + \|u_l - \underline{u}\|_{C^1([0,P])} \leq \varepsilon \leq \varepsilon_0, \tag{1.6}$$

then there is a unique Lipschitz continuous solution (ρ, u) to problem (1.1)(1.4)(1.5), which is C^1 for $t > T_0$, and satisfies that

$$\rho(t+P,x) = \rho(t,x), \quad u(t+P,x) = u(t,x), \quad \forall\, t > T_0, \;\; x \in [0,L], \tag{1.7}$$

$$\|\rho - \underline{\rho}\|_{C^1([T,T+P]\times[0,L])} + \|u - \underline{u}\|_{C^1([T,T+P]\times[0,L])} \leq C_0\varepsilon, \quad \forall T > T_0. \tag{1.8}$$

A weak solution to problem (1.1)(1.4)(1.5) is defined in the standard way via integration by parts after multiplying a test function in $C_0^\infty(\mathbb{R}^2)$ whose support does not meet the line $\{x = L\}$, cf. [2, Section 4.3, p. 82] or [3, Definition 2.1, p. 7]. A weak solution satisfying the following well-known Lax E-condition [2, (8.3.7) in p. 274] is called a weak entropy solution

$$\lambda_i(U_+) < s < \lambda_i(U_-), \quad i = 1, 2, \tag{1.9}$$

here $\lambda_1 = u - c$, $\lambda_2 = u + c$ are characteristics of system (1.1); s is the speed of a shock-front, and U_- (U_+) is the state of the gas on the left-hand (right-hand) side of the shock-front (here as usual the x-axis is placed horizontally; its positive direction is from the left-hand side to the right-hand side. The positive direction of t-axis is upward). From Theorem 8.5.3 in [2, p. 283], we know that for our situation, such weak entropy solutions also satisfy the entropy admissability condition based on entropy-entropy flux pair (see [2, p. 85]).

Theorem 1.2 (Weak solutions) Suppose that $\rho_l(t)$ and $u_l(t)$ are periodic functions with a period $P > 0$. There are positive constants ε_0, T_0 and C_0, such that if

$$\|\rho_l - \underline{\rho}\|_{\mathrm{BV}([0,P])} + \|u_l - \underline{u}\|_{\mathrm{BV}([0,P])} \leq \varepsilon \leq \varepsilon_0, \tag{1.10}$$

where $\|\cdot\|_{\mathrm{BV}(A)}$ is the bounded variation norm of a single-variable function on a set A, then there is a unique weak entropy solution (ρ, u) to problem (1.1)(1.4)(1.5), which satisfies that

$$\rho(t+P,x) = \rho(t,x), \quad u(t+P,x) = u(t,x) \qquad \text{for a.e. } \; t > T_0, \;\; x \in [0,L] \tag{1.11}$$

and

$$\mathrm{TV}_{[T,T+P]}(\rho(\cdot,x) - \underline{\rho}) + \mathrm{TV}_{[T,T+P]}(u(\cdot,x) - \underline{u}) \leq C_0\varepsilon, \;\; \forall T > T_0, x \in [0,L], \tag{1.12}$$

$$\left\|\rho-\underline{\rho}\right\|_{L^\infty([T,T+P]\times[0,L])}+\|u-\underline{u}\|_{L^\infty([T,T+P]\times[0,L])}\le C_0\varepsilon,\qquad \forall T>T_0, \tag{1.13}$$

here $\mathrm{TV}_A(\cdot)$ means total variation of a function on the set A.

Remark 1.3 Due to the finite speed propagation of signals in hyperbolic systems, the appearance of the start-up time T_0 needed for the boundary conditions to effect in the whole duct is natural. We also note that T_0 depends only on L and a bound of the characteristic speed of the Euler system, as can be seen easily from the proof of these theorems.

Theorem 1.4 (High-frequency limit) Suppose that $\rho_l(t)$ and $u_l(t)$ are given bounded periodic functions with a period $P>0$. There are positive constants ε_0 such that if

$$\left\|\rho_l-\underline{\rho}\right\|_{L^\infty([0,P])}+\|u_l-\underline{u}\|_{L^\infty([0,P])}\le\varepsilon\le\varepsilon_0, \tag{1.14}$$

then for any positive number ω, there is a weak entropy solution (ρ^ω,u^ω) to (1.1), (1.4), subjected to the boundary conditions

$$\rho(t,0)=\rho_l(t/\omega),\quad u(t,0)=u_l(t/\omega),\quad \forall t\ge 0. \tag{1.15}$$

There are positive constants T_0, C_0 so that the solution satisfies

$$\rho^\omega(t+P\omega,x)=\rho^\omega(t,x),\quad u^\omega(t+P\omega,x)=u^\omega(t,x)\ \text{for a.e. }\ t>T_0,\ \ x\in[0,L] \tag{1.16}$$

and

$$\left\|\rho^\omega-\underline{\rho}\right\|_{L^\infty([T,T+P\omega]\times[0,L])}+\|u^\omega-\underline{u}\|_{L^\infty([T,T+P\omega]\times[0,L])}\le C_0\varepsilon,\quad \forall T>T_0. \tag{1.17}$$

Furthermore, it holds that

$$\|\rho^\omega(\cdot,x)-\bar\rho\|_{\mathrm{BV}([T,T+P\omega])}+\|u^\omega(\cdot,x)-\bar u\|_{\mathrm{BV}([T,T+P\omega])}\le C_0\frac{P\omega}{x},\quad \forall T>T_0,\ L\ge x>0, \tag{1.18}$$

and particularly,

$$\lim_{\omega\to0}(\|\rho^\omega(t,x)-\bar\rho\|_{L^\infty([T_0,+\infty)\times[\eta,L])}+\|u^\omega(t,x)-\bar u\|_{L^\infty([T_0,+\infty)\times[\eta,L])})=0,\quad \forall\eta\in(0,L) \tag{1.19}$$

with

$$\bar\rho=\frac1P\int_0^P\rho_l(t)\,\mathrm{d}t,\qquad \bar u=\frac1P\int_0^P u_l(t)\,\mathrm{d}t \tag{1.20}$$

being the averages of ρ_l,u_l over a period.

Remark 1.5 Comparing to Theorem 1.2, under the weaker assumption (1.14), we do not claim uniqueness of weak entropy solutions in Theorem 1.4, which is an open problem in the theory of hyperbolic systems of conservation laws. Estimates (1.18), (1.19) demonstrate homogenization in the domain realized by high-frequency, small-amplitude perturbations from the boundary conditions.

In the following Sections 2–4, we prove the above three theorems, by reducing to several Cauchy problems of hyperbolic systems.

There are several reasons why we are interested in time-periodic solutions of Euler equations in ducts, driving by periodic boundary conditions. The main motivation is to understand better some interesting physical phenomena related to periodic motions of fluids in nozzles, see for example, [4], where the authors carried out experiments, trying to control a normal shock to move periodically in a tube, by adjusting the boundary conditions acting at the exit of the tube, while the supersonic flow ahead of the normal shock remained unchanged. It turns out that the mathematical analysis is quite challenging. Up to now there are many impressive progress on the studies of periodic solutions of partial differential equations arising from fluid dynamics, when viscosity, damping, or certain other dissipation effects were taken into account, see, for instance, [5–10] and references therein. However, except the piston problems considered for viscous flows in [9, 10], most of the works considered time-periodic solutions driving only by time-periodic external forces. As known in control theory, it is usually harder to realize periodic motion just by boundary control. There are fewer works on periodic solutions of hyperbolic conservation laws, while most of them concentrate on space-periodic solutions [11–18]. See [19, 20] for some results on the case of time-periodic solutions driving by time-periodic external forces. To our knowledge, it seems that there is no previous result on time-periodic solutions of hyperbolic conservation laws induced by time-periodic boundary conditions. There are so many interesting open problems in this direction, and our investigations of periodic motions of subsonic flows and transonic shocks in nozzles will be reported in other papers.

2 Classical Solutions

For classical solutions, system (1.1) admits the Riemann invariants [1, p. 32]

$$r=\frac{1}{2}\left(u-\frac{2}{\gamma-1}c\right),\quad s=\frac{1}{2}\left(u+\frac{2}{\gamma-1}c\right),\tag{2.1}$$

and (1.1) is equivalent to the following diagonal system

$$\begin{cases} r_t+\lambda_1(r,s)r_x=0,\\ s_t+\lambda_2(r,s)s_x=0,\end{cases}\tag{2.2}$$

where

$$\lambda_1=u-c=\frac{\gamma+1}{2}r-\frac{\gamma-3}{2}s,\quad \lambda_2=u+c=\frac{3-\gamma}{2}r+\frac{\gamma+1}{2}s\tag{2.3}$$

are characteristics of system (1.1), and are both positive for supersonic flows. In an obvious way, the initial data and boundary conditions can be written respectively as

$$r(0,x)=\underline{r},\quad s(0,x)=\underline{s},\qquad x\in[0,L],\tag{2.4}$$

$$r(t,0)=r_l(t),\quad s(t,0)=s_l(t),\qquad t>0,\tag{2.5}$$

here $\underline{r},\underline{s}$ are constants, and $r_l(t),s_l(t)$ are periodic functions with the period P.

We now interchange the role of t and x, and consider the following Cauchy problem in the direction $x>0$, while $t\in\mathbb{R}$,

$$\begin{cases} r_x+\dfrac{1}{\lambda_1}r_t=0,\quad s_x+\dfrac{1}{\lambda_2}s_t=0;\\ r(t,0)=r_*(t)=\begin{cases} r_l(t), & t>0,\\ \underline{r}, & t\le 0,\end{cases}\\ s(t,0)=s_*(t)=\begin{cases} s_l(t), & t>0,\\ \underline{s}, & t\le 0.\end{cases}\end{cases}\tag{2.6}$$

The compatibility condition required in Theorem 1.1 implies that the "initial data" r_* and s_* are Lipschitz continuous and C^1, except at $t=0$. Then Theorem 7.2 in [1] (with $\alpha=0$) guarantees that for given $L>0$, there is a constant ε_0 of the order $1/L$, so that the above problem (2.6) admits a unique classical solution (r,s), provided that

$$\|r_*-\underline{r}\|_{W^{1,\infty}(\mathbb{R})}+\|s_*-\underline{s}\|_{W^{1,\infty}(\mathbb{R})}\le\varepsilon\le\varepsilon_0. \tag{2.7}$$

Furthermore, it holds that

$$\|r-\underline{r}\|_{W^{1,\infty}(\mathbb{R}\times[0,L])}+\|s-\underline{s}\|_{W^{1,\infty}(\mathbb{R}\times[0,L])}\le C_1\varepsilon \tag{2.8}$$

for a constant C_1 depending only on $\underline{\rho}$, $\underline{u}$, γ and L. By the characteristic method, it is easily seen that the solution (r,s) restricted on $(0,\infty)\times[0,L]$ is a classical solution to problem (2.2)(2.4)(2.5). Since ε_0 is chosen small, we conclude that the flow is still supersonic, and

$$\lambda_0\triangleq\inf_{t>0,x\in[0,L]}\lambda_1(r(t,x),s(t,x))>0. \tag{2.9}$$

What left is to show that for

$$T_0=L/\lambda_0, \tag{2.10}$$

there holds

$$r(t+P,x)=r(t,x),\quad s(t+P,x)=s(t,x),\quad \forall t>T_0,\quad x\in[0,L]. \tag{2.11}$$

For simplicity, we set $U=(r,s)^\top$, $\Lambda(t,x)=\mathrm{diag}(\frac{1}{\lambda_1(r(t,x),s(t,x))},\frac{1}{\lambda_2(r(t,x),s(t,x))})$, and

$$V(t,x)=U(t+P,x)-U(t,x). \tag{2.12}$$

Then $U_x+\Lambda U_t=0$, and

$$\begin{cases}V_x+\Lambda(t,x)V_t=F(t,x)\triangleq-(\Lambda(t+P,x)-\Lambda(t,x))U_t(t+P,x),\\ V(t,0)=\begin{cases}(r_l(t+P)-\underline{r},s_l(t+P)-\underline{s})^\top, & -P\le t\le 0,\\ 0, & t>0 \text{ or } t<-P.\end{cases}\end{cases} \tag{2.13}$$

Notice that $\lambda_1(r,s)$, $\lambda_2(r,s)$ are Lipschitz continuous functions of (r,s), simple computation and (2.8) yield the estimates

$$U_t(t+P,x)_{L^\infty([0,\infty)\times[0,L])}\le C_2\varepsilon, \tag{2.14}$$

$$\Lambda_t(r(t,x),s(t,x))_{L^\infty([0,\infty)\times[0,L])}\le C_2, \tag{2.15}$$

$$|\Lambda(t+P,x)-\Lambda(t,x)|\le C_2|V(t,x)|, \tag{2.16}$$

where $C_2>0$ depends only on $\underline{\rho},\underline{u}$ and γ,L. It follows that

$$|F(t,x)|\le C_3|V(t,x)|. \tag{2.17}$$

Now fix a point (t',x') with $t'>T_0$, $0<x'<L$. We solve the slow and fast characteristic curves $\Gamma_1: t=t_1(x)$ and $\Gamma_2: t=t_2(x)$, respectively, by

$$\frac{\mathrm{d}t_1}{\mathrm{d}x}=\frac{1}{\lambda_1(r(t,x),s(t,x))},\quad t_1(x')=t' \tag{2.18}$$

and

$$\frac{\mathrm{d}t_2}{\mathrm{d}x}=\frac{1}{\lambda_2(r(t,x),s(t,x))},\quad t_2(x')=t' \tag{2.19}$$

for $0 < x < x'$. Note that Γ_1 lies below Γ_2.

For $x \in (0, x')$, set

$$I(x) = \frac{1}{2}\int_{t_1(x)}^{t_2(x)} |V(t,x)|^2 \,\mathrm{d}t. \tag{2.20}$$

Then since $t' > T_0$ and $x' \in [0, L]$, by definition of T_0, it follows from uniqueness of solutions of Cauchy problems of ordinary differential equations that $(t_1(0), t_2(0)) \subset (0, +\infty)$, and by (2.13), $V(t, 0) \equiv 0$ there. Hence

$$I(0) = 0. \tag{2.21}$$

Taking derivative with respect to x, we also have

$$\begin{aligned} I'(x) =& \int_{t_1(x)}^{t_2(x)} V(t,x)V_x(t,x)\,\mathrm{d}t + \frac{1}{2}|V(t_2(x),x)|^2 \frac{1}{\lambda_2(t,x)} - \frac{1}{2}|V(t_1(x),x)|^2 \frac{1}{\lambda_1(t,x)} \\ =& -\int_{t_1(x)}^{t_2(x)} V(t,x)^\top \Lambda(t,x) V_t(t,x)\,\mathrm{d}t + \int_{t_1(x)}^{t_2(x)} V(t,x)^\top F(t,x)\,\mathrm{d}t \\ &+ \frac{1}{2}V(t,x)^\top \Lambda(t,x)V(t,x)\Big|_{t=t_1(x)}^{t=t_2(x)} \\ =& -\frac{1}{2}\int_{t_1(x)}^{t_2(x)} \big((V(t,x)^\top \Lambda(t,x)V(t,x))_t - V(t,x)^\top \Lambda(t,x)_t V(t,x)\big)\,\mathrm{d}t \\ &+ \int_{t_1(x)}^{t_2(x)} V(t,x)^\top F(t,x)\,\mathrm{d}t + \frac{1}{2}V(t,x)^\top \Lambda(t,x)V(t,x)\Big|_{t=t_1(x)}^{t=t_2(x)} \\ =& \frac{1}{2}\int_{t_1(x)}^{t_2(x)} V(t,x)^\top \Lambda(t,x)_t V(t,x)\,\mathrm{d}t + \int_{t_1(x)}^{t_2(x)} V(t,x)^\top F(t,x)\,\mathrm{d}t \\ \leq& (\frac{C_2}{2} + C_3)I(x). \end{aligned}$$

In the last inequality we used (2.15) and (2.17). So by Gronwall's inequality, we infer that $I(x) \equiv 0$. Particularly, by continuity of $V(t, x)$, we see $V(t', x') = 0$. This proves (2.11).

Since ρ_l and u_l are C^1 as we assumed in (1.6), by the above choice of T_0 and characteristic method, we see that the solution is actually C^1 for $t > T_0, 0 \leq x \leq L$; and one may replace the $W^{1,\infty}$ norm in (2.8) by C^1 norm to obtain (1.8). The proof of Theorem 1.1 is completed.

3 Weak Solutions

Set $m = \rho u$, $n = \rho u^2 + p$. Then $n = \frac{m^2}{\rho} + a\rho^\gamma$, and since $u > c$, by implicit function theorem, we could solve $\rho = f(m, n)$. Hence the equations (1.1) could be written as

$$\begin{pmatrix} m \\ n \end{pmatrix}_x + \begin{pmatrix} f(m,n) \\ m \end{pmatrix}_t = 0. \tag{3.1}$$

It is easy to check that this system, taking x as "time", is strictly hyperbolic and genuinely nonlinear, with eigenvalues $1/(u + c)$ and $1/(u - c)$.

Now define

$$m_*(t) = \begin{cases} \rho_l(t)u_l(t), & t > 0, \\ \underline{\rho u}, & t \leq 0, \end{cases} \tag{3.2}$$

$$n_*(t) = \begin{cases} \rho_l(t)u_l(t)^2 + a\rho_l(t)^\gamma, & t > 0, \\ \underline{\rho}\underline{u}^2 + a\underline{\rho}^\gamma, & t \le 0, \end{cases} \tag{3.3}$$

and set $B_\delta(\underline{\rho}, \underline{u}) = \{(\rho, u) : |\rho - \underline{\rho}| + |u - \underline{u}| \le \delta\}$. We firstly fix a number $\delta_0 > 0$ so that

$$\lambda_0^* \triangleq \sup_{(\rho,u)\in B_{\delta_0}(\underline{\rho},\underline{u})} \left(\frac{1}{u-c}\right) > 0. \tag{3.4}$$

Then define

$$N = \left[5 + \frac{2L\lambda_0^*}{P}\right], \tag{3.5}$$

where $[k]$ means the largest integer that is smaller than or equal to k.

Now we study a Cauchy problem of (3.1) with the initial data

$$m(t,0) = \mathrm{I}_{\{0<t<NP\}}(t)m_*(t) + (1 - \mathrm{I}_{\{0<t<NP\}}(t))\underline{\rho}\underline{u}, \tag{3.6}$$

$$n(t,0) = \mathrm{I}_{\{0<t<NP\}}(t)n_*(t) + (1 - \mathrm{I}_{\{0<t<NP\}}(t))(\underline{\rho}\underline{u}^2 + a\underline{\rho}^\gamma), \tag{3.7}$$

here $\mathrm{I}_A(\cdot)$ is the characteristic function of the set A: $\mathrm{I}_A(t) = 1$ if $t \in A$ and $\mathrm{I}_A(t) = 0$ otherwise. By either methods of vanishing viscosity or wave front tracking (cf. Theorem 15.1.1 in [2, p. 557]), we know there is a constant ε_0 such that if (1.10) holds, then there is a unique global weak entropy solution $(m, n) \in L^\infty((-\infty, +\infty) \times [0, +\infty))$ that is of bounded variation in t-variable for each fixed $x > 0$,

$$\mathrm{TV}_{\mathbb{R}} m(\cdot, x) + \mathrm{TV}_{\mathbb{R}} n(\cdot, x) \le C_0\varepsilon. \tag{3.8}$$

Also, the solution lies in $B_{\delta_0}(\underline{\rho}, \underline{u})$. These lead to estimates (1.12), (1.13).

For simplicity of writing, we set $U(t,x) = (m(t,x), n(t,x))^\top$, and $\bar{U}(t,x) = U(t+P,x)$. Then by (15.8.10) or (15.8.11) in [2, p. 581] (see also (13.13) in [21, p. 294]), we have, for arbitrary $a < b$, that

$$\int_a^b |\bar{U}(t,x) - U(t,x)|\,\mathrm{d}t \le M \int_{a-\lambda_0^* x}^{b+\lambda_0^* x} |\bar{U}(t,0) - U(t,0)|\,\mathrm{d}t \tag{3.9}$$

for some positive constant M depending only on $\underline{\rho}, \underline{u}$ and γ, δ_0. By (3.6), (3.7), one has

$$\bar{U}(t,0) - U(t,0) = \begin{cases} 0, & t > NP, \\ \underline{U} - U_L(t+p), & (N-1)P \le t \le NP, \\ 0, & 0 \le t < (N-1)P, \\ -\underline{U} + U_L(t+p), & -P \le t < 0, \\ 0, & t < -P. \end{cases} \tag{3.10}$$

Therefore, taking $a = L\lambda_0^*$, $b = (N-1)P - L\lambda_0^*$, by our choice of N, the interval $(a - L\lambda_0^*, b + L\lambda_0^*)$ is contained in $(0, (N-1)P)$, and $|b-a| > 4P$. Then (3.9) implies that $U(t+P,x) = U(t,x)$ a.e. for $(t,x) \in [a, b-P] \times [0, L]$. From this, we may extend U to be a periodic function in t-variable with period P for all $t \ge a = L\lambda_0^*$, and get a global solution defined on $(0, +\infty) \times [0, L]$ for problem (3.1)–(3.3).

We note by definition of weak solutions, a weak solution to problem (3.1)–(3.3) is also a weak solution to problem (1.1)(1.4)(1.5). Since the solution constructed above for problem

(3.1)–(3.3) satisfies the Lax E-condition (observing that the orientation of (x,t)-coordinates is opposite to the (t,x)-coordinates)

$$\frac{1}{\lambda_i(U_-)} < \frac{1}{s} < \frac{1}{\lambda_i(U_+)}, \quad i = 1, 2, \tag{3.11}$$

it also satisfies the Lax E-condition (1.9) for problem (1.1)(1.4)(1.5). We remark that the positiveness of λ_1, λ_2, namely, the flow is supersonic, is crucial here. For more general results on equivalence of weak solutions of hyperbolic conservation laws by interchanging the role of t and x, one may refer to Lemma 2.12 in [3, p. 14].

This establishes existence of weak entropy solutions that are time periodic for $t > T_0$, to problem (1.1)(1.4)(1.5). We may take $T_0 = L\lambda_0^*$. Uniqueness follows directly from the L^1 stability estimate (3.9).

4 High-Frequency Limit

Let $U_l(t)$ be a given bounded periodic function for $t \geq 0$ with period $P > 0$, and ω a positive number. We now consider the following problem

$$\begin{cases} U_x + F(U)_t = 0, & 0 < x < L, \quad 0 < t < +\infty, \\ U(t,0) = U_l(t/\omega), & 0 < t < +\infty, \\ U(0,x) = \underline{U}, & 0 \leq x \leq L, \end{cases} \tag{4.1}$$

where, following notations in the previous section, $U = (m,n)^\top$ and $F(U) = (f(m,n), m)^\top$. A solution of this problem will be denoted as U^ω. We wish to understand limiting behaviour of U^ω as $\omega \downarrow 0$.

Let $\tilde{U}^\omega = U^\omega - \underline{U}$, $\tilde{F}(\tilde{U}^\omega) = F(\tilde{U}^\omega + \underline{U})$ with $\underline{U} = (\underline{m}, \underline{n})^\top$. We consider the problem

$$\begin{cases} \tilde{U}^\omega_x + \tilde{F}(\tilde{U}^\omega)_t = 0, & 0 < x < L, \quad -\infty < t < +\infty, \\ \tilde{U}^\omega(t,0) = \Phi(t/\omega) - \underline{U}, & -\infty < t < +\infty, \end{cases} \tag{4.2}$$

where

$$\Phi(t) = \begin{cases} U_l(t), & t > 0, \\ \underline{U}, & t \leq 0. \end{cases} \tag{4.3}$$

By the assumptions in Theorem 1.4, we have $|\Phi(t/\omega) - \underline{U}| \leq C\varepsilon \leq C\varepsilon_0$ for a constant C depending only on the Euler system. Thanks to Theorem 5.1 in [15, p. 95], problem (4.2) has a weak entropy solution $\tilde{U}^\omega$ for any $\omega > 0$. Then by applying the method of generalized characteristics as in [15] (see also [2, Chapter X]), with similar arguments of the previous section, by restricting the solution to $0 < x < L$, we get a solution $U^\omega = \tilde{U}^\omega + \underline{U}$ to problem (4.1), which is periodic in t-variable for $t > T_0 = L\lambda_0^*$, with a period $P\omega$.

We consider the asymptotic behaviour of $U^\omega(t,x)$ as $\omega \to 0$, where $t > T_0$, and $0 < x \leq L$. Since L is fixed, and $t > T_0$, by finite speed propagation of signals for hyperbolic systems, we could regard U^ω as a weak entropy solution to the following problem

$$\begin{cases} U^\omega_x + F(U^\omega)_t = 0, & 0 < x < L, \quad -\infty < t < +\infty, \\ U^\omega(t,0) = \tilde{\Phi}(t/\omega), & -\infty < t < +\infty, \end{cases} \tag{4.4}$$

where $\tilde{\Phi}(t)$ is obtained by extending $U_l(t)$ periodically to $t<0$ with period P.

Set $\tau = t/\omega, y = x/\omega$, and

$$V(\tau, y) = U^\omega(\omega\tau, \omega y).$$

Then V solves

$$\begin{cases} V_y + F(V)_\tau = 0, & 0 < y < L/\omega, \quad -\infty < \tau < +\infty, \\ V(\tau, 0) = \tilde{\Phi}(\tau), & -\infty < \tau < +\infty. \end{cases} \tag{4.5}$$

Hence, we are led to consider the Cauchy problem

$$\begin{cases} V_y + F(V)_\tau = 0, & 0 < y < +\infty, \quad -\infty < \tau < +\infty, \\ V(\tau, 0) = \tilde{\Phi}(\tau), & -\infty < \tau < +\infty. \end{cases} \tag{4.6}$$

Thanks to Theorem 5.2 in [15, p. 109], not only does V exist as a periodic function in τ-variable with period P, but also it satisfies the estimates

$$\mathrm{TV}_{[\tau_0, \tau_0+P]}(V(\cdot, y)) \le H_1 \frac{P}{y}, \tag{4.7}$$

$$|V(\tau, y) - \bar{V}| \le H_1 \frac{P}{y}, \qquad \forall \tau_0, \tau \in \mathbb{R}, \quad y > 0 \tag{4.8}$$

for a generic constant H_1 depending only on $\underline{U}$ and γ. Here we have set $\bar{V}$ to be the average of $\tilde{\Phi}$ over an interval with length P. It follows immediately that, for any $x \in (0, L]$,

$$\mathrm{TV}_{[t, t+P\omega]}(U^\omega(\cdot, x)) \le H_1 \frac{P\omega}{x}, \qquad \forall t > T_0, \tag{4.9}$$

$$|U^\omega(t, x) - \bar{V}| \le H_1 \frac{P\omega}{x}, \qquad \forall t > T_0. \tag{4.10}$$

These inequalities lead easily to the estimates claimed in Theorem 1.4.

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