Acta Mathematica Scientia, 2019, **39B**(1): 284–296
http://doi.org/s10473-019-0121-x

Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

A NECESSARY CONDITION FOR CERTAIN INTEGRAL EQUATIONS WITH NEGATIVE EXPONENTS*

Jiankai XU (许建开)†
College of Sciences; College of Computer Science, Hunan Agriculture University, Changsha 410128, China
E-mail: *jiankaixu@126.com*

Zhong TAN (谭忠)
School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
E-mail: *ztan85@163.com*

Weiwei WANG (王伟伟)
College of Mathematics and Computer Science, Fuzhou University, Fuzhou 361000, China
E-mail: *wei.wei.84@163.com*

Zepeng XIONG (熊泽鹏)
The First Middle School of Longhui, Longhui 422200, China
E-mail: *770685723@qq.com*

Abstract This paper is devoted to studying the existence of positive solutions for the following integral system

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n} |x-y|^{\lambda} v^{-q}(y)\mathrm{d}y, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} |x-y|^{\lambda} u^{-p}(y)\mathrm{d}y, \end{cases} \quad p,\, q > 0, \;\; \lambda \in (0, \infty),\, n \geq 1.$$

It is shown that if (u, v) is a pair of positive Lebesgue measurable solutions of this integral system, then

$$\frac{1}{p-1} + \frac{1}{q-1} = \frac{\lambda}{n},$$

which is different from the well-known case of the Lane-Emden system and its natural extension, the Hardy-Littlewood-Sobolev type integral equations.

Key words integral equations; Lane-Emden system; conformal invariance; positive solutions; existence

2010 MR Subject Classification 45G05; 45M99

*Received March 8, 2017; revised June 6, 2018. Supported by National Natural Science Foundation of China (11126148, 11501116, 11671086, 11871208), Natural Science Foundation of Hunan Province of China (2018JJ2159), the Project Supported by Scientific Research Fund of Hunan Provincial Education Department (16C0763) and the Education Department of Fujian Province (JA15063).

†Corresponding author: Jiankai XU.

1 Introduction

In this article, we investigate the existence of positive solutions for a non-linear integral equations of the form:

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n} |x-y|^{\lambda}\ v^{-q}(y)\mathrm{d}y, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} |x-y|^{\lambda}\ u^{-p}(y)\mathrm{d}y, \end{cases} \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $\lambda > 0$, $0 < p \neq 1, 0 < q \neq 1$ and $n \geq 1$. Precisely, for system (1.1), we will determine the necessary conditions for the existence of non-trivial positive solutions, that are non-infinity and Lebesgue measurable. The motivation to study this equations comes from the classification of renowned Lane-Emden system and its natural extension, the Hardy-Littlewood-Sobolev type integral equations.

The well known Lane-Emden system, which arises from the chemical, biological and physical studies and has attracted several researchers' attention, can be written as follows

$$\begin{cases} \Delta u(x) + v^q(x) = 0, \\ \Delta v(x) + u^p(x) = 0, \end{cases} \qquad x \in \mathbb{R}^n, \tag{1.2}$$

here $u(x), v(x) \geq 0$, $0 < p,\ q < \infty$. According to the value of exponents $(p,\ q)$, system (1.2) is usually divided into the following three cases. When the pair $(p,\ q)$ lies on the Sobolev hyperbola, i.e.,

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n} \quad (n \geq 3)\ , \tag{1.3}$$

(1.2) is called critical. We also say that system (1.2) is supercritical, or subcritical if $(p,\ q)$ satisfies that

$$\frac{1}{p+1} + \frac{1}{q+1} < \frac{n-2}{n}, \quad \text{or} \quad > \frac{n-2}{n} \quad (n \geq 3). \tag{1.4}$$

The famous Lane-Emden conjecture states that system (1.3) does not admit a positive solution under the subcritical condition. That is to say that (1.3) is a corresponding dividing curve with the property that (1.2) admits positive solutions if and only if $(p,\ q)$ satisfies the critical condition or supercritical condition.

Also, system (1.2) has the natural extension as follows:

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = v^q(x), \\ (-\Delta)^{\alpha/2} v(x) = u^p(x) \end{cases} \qquad \text{for} \ \ x \in \mathbb{R}^n, \quad \alpha \in (0,\ n). \tag{1.5}$$

Under certain regularity conditions, system (1.5) is equivalent to the following integral system, which is closely related to the problem of finding the sharp constant in the Hardy-Littlewood-Sobolev inequality

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n} \frac{v^q(y)}{|x-y|^{\lambda}}\mathrm{d}y, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{\lambda}}\mathrm{d}y, \end{cases} \qquad x \in \mathbb{R}^n, \tag{1.6}$$

here $\lambda \in (0,\ n)$, $n \geq 3$ and $p,\ q > 1$. System (1.5) or (1.6) is not only a natural extension of (1.2), but also has own interest which provides an important way to study the Lane-Emden conjecture. Similarly, the value of exponents $(p,\ q)$ in (1.6) is also divided into three cases and the corresponding Lane-Emden conjecture becomes the Hardy-Littlewood-Sobolev type integral equations conjecture, namely, system (1.6) does not admit a positive solution if and only if these parameters $(p,\ q)$ satisfy the following inequality

$$\frac{1}{p+1}+\frac{1}{q+1}>\frac{\lambda}{n}.$$

Now, we recall some results which are closely related to our topic. In 1998, by the shooting method and the Pohozaev identity, Serrin and Zou [21] showed the existence of a positive solution of (1.2), when $(p,\ q)$ satisfies that

$$\frac{1}{p+1}+\frac{1}{q+1}\leq\frac{n-2}{n}\ (n\geq 3). \tag{1.7}$$

Later on, Mitidieri [17] proved that the Lane-Emden conjecture holds with additional assumption that $(u,\ v)$ is a pair of radial solution of (1.2). Therefore, for radial case of system (1.2), Sobolev hyperbola (1.3) is the dividing curve for the existence and nonexistence of positive solutions. As for the non-radial solutions of (1.2), the Lane-Emden conjecture is still open except for $n\leq 4$. We refer the readers to [11, 17, 19, 20, 23, 24, 27], among numerous references, for more information. When α $(\alpha<n)$ is an even integer, the higher order system (1.5) is defined in pointwise sense. Under this case, Liu, Guo and Zhang [16] showed that if $(p,\ q)$ satisfies

$$\frac{1}{p+1}+\frac{1}{q+1}>\frac{n-\alpha}{n},$$

then system (1.5) has no radial non-negative solutions. With the same assumption to the parameter α, Lei and Li [9] proved that system (1.5) admits a pair of positive radial solutions (u,v), provided

$$\frac{1}{p+1}+\frac{1}{q+1}\leq\frac{n-\alpha}{n}.$$

Moreover, with the help of degree theory, Li and Villavert in [12, 13] considered the more general abstract model and established the existence result under suitable conditions. For higher order system (1.5) or (1.6) with the general parameter $\lambda=n-\alpha\in(0,n)$, Lieb [15] proved that the existence of positive solution for (1.6) in the critical case. Subsequently, Caristi, Dambrosio and Mitidieri [1], under certain smooth condition assumptions, proved the conjecture for the Hardy-Littlewood-Sobolev type integral equations. That is, if $(u,\ v)\in C^2(\mathbb{R}^n)\times C^2(\mathbb{R}^n)$ is a pair of nonnegative radial solutions of (1.6) and $\lambda<n-2$, then system (1.6) has no positive solution in subcritical case. Recently, Lei and Li [9] removed the key assumptions and established the same results (see [9, Theorem 1.2] for the details). These results suggest that under some special cases, the hyperbolic

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{\lambda}{n}\qquad\text{for}\quad\lambda\in(0,n)$$

is the dividing curve in the $(p,\ q)$-plane for the existence and non-existence of Hardy-Littlewood-Sobolev type integral system.

Based on the above, it is natural and interesting to ask whether there exists a corresponding dividing curve in the $(p,\ q)$-plane such that (1.1) for $\lambda\in(0,\ \infty)$ and $p,q>0$, admits positive solutions if and only if $(p,\ q)$ is on or above the curve? The main purpose of this paper is to address this question. Our main result can be formulated as follows.

Theorem 1.1 Suppose that (u, v) is a pair of positive solutions of system (1.1) with $\lambda \in (0, \infty)$ and $0 < q \neq 1, 0 < p \neq 1$. Then the parameters p, q, λ satisfy the following condition
$$\frac{1}{p-1} + \frac{1}{q-1} = \frac{\lambda}{n}. \tag{1.8}$$

Remark 1.2 Comparing Theorem 1.1 with the results of the well-known Lane-Emden system (1.2) and its natural extension to Hardy-Littlewood-Sobolev type integral equations (1.6), system (1.1) has the same radial symmetry solution, provided the exponents (p, q) lies on the hyperbola curve. However, as the pair of parameters (p, q) is not on the hyperbola curve, there is obvious difference between (1.1) and (1.6). Precisely, by Theorem 1.1, we can see that system (1.1) has no positive solution, if (p, q) is not on the hyperbola. But as for Lane-Emden system (1.2) and (1.6), the system has positive solutions when (p, q) is under the supercritical conditions. The essential reasons for the difference between (1.1) and (1.6), comes from the integrability and asymptotic behavior of each system.

Remark 1.3 From an analytical point of view, a natural and interesting question that raised from the above result is whether there exist positive solutions on hyperbola (1.8) for system (1.1) ? The conjecture is not true. An example is that it is easy to see that for p, q near 1, from the asymptotic estimates (2.1) and (2.2), there is no such solution on hyperbola (1.8). On the other hand, as $p = q > 1$ and $\max\{p, q\} > (n + \lambda)/\lambda$, by [7] and [10], system (1.1) admits a pair of radial positive solution.

The rest of this paper is organized as follows. After recalling and establishing some technical lemmas in Section 2, we will prove Theorem 1.1 in Section 3. Throughout this paper, we always use the letter C to denote positive constants that may vary at each occurrence but are independent of the essential variables.

2 Preliminary

In this section, we will recall and establish some standard ingredients needed in the proof of our theorem. These results essentially follow from [7, 10, 14, 25]. Here, for completeness, we will present the corresponding proofs.

Lemma 2.1 For $n \geq 1$, let $u(x), v(x)$ be a pair of positive Lebesgue measurable solutions of (1.1) with $\lambda > 0, q > 0$ and $p > 0$. Then the following properties hold

- equivalents of pointwise
$$C^{-1}(1 + |x|^\lambda) \leq u(x) \leq C(1 + |x|^\lambda), \qquad x \in \mathbb{R}^n, \tag{2.1}$$
$$C^{-1}(1 + |x|^\lambda) \leq v(x) \leq C(1 + |x|^\lambda), \qquad x \in \mathbb{R}^n; \tag{2.2}$$
- integrability of $u(x)$ and $v(x)$
$$C^{-1} \leq \int_{\mathbb{R}^n} u^{1-p}(y)\mathrm{d}y \leq C < \infty, \quad \text{as } 1 - p \neq 0, \tag{2.3}$$
$$C^{-1} \leq \int_{\mathbb{R}^n} v^{1-q}(y)\mathrm{d}y \leq C < \infty, \quad \text{as } 1 - q \neq 0. \tag{2.4}$$

Furthermore,
$$\|v\|_{1-q}^{1-q} = \int_{\mathbb{R}^n} v^{1-q}(y)\mathrm{d}y = \int_{\mathbb{R}^n} u^{1-p}(y)\mathrm{d}y = \|u\|_{1-p}^{1-p}, \tag{2.5}$$

$$\int_{\mathbb{R}^n} (1+|y|)^{\lambda} u^{-p}(y)\mathrm{d}y \le C < \infty, \tag{2.6}$$

$$\int_{\mathbb{R}^n} (1+|y|)^{\lambda} v^{-q}(y)\mathrm{d}y \le C < \infty; \tag{2.7}$$

- asymptotic behavior of u and v

$$\lim_{|x|\to\infty} |x|^{-\lambda} v(x) = \lim_{|x|\to\infty} \int_{\mathbb{R}^n} \left(\frac{|x-y|}{|x|}\right)^{\lambda} u^{-p}(y)\mathrm{d}y = \int_{\mathbb{R}^n} u^{-p}(y)\mathrm{d}y, \tag{2.8}$$

$$\lim_{|x|\to\infty} |x|^{-\lambda} u(x) = \lim_{|x|\to\infty} \int_{\mathbb{R}^n} \left(\frac{|x-y|}{|x|}\right)^{\lambda} v^{-q}(y)\mathrm{d}y = \int_{\mathbb{R}^n} v^{-q}(y)\mathrm{d}y. \tag{2.9}$$

Proof Noting that $p>0, q>0$ and $\lambda \in (0,\infty)$, by (1.1) we conclude that

$$\mu\bigg(\{y \in \mathbb{R}^n : 0 < u(y) < \infty \ \text{ and } \ 0 < v(y) < \infty\}\bigg) > 0, \tag{2.10}$$

here μ denotes the Lebesgue measure of the set. Conversely, if (2.10) does not hold, then

$$\mu\bigg(\{y \in \mathbb{R}^n : 0 < u(y) < \infty \text{ and } 0 < v(y) < \infty\}\bigg) = 0,$$

which implies that

$$u(y) = \infty, \quad v(y) = \infty \quad \text{a.e. } \ y \in \mathbb{R}^n.$$

This is obviously contradictory to (1.1). Therefore, there exist $R>1$ and a non-empty measurable set $E \subset \mathbb{R}^n$ such that

$$E \subset \bigg\{y |\ 0 < u(y) < R \text{ and } 0 < v(y) < R\bigg\} \cap B_R(0), \qquad \mu(E) \ge R^{-1}$$

and

$$u(x) = \int_{\mathbb{R}^n} |x-y|^{\lambda} v^{-q}(y)\mathrm{d}y \ge \int_E |x-y|^{\lambda} v^{-q}(y)\mathrm{d}y \ge R^{-q} \int_E |x-y|^{\lambda}\mathrm{d}y, \ \forall\, x \in \mathbb{R}^n. \tag{2.11}$$

On the other hand, since $|y| \le R$ and $|x| \ge 2R$, we have

$$u(x) \ge R^{-q} \int_E |x-y|^{\lambda}\mathrm{d}x \ge R^{-q-1} \left(\frac{|x|}{2}\right)^{\lambda}. \tag{2.12}$$

Hence, the first inequality in (2.1) follows from (2.11) and (2.12). Similarly, we can get the first inequality in (2.2).

Next, we will prove the second inequality in (2.1) and (2.2). By (2.10), there exist $\bar{x}$ such that $|\overline{x}| \in [1,2]$ and $v(\bar{x}) < \infty, u(\bar{x}) < \infty$. Thus when $|y| \ge 4$, we conclude that

$$\int_{\mathbb{R}^n \setminus B_4(0)} u^{-p}(y)\mathrm{d}y \le \int_{\mathbb{R}^n} |\overline{x}-y|^{\lambda} u^{-p}(y)\mathrm{d}y = v(\bar{x}) < +\infty$$

and

$$\int_{\mathbb{R}^n \setminus B_4(0)} v^{-q}(y)\mathrm{d}y \le \int_{\mathbb{R}^n} |\overline{x}-y|^{\lambda} u^{-p}(y)\mathrm{d}y = u(\bar{x}) < +\infty,$$

which, together with the first inequality in (2.1) and $p>0, q>0$, yields that

$$u^{-p}(x), \quad v^{-q}(y) \in L^1(\mathbb{R}^n). \tag{2.13}$$

Similarly,

$$|y|^{\lambda} u^{-p}(y), \quad |y|^{\lambda} v^{-q}(y) \in L^1(\mathbb{R}^n). \tag{2.14}$$

Indeed, for some $\overline{x}$ and any y satisfying $1 \leq |\overline{x}| \leq 2$ and $|y| \geq 4$, we deduce that

$$2^{-\lambda}|y|^{\lambda} \leq \ |y-\bar{x}|^{\lambda}.$$

This together with (2.13), implies that

$$\begin{aligned}\int_{\mathbb{R}^n} |y|^{\lambda} u^{-p}(y)\mathrm{d}y &= \int_{|y|>4} |y|^{\lambda} u^{-p}(y)\mathrm{d}y + \int_{|y|\leq 4} |y|^{\lambda} u^{-p}(y)\mathrm{d}y \\ &\leq C(\lambda)\int_{\mathbb{R}^n} |\overline{x}-y|^{\lambda} u^{-p}(y)\mathrm{d}y + 4^{\lambda}\int_{\mathbb{R}^n} u^{-p}(y)\mathrm{d}y \\ &\leq C(\lambda)v(\overline{x}) + 4^{\lambda}\|u(x)\|_{-p} < \infty\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} |y|^{\lambda} v^{-q}(y)\mathrm{d}y &= \int_{|y|>4} |y|^{\lambda} v^{-q}(y)\mathrm{d}y + \int_{|y|\leq 4} |y|^{\lambda} v^{-q}(y)\mathrm{d}y \\ &\leq C(\lambda)\int_{\mathbb{R}^n} |\overline{x}-y|^{-\lambda} v^{-q}(y)\mathrm{d}y + 4^{\lambda}\int_{\mathbb{R}^n} v^{-q}(y)\mathrm{d}y \\ &\leq C(\lambda)u(\overline{x}) + 4^{\lambda}\|v(x)\|_{-q} < \infty.\end{aligned}$$

Then (2.6) and (2.7) directly follow from (2.13) and (2.14). Meanwhile, noticing that

$$\left|\frac{|x-y|^{\lambda}}{|x|^{\lambda}}\right| \leq C(1+|y|^{\lambda}) \quad \text{for} \quad |x| \geq 1,$$

and by (2.13), (2.14) and the dominated convergence theorem, we have

$$\lim_{|x|\to\infty} |x|^{-\lambda}\ u(x) = \int_{\mathbb{R}^n} v^{-q}(y)\mathrm{d}y$$

and

$$\lim_{|x|\to\infty} |x|^{-\lambda}\ v(x) = \int_{\mathbb{R}^n} u^{-p}(y)\mathrm{d}y.$$

This shows (2.8) and (2.9) which, furthermore ensures that (2.1) and (2.2) hold.

Now, we turn to (2.3) and (2.4). By (2.1) and (2.14), we get

$$\int_{\mathbb{R}^n\setminus B_R(0)} u^{1-p}(y)\mathrm{d}x = \int_{\mathbb{R}^n\setminus B_R(0)} u^{-p}(y)u(y)\mathrm{d}x \leq C\int_{\mathbb{R}^n\setminus B_R(0)} u^{-p}(y)|y|^{\lambda}\mathrm{d}y < \infty,$$

which together with (2.1) leads to (2.3). Similarly, we can obtain (2.4).

Finally, we verify (2.5). Since $\lambda \in (0,\infty,)$, we have

$$|x-y|^{\lambda} \leq (|x|+|y|)^{\lambda} \leq [(1+|x|)(1+|y|)]^{\lambda}.$$

This together with (2.6) and (2.7), yields that

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^{\lambda} u^{-p}(y)v^{-q}(x)\mathrm{d}x\mathrm{d}y \leq \int_{\mathbb{R}^n} (1+|x|)^{\lambda} v^{-q}(x)\mathrm{d}x \int_{\mathbb{R}^n} (1+|y|)^{\lambda} u^{-p}(y)\mathrm{d}y < \infty.$$

Therefore, by (1.1) and the Lebesgue dominated convergence theorem again, we get

$$\begin{aligned}\|u^{1-p}\|_1 &= \int_{\mathbb{R}^n} u^{-p}(x)\int_{\mathbb{R}^n} |x-y|^{\lambda} v^{-q}(y)\mathrm{d}y\mathrm{d}x \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\lambda}\ v^{-q}(y)\ u^{-p}(x)\ \mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^n} v^{-q}(y)\int_{\mathbb{R}^n} |x-y|^{\lambda} u^{-p}(x)\mathrm{d}x\mathrm{d}y = \|v^{1-q}\|_1.\end{aligned}$$

This is (2.5) and completes the proof of Lemma 2.1. □

Lemma 2.2 For $n \geq 1, \lambda > 0, q > 0, p > 0$, let (u, v) be a pair of positive solutions of system (1.1). Then for any $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} u(x)\ \nabla\varphi(x)\mathrm{d}x = -\lambda \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)v^{-q}(y)\mathrm{d}y\varphi(x)\mathrm{d}x \tag{2.15}$$

and

$$\int_{\mathbb{R}^n} v(x)\ \nabla\varphi(x)\mathrm{d}x = -\lambda \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)u^{-p}(y)\mathrm{d}y\varphi(x)\mathrm{d}x. \tag{2.16}$$

Proof For convenience, we denote $\mathfrak{M}_1(x),\ \mathfrak{M}_2(x), \mathfrak{M}_3(y), \mathfrak{M}_4(y)$ by

$$\mathfrak{M}_1(x) \triangleq \lambda \int_{\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)v^{-q}(y)\mathrm{d}y,$$

$$\mathfrak{M}_2(x) \triangleq \lambda \int_{\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)u^{-p}(y)\mathrm{d}y,$$

$$\mathfrak{M}_3(y) \triangleq \int_{\mathbb{R}^n} |x-y|^{\lambda}\nabla\varphi(x)\mathrm{d}x, \quad \varphi \in C_0^\infty(\mathbb{R}^n),$$

and

$$\mathfrak{M}_4(y) \triangleq -\lambda \int_{\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)\varphi(x)\mathrm{d}x, \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

First, we will show that these functions are well-defined for any given $x,\ y \in \mathbb{R}^n$. To do this, it suffices to prove that

$$\mathfrak{S}_1(x) \triangleq \int_{\mathbb{R}^n} |x-y|^{\lambda-1}[v^{-q}(y) + u^{-p}(y)]\mathrm{d}y < \infty,$$

$$\mathfrak{S}_2(y) \triangleq \int_{\mathbb{R}^n} |x-y|^{\lambda-1}|\varphi(x)|\mathrm{d}x < \infty,$$

and

$$\mathfrak{S}_3(y) \triangleq \int_{\mathbb{R}^n} |x-y|^{\lambda}\ |\nabla\varphi(x)|\mathrm{d}x < \infty.$$

In view of $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\lambda \in (0, \infty)$, it is easy to verify that $\mathfrak{S}_2(y)$ and $\mathfrak{S}_3(y)$ are well-defined. What's more, by integration by part, we have

$$\begin{aligned}\mathfrak{M}_3(y) &= \int_{\mathbb{R}^n} (\nabla\varphi)(x)|x-y|^{\lambda}\mathrm{d}x \\ &= -\lambda \int_{\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)\varphi(x)\mathrm{d}x = \mathfrak{M}_4(y).\end{aligned} \tag{2.17}$$

For $\mathfrak{S}_1(x)$, we can write

$$\begin{aligned}\mathfrak{S}_1(x) &= \int_{\mathbb{R}^n} |x-y|^{\lambda-1}\ [v^{-q}(y) + u^{-p}(y)]\mathrm{d}y \\ &= \int_{|x-y|\leq R} |x-y|^{\lambda-1}\ [v^{-q}(y) + u^{-p}(y)]\mathrm{d}y \\ &\quad + \int_{|x-y|\geq R} |x-y|^{\lambda-1}\ [v^{-q}(y) + u^{-p}(y)]\mathrm{d}y \\ &\triangleq \mathfrak{S}_{1,1}(x) + \mathfrak{S}_{1,2}(x).\end{aligned} \tag{2.18}$$

By (2.1) and (2.2) with $p,\ q > 0$, we have

$$v^{-q}(y) + u^{-p}(y) \leq M < \infty, \quad \forall\, y \in \mathbb{R}^n.$$

Therefore for $\lambda \in (0,1)$,

$$\begin{aligned}\mathfrak{S}_{1,1}(x) &= \int_{|x-y|\le R} |x-y|^{\lambda-1}\ [v^{-q}(y)+u^{-p}(y)]\mathrm{d}y\\ &\le C(M,n)\int_0^R r^{-2+\lambda+n}\mathrm{d}r < \infty,\end{aligned}$$

and by (2.13),

$$\begin{aligned}\mathfrak{S}_{1,2}(x) &= \int_{|x-y|\ge R} |x-y|^{\lambda-1}\ [v^{-q}(y)+u^{-p}(y)]\mathrm{d}y\\ &\le R^{\lambda-1}\int_{\mathbb{R}^n} [v^{-q}(y)+u^{-p}(y)]\ \mathrm{d}y < \infty.\end{aligned}$$

And for $\lambda \in [1,\infty)$, note that

$$|x-y|^{\lambda-1} \le [(1+|x|)(1+|y|)]^{\lambda},$$

by (2.6) and (2.7), we have

$$\begin{aligned}\mathfrak{S}_1(x) &= \int_{\mathbb{R}^n} |x-y|^{\lambda-1}\ [v^{-q}(y)+u^{-p}(y)]\mathrm{d}y\\ &\le (1+|x|)^{\lambda}\int_{\mathbb{R}^n}(1+|y|)^{\lambda}[v^{-q}(y)+u^{-p}(y)]\mathrm{d}y < \infty.\end{aligned}$$

We then know that $\mathfrak{M}_1(y),\ \mathfrak{M}_2(y), \mathfrak{M}_3(x), \mathfrak{M}_4(x)$ are well-posed.

Now we turn to the proofs of (2.15) and (2.16). Note that for any $r>1$

$$\begin{aligned}\|u^{-p}\|_r^r + \|v^{-q}\|_r^r &\le C\int_{\mathbb{R}^n}[(1+|y|)^{-\lambda p\ r} + (1+|y|)^{-\lambda q r}]\mathrm{d}y\\ &\le C\int_{\mathbb{R}^n}[(1+|y|)^{-\lambda p} + (1+|y|)^{-\lambda q}]\mathrm{d}y\\ &= C(\|u^{-p}\|_1 + \|v^{-q}\|_1) < \infty.\end{aligned}$$

Thus when $0<\lambda<1$, by Hardy-Littlewood-Sobolev inequality, (2.1), (2.2) and (2.13), we have

$$\begin{aligned}&\left|\int\int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)[v^{-q}(y)+u^{-p}(y)]\varphi(x)\mathrm{d}x\mathrm{d}y\right|\\ &\le \int\int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^{\lambda-1}[v^{-q}(y)+u^{-p}(y)]|\varphi(x)|\mathrm{d}x\mathrm{d}y\\ &\le C(\lambda,n,r,s)(\|u^{-p}\|_r + \|v^{-q}\|_r)\|\varphi\|_s < \infty,\end{aligned} \tag{2.19}$$

where $1/r+1/s+(1-\lambda)/n = 2, r>1, s>1$.

On the other hand, for $\lambda \in [1,\infty)$, note that

$$|x-y|^{\lambda-1} \le (1+|x|)^{\lambda-1}(1+|y|)^{\lambda-1} \le (1+|x|)^{\lambda}(1+|y|)^{\lambda}.$$

It follows from (2.8) and (2.9) that

$$\begin{aligned}&\left|\int\int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)[u^{-p}(y)+v^{-q}(y)]\ \varphi(x)\mathrm{d}x\mathrm{d}y\right|\\ &\le \int\int_{\mathbb{R}^n\times\mathbb{R}^n} (1+|x|)^{\lambda}\varphi(x)\ (1+|y|)^{\lambda}[u^{-p}(y)+v^{-q}(y)]\mathrm{d}y\mathrm{d}x < \infty,\end{aligned}$$

which combining with (2.19) implies that

$$(x-y)|x-y|^{\lambda-2}\varphi(x)u^{-p}(y) \quad \text{and} \quad (x-y)|x-y|^{\lambda-2}\varphi(x)v^{-q}(y)$$

are absolutely integrable on $\mathbb{R}^n \times \mathbb{R}^n$. Hence, by (2.17) and Fubini's theorem, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^n} u(x)\nabla\varphi(x)\mathrm{d}x &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\lambda} v^{-q}(y)\ \nabla\varphi(x)\mathrm{d}x\mathrm{d}y \\ &= -\lambda\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)v^{-q}(y)\varphi(x)\mathrm{d}x\mathrm{d}y \\ &= -\lambda\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)v^{-q}(y)\mathrm{d}y\varphi(x)\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} v(x)\nabla\varphi(x)\mathrm{d}x &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\lambda} u^{-p}(y)\ \nabla\varphi(x)\mathrm{d}x\mathrm{d}y \\ &= -\lambda\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)u^{-p}(y)\varphi(x)\mathrm{d}x\mathrm{d}y \\ &= -\lambda\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)u^{-p}(y)\mathrm{d}y\varphi(x)\mathrm{d}x,\end{aligned}$$

which completes the proof of Lemma 2.2. □

Lemma 2.3 Suppose that $(u(x),\ v(x))$ is a pair of positive solutions of (1.1) for $\lambda > 0$, $q > 0, p > 0$. Then $(u(x),\ v(x)) \in C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n)$.

Proof Given $R > 0$, we rewrite (1.1) as follows

$$\begin{aligned}u(x) &= \int_{|y|\le 2R} |x-y|^{\lambda} v^{-q}(y)\mathrm{d}y + \int_{|y|\ge 2R} |x-y|^{\lambda} v^{-q}(y)\mathrm{d}y \\ &\triangleq \mathcal{I}_u(x) + \mathcal{II}_u(x)\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}v(x) &= \int_{|y|\le 2R} |x-y|^{\lambda} u^{-p}(y)\mathrm{d}y + \int_{|y|\ge 2R} |x-y|^{\lambda} u^{-p}(y)\mathrm{d}y \\ &\triangleq \mathcal{I}_v(x) + \mathcal{II}_v(x).\end{aligned} \tag{2.21}$$

Set

$$g(x) = |x-y|^{\lambda}.$$

It is easy to check that for any $x \in \mathbb{R}^n$ and $x \neq y$

$$|D^{\beta} g(x)| \le C(\lambda, n, \beta)|x-y|^{\lambda-|\beta|}, \quad |\beta| \ge 1.$$

On the other hand, noting that for $|x| \le R$ and $|y| \ge 2R$, $|x-y| \ge |y| - |x| \ge R$. Then for any $\beta \in \mathbb{Z}^n_+$,

$$\begin{aligned}&\int_{|y|\ge 2R} D^{\beta} g(x)[v^{-q}(y) + u^{-p}(y)]\mathrm{d}y \\ &\le C(\lambda, n, \beta, R)\int_{|y|\ge 2R}[u^{-p}(y) + v^{-q}(y)]\mathrm{d}y < \infty.\end{aligned}$$

Therefore, we can differentiate $\mathcal{II}_u(x)$ and $\mathcal{II}_v(x)$ under the integral for $|x| < R$ and show that $\mathcal{II}_u(x), \mathcal{II}_v(x) \in C^\infty(B_R)$.

Next, we verify the smooth property of $\mathcal{I}_u(x)$ and $\mathcal{I}_v(x)$. Note that

$$\int_{|y|\le 2R} |x-y|^{\lambda-1}[v^{-q}(y) + u^{-p}(y)]\mathrm{d}y \le C(M)\int_{|x-y|\le 3R} |x-y|^{\lambda-1}$$

$$\leq C(M,n)\int_0^{3R} r^{\lambda-2+n}\mathrm{d}r < \infty.$$

We conclude that $\mathcal{I}_u(x)$, $\mathcal{I}_v(x) \in C^1(B_R)$. This, together with $\mathcal{II}_u(x), \mathcal{II}_v(x) \in C^\infty(B_R)$, (2.20) and (2.21), implies that $(u(x),\ v(x)) \in C^1(B_R) \times C^1(B_R)$. Meanwhile, by chain rule of derivatives and the arbitray of R, it is easy to check that $v^{-q}(x)$ and $u^{-p}(x)$ are derivative on $\mathbb{R}^n$. Therefore, we can improve the regularity of $\mathcal{I}_u(x)$ and $\mathcal{I}_v(x)$ to C^2 in $x \in B_R(0)$ which with (2.20) and (2.21), implies that $(u(x),\ v(x)) \in C^2(\mathbb{R}^n)\times C^2(\mathbb{R}^n)$. Similarly, by the bootstrapping arguments, we eventually get that $u(x), v(x) \in C^\infty(\mathbb{R}^n)$. The proof of Lemma 2.3 is completed. □

3 Proof of Theorem 1.1

Proof of Theorem 1.1 By Lemma 2.2, $f(t) = t^{1-p}\,(p \neq 1)$ being a C^1 function and the chain rule of weak derivatives, we conclude, in the sense of distribution, that

$$\frac{\nabla u^{1-p}(x)}{1-p} = \lambda\ u^{-p}(x)\int_{\mathbb{R}^n} |x-y|^{\lambda-2}(x-y)\ v^{-q}(y)\mathrm{d}y. \tag{3.1}$$

Chooses $\eta \in C_0^\infty(\mathbb{R})$ such that

$$0 \leq \eta \leq 1, \quad \eta(t) \equiv 1 \quad \text{for } |t| \leq 1$$

and

$$\mathrm{supp}(\eta) \subset [0,2), \quad |\eta'(t)| \leq 2, \quad \forall\, t \in \mathbb{R}.$$

For any $R > 1$, multiplying by $\eta(\frac{|x|}{R})x \in C_0^\infty(\mathbb{R}^n)$ on both side of (3.1), we have

$$\begin{aligned}&\int_{\mathbb{R}^n} \eta(\frac{|x|}{R})\ u^{-p}(x)\ x\cdot\nabla u(x)\ \mathrm{d}x\\ =\,&\lambda\int_{\mathbb{R}^n}\eta(\frac{|x|}{R})u^{-p}(x)\left\{\int_{\mathbb{R}^n}|x-y|^{\lambda-2}x\cdot(x-y)v^{-q}(y)\mathrm{d}y\right\}\mathrm{d}x \triangleq \mathscr{E}_1.\end{aligned} \tag{3.2}$$

By integration by parts, we rewrite the left hand side of (3.2) as follows

$$\begin{aligned}&\frac{1}{1-p}\int_{\mathbb{R}^n}\eta(\frac{|x|}{R})x\cdot\nabla u^{1-p}(x)\mathrm{d}x\\ =&-\frac{n}{1-p}\int_{\mathbb{R}^n}\eta(\frac{|x|}{R})u^{1-p}(x)\mathrm{d}x - \frac{1}{1-p}\int_{\mathbb{R}^n}\nabla(\eta(\frac{|x|}{R}))\cdot x u^{1-p}(x)\ \mathrm{d}x\\ \triangleq\,&\mathbb{A}+\mathbb{B}.\end{aligned} \tag{3.3}$$

By Lemma 2.1, we have $u^{1-p}(x) \in L^1(\mathbb{R}^n)$, which together with $0 \leq \eta \leq 1$ implies that

$$\lim_{R\to\infty}\mathbb{A} = \int_{\mathbb{R}^n}\frac{-n}{1-p}u^{1-p}(x)\ \mathrm{d}x. \tag{3.4}$$

Noting that $|\nabla(\eta(|x|/R))\cdot x| \leq 2|x|/R$, we can conclude that

$$\Big|\int_{\mathbb{R}^n}\nabla(\eta(\frac{|x|}{R}))\cdot x\ u^{1-p}(x)\ \mathrm{d}x\Big| \leq 4\int_{R\leq|x|\leq 2R} u^{1-p}(x)\mathrm{d}x,$$

which together with $u^{1-p} \in L^1(\mathbb{R}^n)$ implies that

$$\lim_{R\to\infty}\mathbb{B} = 0.$$

This combining with (3.3) and (3.4) leads to

$$\lim_{R\to\infty}\int_{\mathbb{R}^n}\eta(\frac{|x|}{R})u^{-p}(x)x\cdot\nabla u(x)\mathrm{d}x=\int_{\mathbb{R}^n}\frac{-n}{1-p}u^{1-p}(x)\mathrm{d}x. \tag{3.5}$$

Similarly, for $q\neq 1$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\eta(\frac{|x|}{R})v^{-q}(x)x\cdot\nabla v(x)\ \mathrm{d}x\\ =\ &\lambda\int_{\mathbb{R}^n}\eta(\frac{|x|}{R})v^{-q}(x)\Big\{\int_{\mathbb{R}^n}x\cdot(x-y)|x-y|^{\lambda-2}u^{-p}(y)\mathrm{d}y\Big\}\mathrm{d}x\triangleq\mathscr{E}_2\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\lim_{R\to\infty}\int_{\mathbb{R}^n}\eta(\frac{|x|}{R})v^{-q}(x)x\cdot\nabla v(x)\mathrm{d}x\\ =\ &\int_{\mathbb{R}^n}\frac{-n}{1-q}v^{1-q}(x)\mathrm{d}x-\lim_{R\to\infty}\int_{\mathbb{R}^n}\frac{1}{1-q}\nabla(\eta(\frac{|x|}{R}))\cdot xv^{1-q}(x)\mathrm{d}x\\ =\ &\int_{\mathbb{R}^n}\frac{-n}{1-q}v^{1-q}(x)\mathrm{d}x.\end{aligned} \tag{3.7}$$

Now we consider $\mathscr{E}_1$ and $\mathscr{E}_2$. For $\lambda\in(0,1)$, by Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n}\Big\{u^{-p}(y)|x-y|^{-(1-\lambda)}\eta(\frac{|x|}{R})|x|v^{-q}(x)\Big\}\mathrm{d}y\mathrm{d}x\\ \le\ &C(n,\lambda,r,s)\|u^{-p}\|_r\|\eta(\frac{|x|}{R})|x|v^{-q}(x)\|_s<\infty,\end{aligned} \tag{3.8}$$

where $1/r+1/s+(1-\lambda)/n=2$, $r>1$, $s>1$.

For $\lambda\in[1,\infty)$, we have

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n}\Big\{u^{-p}(y)|x-y|^{\lambda-1}\eta(\frac{|x|}{R})\ |x|\ v^{-q}(x)\Big\}\mathrm{d}y\mathrm{d}x\\ \le\ &C\iint_{\mathbb{R}^n\times\mathbb{R}^n}(|x|+|y|+1+|x||y|)^{\lambda-1}|x|v^{-q}(x)u^{-p}(y)\mathrm{d}x\mathrm{d}y\\ \le\ &C\iint_{\mathbb{R}^n\times\mathbb{R}^n}(1+|x|)^{\lambda}v^{-q}(x)(1+|y|)^{\lambda}u^{-p}(y)\mathrm{d}x\mathrm{d}y<\infty.\end{aligned}$$

This, with (3.8) yields that $u^{-p}(y)|x-y|^{\lambda-2}x\cdot(x-y)\eta(|x|/R)\ v^{-q}(x)$ is absolutely integrable on $\mathbb{R}^n\times\mathbb{R}^n$.

Similarly, the function $v^{-q}(y)|x-y|^{\lambda-2}x\cdot(x-y)\eta(|x|/R)\ u^{-p}(x)$ on $\mathbb{R}^n\times\mathbb{R}^n$ is also absolutely integrable.

Therefore, by the Fubini theorem, we conclude that

$$\begin{aligned}\mathscr{E}_1+\mathscr{E}_2=\ &\int_{\mathbb{R}^n}\lambda\ \eta(\frac{|x|}{R})\Big\{\int_{\mathbb{R}^n}u^{-p}(x)|x-y|^{\lambda-2}x\cdot(x-y)v^{-q}(y)\mathrm{d}y\Big\}\mathrm{d}x\\ &+\lambda\int_{\mathbb{R}^n}\eta(\frac{|x|}{R})\ v^{-q}(x)\Big\{\int_{\mathbb{R}^n}x\cdot(x-y)|x-y|^{\lambda-2}u^{-p}(y)\mathrm{d}y\Big\}\mathrm{d}x\\ =\ &\lambda\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}u^{-p}(x)v^{-q}(y)|x-y|^{\lambda-2}(x-y)\Big\{x\eta(\frac{|x|}{R})-y\eta(\frac{|y|}{R})\Big\}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

To pass to the limit on the above, we need to build up a prior estimate of $\mathscr{E}_1+\mathscr{E}_2$. Let $\tilde{\mathbf{f}}(x)$ be a vector-value function from $\mathbb{R}^n$ to $\mathbb{R}^n$, given by

$$\tilde{\mathbf{f}}(x)=(f_1(x),f_2(x),\cdots,f_n(x))\triangleq\eta(\frac{|x|}{R})\ x.$$

It is easy to check that

$$\frac{\partial f_k(x)}{\partial x_l} = \eta'(\frac{|x|}{R})\frac{x_l}{|x|}\ \frac{x_k}{R} + \eta(\frac{|x|}{R})\ \delta_k^l, \qquad |x| \neq 0,$$

where δ_k^l is the Kronecker function. In view of the definition of η, we know that $|\frac{\partial}{\partial x_l} f_k(x)| \leq 5$ and

$$|\tilde{\mathbf{f}}(x) - \tilde{\mathbf{f}}(y)| \leq |\nabla \tilde{\mathbf{f}}(\xi)\ (x-y)| \leq C(n)|x-y|, \qquad x,\, y \in \mathbb{R}^n.$$

Therefore

$$\begin{aligned} &\left| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u^{-p}(x)\ v^{-q}(y)|x-y|^{\lambda-2}(x-y)\ \cdot \left\{ x\ \eta(\frac{|x|}{R}) - y\eta(\frac{|y|}{R}) \right\} \mathrm{d}y\mathrm{d}x \right| \\ &\leq C(n,\lambda)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|x|)^{\lambda} u^{-p}(x)\mathrm{d}x(1+|y|)^{\lambda} v^{-q}(y)\mathrm{d}y < \infty, \end{aligned}$$

and by (1.1)

$$\begin{aligned} \lim_{R\to\infty} \mathscr{E}_1 + \mathscr{E}_2 &= \lambda \lim_{R\to\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u^{-p}(x)\ v^{-q}(y)|x-y|^{\lambda-2}(x-y) \\ &\quad \times \left\{ x\ \eta(\frac{|x|}{R}) - y\eta(\frac{|y|}{R}) \right\} \mathrm{d}y\mathrm{d}x \\ &= \lambda \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\lambda-2} u^{-p}(x)\ v^{-q}(y)\ \ (x-y)\ \cdot (x-y)\mathrm{d}y\mathrm{d}x \\ &= \lambda\ \|u^{1-q}\|_1. \end{aligned}$$

This together with (2.5), (3.2) and (3.5)–(3.7) yields that

$$\frac{1}{1-p} + \frac{1}{1-q} = -\frac{\lambda}{n}.$$

Theorem 1.1 is proved. □

References

[1] Gabriella C, Lorenzo D, Enzo M. Representation formulae for solutions to some classes of higher order systems and related Liouville theorem. Milan J Math, 2008, 76: 27–67

[2] Chen W, Li C. An integral system and Lane-Emden conjecture. Discrete Contin Dyn Syst, 2008, **4**: 1167–1184

[3] Chen W, Li C. Regularity of solutions for a system of integral equations. Comm Pure Appl Anal, 2005, **4**(1): 1–8

[4] Chen W, Li C, Ou B. Classification of solutions for an integral equation. Comm Pure Appl Math, 2006, **59**(3): 330–343

[5] Chen W, Li C, Ou B. Classification of solutions for a system of integral equation. Comm Partial Differential Equations, 2005, **30**: 59–65

[6] Cheng Z, Huang G, Li C. A Liouville theorem for subcritical Lane-Emden system. arXiv:1412.7275v1

[7] Dou J, Zhu M. Reversed Hardy-Littewood-Sobolev inequality. Int Math Res Not, 2015, **19**: 9696–9726

[8] Jin C, Li C. Qualitative analysis of some systems of integral equations. Calc Var Partial Differential Equations, 2006, **26**: 447–457

[9] Lei Y, Li C. Sharp criteria of Liouville type for some nonlinear systems. Discrete Contin Dyn Syst, 2016, **36**: 3277–3315

[10] Lei Y. On the integral systems with negative exponents. Discrete Contin Dyn Syst, 2015, **35**: 1039–1057

[11] Li C. A degree theory approach for the shooting method. arXiv:1301.6232

[12] Li C, Villavert J. Existence of positive solutions to semilinear elliptic systems with supercritical growth. Comm Partial Differential Equations, 2016, **41**: 1029–1039

[13] Li C, Villavert J. A degree theory framework for semilinear elliptic systems. Proc Amer Math Soc, 2016, **144**: 3731–3740

[14] Li Y. Remark on some conformally invariant integral equations: the method of moving spheres. J Eur Math Soc, 2004, **6**: 153–180

[15] Lieb E H. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann Math, 1983, **114**: 349–374

[16] Liu J, Guo Y, Zhang Y. Liouville type theorems for polyharmonic systems in $\mathbb{R}^n$. J Differential Equations, 2006, **225**: 685–709

[17] Mitidieri E. Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^n$. Differential Integral Equations, 2009, **9**: 465–479

[18] Ngó Q A, Nguyen V H. Sharp reversed Hardy Littlewood Sobolev inequality on $\mathbb{R}^n$. Israel J Math, 2017, **220**(1): 189–223

[19] Polacik P, Quittner P, Souplet P. Singularity and decay esimates in superlinear problems via Liouville-type theorems. part I: Elliptic equations and systems. Duke Math J, 2007, **139**: 555–579

[20] Reichel W, Zou H. Non-existence results for semilinear cooperative elliptic systems via moving spheres. J Differential Equations, 2004, **161**: 219–243

[21] Serrin J, Zou H. Existence of positive solutions of the Lane-Emden systems. Atti Semin Mat Fis Univ Modena Reggio Emilia, 1998, **46**: 369–380

[22] Souplet P. The proof of the Lane-Emden conjecture in 4 space dimension. Adv Math, 2009, **221**(5): 1409–1427

[23] Souplet P. Liouville-type theorems for elliptic Schròdinger systems associated with copositive matrices. Networks and Heterogeneous Media, 2012, **7**(4): 967–988

[24] Xu J, Tan Z. Classification of solutions for a class of singular integral system. Acta Math Sci, 2011, **31B**(4): 1449–1456

[25] Xu J, Wu H, Tan Z. The non-existence results for a class of integral equation. J Differential Equations, 2014, **256**: 1873–1902

[26] Xu X. Exact solutions of nonlinear conformally invariant integral equations in $\mathbb{R}^3$. Adv Math, 2005, **194**: 485–503

[27] Xu X. Uniqueness and non-existence theorem for conformally invariant equations. J Funct Anal, 2005, **222**: 1–28