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FINITE TIME EMERGENCE OF A SHOCK WAVE FOR SCALAR CONSERVATION LAWS VIA LAX-OLEINIK FORMULA*

Zejun WANG (王泽军)† *Qi ZHANG* (张琦)
Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China
E-mail: *wangzejun@gmail.com; zhangqinju@gmail.com*

Abstract In this paper, we use Lax-Oleinik formula to study the asymptotic behavior for the initial problem of scalar conservation law $u_t + F(u)_x = 0$. First, we prove a simple but useful property of Lax-Oleinik formula (Lemma 2.7). In fact, denote the Legendre transform of $F(u)$ as $L(\sigma)$, then we can prove that the quantity $F(q) - qF'(q) + L(F'(q))$ is a constant independent of q. As a simple application, we first give the solution of Riemann problem without using of Rankine-Hugoniot condition and entropy condition. Then we study the asymptotic behavior of the problem with some special initial data and prove that the solution contains only a single shock for $t > T^*$. Meanwhile, we can give the equation of the shock and an explicit value of T^*.

Key words scalar conservation law; Lax-Oleinik formula; Riemann problem; asymptotic behavior.

2010 MR Subject Classification 35B40; 35L65; 76Y05

1 Introduction

In this paper, we study the following scalar conservation law

$$u_t + F(u)_x = 0, \tag{1.1}$$

where $F(u): \mathbb{R} \to \mathbb{R}$ is a C^2 function and is uniformly convex. The initial data is given by

$$u(x, 0) = u_0(x), \tag{1.2}$$

where $u_0 \in L^\infty$ and measurable. This problem was widely studied by many authors. The existence of global weak solution to (1.1)–(1.2) can be proved by Lax-Friendrich scheme [1, 2], viscosity method [3–6], Glimm Scheme [7] and compensated compactness method [8]. The uniqueness of solution to (1.1)–(1.2) is due to Oleinik [9] where the entropy condition was given (see (2.15) below). The asymptotic behavior of (1.1)–(1.2) was also studied by many authors.

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†Corresponding author: Zejun WANG.

Earlier studies can be found in [10, 11]. When $u_0(x)$ is bounded and measurable, then $u(x,t)$ decays to 0 at a rate $O(t^{-1/2})$ and when the support of $u_0(x)$ is compact, $u(x,t)$ decays to an N-wave at a rate $O(t^{-1/2})$. This can be proved by characteristic method [2] or by Lax-Oleinik formula [12]. Lax-Oleinik formula is an explicit formula of the solution to (1.1)–(1.2) which was initially obtained by Lax [11] and Oleinik [9], also see [12, 13].

In Section 2 of this paper, we'll recall some basic facts of Lax-Oleinik formula and give two related properties. As an example, we use these properties to study Riemann problem and give the exact solution without help of the Rankine-Hugoniot condition and entropy condition. In fact, the weak solution of this Riemann problem is well known and can be easily given (see [2, 12, 14]). Then in Section 3, we will go on to study the finite time emergence of a shock when

$$u_0(x) = \begin{cases} u_-, \ x < -R, \\ u_+, \ x > R \end{cases} \tag{1.3}$$

with $u_- > u_+$, and $u_0(x)$ is bounded measurable in $[-R, R]$. To study this problem, an efficient method is generalized characteristics (see [15–17]). It was shown that a single shock must appear in finite time. For some generalization of generalized characteristics to inhomogeneous conservation laws, see [18, 19]. In [13], Lax-Oleinik formula was used in the special case of stationary shock solutions of Burgers equation where $F(u) = u^2/2$. A simplified proof by using of generalized characteristics can be found in [20]. In Section 3 of this paper, we use Lax-Oleinik formula to give another proof, and we also give the equation of the shock and an estimate on the time of the emergency of the shock.

2 Preliminary

Denote L as the Legendre transform of F, that is,

$$L(\sigma) \triangleq \sup_{q\in\mathbb{R}}\{\sigma q - F(q)\}, \tag{2.1}$$

and it can be given explicitly as (see [12])

$$L(\sigma) = \sigma G(\sigma) - F(G(\sigma)), \tag{2.2}$$

where

$$G = (F')^{-1}. \tag{2.3}$$

Simple analysis gives

$$L'(\sigma) = G(\sigma). \tag{2.4}$$

Since F is uniformly convex, that is

$$F''(u) \geq \alpha > 0. \tag{2.5}$$

From $F'(G(\sigma)) = \sigma$, we have $F''(G(\sigma))G'(\sigma) = 1$. Thus from (2.4), we can get $F''(G(\sigma))L''(\sigma) = 1$. Thus L is also uniformly convex. Denote $\theta = \frac{1}{\alpha} > 0$, then we have

$$L''(\sigma) \geq \theta > 0. \tag{2.6}$$

Denote

$$A(y; x, t) \triangleq tL(\frac{x-y}{t}) + h(y), \tag{2.7}$$

where $h(y) = \int_0^y u_0(z)\mathrm{d}z$. $A(y;x,t)$ is also called action function. It is easy to get

$$\frac{\partial A}{\partial y} = -G(\frac{x-y}{t}) + u_0(y). \tag{2.8}$$

Thus from $\frac{\partial A}{\partial y} = 0$, we can get $-G(\frac{x-y}{t}) + u_0(y) = 0$, which, with the help of (2.3), is equivalent to

$$x = y + g(y)t. \tag{2.9}$$

Here for simplicity, we denote $g(y) = F'(u_0(y))$. This equation gives a relation of x, t and y, which is a necessary condition for y to be a minimum point of $A(y;x,t)$.

Remark 2.1 (2.9) has the same form as the characteristic of equation (1.1). However, they are not exactly the same. For fixed y, every line satisfying (2.9) is called a characteristic. But here (2.9) is only a necessary condition that y is the minimum point of $A(y;x,t)$ since y can also be a maximum point.

Lemma 2.2 For any x, t, y satisfying (2.9), we have

$$G'(\frac{x-y}{t})F''(u_0(y)) = 1. \tag{2.10}$$

Proof Differentiating identical equation $G(F'(u)) = u$ with respect to u, we get $G'(F'(u))F''(u) = 1$. Take $u = u_0(y)$ and use (2.9), then we get (2.10). □

Lemma 2.3 If $u_0(y) \in C^1$, then for any x, t, y satisfying (2.9), we have

(i) if $1 + g'(y)t > 0$, then y is a minimum point of $A(y;x,t)$;

(ii) if $1 + g'(y)t < 0$, then y is a maximum point of $A(y;x,t)$.

Proof If $u_0(y) \in C^1$, we can easily get

$$\frac{\partial^2 A}{\partial y^2} = \frac{1}{t}G'(\frac{x-y}{t}) + u_0'(y). \tag{2.11}$$

Thus, to assure that y is a minimum point of $A(y;x,t)$, a sufficient condition is $\frac{1}{t}G'(\frac{x-y}{t}) + u_0'(y) > 0$, which, by using of Lemma 2.2, is equivalent to $1 + g'(y)t > 0$. Thus (i) is proved.

(ii) can be proved analogously. □

Theorem 2.4 (see [12]) Assume $F : \mathbb{R} \to \mathbb{R}$ is smooth, uniformly convex, and $u_0(x) \in L^\infty(\mathbb{R})$.

(1) For each time $t > 0$, there exists, for all but at most countably many values of $x \in \mathbb{R}$, a unique point $y(x,t)$ such that

$$\min_{y\in\mathbb{R}} \left\{ tL(\frac{x-y}{t}) + h(y) \right\} = tL\left(\frac{x-y(x,t)}{t}\right) + h(y(x,t)). \tag{2.12}$$

(2) The mapping $x \to y(x,t)$ is nondecreasing.

(3) For each time $t > 0$, the function u defined by

$$u(x,t) = G(\frac{x-y}{t}) \tag{2.13}$$

is a weak solution of the initial-value (1.1)–(1.2).

(2.13) is called Lax-Oleinik formula. This theorem shows that the solution of (1.1)–(1.2) can be given as in (2.13) and y belongs to the set of minimum points of the following problem

$$\min_{y\in\mathbb{R}} \left\{ tL(\frac{x-y}{t}) + h(y) \right\}. \tag{2.14}$$

Theorem 2.5 (see [12]) Under the assumptions of Theorem 2.4, there exists a constant C such that the function u defined by the Lax-Oleinik formula satisfies the entropy inequality

$$u(x+z,t)-u(x,t)\le \frac{C}{t}z \tag{2.15}$$

for all $t>0$ and $x,z\in\mathbb{R},\ z>0$.

Theorem 2.5 shows that Lax-Oleinik formula satisfies the entropy condition, which assure the uniqueness of the solution to (1.1)–(1.2), so we firmly believe that Lax-Oleinik formula is a useful tool to study the scalar conservation law.

Before going to our main results, we give two properties of Lax-Oleinik formula. Since $u_0(x)\in L^\infty$, we have for some constant M,

$$N\le F'(u_0(x))\le M. \tag{2.16}$$

Lemma 2.6 For every $y<x-Mt$, we have

$$tL(\frac{x-y}{t})+h(y)>tL(M)+h(x-Mt), \tag{2.17}$$

and for every $y>x-Nt$, we have

$$tL(\frac{x-y}{t})+h(y)>tL(N)+h(x-Nt). \tag{2.18}$$

Proof Assume that $y<x-Mt$. Since $F'(u_0(z))\le M$, we can get $u_0(z)\le G(M)$. Since L is strictly convex, we have

$$\begin{aligned}
&tL(\frac{x-y}{t})+h(y)-tL(M)-h(x-Mt)\\
&>L'(M)(x-Mt-y)+\int_{x-Mt}^{y}u_0(z)\mathrm{d}z\\
&=\int_{y}^{x-Mt}(G(M)-u_0(z))\mathrm{d}z\ge 0.
\end{aligned}$$

Thus we get (2.17). (2.18) can also be proved similarly. □

This lemma shows that to study the minimum point in problem (2.14), we need only consider the values of y in between $[x-Mt,x-Nt]$.

In this paper, for any two given constant states u_- and u_+, we always denote

$$\sigma_\pm=F'(u_\pm). \tag{2.19}$$

Thus we have

$$L'(\sigma_\pm)=G(F'(u_\pm))=u_\pm. \tag{2.20}$$

The following lemma is important in our analysis below. It gives a closed relation between L and F.

Lemma 2.7 For any $q\in\mathbb{R}$, the quantity $F(q)-qF'(q)+L(F'(q))$ is a constant. Specially, for any constant states u_- and u_+, we have

$$F(u_-)-u_-\sigma_-+L(\sigma_-)=F(u_+)-u_+\sigma_++L(\sigma_+). \tag{2.21}$$

Proof Denote $Q(q)=F(q)-qF'(q)+L(F'(q))$. It is easy to verify that $Q'(q)=0$ with the help of (2.3) and (2.4). □

Next, as a simple application, we use Lax-Oleinik formula to study the Riemann problem of (1.1), i.e., the initial data is given by

$$u(x,0)=\begin{cases}u_-,\ x<0,\\ u_+,\ x>0.\end{cases}\tag{2.22}$$

Riemann problem for scalar conservation law is well known. As stated in Section 1, when $u_- > u_+$, with the help of Rankine-Hugoniot condition and the entropy condition, we can get the solution. The solution contains a single shock which divide the upper plane into two constant states (see (2.23)). When $u_- < u_+$, we first observe that the problem admits a self-similar solution depending only on $\sigma = \frac{x}{t}$, then try to find a self-similar solution of this form. The solution contains a single rarefaction wave between two constant states u_- and u_+ (see (2.30)).

The following theorem gives the solution of the Riemann problem (1.1)–(2.22) for the case $u_- > u_+$ by Lax-Oleinik formula without using of Rankine-Hugoniot condition and the entropy condition.

Theorem 2.8 For Riemann problem (1.1)–(2.22), if $u_- > u_+$, the solution can be written as

$$u(x,t)=\begin{cases}u_-,\ \ x<st,\\ u_+,\ \ x>st,\end{cases}\tag{2.23}$$

where

$$s=\frac{F(u_-)-F(u_+)}{u_- - u_+}.\tag{2.24}$$

Proof Let's first calculate the value of the minimum point y in (2.14). From (2.9), and with the help of Lemma 2.6, we can get

$$x=y+F'(u_0(y))t=\begin{cases}y+\sigma_-t,\ \ x<\sigma_+t,\\ y+\sigma_+t,\ \ x>\sigma_-t.\end{cases}\tag{2.25}$$

Since $u_- > u_+$, we have $\sigma_- > \sigma_+$. See Fig. 2.1 and Fig. 2.2.

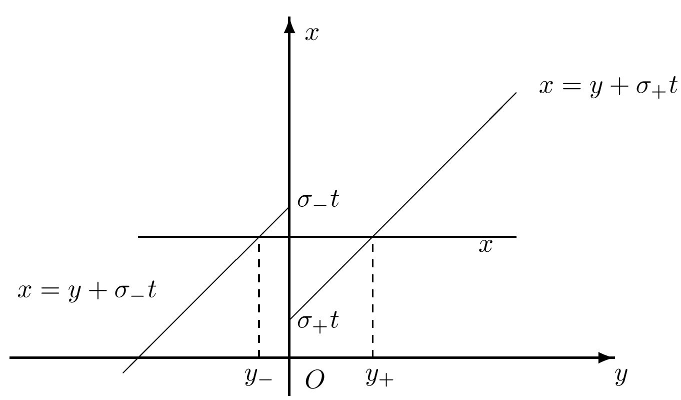

Fig. 2.1

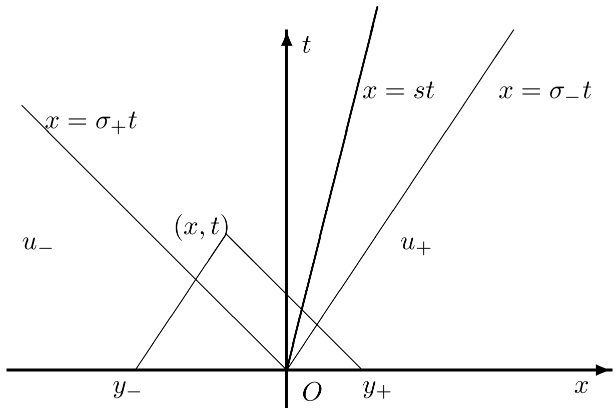

Fig. 2.2

Obviously, if $x < \sigma_+ t$(or $x > \sigma_- t$), there exists a unique $y_- = x - \sigma_- t$ (or $y_+ = x - \sigma_+ t$) satisfying (2.25). Since $u_0'(y_-) = u_0'(y_+) = 0$, from (i) of Lemma 2.3, we know that both y_- and y_+ are the minimum points of problem (2.14). Thus

$$y = \begin{cases} x - \sigma_- t, & x < \sigma_+ t, \\ x - \sigma_+ t, & x > \sigma_- t. \end{cases} \tag{2.26}$$

Thus when $x < \sigma_+ t$, from (2.13), we can get

$$u(x,t) = G\left(\frac{x - (x - \sigma_- t)}{t}\right) = u_-.$$

By the same way, we can get $u(x,t) = u_+$ for $x > \sigma_- t$. In summary, (2.23) holds for $x > \sigma_- t$ and $x < \sigma_+ t$. For the case of $\sigma_+ t < x < \sigma_- t$, we will prove it in the next lemma. The proof is completed. □

Lemma 2.9 For (x,t) satisfying $\sigma_+ t < x < \sigma_- t$, a necessary and sufficient condition of

$$tL(\sigma_-) + h(x - \sigma_- t) \lesseqqgtr tL(\sigma_+) + h(x - \sigma_+ t) \tag{2.27}$$

is $x \lesseqqgtr st$, where s is given by (2.24).

Proof We only prove the case of "<" since the other cases can be proved similarly. When $\sigma_+ t < x < \sigma_- t$, as shown in Fig. 2.1, for every (x,t), we have two possible minimum points $y_- = x - \sigma_- t$ and $y_+ = x - \sigma_+ t$. Obviously, $y_- < 0$ and $y_+ > 0$, and we can easily know that y_- and y_+ are both minimum points. Thus (2.27) is equivalent to

$$tL(\sigma_-) + u_-(x - \sigma_- t) \lesseqqgtr tL(\sigma_+) + u_+(x - \sigma_+ t). \tag{2.28}$$

By using of (2.21) in Lemma 2.7, $L(\sigma_-) - L(\sigma_+) = F(u_+) - F(u_-) + \sigma_- u_- - \sigma_+ u_+$, then (2.28) is equivalent to

$$x \lesseqqgtr \frac{F(u_-) - F(u_+)}{u_- - u_+} t. \tag{2.29}$$

The proof of Lemma 2.9 is completed. □

From Lemma 2.9, we know that when $\sigma_+ t < x < st$, we should choose the minimum point $y_- = x - \sigma_- t$, thus $u(x,t) = G(\frac{x-y_-}{t}) = G(\sigma_-) = u_-$. When $st < x < \sigma_- t$, we should choose the minimum point $y_+ = x - \sigma_+ t$, and $u(x,t) = u_+$. Thus we prove Theorem 2.8.

Remark 2.10 On the line $x = st$, $y_- = x - \sigma_- t$ and $y_+ = x - \sigma_+ t$ are both minimum points, thus on this line, $u(x,t)$ is double valued. As is well known, the line $x = st$ is called shock wave, and s is its velocity, $s = \frac{F(u_-)-F(u_+)}{u_- - u_+}$ is the well knows called Rankine-Hugoniot condition.

Remark 2.11 By similar analysis, for the case $u_- < u_+$, the unique solution of Riemann problem (1.1), (2.22) can be given as

$$u(x,t) = \begin{cases} u_-, & x < \sigma_- t, \\ (F')^{-1}(\dfrac{x}{t}), & \sigma_- t < x < \sigma_+ t, \\ u_+, & x > \sigma_+ t. \end{cases} \tag{2.30}$$

In fact, since $u_- < u_+$, we have $\sigma_- < \sigma_+$. By similar analysis as in Theorem 2.8, we can get

$$u(x,t) = \begin{cases} u_-, \ x < \sigma_- t, \\ u_+, \ x > \sigma_+ t. \end{cases} \tag{2.31}$$

For all x which are in between $\sigma_+ t < x < \sigma_- t$, the minimum point is always the same point: $y = 0$. Thus from (2.13), we get $u(x,t) = G(\frac{x}{t}) = (F')^{-1}(\frac{x}{t})$.

3 Asymptotic Behavior

In this section, we study the asymptotic behavior of the Cauchy problem (1.1)–(1.2) where $u_0(x)$ is in L^∞, measurable and satisfies (1.3). When $u_- > u_+$, we have $\sigma_+ < s < \sigma_-$. Denote $a = \min(s - \sigma_+, \sigma_- - s)$, $\mu = \max(|u_-|, |u_+|)$ and $M = ||u_0||_{L^\infty}$. Set

$$T^* = 2R(\mu + M)\left(\frac{4}{a(u_- - u_+)} + \frac{\alpha}{a^2}\right), \tag{3.1}$$

where α is given in (2.5). The following is the main theorem in this section which shows that a single shock must appear in finite time.

Theorem 3.1 For Cauchy problem (1.1)–(1.2) satisfying (1.3), if $u_- > u_+$, then for $t > T^*$, the solution can be written as

$$u(x,t) = \begin{cases} u_-, \ x < st + x_0, \\ u_+, \ x > st + x_0, \end{cases} \tag{3.2}$$

where s is given by (2.24) and

$$x_0 = \frac{(u_+ + u_-)R - \int_{-R}^{R} u_0(x)\mathrm{d}x}{u_+ - u_-}. \tag{3.3}$$

We will prove Theorem 3.1 by several lemmas.

Lemma 3.2 Given $(x,t) \in \mathbb{R} \times (0, +\infty)$, then the following hold

(i) if $x - \sigma_+ t > R$, then for any $y \in [R, +\infty)$, we have

$$tL(\frac{x-y}{t}) + h(y) > tL(\sigma_+) + h(x - \sigma_+ t); \tag{3.4}$$

(ii) if $x-\sigma_+t<R$, then for any $y\in[R,+\infty)$, we have

$$tL(\frac{x-y}{t})+h(y)>tL(\frac{x-R}{t})+h(R); \tag{3.5}$$

(iii) if $x-\sigma_-t<-R$, then for any $y\in(-\infty,-R]$, we have

$$tL(\frac{x-y}{t})+h(y)>tL(\sigma_-)+h(x-\sigma_-t); \tag{3.6}$$

(iv) if $x-\sigma_-t>-R$, then for any $y\in(-\infty,-R]$, we have

$$tL(\frac{x-y}{t})+h(y)>tL(\frac{x+R}{t})+h(-R). \tag{3.7}$$

Proof We only prove (i) and (ii), for the other two cases can be proved similarly. If $x-\sigma_+t>R$, then by using of the convexity of L and $L'(\sigma_+)=u_+$, and direct calculation gives

$$\begin{aligned}&tL(\frac{x-y}{t})+h(y)-tL(\sigma_+)-h(x-\sigma_+t)\\ \geq\ & t(L(\sigma_+)+L'(\sigma_+)(\frac{x-y}{t}-\sigma_+))-tL(\sigma_+)+\int_{x-\sigma_+t}^{y}u_+\mathrm{d}z=0,\end{aligned}$$

which proves (i). If $x-\sigma_+t<R$, we have $\frac{x-R}{t}<\sigma_+$, then $L'(\frac{x-R}{t})<L'(\sigma_+)=u_+$. Thus we have

$$\begin{aligned}&tL(\frac{x-y}{t})+h(y)-tL(\frac{x-R}{t})-h(R)\\ \geq\ & t\left(L(\frac{x-R}{t})+L'(\frac{x-R}{t})(\frac{R-y}{t})\right)-tL(\frac{x-R}{t})+\int_{R}^{y}u_+\mathrm{d}z\\ =\ &\left(u_+-L'(\frac{x-R}{t})\right)(y-R)\geq 0.\end{aligned}$$

Thus (ii) is proved. □

Remark 3.3 (i) and (ii) of Lemma 3.2 show that if $x-\sigma_+t$ lies in $[R,+\infty)$, then it is the unique minimum point of (2.14) in $[R,+\infty)$. If $x-\sigma_+t<R$, then R is the unique minimum point of (2.14) in $[R,+\infty)$. The cases of (iii) and (iv) are the same.

Lemma 3.4 Given $(x,t)\in\mathbb{R}\times[0,+\infty)$ satisfying $x-\sigma_-t<-R,\ x-\sigma_+t>R$, a sufficient and necessary condition of

$$tL(\sigma_-)+h(x-\sigma_-t)\lesseqqgtr tL(\sigma_+)+h(x-\sigma_+t) \tag{3.8}$$

is $x\lesseqqgtr x_0+st$, where x_0 is given in (3.3).

Proof We only prove the case of "<" since the other cases can be proved similarly. In this case, (3.8) is equivalent to

$$\int_{x-\sigma_-t}^{x-\sigma_+t}u_0(y)\mathrm{d}y+t(L(\sigma_+)-L(\sigma_-))>0.$$

Since $x-\sigma_-t<-R,\ x-\sigma_+t>R$, it can be simplified as

$$\int_{-R}^{R}u_0(y)\mathrm{d}y+(x-\sigma_+t-R)u_++(-R-x+\sigma_-t)u_-+t(L(\sigma_+)-L(\sigma_-))>0,$$

or

$$\int_{-R}^{R}u_0(y)\mathrm{d}y-x(u_--u_+)-R(u_-+u_+)+t(\sigma_-u_--\sigma_+u_++L(\sigma_+)-L(\sigma_-))>0.$$

By using of Lemma 2.7, and the definition of s in (2.24), it is equivalent to

$$x(u_- - u_+) - t(F(u_-) - F(u_+)) < \int_{-R}^{R} u_0(y)\mathrm{d}y - R(u_+ + u_-),$$

or

$$x < \frac{(u_+ + u_-)R - \int_{-R}^{R} u_0(x)\mathrm{d}x}{u_+ - u_-} + st.$$

The proof is completed. □

For definiteness, we denote three lines: $l_1:\ x = -R+\sigma_- t,\ l_2:\ x = x_0+st,\ l_3: x = R+\sigma_+ t$ (see Fig. 3.1), then $x - \sigma_- t < -R$ or $x - \sigma_+ t > R$ means on the left side of l_1 or on the right side of l_3 respectively.

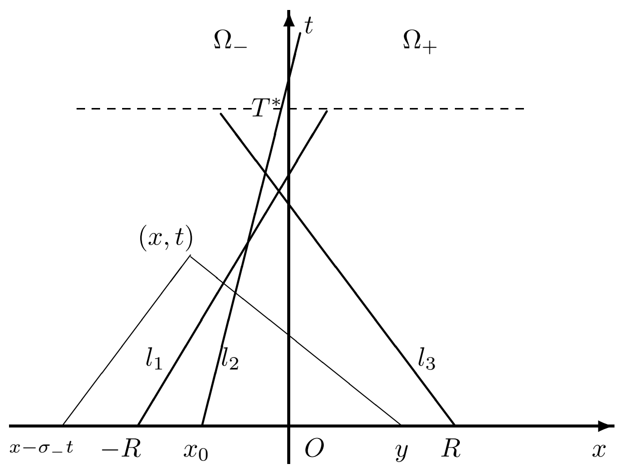

Fig. 3.1

Lemma 3.5 Assume that $t > T^*$, then the following hold

(1) if $x < -R + \sigma_- t$ and $x < x_0 + st$, then for any $y \in \mathbb{R}$,

$$tL(\frac{x-y}{t}) + h(y) \geq tL(\sigma_-) + h(x - \sigma_- t); \tag{3.9}$$

(2) if $x > R + \sigma_+ t$ and $x > x_0 + st$, then for any $y \in \mathbb{R}$,

$$tL(\frac{x-y}{t}) + h(y) \geq tL(\sigma_+) + h(x - \sigma_+ t). \tag{3.10}$$

Proof We only prove (3.9) since (3.10) can be proved similarly. Since $x < -R + \sigma_- t$, from (iii) of Lemma 3.2, we know (3.9) holds for $y < -R$. If $y \geq R$ and $x - \sigma_+ t > R$, from (i) of Lemma 3.2, we need only to show that

$$tL(\sigma_-) + h(x - \sigma_- t) < tL(\sigma_+) + h(x - \sigma_+ t). \tag{3.11}$$

Since $x < x_0 + st$, from Lemma 3.4, (3.11) holds. If $y \geq R$ and $x - \sigma_+ t \leq R$, from (ii) of Lemma 3.2, we need only to show that

$$tL(\sigma_-) + h(x - \sigma_- t) < tL(\frac{x-R}{t}) + h(R).$$

In other words, we need only consider the case $y = R$, which can be included in the case $y \in [-R, R]$.

When $y \in [-R, R]$, denote $p(y) = \int_{-R}^{y}(u_0(z) - u_-)\mathrm{d}z$, $q(y) = \int_{y}^{R}(u_+ - u_0(z))\mathrm{d}z$, then it is easy to verify that

$$|p(y)| \le 2R(\mu + M),\ |q(y)| \le 2R(\mu + M). \tag{3.12}$$

Since $x_0 - y = \frac{1}{u_- - u_+}(p(y) - q(y))$, we have

$$|x_0 - y| \le \frac{4R(\mu + M)}{u_- - u_+}. \tag{3.13}$$

Since $t > T^*$, we can deduce that $t > \frac{4R(\mu+M)}{(u_- - u_+)a}$, which means that

$$at - x_0 + y > 0. \tag{3.14}$$

From (2.6), we have $L(\frac{x-y}{t}) \ge L(\sigma_-) + u_-(\frac{x-y}{t} - \sigma_-) + \theta(\frac{x-y}{t} - \sigma_-)^2$. Thus to prove (3.9), we need only to prove that

$$\theta(x - y - \sigma_- t)^2 + tp(y) \ge 0. \tag{3.15}$$

$$\theta(at - x_0 + y)^2 + tp(y) \ge 0. \tag{3.16}$$

Since $y \in [-R, R]$ and $a > 0$, (3.16) obviously holds when T is large enough. Next we will give an estimate of T^*. With the help of (3.12) and (3.13), we know that a sufficient condition of (3.16) is

$$\left(at - \frac{4R(\mu + M)}{u_- - u_+}\right)^2 - 2\alpha R(\mu + M)t \ge 0, \tag{3.17}$$

or

$$at - \frac{4R(\mu + M)}{u_- - u_+} > \sqrt{2\alpha R(\mu + M)t}. \tag{3.18}$$

Condition $t > T^*$ can be rewritten as

$$at > 2(\mu + M)R\left(\frac{4}{u_- - u_+} + \frac{\alpha}{a}\right). \tag{3.19}$$

By using of ε-Cauchy's inequality, we have $\sqrt{2R(\mu + M)t} \le \frac{at}{2\sqrt{\alpha}} + \frac{\sqrt{\alpha}R(\mu+M)}{a}$, or

$$\frac{2\alpha R(\mu + M)}{a} \ge 2\sqrt{2\alpha R(\mu + M)t} - at. \tag{3.20}$$

(3.19) and (3.20) imply (3.9). The proof is completed. □

From Lemmas 3.2 to 3.5, we know that when $t > T^*$ and (x, t) is on the left side of both l_1 and l_2, we can choose $y = x - \sigma_- t$ and then $u = u_-$. When $t > T^*$ and (x, t) is on the right side of both l_2 and l_3, we can choose $y = x - \sigma_+ t$ and then $u = u_+$. Thus when $t > T^*$, the solution is given as in (3.2), and $x = x_0 + xt$ is the unique the shock wave. As shown in Fig. 3.1, $u = u_-$ in the region Ω_- and $u = u_+$ in the region Ω_+.

Remark 3.6 If $R = 0$, we can easily know that $x_0 = 0$ and $T^* = 0$. This is exactly the case in Theorem 2.8.

Remark 3.7 Denote the time when l_1 and l_2 intersects as T_1, and the time when l_2 and l_3 intersect as T_2, then $t > T^*$ implies $t > T_1$ and $t > T_2$ since $\max(T_1, T_2) \le \frac{2R(\mu+M)}{a(u_- - u_+)}$.

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