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Controlling weighted voting games by deleting or adding players with or without changing the quota

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Abstract

Weighted voting games are a well-studied class of succinct simple games that can be used to model collective decision-making in, e.g., legislative bodies such as parliaments and shareholder voting. Power indices [1–4] are used to measure the influence of players in weighted voting games. In such games, it has been studied how a distinguished player's power can be changed, e.g., by merging or splitting players (the latter is a.k.a. false-name manipulation) [5, 6], by changing the quota [7], or via structural control by adding or deleting players [8]. We continue the work on the structural control initiated by Rey and Rothe [8] by solving some of their open problems. In addition, we also modify their model to a more realistic setting in which the quota is indirectly changed during the addition or deletion of players (in a different sense than that of Zuckerman et al. [7] who manipulate the quota directly without changing the set of players), and we study the corresponding problems in terms of their computational complexity.

Keywords Cooperative game theory · Weigthed voting game · Computational complexity

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1 Introduction

Weighted voting games are an important class of compactly representable simple games and have been thoroughly studied in cooperative game theory (see, e.g., the textbooks [9–11] and the book chapter [12]). Most crucially, WVGs have been analyzed in terms of power indices that describe how much influence a player has in a game. Well-known power indices are the *normalized Penrose-Banzhaf index* due to Penrose [3] and Banzhaf [1], the *probabilistic*

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Penrose-Banzhaf index due to Dubey and Shubik [2], and the *Shapley-Shubik index* due to Shapley and Shubik [4]. We will focus on the latter two.

There are many applications of WVGs. They can be used for collective decision-making in legislative bodies (e.g., in parliamentary voting), in order to analyze the voting structures of the European Union Council of Ministers and the International Monetary Fund [13, 14], they are applied in joint stock companies where each shareholder gets votes in proportion to the ownership of a stock and in automated stock-trading systems [15, 16], and widely used in many practical application areas beyond social choice theory and game theory.

Just as for voting rules in computational social choice [17–19], for judgment aggregation procedures [20], and for algorithms and protocols in fair division [21, 22], strategic behavior has attracted much attention for WVGs. Bachrach and Elkind [23] were the first to study the complexity of *false-name manipulation*, i.e., changing the players' power indices by splitting a player into several players (distributing the weight among them), or by merging several players into one (adding up their weights). These problems have been further analyzed by Aziz et al. [5, 24], Faliszewski and Hemaspaandra [25], and Rey and Rothe [6]. Zuckerman et al. [7] studied the problem of influencing power indices in WVGs by directly *manipulating the quota*. Inspired by electoral control of voting rules [26, 27], Rey and Rothe [8] introduced problems of *structural control by adding players to and by deleting players from WVGs* and studied them in terms of their computational complexity. Continuing their analysis, in Section 3 we solve some of their open problems regarding control by deleting players from WVGs, also fixing a minor flaw in their paper [8] for bounds of how much the Shapley-Shubik index can change by deleting players.

In Section 4, we modify the model presented by Rey and Rothe [8] in a natural way: While they assume that the quota remains the same even though players have been added to or deleted from a weighted voting game, we will assume that the quota will change accordingly in the modified game, i.e., the quota will be a fraction of the players' total weight. This way of modifying the quota, however, differs from the model of Zuckerman et al. [7] who manipulate the quota directly. We define the corresponding problems of control by adding or deleting players *with changing the quota*, with the goal to increase, to decrease, or to maintain a distinguished player's power index. We study these problems for the probabilistic Penrose-Banzhaf index and the Shapley-Shubik index in terms of their computational complexity.

We conclude in Section 5 and mention some open problems for future work.

This work extends a preliminary version that appeared in the proceedings of the *33rd International Workshop on Combinatorial Algorithms* (IWOCA'22) [28] and has previously also been presented at the 17th International Symposium on Artificial Intelligence and Mathematics (ISAIM'22).

2 Preliminaries

In this section, we provide the needed notions from cooperative game theory and computational complexity theory.

Definition 2.1 A *coalitional game* is a pair $\mathcal{G} = (N, v)$, where $N = \{1, 2, \ldots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$, is a characteristic function that assigns a payoff to every coalition of players (i.e., subset of N). $\mathcal{G} = (N, v)$ is called *simple* if $v(C) \in \{0, 1\}$ for every coalition $C \subseteq N$ and v is *monotonic*, i.e., $v(A) \leq v(B)$ whenever $A \subseteq B \subseteq N$.

We focus on a special class of simple coalitional games: weighted voting games.

Definition 2.2 A *weighted voting game* (*WVG*, for short) $\mathcal{G} = (w_1, \ldots, w_n; q)$ is a simple coalitional game that consists of a quota $q \in \mathbb{R}_{\geq 0}$ and weights $w_i \in \mathbb{R}_{\geq 0}$, where w_i is the i-th player's weight, $i \in N$, and $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. For each coalition $S \subseteq N$, letting $w_S = \sum_{i \in S} w_i$, S *wins* if $w_S \geq q$, and *loses* otherwise:

$$v(S) = \begin{cases} 1 \text{ if } w_S \geq q, \\ 0 \text{ otherwise.} \end{cases}$$

In Section 4, we will use the quota depending on the players' total weight as $q = r \sum_{i \in N} w_i$ for a parameter $r \in [0, 1]$.

We now define two of the most popular power indices that can be used to measure a player's significance in a simple game, the *probabilistic Penrose-Banzhaf index* (introduced by Dubey and Shapley [2] as an alternative to the normalized Penrose-Banzhaf index that was originally introduced by Penrose [3] and later re-invented by Banzhaf [1]) and the *Shapley-Shubik index* due to Shapley and Shubik [4].

Definition 2.3 Let n be the number of players in a simple game $\mathcal{G} = (N, v)$ and let $i \in N$ be a player. The *probabilistic Penrose-Banzhaf index of player* i *in* $\mathcal{G}$ is defined by

$$\beta(\mathcal{G}, i) = \frac{\sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S))}{2^{n-1}}.$$

The *Shapley-Shubik index of player* i *in* $\mathcal{G}$ is defined by

$$\varphi(\mathcal{G}, i) = \frac{\sum_{S \subseteq N \setminus \{i\}} \|S\|!(n - 1 - \|S\|)!(v(S \cup \{i\}) - v(S))}{n!}.$$

If $v(S \cup \{i\}) - v(S) = 1$, we say that i is *pivotal for* S. If a player is pivotal for all coalitions, we call it a *dictator*, and if it is not pivotal for any set, we call it a *dummy player*.

We will study structural control by adding and deleting players in WVGs, and we adopt the notation of Rey and Rothe [8] who introduced these concepts. For control by adding players, let $\mathcal{G} = (w_1, \ldots, w_n; q)$ be a given WVG and $N = \{1, \ldots, n\}$ and let $M = \{n+1, \ldots, n+m\}$ be a set of m unregistered players with weights $w_{n+1}, \ldots, w_{n+m}$. Adding M to $\mathcal{G}$ yields a new WVG that is denoted by $\mathcal{G}_{\cup M} = (w_1, \ldots, w_{n+m}; q)$. Similarly, if $M \subseteq N$, deleting M from $\mathcal{G}$ yields a new WVG $\mathcal{G}_{\setminus M} = (w_{j_1}, \ldots, w_{j_{n-m}}; q)$, where $\{j_1, \ldots, j_{n-m}\} = N \setminus M$. For more background on cooperative game theory, we refer to the books by Chalkiadakis et al. [9], Peleg and Sudhölter [10], and Taylor and Zwicker [11], and to the chapter by Elkind and Rothe [12].

We assume familiarity with the most fundamental notions of computational complexity, in particular with the complexity classes P (deterministic polynomial time), NP (nondeterministic polynomial time), and PP (probabilistic polynomial time). Moreover, we will also use the well-known complexity classes DP (consisting of differences of NP sets, as introduced by Papadimitriou and Yannakakis [29]) and Θ_2^p (a.k.a. $\mathrm{P}^{\mathrm{NP}[\log]}$, the class of sets accepted by a P algorithm accessing its NP oracle logarithmically often, see [30]). The notion of *hardness* for these classes is based on the *polynomial-time many-one reducibility*: $X \leq_{\mathrm{m}}^{\mathrm{p}} Y$ if there is a polynomial-time computable, total function f such that for each input x, $x \in X$ if and only if $f(x) \in Y$. We refer the reader to the textbooks by Garey and Johnson [31], Papadimitriou [32], and Rothe [33] for more background on complexity theory.

We use the following two well-known NP-complete problems (see, e.g., [31]).

	PARTITION
Given:	A set $I = \{1, \ldots, n\}$, a function $a : I \to \mathbb{N} \setminus \{0\}$, $i \mapsto a_i$, such that $\sum_{i=1}^{n} a_i$ is even.
Question:	Does there exist a partition of I into two subsets of equal weight, that is, does there exist a subset $I' \subseteq I$ such that $\sum_{i \in I'} a_i = \sum_{i \in I \setminus I'} a_i$?

	SUBSETSUM
Given:	A set $I = \{1, \ldots, n\}$, a function $a : I \to \mathbb{N} \setminus \{0\}$, $i \mapsto a_i$, and a positive integer q.
Question:	Does there exist a subset $I' \subseteq I$ such that $\sum_{i=I'} a_i = q$?

We also use the following two PP-complete problems that Rey and Rothe [6] used in their work on false-name manipulation in WVGs.

	COMPARE- #SUBSETSUM- RR
Given:	A set $I = \{1, \ldots, n\}$, a function $a : I \to \mathbb{N} \setminus \{0\}$, $i \mapsto a_i$, where $\alpha = \sum_{i=1}^{n} a_i$.
Question:	Is the number of subsets of I with values summing up to $\frac{\alpha}{2} - 2$ greater than the number of subsets of I with values summing up to $\frac{\alpha}{2} - 1$, i.e., is $\#\text{SUBSETSUM}((a_1, \ldots, a_n), \frac{\alpha}{2} - 2) > \#\text{SUBSETSUM}((a_1, \ldots, a_n), \frac{\alpha}{2} - 1)$?

	COMPARE- #SUBSETSUM- ЯЯ
Given:	A set $I = \{1, \ldots, n\}$, a function $a : I \to \mathbb{N} \setminus \{0\}$, $i \mapsto a_i$, where $\alpha = \sum_{i=1}^{n} a_i$.
Question:	Is the number of subsets of I with values summing up to $\frac{\alpha}{2} - 2$ smaller than the number of subsets of I with values summing up to $\frac{\alpha}{2} - 1$, i.e., is $\#\text{SUBSETSUM}((a_1, \ldots, a_n), \frac{\alpha}{2} - 2) < \#\text{SUBSETSUM}((a_1, \ldots, a_n), \frac{\alpha}{2} - 1)$?

We also use the fact that there exists a reduction to SUBSETSUM from the following NP-complete problem (see, e.g., [31]).

	X3C
Given:	A set of elements $\mathcal{B}$, $\|\mathcal{B}\| = 3k$ for some $k \in \mathbb{N}$, and a family $\mathcal{S}$ of three-element subsets of $\mathcal{B}$.
Question:	Does there exist a subfamily $\mathcal{S}^*$ of $\mathcal{S}$ such that each element from $\mathcal{B}$ is contained in exactly one set in $\mathcal{S}^*$?

Faliszewski and Hemaspaandra [25] proved the following useful property about X3C applied by them and by Rey and Rothe [6] and to be applied here as well later on.

Lemma 2.1 *Every* X3C *instance* $(\mathcal{B}', \mathcal{S}')$ *can be transformed into an* X3C *instance* $(\mathcal{B}, \mathcal{S})$*, where* $\|\mathcal{B}\| = 3k$ *and* $\|\mathcal{S}\| = n$*, such that* $\frac{k}{n} = \frac{2}{3}$ *without changing the number of solutions. Consequently, we can assume that the size of each solution in a* SUBSETSUM *instance is* $\frac{2n}{3}$*, that is, each subsequence summing up to the given quota contains the same number of elements.*

In our proofs, we will apply the following two lemmas due to Wagner [34].

Lemma 2.2 *Let* A *be some* NP*-complete problem and let* B *be an arbitrary problem. If there exists a polynomial-time computable function* f *such that, for all input strings* x_1 *and* x_2 *for which* $x_2 \in A$ *implies* $x_1 \in A$*, we have*

$$(x_1 \in A \wedge x_2 \notin A) \iff f(x_1, x_2) \in B,$$

then B *is* DP*-hard.*

Lemma 2.3 *Let A be some* NP*-complete problem and let B be an arbitrary problem. If there exists a polynomial-time computable function g such that, for all $k \geq 1$ and all input strings $x_1, \dots, x_{2k}$ satisfying $\chi_A(x_1) \geq \cdots \geq \chi_A(x_{2k})$ (where $\chi_A(x_i) = 1$ if $x_i \in A$, and $\chi_A(x_i) = 0$ if $x_i \notin A$), it holds that*

$$\|\{i \mid x_i \in A\}\| \text{ is odd} \iff g(x_1, \dots, x_{2k}) \in B,$$

then B is Θ_2^p-hard.

3 Deleting players without changing the quota

Let us start with an example of deleting players from a weighted voting game without changing its quota and let us see how power indices can change due to this operation.

Example 3.1 Consider the weighted voting game without changing quota $\mathcal{G} = (3, 3, 2, 1; 6)$. The players have the following Penrose-Banzhaf indices: $\beta(\mathcal{G}, 1) = \beta(\mathcal{G}, 2) = \frac{1}{2}$ and $\beta(\mathcal{G}, 3) = \beta(\mathcal{G}, 4) = \frac{1}{4}$; and the following Shapley-Shubik indices: $\varphi(\mathcal{G}, 1) = \varphi(\mathcal{G}, 2) = \frac{1}{3}$ and $\varphi(\mathcal{G}, 3) = \varphi(\mathcal{G}, 4) = \frac{1}{6}$. If we remove player 4, we obtain the new game $\mathcal{G}_{\setminus\{4\}} = (3, 3, 2; 6)$ with the players' Penrose-Banzhaf indices $\beta(\mathcal{G}, 1) = \beta(\mathcal{G}, 2) = \frac{1}{2}$ and $\beta(\mathcal{G}, 3) = 0$; and the players' Shapley-Shubik indices $\varphi(\mathcal{G}, 1) = \varphi(\mathcal{G}, 2) = \frac{1}{2}$ and $\varphi(\mathcal{G}, 3) = 0$. So, the Shapley-Shubik indices of players 1 and 2 have increased while their Penrose-Banzhaf indices have not changed. At the same time, both power indices of player 3 have decreased to 0, so 3 has become a dummy player.

In this section, we consider the model of structural control by deleting players where the goal is to increase, to decrease, to nonincrease, to nondecrease, or to maintain a power index, as proposed by Rey and Rothe [8]. Specifically, we consider the following decision problem for a given power index PI (which will be either the Penrose-Banzhaf index β or the Shapley-Shubik index φ):

	Control by Deleting Players to Increase PI
Given:	A WVG $\mathcal{G}$ with players $N = \{1, \dots, n\}$, a distinguished player $i \in N$, and a positive integer $k < n$.
Question:	Can at most k players $M \subseteq N \setminus \{i\}$ be deleted from $\mathcal{G}$ such that for the new game $\mathcal{G}_{\setminus M}$, it holds that $\mathrm{PI}(\mathcal{G}_{\setminus M}, i) > \mathrm{PI}(\mathcal{G}, i)$?

Like Rey and Rothe [8], we will also study its analogous variants where the goal is to *decrease* a power index PI $\in \{\beta, \varphi\}$ by deleting players (replacing "$\mathrm{PI}(\mathcal{G}_{\setminus M}, i) > \mathrm{PI}(\mathcal{G}, i)$" by "$\mathrm{PI}(\mathcal{G}_{\setminus M}, i) < \mathrm{PI}(\mathcal{G}, i)$"), to *nonincrease* it (replacing by "$\mathrm{PI}(\mathcal{G}_{\setminus M}, i) \leq \mathrm{PI}(\mathcal{G}, i)$"), to *nondecrease* it (replacing by "$\mathrm{PI}(\mathcal{G}_{\setminus M}, i) \geq \mathrm{PI}(\mathcal{G}, i)$"), or to *maintain* it (replacing by "$\mathrm{PI}(\mathcal{G}_{\setminus M}, i) = \mathrm{PI}(\mathcal{G}, i)$"). Note that when the goal is to nonincrease, nondecrease, or maintain a player's power index, these three problems would actually be always trivial to solve if we would merely ask whether at most k players can be deleted from the given WVG to reach these goals; deleting no players would always reach these goals. Therefore, we instead ask whether *at least one player and* at most k players can be deleted from the given WVG to reach these goals when defining these three problems. In fact, in our proofs we will always consider the special problem variants where only one player can be deleted, i.e., the deletion limit is always $k = 1$; this is justified since we only prove lower bounds of the computational complexity of these problems, which thus immediately transfer to the more general problem variants. We focus on deleting players here; the analogous problems for adding players have also been studied by Rey and Rothe [8].

First, we will show upper and lower bounds of how much the Penrose-Banzhaf index and the Shapley-Shubik index can change when players are deleted. Then we will study the problems CONTROL BY DELETING PLAYERS TO INCREASE PI, CONTROL BY DELETING PLAYERS TO DECREASE PI, etc. in terms of their complexity, solving open problems of Rey and Rothe [8].

3.1 Change of power indices by deleting players

Rey and Rothe [8] analyzed how deleting players can change the Penrose-Banzhaf and the Shapley-Shubik index, by providing upper and lower bounds for both power indices. Unfortunately, their result on the lower bound of the Shapley-Shubik index is not correct[1] and we fix it in Theorem 3.2 below (which, for completeness, also contains the correct upper bound for the Shapley-Shubik index and both bounds for the Penrose-Banzhaf index due to Rey and Rothe [8]).

Theorem 3.2 *After deleting the players of a subset $M \subseteq N \setminus \{i\}$ of size $m \geq 1$ from a WVG $\mathcal{G}$ with $n = \|N\|$ players, the difference between player i's old and new*

1. *Penrose-Banzhaf index is at most $1 - 2^{-m}$ and at least $-1 + 2^{-m}$ (as shown by Rey and Rothe [8]);*
2. *Shapley-Shubik index is at most $1 - \frac{(n-m+1)!}{2n!}$ (see [8]) and at least $-1 + \frac{(n-m+1)!}{2n!}$.*

Proof Consider player $i \in N \setminus M$. We have

$$\begin{aligned}
\varphi(\mathcal{G}, i) - \varphi(\mathcal{G}_{\setminus M}, i) &= \frac{\sum_{C \subseteq N \setminus \{i\}} \|C\|!(n-1-\|C\|)!(v(C \cup \{i\}) - v(C))}{n!} \\
&\quad - \frac{\sum_{C \subseteq N \setminus (M \cup \{i\})} \|C\|!(n-m-1-\|C\|)!(v(C \cup \{i\}) - v(C))}{(n-m)!} \\
&= \frac{1}{n!}\Big[\sum_{\substack{C \subseteq N \setminus \{i\} \\ C \cap M \neq \emptyset}} \|C\|!(n-1-\|C\|)!(v(C \cup \{i\}) - v(C)) \\
&\quad - \sum_{C \subseteq N \setminus (M \cup \{i\})} \|C\|!\Big(\frac{n!}{(n-m)!}(n-m-1-\|C\|)! \\
&\quad - (n-1-\|C\|)!\Big)\Big(v(C \cup \{i\}) - v(C)\Big)\Big].
\end{aligned}$$

The proof of the correct lower bound of the Shapley-Shubik index is analogous to the proof of the upper bound from the original proof by Rey and Rothe [8], we just need to change the signs and inequalities as follows:

$$\begin{aligned}
\varphi(\mathcal{G}, i) - \varphi(\mathcal{G}_{\setminus M}, i) &\geq -\frac{1}{n!} \sum_{C \subseteq N \setminus (M \cup \{i\})} \|C\|!\Big(\frac{n!}{(n-m)!}(n-m-1-\|C\|)! - (n-1-\|C\|)!\Big) \\
&\geq -1 + \frac{(n-m+1)!}{2n!}.
\end{aligned}$$

[1] Under the assumptions of Theorem 3.2, their incorrect lower bound of the Shapley-Shubik index [8] is $-1 + \frac{(n-m-1)!}{2(n-2)!}$.

This completes the proof. □

Let us look at a counterexample for the wrong lower bound from [8, Theorem 7] for the difference between a player's old and new Shapley-Shubik power index.

Example 3.3 Consider the game $\mathcal{G} = (2, 2, 2; 2)$ with distinguished player 1. Obviously, $\varphi(\mathcal{G}, 1) = \frac{1}{3}$, and if we remove the other two players from the game, 1's Shapley-Shubik index will increase to 1, so $\varphi(\mathcal{G}, 1) - \varphi(\mathcal{G}_{\setminus\{2,3\}}, 1) = -\frac{2}{3}$. This fits the lower bound according to Theorem 3.2 because $-1 + \frac{(3-2+1)!}{2\cdot 3!} = -\frac{5}{6} < -\frac{2}{3}$, but contradicts the wrong lower bound of [8, Theorem 7] (here stated in Footnote 1) because $-1 + \frac{(3-2-1)!}{2(3-2)!} = -\frac{1}{2} > -\frac{2}{3}$.

Let us now consider the game $\mathcal{H} = (4, 1, 1; 5)$, again with distinguished player 1. Then $\varphi(\mathcal{H}, 1) = \frac{2}{3}$, and if we remove the other players, player 1's Shapley-Shubik index will decrease to 0, so $\varphi(\mathcal{H}, 1) - \varphi(\mathcal{H}_{\setminus\{2,3\}}, 1) = \frac{2}{3}$. If we consider the upper bound $1 - \frac{(3-2+1)!}{2\cdot 3!} = \frac{5}{6}$ from Theorem 3.2, we have $\frac{2}{3} < \frac{5}{6}$, so this value belongs to the stated range. But if we assumed that $1 - \frac{(3-2-1)!}{2(3-2)!} = \frac{1}{2}$ were the correct upper bound, we would get a contradiction because $\frac{2}{3} > \frac{1}{2}$.

The previous theorem gives the bounds of how much the power indices can change depending only on the number of deleted players. In the next theorems, we will see the bounds of changes for a given player which depend not only on the number of deleted players but also on the power indices of the given player and of the deleted players from the initial game. We start with the lower bounds.

Theorem 3.4 *Let $\mathcal{G} = (w_1, \ldots, w_n; q)$ be a WVG with the set of players N and let $i \in N$. Let $M \subseteq N \setminus \{i\}$ be the set of players to be deleted and $m = \|M\|$.*

1. $\beta(\mathcal{G}, i) - \beta(\mathcal{G}_{\setminus M}, i) \geq \max((1 - 2^m)\beta(\mathcal{G}, i), \beta(\mathcal{G}, i) - 1)$.
2. $\varphi(\mathcal{G}, i) - \varphi(\mathcal{G}_{\setminus M}, i) \geq \max((1 - \binom{n}{m})\varphi(\mathcal{G}, i), \varphi(\mathcal{G}, i) - 1)$.

Proof Consider $i \in N \setminus M$. Let x be the number of coalitions for which player i is pivotal (i.e., $\beta(\mathcal{G}, i) = \frac{x}{2^{n-1}}$). After deleting M, the Penrose-Benzhaf index of i increases maximally if i is still pivotal for the same x coalitions or if i's index achieves the maximal value of 1, i.e.:

$$\beta(\mathcal{G}_{\setminus M}, i) \leq \frac{x}{2^{n-m-1}} = 2^m \beta(\mathcal{G}, i) \quad \text{or} \quad \beta(\mathcal{G}_{\setminus M}, i) = 1.$$

Therefore, $\beta(\mathcal{G}, i) - \beta(\mathcal{G}_{\setminus M}, i) \geq \max((1 - 2^m)\beta(\mathcal{G}, i), \beta(\mathcal{G}, i) - 1)$.

Now, let

$$\varphi(\mathcal{G}, i) = \frac{\sum_{S \subseteq N\setminus\{i\}} \|S\|!(n - 1 - \|S\|)!(v(S \cup \{i\}) - v(S))}{n!} = \frac{x}{n!}.$$

Similarly to the Penrose-Banzhaf index, the Shapley-Shubik index of i increases the most if i is still pivotal for the same coalitions as in the old game (notice that each of these coalitions does not contain more than $n - m - 1$ players):

$$
\begin{aligned}
\varphi(\mathcal{G}_{\setminus M}, i) &= \frac{\sum_{S\subseteq N\setminus(M\cup\{i\})} \|S\|!(n-m-1-\|S\|)!(v(S\cup\{i\})-v(S))}{(n-m)!}\\
&= \frac{\sum_{S\subseteq N\setminus(M\cup\{i\})} \|S\|!(n-m-1-\|S\|)!\frac{(n-1-\|S\|)!}{(n-1-\|S\|)!}(v(S\cup\{i\})-v(S))}{(n-m)!}\\
&= \frac{\sum_{S\subseteq N\setminus(M\cup\{i\})} \frac{(n-m-1-\|S\|)!}{(n-1-\|S\|)!}\|S\|!(n-1-\|S\|)!(v(S\cup\{i\})-v(S))}{(n-m)!}\\
&\le \frac{(n-m-1-(n-m-1))!}{(n-1-(n-m-1))!}\\
&\quad\cdot \frac{\sum_{S\subseteq N\setminus(M\cup\{i\})} \|S\|!(n-1-\|S\|)!(v(S\cup\{i\})-v(S))}{(n-m)!}\\
&= \frac{1}{m!}\frac{n!}{n!}\frac{\sum_{S\subseteq N\setminus(M\cup\{i\})} \|S\|!(n-1-\|S\|)!(v(S\cup\{i\})-v(S))}{(n-m)!}\\
&= \frac{n!}{m!(n-m)!}\frac{\sum_{S\subseteq N\setminus(M\cup\{i\})} \|S\|!(n-1-\|S\|)!(v(S\cup\{i\})-v(S))}{n!}\\
&= \binom{n}{m}\varphi(\mathcal{G}, i)
\end{aligned}
$$

or if its index achieves the maximal value:

$$\varphi(\mathcal{G}_{\setminus M}, i) \le \min\left(\binom{n}{m}\varphi(\mathcal{G}, i), 1\right),$$

and therefore,

$$\varphi(\mathcal{G}, i) - \varphi(\mathcal{G}_{\setminus M}, i) \ge \max\left(\left(1-\binom{n}{m}\right)\varphi(\mathcal{G}, i), \varphi(\mathcal{G}, i) - 1\right).$$

This completes the proof. □

The following theorem shows the corresponding upper bounds, i.e., how much smaller the power indices can be in new games after deleting players.

Theorem 3.5 *Let $\mathcal{G} = (w_1, \ldots, w_n; q)$ be a WVG with the set of players N and let $i \in N$. Let $M \subseteq N \setminus \{i\}$ be the set of players to be deleted and $m = \|M\|$.*

1. $\beta(\mathcal{G}, i) - \beta(\mathcal{G}_{\setminus M}, i) \le \min\left(\beta(\mathcal{G}, i), \sum_{j\in M}\beta(\mathcal{G}, j) + \frac{(2^m-1)^2}{2^{n-1}}\right)$.
2. $\varphi(\mathcal{G}, i) - \varphi(\mathcal{G}_{\setminus M}, i) \le \min\left(\varphi(\mathcal{G}, i), \sum_{j\in M}\varphi(\mathcal{G}, j) + \frac{1}{(n-m)!}\right)$.

Proof Consider again our given player $i \in N \setminus M$, and let x be the number of coalitions i is pivotal for. Consider two players k_1, k_2 with weights $w_{k_1} \ge w_{k_2}$. Let

$$\beta(\mathcal{G}, k_1) = \frac{z_1}{2^{n-1}} \quad\text{and}\quad \beta(\mathcal{G}, k_2) = \frac{z_2}{2^{n-1}}.$$

Let $S \subseteq N \setminus \{k_1, k_2\}$. If

$$\sum_{j\in S} w_j \in [q - w_{k_2}, q) \tag{1}$$

then both k_1 and k_2 are pivotal for S. If

$$\sum_{j \in S} w_j \in [q - w_{k_1}, q - w_{k_2}) \tag{2}$$

then k_1 is pivotal for S and $S \cup \{k_2\}$, and k_2 is not pivotal for any of them. If

$$\sum_{j \in S} w_j \in [q - w_{k_1} - w_{k_2}, q - w_{k_1}) \tag{3}$$

then k_1 is pivotal for $S \cup \{k_2\}$ and k_2 is pivotal for $S \cup \{k_1\}$. Otherwise, neither k_1 nor k_2 is pivotal.

All coalitions that meet conditions (1)–(3) are counted in z_1 but only the coalitions meeting (1) and (3) are counted in z_2. However, the coalitions whose total weight falls into the interval (2) are counted twice in z_1.

If we delete player k_2, player k_1 will still be pivotal in the new game for all coalitions $S \subseteq N \setminus \{k_1, k_2\}$ meeting the condition (1). If the value w_S is from the interval (2), k_1 will be pivotal for S but will not be pivotal for $S \cup \{k_2\}$ anymore because this coalition will not exist in the new game. Finally, player k_1 is pivotal for all coalitions $S \cup \{k_2\}$ if S meets the condition (3), so they will not matter for k_1 in the game without the player k_2.

If we delete the player k_1, the situation will be analogously to the previous one, the difference is we do not have to consider the coalitions whose total weight falls into the interval (2).

Recall that $\|M\| = m \geq 1$ players are deleted and consider $y_1, \ldots, y_m$, where y_j is the number of coalitions the j-th player from M is pivotal for. We assume that our given player i shares as many coalitions as possible with the players from M, i.e., we assume that $y_1, \ldots, y_m$ also count different coalitions. Let us assume next that all these sets for which the players from M are pivotal contain the player i and i is also pivotal for them. If $x - \sum_{j \in M} y_j > 0$, there can be still coalitions for which i is pivotal and they can contain the players from M (and these players are not pivotal for them at the same time). Let S be some such coalition. The maximal number of possible coalitions containing a set $S \setminus M$ is 2^m, and only one of them does not contain any player from M—and this coalition can be counted by player i's new power indices. Hence,

$$\begin{aligned}\beta(\mathcal{G}_{\setminus M}, i) &\geq \frac{1}{2^{n-m-1}} \left(\max \left(x - \sum_{j \in M} y_j - \frac{2^m - 1}{2^m} \left(x - \sum_{j \in M} y_j + 2^m - 1 \right), 0 \right) \right) \\ &= \max \left(\beta(\mathcal{G}, i) - \sum_{j \in M} \beta(\mathcal{G}, j) - \frac{(2^m - 1)^2}{2^{n-1}}, 0 \right),\end{aligned}$$

and therefore,

$$\beta(\mathcal{G}, i) - \beta(\mathcal{G}_{\setminus M}, i) \leq \min \left(\beta(\mathcal{G}, i), \sum_{j \in M} \beta(\mathcal{G}, j) + \frac{(2^m - 1)^2}{2^{n-1}} \right).$$

Now, let $y_1, \ldots, y_m$ be the numerators of the Shapley-Shubik indices of the players from M and assume, without loss of generality, that $w_{j_1} \leq \cdots \leq w_{j_m}$. If $x - \sum_{j=1}^{m} y_j > 0$, we have

$$
\begin{aligned}
x - \sum_{j=1}^{m} y_j &\le \sum_{\substack{S \subseteq N \setminus (M \cup \{i\}),\\ w_S \in [q - w_i, q - w_{j_1})}} \sum_{k=0}^{m} \binom{m}{k} (\|S\| + k)!(n - 1 - \|S\| - k)! \\
&= \sum_{\substack{S \subseteq N \setminus (M \cup \{i\}),\\ w_S \in [q - w_i, q - w_{j_1})}} \|S\|!(n - m - 1 - \|S\|)! \\
&\qquad \sum_{k=0}^{m} \binom{m}{k} \frac{(\|S\| + k)!}{\|S\|!} \frac{(n - 1 - \|S\| - k)!}{(n - m - 1 - \|S\|)!} \\
&\le \sum_{\substack{S \subseteq N \setminus (M \cup \{i\}),\\ w_S \in [q - w_i, q - w_{j_1})}} \|S\|!(n - m - 1 - \|S\|)! \\
&\qquad \sum_{k=0}^{m} \binom{m}{k} \frac{(n - m - 1 + k)!}{(n - m - 1)!} \frac{(n - 1 - n + m + 1 - k)!}{(n - m - 1 - n + m + 1)!} \\
&= \sum_{\substack{S \subseteq N \setminus (M \cup \{i\}),\\ w_S \in [q - w_i, q - w_{j_1})}} \|S\|!(n - m - 1 - \|S\|)! \\
&\qquad \sum_{k=0}^{m} \frac{m!}{k!(m - k)!} \frac{(n - m - 1 + k)!}{(n - m - 1)!} \frac{(m - k)!}{0!} \\
&= \sum_{\substack{S \subseteq N \setminus (M \cup \{i\}),\\ w_S \in [q - w_i, q - w_{j_1})}} m!\|S\|!(n - m - 1 - \|S\|)! \sum_{k=0}^{m} \binom{n - m - 1 + k}{k} \\
&= m!\binom{n}{m} \sum_{\substack{S \subseteq N \setminus (M \cup \{i\}),\\ w_S \in [q - w_i, q - w_{j_1})}} \|S\|!(n - m - 1 - \|S\|)!.
\end{aligned}
$$

Now, define the shorthand

$$
z = \sum_{\substack{S \subseteq N \setminus (M \cup \{i\}),\\ w_S \in [q - w_i, q - w_{j_1})}} \|S\|!(n - m - 1 - \|S\|)!
$$

and note that

$$
z \ge \frac{x - \sum_{j=1}^{m} y_j}{m!\binom{n}{m}}.
$$

Thus we get

$$
\begin{aligned}
\varphi(\mathcal{G}_{\setminus M}, i) \ge \frac{\max(z, 0)}{(n - m)!} &\ge \frac{\max(x - \sum_{j=1}^{m} y_j - \frac{n!}{(n-m)!}, 0)}{(n - m)!m!\binom{n}{m}} \\
&= \frac{\max(x - \sum_{j=1}^{m} y_j - \frac{n!}{(n-m)!}, 0)}{n!} \\
&= \max\left(\varphi(\mathcal{G}, i) - \sum_{j \in M} \varphi(\mathcal{G}, j) - \frac{1}{(n - m)!}, 0\right),
\end{aligned}
$$

Table 1 Overview of complexity results for control by deleting players from weighted voting games with respect to various goals for the Shapley-Shubik (φ) and the probabilistic Penrose-Banzhaf index (β)

Goal	Complexity for β	Complexity for φ
Decrease	Θ_2^p-hard *	NP-hard
Nonincrease	coNP-hard [8]	NP-hard
Increase	DP-hard	NP-hard [8]
Nondecrease	coNP-hard	?
Maintain	coNP-hard [8]	coNP-hard [8]

*coNP-hardness was proven by Rey and Rothe [8]

and finally,

$$\varphi(\mathcal{G}, i) - \varphi(\mathcal{G}_{\setminus M}, i) \leq \min\left(\varphi(\mathcal{G}, i), \sum_{j \in M} \varphi(\mathcal{G}, j) + \frac{1}{(n-m)!}\right),$$

completing the proof. □

Example 3.6 Let $\mathcal{G} = (4, 2, 1, 1, 1; 4)$ be a WVG. We are going to remove player 5 with weight 1 (i.e., the subset $M = \{5\}$) from the set of players. Let us consider player 2 having weight 2 in $\mathcal{G}$, with old and new Penrose-Banzhaf indices of $\beta(\mathcal{G}, 2) = \frac{1}{4}$ and $\beta(\mathcal{G}_{\setminus M}, 2) = \frac{1}{8}$, so the index decreases by $\frac{1}{8}$. The upper bound from Theorem 3.2 is $\beta(\mathcal{G}, 2) - \beta(\mathcal{G}_{\setminus M}, 2) \leq 1 - \frac{1}{2} = \frac{1}{2}$ and that from Theorem 3.5 is $\beta(\mathcal{G}, 2) - \beta(\mathcal{G}_{\setminus M}, 2) \leq \min(\frac{1}{4}, \frac{1}{8} + \frac{1}{16}) = \frac{3}{16}$, so both upper bounds are greater than the actual difference but the second one is more exact.

Now, consider player 2's old and new Shapley-Shubik index: $\varphi(\mathcal{G}, 2) = \frac{11}{60}$ and $\varphi(\mathcal{G}_{\setminus M}, 2) = \frac{5}{60}$, so it decreases by $\frac{1}{10}$. The upper bound from Theorem 3.2 is $\varphi(\mathcal{G}, 2) - \varphi(\mathcal{G}_{\setminus M}, 2) \leq 1 - \frac{(5-1+1)!}{2 \cdot 5!} = \frac{1}{2}$ and that from Theorem 3.5 is $\varphi(\mathcal{G}, 2) - \varphi(\mathcal{G}_{\setminus M}, 2) \leq \min(\frac{11}{60}, \frac{1}{10} + \frac{1}{4!}) = \frac{17}{120}$, which are greater again, but the second one is much closer to the true difference.

3.2 Control by deleting players

Rey and Rothe [8] analyzed the problems of control by adding and by deleting players in WVGs in terms of their complexity. While they obtained many results for the case of control by adding players, they left many problems open for control by deleting players. In the next two theorems, we solve all their open problems but one.

All currently known results about the lower bounds of these problems are summarized in Table 1. As already noted by Rey and Rothe [8], the best known upper bound for each of these problems is the complexity class $\mathrm{NP}^{\mathrm{PP}}$, which belongs to the second level of Wagner's counting hierarchy [35] and is defined as the class of problems that can be solved by an NP oracle machine accessing a PP oracle. Not many natural problems are known to be complete for this class: Some are related to finite-horizon Markov decision processes [36] and others to a variant of the satisfiability problem and certain tasks involving probabilistic planning [37]. Of course, there is a large gap between each lower bound listed in Table 1—note that each complexity class in this table is contained in PP—and the upper bound of $\mathrm{NP}^{\mathrm{PP}}$. Closing these gaps would be an interesting task for future research.

Our first result provides DP-hardness for increasing a player's Penrose-Banzhaf index by deleting players. Rey and Rothe [8] showed that the corresponding control problem is NP-hard for the Shapley-Shubik index.

Theorem 3.7 *Control by deleting players to increase a distinguished player's Penrose-Banzhaf index in a WVG is* DP*-hard.*

Proof To apply Lemma 2.2, let us define a reduction from the NP-complete problem PARTITION (which we will call A, just as the problem from Lemma 2.2). Let $x_1 = (a_1, \ldots, a_{n_1})$ and $x_2 = (b_1, \ldots, b_{n_2})$ be two instances of PARTITION, let $a = \sum_{i=1}^{n_1} a_i$ and $b = \sum_{i=1}^{n_2} b_i$, and let ξ_1 be the number of x_1's solutions for PARTITION and let ξ_2 be the number x_2's solutions for PARTITION. Consider the weighted voting game

$$\mathcal{G} = \left(1, a_1 \cdot 10^{\ell}, \ldots, a_{n_1} \cdot 10^{\ell}, b_1, \ldots, b_{n_2}, \frac{b}{2}; \frac{a}{2} \cdot 10^{\ell} + b + 1\right),$$

where $\ell \in \mathbb{N}$ and $10^{\ell} > \frac{3}{2}b$. Let 1 be the distinguished player and let the deletion limit be $k = 1$. Assume that $\chi_A(x_1) \geq \chi_A(x_2)$ (recall that this means that $x_2 \in A$ implies $x_1 \in A$).

We will now prove that

$$(\exists i \in \{2, \ldots, n_1 + n_2 + 2\}) \left[\beta(\mathcal{G}, 1) - \beta(\mathcal{G}_{\setminus\{i\}}, 1) < 0\right] \iff (x_1 \in A \wedge x_2 \notin A).$$

If $x_1 \notin A \wedge x_2 \notin A$, then for all $i \in \{2, \ldots, n_1 + n_2 + 2\}$, $\beta(\mathcal{G}, 1) = \beta(\mathcal{G}_{\setminus\{i\}}, 1) = 0$, so the index does not increase.

If $x_1 \in A \wedge x_2 \notin A$, then $\beta(\mathcal{G}, 1) = \frac{\xi_1}{2^{n_1+n_2+1}}$ and $\beta(\mathcal{G}_{\setminus\{n_1+n_2+2\}}, 1) = \frac{\xi_1}{2^{n_1+n_2}} > \beta(\mathcal{G}, 1)$, so it is possible to increase the index of player 1 by deleting player $n_1 + n_2 + 2$.

If $x_1 \in A \wedge x_2 \in A$, then $\beta(\mathcal{G}, 1) = \frac{\xi_1 + \xi_1 \xi_2}{2^{n_1+n_2+1}}$. If we delete the player with weight $\frac{b}{2}$, then $\beta(\mathcal{G}_{\setminus\{n_1+n_2+2\}}, 1) = \frac{\xi_1}{2^{n_1+n_2}} < \beta(\mathcal{G}, 1)$. If we remove a player j with any weight b_i, then $\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{\xi_1 \frac{\xi_2}{2}}{2^{n_1+n_2}} < \beta(\mathcal{G}, 1)$. Finally, if we delete a player j with any weight $a_i \cdot 10^{\ell}$, then $\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{\frac{\xi_1}{2} + \frac{\xi_1}{2}\xi_2}{2^{n_1+n_2}} = \beta(\mathcal{G}, 1)$. So, the index decreases or stays unchanged. □

Next, we show coNP-hardness for the goal of nondecreasing a distinguished player's Penrose-Banzhaf index by deleting players. The complexity of the corresponding control problem for the Shapley-Shubik index remains open.

Theorem 3.8 *Control by deleting players to nondecrease a distinguished player's Penrose-Banzhaf index in a WVG is* coNP*-hard.*

Proof We provide a reduction from the complement of the NP-complete problem PARTITION. Let $x = (a_1, \ldots, a_n)$ be a given instance of PARTITION, let $\alpha = \sum_{j=1}^{n} a_j$, and let ξ be the number of x's solutions for PARTITION. Consider the weighted voting game

$$\mathcal{G} = \left(1, a_1, \ldots, a_n, \frac{\alpha}{2}; \alpha + 1\right)$$

with the distinguished player 1 and the deletion limit $k = 1$.

We will show that

$$(\exists i \in \{2, \ldots, n + 2\}) \left[\beta(\mathcal{G}, 1) - \beta(\mathcal{G}_{\setminus\{i\}}, 1) \leq 0\right] \iff \xi = 0.$$

Let $\xi = 0$. Then

$$\beta(\mathcal{G}, 1) = \frac{1}{2^{n+1}},$$

and if we remove the player $n + 2$ with weight $\frac{\alpha}{2}$, the Penrose-Banzhaf index of player 1 will increase (i.e., also nondecrease), since the player is not in any coalition for which 1 is pivotal.

Let $\xi > 0$. Then

$$\beta(\mathcal{G}, 1) = \frac{1+\xi}{2^{n+1}}.$$

Now, if we delete the player $n+2$ with weight $\frac{\alpha}{2}$, the Penrose-Banzhaf index of player 1 will change to

$$\beta(\mathcal{G}_{\setminus\{n+2\}}, 1) = \frac{1}{2^n} = \frac{2}{2^{n+1}} < \frac{1+\xi}{2^{n+1}} = \beta(\mathcal{G}, 1)$$

because the number of solutions ξ is even, so $\xi \geq 2$. Finally, if we remove any one of the other players, say i, then we have

$$\beta(\mathcal{G}_{\setminus\{i\}}, 1) = \frac{\frac{1}{2}\xi}{2^n} = \frac{\xi}{2^{n+1}} < \beta(\mathcal{G}, 1),$$

which completes the proof. □

For the Shapley-Shubik index, we show NP-hardness for the goals of decreasing or nonincreasing a player's power by deleting players.

Theorem 3.9 *Control by deleting players to decrease or to nonincrease a player's Shapley-Shubik index in a WVG is* NP-*hard.*

Proof We show NP-hardness for the Shapley-Shubik power index by means of a reduction from the SUBSETSUM problem. Let $(a_1, \ldots, a_n; q)$ be a SUBSETSUM instance with $\alpha = \sum_{i=1}^{n} a_i$, denote by ξ the number of its solutions and note that, due to Lemma 2.1, we can assume that each of these solutions has its size equal to $\frac{2}{3}n$.

Construct the control problem instance consisting of a game

$$\mathcal{G} = \big(1, a_1 \cdot 10^s, \ldots, a_n \cdot 10^s, t - q \cdot 10^s - x, t - 2ny_1, \ldots, t - 2ny_{n+3}, \underbrace{y_1, \ldots, y_1}_{2n}, \ldots, \underbrace{y_{n+3}, \ldots, y_{n+3}}_{2n}; t+1\big)$$

with $2n^2 + 8n + 5$ players, where

$$x = y_1 + \cdots + y_n,$$

$$y_i > 2n \sum_{j=i+1}^{n+3} y_j \quad \text{for } i \in \{1, \ldots, n+2\},$$

$$10^s > 2n \sum_{j=1}^{n+3} y_j, \quad \text{and}$$

$$t > 2\alpha \cdot 10^s + 4n \sum_{j=1}^{n+3} y_j.$$

Finally, let 1 be the distinguished player and the deletion limit be $k = 1$.

We will now show that the following three statements are pairwise equivalent:

1. $(a_1, \ldots, a_n; q)$ is a yes-instance of SUBSETSUM, i.e., $\xi > 0$.
2. There is a player $j > 1$ whose deletion decreases the Shapley-Shubik index of player 1: $(\exists j \in \{2, \ldots, 2n^2 + 8n + 5\})\left[\varphi(\mathcal{G}_{\setminus\{j\}}, 1) - \varphi(\mathcal{G}, 1) < 0\right]$.
3. There is a player $j > 1$ whose deletion nonincreases the Shapley-Shubik index of player 1: $(\exists j \in \{2, \ldots, 2n^2 + 8n + 5\})\left[\varphi(\mathcal{G}_{\setminus\{j\}}, 1) - \varphi(\mathcal{G}, 1) \leq 0\right]$.

(1) implies (2): Suppose that $\xi > 0$. Then

$$\varphi(\mathcal{G}, 1) = (n+3)\frac{(2n+1)!(2n^2+6n+3)!}{(2n^2+8n+5)!} + \xi(2n)^n \frac{(\frac{5}{3}n+1)!(2n^2+\frac{19}{3}n+3)!}{(2n^2+8n+5)!}.$$

If we delete the player $n+2$ with weight $t-q-x$, then

$$\begin{aligned}\varphi(\mathcal{G}_{\setminus\{n+2\}}, 1) &= (n+3)\frac{(2n+1)!(2n^2+6n+2)!}{(2n^2+8n+4)!}\\ &= (n+3)\frac{(2n+1)!(2n^2+6n+3)!}{(2n^2+8n+5)!}\frac{2n^2+8n+5}{2n^2+6n+3}\end{aligned}$$

and thus

$$\begin{aligned}\varphi(\mathcal{G}_{\setminus\{n+2\}}, 1) - \varphi(\mathcal{G}, 1) \;=\; & (n+3)\frac{(2n+1)!(2n^2+6n+3)!}{(2n^2+8n+5)!}\left(1+\frac{2n+2}{2n^2+6n+3}\right)\\ & - (n+3)\frac{(2n+1)!(2n^2+6n+3)!}{(2n^2+8n+5)!}\\ & - \xi(2n)^n\frac{(\frac{5}{3}n+1)!(2n^2+\frac{19}{3}n+3)!}{(2n^2+8n+5)!}\\ =\; & (n+3)\frac{(2n+2)!(2n^2+6n+2)!}{(2n^2+8n+5)!}\\ & - \xi(2n)^n\frac{(\frac{5}{3}n+1)!(2n^2+\frac{19}{3}n+3)!}{(2n^2+8n+5)!} \;<\; 0,\end{aligned}$$

so the Shapley-Shubik index of player 1 decreases.

(2) implies (3): is obvious.

(3) implies (1): To prove the contrapositive, suppose that $\xi = 0$. Then

$$\varphi(\mathcal{G}, 1) = (n+3)\frac{(2n+1)!(2n^2+6n+3)!}{(2n^2+8n+5)!}.$$

If we delete any player that is not a part of any coalition for which player 1 is pivotal (i.e., any of the players $2, \dots, n+2$), then the Shapley-Shubik index of 1 will increase. Considering the other players, each player is in exactly one coalition counted in the index, i.e., for each $i \in \{n+3, \dots, 2n^2+8n+5\}$, we have

$$\begin{aligned}\varphi(\mathcal{G}_{\setminus\{i\}}, 1) &= (n+2)\frac{(2n+1)!(2n^2+6n+2)!}{(2n^2+8n+4)!}\\ &= (n+2)\frac{(2n+1)!(2n^2+6n+3)!}{(2n^2+8n+5)!}\frac{2n^2+8n+5}{2n^2+6n+3}.\end{aligned}$$

Because

$$(n+2)\frac{2n^2+8n+5}{2n^2+6n+3} = n+2+(n+2)\frac{2n+2}{2n^2+6n+3} = n+2+\frac{2n^2+6n+4}{2n^2+6n+3} > n+3$$

the new Shapley-Shubik index of player 1 increases. □

Rey and Rothe [8] also showed that the problem of control by deleting a single player to decrease a distinguished player's Penrose-Banzhaf index is coNP-hard. We improve this lower bound to Θ_2^p-hardness.

Theorem 3.10 *Control by deleting players to decrease a distinguished player's Penrose-Banzhaf index in a WVG is* Θ_2^p*-hard.*

Proof To apply Lemma 2.3, we provide a reduction from the NP-complete problem PARTITION (again called A but this time playing the role of the problem from Lemma 2.3). Let $x_i = (a_{i,1}, \dots, a_{i,m_i})$ be an instance of PARTITION for $i \in \{1, \dots, 2n\}$, let $\alpha_i = \sum_{j=1}^{m_i} a_{i,j}$, and let ξ_i be the number of x_i's solutions for PARTITION.

Let $\ell_1, \dots, \ell_{2n} \in \mathbb{N}$ be chosen such that for all $i \in \{1, \dots, 2n-1\}$, we have

$$10^{\ell_i} > \sum_{j=1}^{2n-i} \alpha_{2n+1-j} \cdot 10^{i+1},$$

let $y_1 = 1$, $y_2 = 2$, and for all $i \in \{3, \dots, 2n\}$, let

$$y_i = \begin{cases} \sum_{j=1}^{\frac{i-1}{2}} y_{2j} & \text{if } i \text{ is odd,} \\ y_{i-1} & \text{if } i \text{ is even.} \end{cases}$$

Furthermore, choose $z \in \mathbb{N}$ so that $y_{2n} \cdot z < 10^{\ell_{2n}}$, and define

$$q = \frac{\alpha_1}{2} \cdot 10^{\ell_1} + \frac{\alpha_2}{2} \cdot 10^{\ell_2} + \dots + \frac{\alpha_{2n}}{2} \cdot 10^{\ell_{2n}} + z + 1$$

and $q' = q - 1$. Consider the weighted voting game

$$\mathcal{G} = \Big(1, a_{1,1} \cdot 10^{\ell_1}, \dots, a_{1,m_1} \cdot 10^{\ell_1}, \dots, a_{2n,1} \cdot 10^{\ell_{2n}}, \dots, a_{2n,m_{2n}} \cdot 10^{\ell_{2n}},$$
$$x, r_1, r_2, r_2, \underbrace{r_3, \dots, r_3}_{y_3}, \dots, \underbrace{r_{2n-1}, \dots, r_{2n-1}}_{y_{2n-1}}, \underbrace{r_{2n}, \dots, r_{2n}}_{y_{2n}}; q\Big)$$

with $\tilde{n} = \sum_{i=1}^{2n}(m_i + y_i) + 2$ players, where $x \in \mathbb{N}$, $x < z$, and for all $i \in \{1, \dots, 2n\}$,

$$r_i = \begin{cases} q' - (\sum_{j=1}^{i} \frac{\alpha_j}{2} \cdot 10^{\ell_j}) - x & \text{if } i \text{ is odd,} \\ q' - \sum_{j=1}^{i} \frac{\alpha_j}{2} \cdot 10^{\ell_j} & \text{if } i \text{ is even.} \end{cases}$$

Let the first player be the distinguished player and let the deletion limit be $k = 1$. Assume that $\chi_A(x_1) \geq \chi_A(x_2) \geq \dots \geq \chi_A(x_{2n})$. We will now prove that

$$(\exists i \in \{2, \dots, \tilde{n}\}) \left[\beta(\mathcal{G}, 1) - \beta(\mathcal{G}_{\setminus\{i\}}, 1) > 0\right] \iff \|\{i \mid \chi_A(x_i) = 1\}\| \text{ is odd.}$$

First, suppose that $\|\{i \mid \chi_A(x_i) = 1\}\|$ is even. If $\|\{i \mid \chi_A(x_i) = 1\}\| = 0$, then for all $i \in \{2, \dots, \tilde{n}\}$, $\beta(\mathcal{G}, 1) = \beta(\mathcal{G}_{\setminus\{i\}}, 1) = 0$. If $\|\{i \mid \chi_A(x_i) = 1\}\| > 0$, then there exists some i such that $\chi_A(x_{2i}) = 1$ and $\chi_A(x_{2i+1}) = 0$ (or $i = 2n$) and

$$\beta(\mathcal{G}, 1) = \frac{\xi_1 + 2\xi_1\xi_2 + \dots + y_{2i}\xi_1 \cdots \xi_{2i}}{2^{\tilde{n}-1}}.$$

If we delete any player j with weight $a_k^j \cdot 10^{\ell_j}$ or r_j for $j > 2i$, then the index will increase:

$$\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{\xi_1 + 2\xi_1\xi_2 + \dots + y_{2i}\xi_1 \cdots \xi_{2i}}{2^{\tilde{n}-2}}.$$

If we delete any player j with weight $a_k^j \cdot 10^{\ell_j}$ for $j \leq 2i$, then

$$\begin{aligned}\beta(\mathcal{G}_{\setminus\{j\}}, 1) &= \frac{\xi_1 + \cdots + y_j \xi_1 \cdots \frac{\xi_j}{2} + \cdots + y_{2i}\xi_1 \cdots \frac{\xi_j}{2} \cdots \xi_{2i}}{2^{\tilde{n}-2}} \\ &= \frac{\xi_1 + \cdots + y_{j-1}\xi_1 \cdots \xi_{j-1}}{2^{\tilde{n}-2}} + \frac{y_j \xi_1 \cdots \xi_j + \cdots + y_{2i}\xi_1 \cdots \xi_{2i}}{2^{\tilde{n}-1}} \geq \beta(\mathcal{G}, 1).\end{aligned}$$

If we remove any player j with weight r_j for $j \leq 2i$, then

$$\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{\xi_1 + \cdots + (y_j - 1)\xi_1 \cdots \xi_j + \cdots + y_{2i}\xi_1 \cdots \xi_{2i}}{2^{\tilde{n}-2}},$$

so the index does not decrease because $2y_j - 2 \geq y_j$ for $j \geq 2$, as $y_j \geq 2$ and $2\xi_1\xi_2 > \xi_1$. Finally, if we delete the player with weight x, we have

$$\begin{aligned}\beta(\mathcal{G}_{\setminus\{\sum_{j=1}^{2n} m_j + 2\}}, 1) &= \frac{2\xi_1\xi_2 + y_4\xi_1\xi_2\xi_3\xi_4 + \cdots + y_{2i}\xi_1 \cdots \xi_{2i}}{2^{\tilde{n}-2}} \\ &= \frac{2\xi_1\xi_2 + 2\xi_1\xi_2 + \cdots + y_{2i}\xi_1 \cdots \xi_{2i} + y_{2i}\xi_1 \cdots \xi_{2i}}{2^{\tilde{n}-1}} > \beta(\mathcal{G}, 1).\end{aligned}$$

Summing up, if $\|\{i \mid \chi_A(x_i) = 1\}\|$ is even, the Penrose-Banzhaf power index of the first player increases or stays the same after removing a player from the game.

Let us assume now that $\|\{i \mid \chi_A(x_i) = 1\}\|$ is odd. If $\|\{i \mid \chi_A(x_i) = 1\}\| = 1$, then $\beta(\mathcal{G}, 1) = \frac{\xi_1}{2^{\tilde{n}-1}}$, and after removing the player with weight x, the index decreases to 0. If $\|\{i \mid \chi_A(x_i) = 1\}\| > 1$, there exists some i such that $\chi_A(x_{2i-1}) = 1$ and $\chi_A(x_{2i}) = 0$ and

$$\beta(\mathcal{G}, 1) = \frac{\xi_1 + 2\xi_1\xi_2 + \cdots + y_{2i-1}\xi_1 \cdots \xi_{2i-1}}{2^{\tilde{n}-1}}.$$

After removing the player with weight x, we have

$$\beta(\mathcal{G}_{\setminus\{\sum_{j=1}^{2n} m_j + 2\}}, 1) = \frac{2\xi_1\xi_2 + y_4\xi_1\xi_2\xi_3\xi_4 + \cdots + y_{2i-2}\xi_1 \cdots \xi_{2i-2}}{2^{\tilde{n}-2}}$$

and

$$\begin{aligned}&\beta(\mathcal{G}, 1) - \beta(\mathcal{G}_{\setminus\{\sum_{j=1}^{2n} m_j + 2\}}, 1) \\ &= \frac{\xi_1 - 2\xi_1\xi_2 + y_3\xi_1\xi_2\xi_3 - \cdots - y_{2i-2}\xi_1 \cdots \xi_{2i-2} + y_{2i-1}\xi_1 \cdots \xi_{2i-1}}{2^{\tilde{n}-1}} \\ &> \frac{\xi_1 + \cdots + y_{2i-1}\xi_1 \cdots \xi_{2i-1} - \sum_{j=1}^{i-1} y_{2j}\xi_1 \cdots \xi_{2i-1}}{2^{\tilde{n}-1}} > 0,\end{aligned}$$

since $y_{2i-1} = \sum_{j=1}^{i-1} y_{2j}$. Therefore, if $\|\{i \mid \chi_A(x_i) = 1\}\|$ is odd, it is possible to decrease the Penrose-Banzhaf index of the first player. □

4 Deleting or adding players with changing the quota

From now on, we define the quota of a WVG depending on the players' total weight. With this assumption, we modify the model of Rey and Rothe [8] in a natural way: While they assume that the quota remains the same after players have been added or deleted, we now assume that the quota will change accordingly in the modified game. That way, games can keep important properties. For example, in the case of adding new players to a WVG, suppose we want to have at most one winning coalition in each partition of the players; if the quota would stay unchanged after the manipulation, however, it will be easy to get a game with two or more winning coalitions at a time, so we would lose the desired property. In the case of deleting players from a WVG, it can happen that it is impossible for any coalition to win in the new game after the manipulation because the unchanged quota could be greater than the total sum of the players' weights, i.e., the weight of the grant coalition.

Example 4.1 Let us consider the weighted voting game with changing quota $\mathcal{G} = (3, 3, 2; 6)$, so the parameter for the quota is $\frac{3}{4}$. The power indices of the players are as follows: $\beta(\mathcal{G}, 1) = \beta(\mathcal{G}, 2) = \varphi(\mathcal{G}, 1) = \varphi(\mathcal{G}, 2) = \frac{1}{2}$ and $\beta(\mathcal{G}, 3) = \varphi(\mathcal{G}, 3) = 0$.

Now, let us add to this game one new player with weight 1. Then we get the new game $\mathcal{G}_{\cup\{4\}} = (3, 3, 2, 1; \frac{27}{4})$. Since all the players have nonnegative integer weights, all coalitions formed by them also have nonnegative integer weights and the new quota is equivalent[2] to 7. The power indices of players 1, 2, and 3 in the new game are as follows: $\beta(\mathcal{G}_{\cup\{4\}}, 1) = \beta(\mathcal{G}_{\cup\{4\}}, 2) = \frac{3}{8}$ and $\beta(\mathcal{G}_{\cup\{4\}}, 3) = \frac{1}{8}$ for the Penrose-Banzhaf index and $\varphi(\mathcal{G}_{\cup\{4\}}, 1) = \varphi(\mathcal{G}_{\cup\{4\}}, 2) = \frac{5}{12}$ and $\varphi(\mathcal{G}_{\cup\{4\}}, 3) = \frac{1}{12}$ for the Shapley-Shubik. So, both power indices of players 1 and 2 have decreased and player 3 no longer is a dummy player.

On the other hand, if we delete player 2 from the game $\mathcal{G}$ instead of adding a new player, we get the new game $\mathcal{G}_{\setminus\{2\}} = (3, 2; \frac{15}{4})$ in which both power indices of both players are $\frac{1}{2}$, so they are unchanged for player 1 but have increased for the other player. Note that if we had not changed the quota, there would be no winning coalitions in the game after the deletion.

4.1 Change of power by adding or deleting players with changing the quota

As we have already mentioned in the introduction, Zuckerman et al. [7] studied manipulation of the quota in WVGs without any structural changes in the set of players. They presented upper and lower bounds for how much the power index of a single player can change when the quota is manipulated.

Our next two theorems present the bounds in situations where quotas are changed not directly but they change as a consequence of adding or deleting players: Recall that from now on, in a WVG $\mathcal{G} = (w_1, \ldots, w_n; q)$, the quota will depend on the players' total weight as $q = r \sum_{i=1}^{n} w_i$ for a parameter $r \in [0, 1]$, thus changing the quota by adding or deleting players. In these cases, the power of a player can change much more extremely than in the games where quota remains the same after our manipulation—for example, a player with no power at all can become the most powerful one and the other way around.

We start with the case when we add some new players to a WVG. Theorem 4.2 shows how the power indices can change depending on the number of added players.

Theorem 4.2 *Let $\mathcal{G} = (w_1, \ldots, w_n; q_1)$ be a WVG with set N of players and quota $q_1 = r \sum_{j=1}^{n} w_j$ for some $r \in [0, 1]$. Let M, $m = \|M\|$, be a set of players that are to be added to the game $\mathcal{G}$. Let $\mathcal{G}_{\cup M}$ be the new game with players $N \cup M$ and quota $q_2 = r \sum_{j \in N \cup M} w_j$. Then, for $i \in N$:*

1. $-1 + 2^{-m} \leq \beta(\mathcal{G}, i) - \beta(\mathcal{G}_{\cup M}, i) \leq 1$,
2. $-1 + \frac{(n+1)!}{2(n+m)!} \leq \varphi(\mathcal{G}, i) - \varphi(\mathcal{G}_{\cup M}, i) \leq 1$.

Proof Let us start with the upper bounds. The power indices of a player can differ by 1 before and after adding new players only if

$$\beta(\mathcal{G}, i) = \varphi(\mathcal{G}, i) = 1 \quad \text{and} \quad \beta(\mathcal{G}_{\cup M}, i) = \varphi(\mathcal{G}_{\cup M}, i) = 0.$$

[2] By "equivalent" we mean that we have the same winning and losing coalitions for the quota 7 and the quota $\frac{27}{4}$.

That means, player i was a so-called dictator in the old game, i.e., i was pivotal for each coalition, so $v(\{i\}) = 1$ implies that $w_i \geq q_1 = rw_i + r\sum_{j \in N \setminus \{i\}} w_j$ and $v(N \setminus \{i\}) = 0$.

A dictator can become a dummy (i.e., has no power at all) when a new dictator is added to the game. Let $r \in (0, 1)$. If there is a new dictator k, then

$$w_i < q_2 = q_1 + rw_k \Rightarrow w_k > \frac{1}{r}(w_i - q_1)$$

and

$$w_k \geq q_2 = q_1 + rw_k \Rightarrow w_k \geq \frac{1}{1-r} q_1.$$

It follows that if we add a player with weight greater than $\max(\frac{1}{r}(w_i - q_1), \frac{1}{1-r} q_1)$, player i will become a dummy.

Player i is pivotal for a coalition $S \subseteq N \setminus \{i\}$ in $\mathcal{G}$ if and only if

$$\sum_{j \in S} w_j \in \left[r \sum_{j \in N} w_j - w_i, r \sum_{j \in N} w_j \right).$$

Analogously, i is pivotal for $S \subseteq (N \cup M) \setminus \{i\}$ in $\mathcal{G}_{\cup M}$ if and only if

$$\sum_{j \in S} w_j \in \left[r \sum_{j \in N} w_j + r \sum_{j \in M} w_j - w_i, r \sum_{j \in N} w_j + r \sum_{j \in M} w_j \right).$$

Let $S \subseteq N \setminus \{i\}$.

- If $w_i > r \sum_{j \in M} w_j$, then

$$\sum_{j \in S} w_j \in \left[r \sum_{j \in N} w_j, r \sum_{j \in N} w_j + r \sum_{j \in M} w_j \right).$$

  So i is not pivotal for S in $\mathcal{G}$ but i is pivotal for S in $\mathcal{G}_{\cup M}$. For $S \cup M$:

$$\sum_{j \in S \cup M} w_j \notin \left[r \sum_{j \in N} w_j, r \sum_{j \in N} w_j + r \sum_{j \in M} w_j \right).$$

- If $w_i \leq r \sum_{j \in M} w_j$, then

$$\sum_{j \in S} w_j \in \left[r \sum_{j \in N} w_j + r \sum_{j \in M} w_j - w_i, r \sum_{j \in N} w_j + r \sum_{j \in M} w_j \right),$$

  so i is not pivotal for S in $\mathcal{G}$ but i is pivotal for S in $\mathcal{G}_{\cup M}$. For $S \cup M$:

$$\sum_{j \in S \cup M} w_j \notin \left[r \sum_{j \in N} w_j + r \sum_{j \in M} w_j - w_i, r \sum_{j \in N} w_j + r \sum_{j \in M} w_j \right).$$

Therefore, i cannot be pivotal for $S \cup M$ in $\mathcal{G}_{\cup M}$.

Let v be the characteristic function for $\mathcal{G}$ and v' that for $\mathcal{G}_{\cup M}$ (because these games have different quotas, their characteristic functions can also differ from each other). For the

Penrose-Banzhaf index, we have

$$\begin{aligned}
\beta(\mathcal{G}, i) - \beta(\mathcal{G}_{\cup M}, i) = &\ \frac{\sum_{C \subseteq N \setminus \{i\}} (v(C \cup \{i\}) - v(C))}{2^{n-1}} \\
&- \frac{\sum_{C \subseteq (N \cup M) \setminus \{i\})} (v'(C \cup \{i\}) - v'(C))}{2^{n+m-1}} \\
= &\ \frac{\sum_{C \subseteq N \setminus \{i\}} (v(C \cup \{i\}) - v(C))}{2^{n-1}} \\
&- \frac{\sum_{C \subseteq N \setminus \{i\}} (v'(C \cup \{i\}) - v'(C))}{2^{n+m-1}} \\
&- \frac{\sum_{\substack{C \subseteq (N \cup M) \setminus \{i\}) \\ C \cap M \neq \emptyset}} (v'(C \cup \{i\}) - v'(C))}{2^{n+m-1}} \\
\geq &\ 0 - \frac{2^{n-1}}{2^{n+m-1}} - \frac{2^{n+m-1} - 2 \cdot 2^{n-1}}{2^{n+m-1}} = -1 + 2^{-m}.
\end{aligned}$$

For the Shapley-Shubik index, we have

$$\begin{aligned}
\varphi(\mathcal{G}, i) - \varphi(\mathcal{G}_{\cup M}, i) \geq &\ 0 - \frac{\sum_{C \subseteq N \setminus \{i\}} \|C\|!(n-1-\|C\|)!(v'(C \cup \{i\}) - v'(C))}{(n+m)!} \\
&- \frac{\sum_{\substack{C \subseteq (N \cup M) \setminus \{i\}) \\ C \cap M \neq \emptyset}} \|C\|!(n-1-\|C\|)!(v'(C \cup \{i\}) - v'(C))}{(n+m)!} \\
\geq &\ - \frac{\sum_{k=0}^{n-1} \binom{n-1}{k} k!(n+m-1-k)!}{(n+m)!} \\
&- \frac{\sum_{k=0}^{n+m-1} \binom{n+m-1}{k} k!(n+m-1-k)!}{(n+m)!} \\
&- \frac{\sum_{k=0}^{n-1} \binom{n-1}{k} [k!(n+m-1-k)! + (k+m)!(n-1-k)!]}{(n+m)!} \\
= &\ - \frac{\sum_{k=0}^{n+m-1} \binom{n+m-1}{k} k!(n+m-1-k)!}{(n+m)!} \\
&- \frac{\sum_{k=0}^{n-1} \binom{n-1}{k} (k+m)!(n-1-k)!}{(n+m)!} \\
= &\ \frac{1}{(n+m)!} \left((n-1)! \sum_{k=0}^{n-1} \frac{(k+m)!}{k!} - \sum_{k=0}^{n+m-1} (n+m-1)! \right) \\
\geq &\ \frac{1}{(n+m)!} \left((n-1)! \sum_{k=0}^{n-1} (k+m) - (n+m)! \right) \\
= &\ \frac{(n-1)!}{(n+m)!} \frac{n(n+2m-1)}{2} - 1 \geq \frac{(n+1)!}{2(n+m)!} - 1,
\end{aligned}$$

which completes the proof. □

Interestingly, it is possible for the strongest player to become a dummy by adding even one new player but it is impossible to turn a dummy into a dictator. The following example shows an extreme change of a player's power in a game.

Example 4.3 Let $\mathcal{G} = (5, 1, 1; 4)$ be a WVG with $r = \frac{4}{7}$. It is easy to see that player 1 with weight 5 is a dictator, so $\beta(\mathcal{G}, 1) = \varphi(\mathcal{G}, 1) = 1$. Let us add to the game a new player with weight 10. In this way, we get a new game: $\mathcal{G}_{\cup\{4\}} = (5, 1, 1, 10; \frac{68}{7})$ and the new quota is equivalent to 10. Therefore, the new player becomes the new dictator in the game $\mathcal{G}_{\cup\{4\}}$ and player 1's power indices decrease to 0.

Similarly, in the game $\mathcal{H} = (2, 1; 2)$ with $r = \frac{2}{3}$, $\beta(\mathcal{H}, 1) = \varphi(\mathcal{H}, 1) = 1$ and after adding two new players 3 and 4 with $w_3 = w_4 = 4$, we get the new quota equivalent to 8. Then, in the game $\mathcal{H}_{\cup\{3,4\}}$, player 1's power indices decrease to 0, too.

The changes of the power indices by deletion of players were presented by Rey and Rothe [8]. Those changes were derived for the case of the structural manipulation without changing the quota of a game. As we can see in Theorem 4.4, the Penrose-Banzhaf index and the Shapley-Shubik index can decrease by at most the same value with and without the change of quotas while the indices can increase more when the quota changes.

Theorem 4.4 *Let $\mathcal{G} = (w_1, \ldots, w_n; q_1)$ be a WVG with set N of players and quota $q_1 = r\sum_{j=1}^{n} w_j$ for some $r \in [0, 1]$. Let $M \subseteq N \setminus \{i\}$, $m = \|M\|$, be a set of players that are to be deleted from $\mathcal{G}$. Let $\mathcal{G}_{\setminus M}$ be the new game with players $N \setminus M$ and quota $q_2 = r\sum_{j\in N\setminus M} w_j$.*

Then, for $i \in N$:

1. $-1 \leq \beta(\mathcal{G}, i) - \beta(\mathcal{G}_{\setminus M}, i) \leq 1 - 2^{-m}$,
2. $-1 \leq \varphi(\mathcal{G}, i) - \varphi(\mathcal{G}_{\setminus M}, i) \leq 1 - \frac{(n-m+1)!}{2n!}$.

Proof Both the Penrose-Banzhaf and the Shapley-Shubik index can differ before and after deleting players by -1 for a player i only if

$$\beta(\mathcal{G}, i) = \varphi(\mathcal{G}, i) = 0 \quad \text{and} \quad \beta(\mathcal{G}_{\setminus M}, i) = \varphi(\mathcal{G}_{\setminus M}, i) = 1,$$

which is possible for any game after deleting a set $M = N \setminus \{i\}$ (then $q_2 \in (\frac{1}{2}w_i, w_i]$). Of course, since both the Penrose-Banzhaf and the Shaplay-Shubik index have values from the interval $[0, 1]$, the maximal difference is reached when the player's index increases.

Let $q_1 = r\sum_{j\in N} w_j$ be the quota in $\mathcal{G}$ and then $q_2 = r\sum_{j\in N\setminus M} w_j = r\sum_{j\in N} w_j - r\sum_{j\in M} w_j$ is the quota in $\mathcal{G}_{\setminus M}$. Player i is pivotal for a coalition $S \subseteq N \setminus \{i\}$ in $\mathcal{G}$ if and only if

$$\sum_{j\in S} w_j \in \left[r\sum_{j\in N} w_j - w_i, r\sum_{j\in N} w_j \right).$$

Similarly, i is pivotal for $S \subseteq N \setminus (M \cup \{i\})$ in $\mathcal{G}_{\setminus M}$ if and only if

$$\sum_{j\in S} w_j \in \left[r\sum_{j\in N} w_j - r\sum_{j\in M} w_j - w_i, r\sum_{j\in N} w_j - r\sum_{j\in M} w_j \right).$$

Let $S \subseteq N \setminus (M \cup \{i\})$.

If $w_i > r\sum_{j\in M} w_j$, then $\sum_{j\in S} w_j \in \left[r\sum_{j\in N} w_j - r\sum_{j\in M} w_j, r\sum_{j\in N} w_j \right)$. So i is pivotal for S in $\mathcal{G}$ and i is not pivotal for S in $\mathcal{G}_{\setminus M}$. On the other hand, for $S \cup M$, we have $\sum_{j\in S\cup M} w_j \notin \left[r\sum_{j\in N} w_j - r\sum_{j\in M} w_j, r\sum_{j\in N} w_j \right)$.

If $w_i \leq r\sum_{j\in M} w_j$, then we have $\sum_{j\in S} w_j \in \left[r\sum_{j\in N} w_j - w_i, r\sum_{j\in N} w_j \right)$ and we have $\sum_{j\in S\cup M} w_j \notin \left[r\sum_{j\in N} w_j - w_i, r\sum_{j\in N} w_j \right)$.

Therefore, i cannot be pivotal for $S \cup M$ in $\mathcal{G}$.

If i is pivotal for S and $S \cap M = \emptyset$, then i is not pivotal for $S \cup M$ in $\mathcal{G}$. Let v be the characteristic function of $\mathcal{G}$ and v' that of $\mathcal{G}_{\setminus M}$ (again, because these games have different quotas, their characteristic functions can also differ from each other).

For the Penrose-Banzhaf index, we have

$$\begin{aligned}\beta(\mathcal{G}, i) - \beta(\mathcal{G}_{\setminus M}, i) = & \frac{\sum_{C \subseteq N \setminus \{i\}} (v(C \cup \{i\}) - v(C))}{2^{n-1}} \\ & - \frac{\sum_{C \subseteq N \setminus (M \cup \{i\})} (v'(C \cup \{i\}) - v'(C))}{2^{n-m-1}} \\ \leq & \frac{2^{n-1} - 2^{n-m-1}}{2^{n-1}} - 0 = 1 - 2^{-m}.\end{aligned}$$

For the Shapley-Shubik index, we have

$$\begin{aligned}& \varphi(\mathcal{G}, i) - \varphi(\mathcal{G}_{\setminus M}, i) \\ \leq & \frac{\sum_{C \subseteq N \setminus \{i\}} \|C\|!(n-1-\|C\|)! - \sum_{C \subseteq N \setminus (M \cup \{i\})} \|C\|!(n-1-\|C\|)!}{n!} - 0 \\ = & \frac{\sum_{k=0}^{n-1} \binom{n-1}{k} k!(n-1-k)! - \sum_{k=0}^{n-m-1} \binom{n-m-1}{k} k!(n-1-k)!}{n!} \\ = & \frac{1}{n!}\left(\sum_{k=0}^{n-1}(n-1)! - (n-m-1)! \sum_{k=0}^{n-m-1} \frac{(n-1-k)!}{(n-m-1-k)!}\right) \\ = & \frac{1}{n!}\left(n! - (n-m-1)! \sum_{k=0}^{n-m-1} \frac{(m+k)!}{k!}\right) \\ \leq & 1 - \frac{(n-m-1)!}{n!} \sum_{k=0}^{n-m-1} (m+k) \\ = & 1 - \frac{(n-m-1)!}{n!} \frac{(n-m)(n+m-1)}{2} \\ \leq & 1 - \frac{(n-m)!(n-m+1)}{2n!} = 1 - \frac{(n-m+1)!}{2n!}.\end{aligned}$$

This completes the proof. □

We give two simple examples of games that meet the upper bounds for how much the Penrose-Banzhaf power index can change by deleting players, the first one without changing the quota (recall Theorem 3.2) and the second one with changing the quota (as provided by the previous theorem). Note that in both cases we have the same upper bound, $1 - 2^{-m}$, when m players are deleted.

Example 4.5 Consider the two-player game $\mathcal{G} = (2, 2; 3)$. The players have the same Penrose-Banzhaf power index of $\frac{1}{2}$ and after removing one of them, the other player's power decreases to 0 if the quota does not change. That means that the difference between this player's old and new index is $\frac{1}{2}$, which equals the upper bound from Theorem 3.2 for the number $m = 1$ of deleted players.

Now, let $\mathcal{H} = (1, 2, 2; 3)$ with $r = \frac{3}{5}$ be a WVG with changing quota. Player 1 has a Penrose-Banzhaf index of $\beta(\mathcal{H}, 1) = \frac{1}{2}$. If we delete one of the players with weight 2, the quota changes to $\frac{9}{5}$, which is equivalent to 2 since all players' weights are integers. In that case, the other weight-2 player becomes a dictator and player 1's power decreases to 0. Thus the difference between the old and the new Penrose-Banzhaf index of player 1 equals the upper bound of $\frac{1}{2}$ stated in Theorem 4.4.

Analogously to control by adding new players to a WVG, it is possible for a dummy player to become a dictator when we delete some other players from a game. We now give an example that illustrates how the power indices can change when we delete some players from a game and the new game has an accordingly changed quota.

Example 4.6 Let $\mathcal{G} = (5, 5, 3, 3, 1, 1; 10)$ be a WVG with $r = \frac{5}{9}$. Let us start with the Penrose-Banzhaf indices of the players: $\beta(\mathcal{G}, 1) = \beta(\mathcal{G}, 2) = \frac{1}{2}$, $\beta(\mathcal{G}, 3) = \beta(\mathcal{G}, 4) = \frac{1}{4}$, and $\beta(\mathcal{G}, 5) = \beta(\mathcal{G}, 6) = \frac{1}{8}$. Now, we are going to create a new game by deleting one player with weight 5 and one player with weight 3, so $\mathcal{G}_{\setminus\{1,3\}} = (5, 3, 1, 1; \frac{50}{9})$ with the new quota equivalent to 6. The Penrose-Banzhaf indices in the new game are as follows: $\beta(\mathcal{G}_{\setminus\{1,3\}}, 1) = \frac{7}{8}$, $\beta(\mathcal{G}_{\setminus\{1,3\}}, 2) = \frac{1}{8}$, and $\beta(\mathcal{G}_{\setminus\{1,3\}}, 3) = \beta(\mathcal{G}_{\setminus\{1,3\}}, 4) = \frac{1}{8}$. The index of the player with weight 5 has increased by $\frac{3}{8}$ and at the same time the index of the player with weight 3 has decreased by $\frac{1}{8} < 1 - 2^{-2} = \frac{3}{4}$. Finally, although the new quota is smaller than the old one, the Penrose-Banzhaf index of the players with weight 1 is unchanged.

Let us now analyze the Shapley-Shubik indices in these two games. The indices in $\mathcal{G}$ are: $\varphi(\mathcal{G}, 1) = \varphi(\mathcal{G}, 2) = \frac{3}{10}$, $\varphi(\mathcal{G}, 3) = \varphi(\mathcal{G}, 4) = \frac{2}{15}$, and $\varphi(\mathcal{G}, 5) = \varphi(\mathcal{G}, 6) = \frac{1}{15}$; and in $\mathcal{G}_{\setminus\{1,3\}}$: $\varphi(\mathcal{G}_{\setminus\{1,3\}}, 1) = \frac{3}{4}$ and $\varphi(\mathcal{G}_{\setminus\{1,3\}}, 2) = \varphi(\mathcal{G}_{\setminus\{1,3\}}, 3) = \varphi(\mathcal{G}_{\setminus\{1,3\}}, 4) = \frac{1}{12}$. The Shapley-Shubik indices of the player with weight 5 and of the players with weight 1 have increased, whereas the index of the player with weight 3 has decreased.

4.2 Control by adding or deleting players with changing the quota

We start by defining our problems of control by adding or by deleting players *with changing the quota* in WVGs, where the goals again are to increase, to decrease, to nonincrease, to nondecrease, or to maintain a distinguished player's power. Specifically, for the goal of increasing a player's power index PI with changing the quota, we consider the following decision problems that slightly modify the problems introduced and studied by Rey and Rothe [8]:

	Control by Adding Players with Changing Quota to Increase PI
Given:	A WVG $\mathcal{G}$ with players $N = \{1, \dots, n\}$, a quota $r \sum_{j=1}^{n} w_j$ for some real parameter $r \in [0, 1]$, a set M of unregistered players with weights $w_{n+1}, \dots, w_{n+m}$, a distinguished player $i \in N$, and a positive integer k.
Question:	Can at most k players $M' \subseteq M$ be added to $\mathcal{G}$ such that for the new game $\mathcal{G}_{\cup M}$ with the new quota $r \sum_{j \in N \cup M'} w_j$, it holds that $\mathrm{PI}(\mathcal{G}_{\cup M'}, i) > \mathrm{PI}(\mathcal{G}, i)$?

	Control by Deleting Players with Changing Quota to Increase PI
Given:	A WVG $\mathcal{G}$ with players $N = \{1, \dots, n\}$, a quota $r \sum_{j=1}^{n} w_j$ for some real parameter $r \in [0, 1]$, a distinguished player $i \in N$, and a positive integer k.
Question:	Can at most k players $M \subseteq N \setminus \{i\}$ be deleted from $\mathcal{G}$ such that for the new game $\mathcal{G}_{\setminus M}$ with the new quota $r \sum_{j \in N \setminus M} w_j$, it holds that $\mathrm{PI}(\mathcal{G}_{\setminus M}, i) > \mathrm{PI}(\mathcal{G}, i)$?

The problems for the goals of decreasing, nonincreasing, nondecreasing, and maintaining a distinguished player's power by adding or deleting players with changing the quota, in

Table 2 Overview of complexity results for control problems in WVGs with changing the quota with respect to various goals for the Shapley-Shubik (φ) and the probabilistic Penrose-Banzhaf index (β)

Goal	Control by adding players for β and φ	Control by deleting players for β	Control by deleting players for φ
Decrease	PP-hard	DP-hard	NP-hard
Nonincrease	PP-hard	NP-hard	NP-hard
Increase	PP-hard	DP-hard	NP-hard
Nondecrease	PP-hard	coNP-hard	NP-hard
Maintain	coNP-hard	coNP-hard	coNP-hard

relation to the original game, are defined analogously. Again, when the goal is to nonincrease, nondecrease, or maintain a player's power index, we require that *at least one player* must be added or deleted, so as to avoid that the problem becomes trivial.

In fact, in our proofs we will always consider the special problem variants where only one player can be added or deleted, i.e., the addition or deletion limit is always $k = 1$; this is again justified since we only prove lower bounds of the computational complexity of these problems, which thus immediately transfer to the more general problem variants. The best known upper bound for the computational complexity of our problems will only briefly be mentioned at the end of this section.

As one may guess, an additionally varying parameter will not make the decision problems easier: The problems with changing quotas caused by structural control remain hard when the original problems were hard. However, the problems *without changing the quota* defined and studied in Section 3 are not simply special cases of the corresponding problems *with changing the quota* defined above: While the former problems consider WVGs whose quota is fixed at will, the latter problems always have a quota that depends on the parameter r and the total weight of the players in the game. Therefore, there is no obvious reduction from the problems *without changing the quota* to the corresponding problems *with changing the quota*, and lower bounds for the former do not straightforwardly transfer to the lower bounds for the latter.

Table 2 presents a summary of our complexity results. Note that we list only lower bounds of these problems, i.e., we will prove only hardness results for them. As to their best known upper bounds, again, all these problems belong to the complexity class $\mathrm{NP}^{\mathrm{PP}}$. It would be an interesting task for future research to close these gaps by providing matching upper and lower bounds.

4.2.1 Control by adding players with changing the quota

We start with control by adding players with changing the quota. Just as Rey and Rothe [8] do for the corresponding control problems without changing the quota, we obtain PP-hardness for four of our goals.

Theorem 4.7 *For both the Penrose-Banzhaf and the Shapley-Shubik index, control by adding players to decrease, to nonincrease, to increase, or to nondecrease a distinguished player's power index in a WVG with changing the quota is* PP-*hard.*

Proof We only show PP-hardness of control by adding players to decrease the two power indices by reducing from the PP-complete COMPARE- #SUBSETSUM- RR problem. PP-hardness for the goal of nondecreasing either of the two power indices, which give rise to the complementary problems, follows immediately since PP is closed under complementation. PP-hardness for the goals of increasing or nonincreasing either of these two indices can be proven analogously with exactly the same reduction but starting from COMPARE- #SUBSETSUM- ЯЯ instead.

Let $(a_1, \ldots, a_n)$ be a COMPARE- #SUBSETSUM- RR instance with $\alpha = \sum_{i=1}^{n} a_i$. Let ξ_1 and ξ_2, respectively, be the number of SUBSETSUM solutions for $((a_1, \ldots, a_n), \frac{\alpha}{2} - 1)$ and $((a_1, \ldots, a_n), \frac{\alpha}{2} - 2)$, respectively. Now, construct the control problem instance consisting of a game

$$\mathcal{G} = (1, a_1, \ldots, a_n; \frac{\alpha}{2} - 1)$$

with $n+1$ players, the parameter $r = \frac{\frac{\alpha}{2}-1}{\alpha+1}$, and the distinguished player 1. Its power indices are $\beta(\mathcal{G}, 1) = \frac{\xi_2}{2^n} = \frac{2\xi_2}{2^{n+1}}$ and $\varphi(\mathcal{G}, 1) = \xi_2 \frac{t!(n-t)!}{(n+1)!}$, the latter because, using Lemma 2.1, we can assume that each coalition for which player 1 is pivotal has the same size t. Let the addition limit be $k = 1$, and let $n+2$ be the new player with weight $w_{n+2} = 1$. So, the quota in the new game after adding the player $n+2$ is equivalent to $\frac{\alpha}{2}$, since all players' weights are integers.

For PI $\in \{\beta, \varphi\}$, we will show that

$$\mathrm{PI}(\mathcal{G}_{\cup\{n+2\}}, 1) - \mathrm{PI}(\mathcal{G}, 1) < 0 \iff \xi_1 < \xi_2.$$

Assume that $\xi_1 < \xi_2$. Then, after adding the new player, the indices will change to

$$\beta(\mathcal{G}_{\cup\{n+2\}}, 1) = \frac{\xi_1 + \xi_2}{2^{n+1}} < \frac{2\xi_2}{2^{n+1}} = \beta(\mathcal{G}, 1)$$

and

$$\begin{aligned}\varphi(\mathcal{G}_{\cup\{n+2\}}, 1) &= \xi_1 \frac{(t+1)!(n-t)!}{(n+2)!} + \xi_2 \frac{t!(n-t+1!)}{(n+2)!} \\ &< \xi_2 \frac{t!(n-t)!}{(n+2)!}(t+1+n-t+1) = \varphi(\mathcal{G}, 1),\end{aligned}$$

so they both decrease.

Conversely, assume now that $\xi_1 \geq \xi_2$. Then we have $\beta(\mathcal{G}_{\cup\{n+2\}}, 1) \geq \beta(\mathcal{G}, 1)$ and $\varphi(\mathcal{G}_{\cup\{n+2\}}, 1) \geq \varphi(\mathcal{G}, 1)$. So both power indices do not decrease. □

Next, we turn to the goal of maintaining the two power indices.

Theorem 4.8 *For both the Penrose-Banzhaf and the Shapley-Shubik index, control by adding players to maintain a distinguished player's power index in a WVG with changing the quota is* coNP*-hard.*

Proof We show coNP-hardness by means of a reduction from the complement of the PARTITION problem. Let $(a_1, \ldots, a_n)$ be a PARTITION instance with $n > 1$, let $\alpha = \sum_{i=1}^{n} a_i$, and let ξ denote the number of its solutions. Construct the control problem instance consisting of a game

$$\mathcal{G} = (1, 2a_1, \ldots, 2a_n; \alpha)$$

with $n+1$ players and the distinguished player 1. Note that $\alpha = \frac{\alpha}{2\alpha+1} \sum_{i \in N} w_i$. Let us add a new player with weight 1. The quota in the new game will be $q_{\cup\{n+2\}} = \alpha + \frac{\alpha}{2\alpha+1} < \alpha + 1$ and is equivalent to $\alpha + 1$.

For PI $\in \{\beta, \varphi\}$, we will prove that

$$\mathrm{PI}(\mathcal{G}_{\cup\{n+2\}}, 1) - \mathrm{PI}(\mathcal{G}, 1) = 0 \iff \xi = 0.$$

From right to left, suppose that $\xi = 0$. Then $\beta(\mathcal{G}, 1) = \varphi(\mathcal{G}, 1) = 0$. When we add player $n+2$, player 1 is pivotal for coalitions with weight α, which is an even number and therefore the power indices in the new game remain equal to 0. So, if $\xi = 0$ then both power indices remain the same.

From left to right, we show the contrapositive. Suppose that $\xi > 0$. Note that player 1 is pivotal for the coalitions with weight $\alpha - 1$, which is an odd number, and each player $j \in \{2, \ldots, n+1\}$ has an even weight, so there exists no such coalition and $\beta(\mathcal{G}, 1) = \varphi(\mathcal{G}, 1) = 0$. In the new game $\mathcal{G}_{\cup\{n+2\}}$, player 1 is pivotal for the coalitions with weight α, so $\beta(\mathcal{G}_{\cup\{n+2\}}, 1) = \frac{\xi}{2^{n+1}} > \beta(\mathcal{G}, 1)$ and $\varphi(\mathcal{G}_{\cup\{n+2\}}, 1) > 0 = \varphi(\mathcal{G}, 1)$, so both indices increase and, therefore, our statement is true and control by adding players to maintain a distinguished player's power index is coNP-hard.

4.2.2 Control by deleting players with changing the quota

We now turn to control by deleting players with changing the quota, starting with the Penrose-Banzhaf index. The first goal we consider is to decrease this power index.

Theorem 4.9 *Control by deleting players to decrease a distinguished player's Penrose-Banzhaf index in a WVG with changing the quota is* DP*-hard.*

Proof As in Theorem 3.7, we again apply Lemma 2.2 to show DP-hardness and we again use the NP-complete PARTITION problem (which we again will call A as in that lemma). Let $x_1 = (a_1, \ldots, a_{n_1})$ and $x_2 = (b_1, \ldots, b_{n_2})$ be two instances of PARTITION, let $a = \sum_{i=1}^{n_1} a_i$ and $b = \sum_{i=1}^{n_2} b_i$, and let ξ_j be the number of x_j's solutions for PARTITION, $j \in \{1, 2\}$.

Let $W_{\mathcal{G}}$ be the players' total weight in a given game $\mathcal{G}$. Choose $\ell \in \mathbb{N}$ so that $10^\ell > 22b$, and let

$$r = \frac{3a \cdot 10^\ell + 6b + 1}{21a^2 \cdot 10^{2\ell} + \left(10 + \frac{213}{2}b\right) a \cdot 10^\ell + 129b^2 + \frac{55}{2}b + 1} \quad \text{and}$$
$$w_{\max} = \frac{21}{2}a^2 \cdot 10^{2\ell} + \left(\frac{3}{2} + \frac{213}{4}b\right) a \cdot 10^\ell + \frac{129}{2}b^2 + 3b.$$

Consider the weighted voting game

$$\mathcal{G} = \Big(1, a_1 \cdot 10^\ell, \ldots, a_{n_1} \cdot 10^\ell, b_1, \ldots, b_{n_2}, w_{\max}, w_{\max}, a \cdot 10^\ell, \frac{5}{2}a \cdot 10^\ell, \frac{5}{2}a \cdot 10^\ell, 6b, \frac{11}{2}b, \frac{11}{2}b, b, b, \frac{3}{2}b; rW_{\mathcal{G}}\Big).$$

Let player 1 be our distinguished player and let the deletion limit be $k = 1$.

The quota before deleting a player is $q(\mathcal{G}) = 3a \cdot 10^\ell + 6b + 1$.

Now, let j be a player with largest weight, $w_{\max}$; deleting j from $\mathcal{G}$ changes the quota to

$$
\begin{aligned}
q(\mathcal{G}_{\setminus\{j\}}) = rW_{\mathcal{G}_{\setminus\{j\}}} \;=\; & \frac{3a\cdot 10^{\ell}+6b+1}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
& \cdot\left(\frac{21}{2}a^2\cdot 10^{2\ell}+\left(\frac{17}{2}+\frac{213}{4}b\right)a\cdot 10^{\ell}+\frac{129}{2}b^2+\frac{49}{2}b+1\right) \\
= & \frac{\frac{63}{2}a^3\cdot 10^{3\ell}+\left(\frac{51}{2}+\frac{639}{4}b\right)a^2\cdot 10^{2\ell}}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
& +\frac{\frac{387}{2}ab^2\cdot 10^{\ell}+\frac{147}{2}ab\cdot 10^{\ell}+3a\cdot 10^{\ell}}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
& +\frac{63a^2b\cdot 10^{2\ell}+\left(51+\frac{639}{2}b\right)ab\cdot 10^{\ell}+387b^3+147b^2+6b}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
& +\frac{\frac{21}{2}a^2\cdot 10^{2\ell}+\left(\frac{17}{2}+\frac{213}{4}b\right)a\cdot 10^{\ell}+\frac{129}{2}b^2+\frac{49}{2}b+1}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
= & \frac{\frac{63}{2}a^3\cdot 10^{3\ell}+\left(36+\frac{891}{4}b\right)a^2\cdot 10^{2\ell}}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
& +\frac{\left(\frac{23}{2}+\frac{711}{4}b+513b^2\right)a\cdot 10^{\ell}+387b^3+\frac{423}{2}b^2+\frac{61}{2}b+1}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
= & \frac{\frac{3}{2}a\cdot 10^{\ell}\left(21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1\right)}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
& +\frac{(21+63b)\,a^2\cdot 10^{2\ell}+\left(10+\frac{273}{2}b+\frac{639}{2}b^2\right)a\cdot 10^{\ell}}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
& +\frac{387b^3+\frac{423}{2}b^2+\frac{61}{2}b+1}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
= & \frac{3}{2}a\cdot 10^{\ell}+\frac{3b\left(21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1\right)}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
& +\frac{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
= & \frac{3}{2}a\cdot 10^{\ell}+3b+1.
\end{aligned}
$$

The next largest weight is $\frac{5}{2}a\cdot 10^{\ell}$, so for a player j with this weight, we have

$$
\begin{aligned}
rw_j &= \frac{\left(3a\cdot 10^{\ell}+6b+1\right)\frac{5}{2}a\cdot 10^{\ell}}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} \\
&= \frac{\frac{15}{2}a^2\cdot 10^{2\ell}+\left(\frac{5}{2}+15b\right)a\cdot 10^{\ell}}{21a^2\cdot 10^{2\ell}+\left(10+\frac{213}{2}b\right)a\cdot 10^{\ell}+129b^2+\frac{55}{2}b+1} < 1.
\end{aligned}
$$

Thus, for any player j' whose weight is different from $w_{\max}$, the quota $q(\mathcal{G}_{\setminus\{j'\}}) = q(\mathcal{G}) - r w_{j'}$ of the game resulting from $\mathcal{G}$ by deleting j' must belong to the interval $(q(\mathcal{G}) - 1, q(\mathcal{G}))$.

Let $n = n_1 + n_2 + 12$ be the number of players in $\mathcal{G}$, and assume that $\chi_A(x_1) \geq \chi_A(x_2)$. We will now show that there exists some $i \in \{2, \ldots, n\}$ such that

$$\beta(\mathcal{G}, 1) - \beta(\mathcal{G}_{\setminus\{i\}}, 1) > 0 \iff x_1 \in A \wedge x_2 \notin A.$$

If $x_1 \notin A \wedge x_2 \notin A$, then for all $i \in \{2, \ldots, n\}$, $\beta(\mathcal{G}, 1) = \beta(\mathcal{G}_{\setminus\{i\}}, 1) = 0$, so the index does not decrease.

If $x_1 \in A \wedge x_2 \notin A$, then $\beta(\mathcal{G}, 1) = \frac{2\xi_1}{2^{n-1}}$, and after removing the player j with weight $6b$, we have $\beta(\mathcal{G}_{\setminus\{j\}}, 1) = 0$, so the index indeed can decrease by deleting a player in this case.

Finally, if $x_1 \in A \wedge x_2 \in A$, then $\beta(\mathcal{G}, 1) = \frac{2\xi_1+4\xi_1\xi_2}{2^{n-1}}$. If we delete any player j with weight $a \cdot 10^\ell$, b, or $\frac{3}{2}b$, then $\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{2\xi_1+4\xi_1\xi_2}{2^{n-2}}$, so the index increases. If we remove the player j with weight $\frac{11}{2}b$, then $\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{2\xi_1+2\xi_1\xi_2}{2^{n-2}} = \frac{4\xi_1+4\xi_1\xi_2}{2^{n-1}}$, so the index also increases. The index increases again when we delete player j with weight $6b$: $\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{4\xi_1\xi_2}{2^{n-2}} = \frac{4\xi_1\xi_2+4\xi_1\xi_2}{2^{n-1}}$. If we delete any player j with weight $\frac{5}{2}a \cdot 10^\ell$, the index stays unchanged: $\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{\xi_1+2\xi_1\xi_2}{2^{n-2}} = \frac{2\xi_1+4\xi_1\xi_2}{2^{n-1}}$. The same will happen if we delete any player $j \in \{2, \ldots, n_1+1\}$: $\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{2\frac{\xi_1}{2}+4\frac{\xi_1}{2}\xi_2}{2^{n-2}} = \frac{2\xi_1+4\xi_1\xi_2}{2^{n-1}}$. If we remove a player $j \in \{n_1+2, \ldots, n_1+n_2+1\}$, then $\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{2\xi_1+4\xi_1\frac{\xi_2}{2}}{2^{n-2}} = \frac{4\xi_1+4\xi_1\xi_2}{2^{n-1}}$, so the index increases. Finally, if we delete a player j with weight $w_{\max}$, then $\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{\xi_1+2\xi_1\xi_2}{2^{n-2}} = \frac{2\xi_1+4\xi_1\xi_2}{2^{n-1}}$, so the index does not decrease either in this case.

Summing up, in this case (where we have two yes-instances of PARTITION) the Penrose-Banzhaf index of our distinguished player 1 does not decrease, no matter which player is deleted.

By Lemma 2.2, DP-hardness of our control problem follows. □

Next, we consider the goal of nonincreasing the Penrose-Banzhaf index.

Theorem 4.10 *Control by deleting players to nonincrease a distinguished player's Penrose-Banzhaf index in a WVG with changing the quota is* NP*-hard.*

Proof We provide a reduction from the NP-complete SUBSETSUM problem. Let $x = (a_1, \ldots, a_n; q)$ be an instance of SUBSETSUM, let $\alpha = \sum_{i=1}^n a_i$, and let ξ be the number of x's solutions for SUBSETSUM. Consider the weighted voting game (with changing the quota) defined by

$$\mathcal{G} = (1, 4a_1, \ldots, 4a_n, 5\alpha, 5\alpha, 5\alpha, 1, 1, 1, 4q + 5\alpha, 4q + 5\alpha, 4q + 5\alpha, 85\alpha^2 + 98q \cdot \alpha + 24q^2 + 2q; 5\alpha + 4q + 1)$$

with $n+11$ players, the parameter $r = \frac{5\alpha+4q+1}{85\alpha^2+(34+98q)\alpha+24q^2+14q+4}$, the distinguished player 1, and the deletion limit $k = 1$. After removing any one of the players $i \in \{2, \ldots, n+10\}$ from $\mathcal{G}$, we get the following quota:

$$\begin{aligned} q(\mathcal{G}_{\setminus\{i\}}) &\geq 5\alpha + 4q + 1 - r(5\alpha + 4q) \\ &= 5\alpha + 4q + 1 - \frac{25\alpha^2 + (5 + 40q)\alpha + 16q^2 + 4q}{85\alpha^2 + (34 + 98q)\alpha + 24q^2 + 14q + 4}, \end{aligned}$$

so $q(\mathcal{G}_{\setminus\{i\}})$ is in the range $(5\alpha + 4q; 5\alpha + 4q + 1)$, and by deleting the last player, we obtain

$$\begin{aligned} q(\mathcal{G}_{\setminus\{n+11\}}) &= r(34\alpha + 12q + 4) \\ &= \frac{170\alpha^2 + (54 + 196q)\alpha + 48q^2 + 28q + 4}{85\alpha^2 + (34 + 98q)\alpha + 24q^2 + 14q + 4} \\ &= 1 + \frac{85\alpha^2 + (20 + 98q)\alpha + 24q^2 + 14q}{85\alpha^2 + (34 + 98q)\alpha + 24q^2 + 14q + 4}, \end{aligned}$$

so $q(\mathcal{G}_{\setminus\{n+11\}})$ is in the range $(1, 2)$.

Suppose $\xi = 0$. Then

$$\beta(\mathcal{G}, 1) = \frac{3}{2^{n+10}}.$$

If we delete one of the players from $\{2, \dots, n + 7\}$, the index will increase because player 1 will stay pivotal for the same coalitions and will not become pivotal for any other one. If we remove one of the players $n + 8$, $n + 9$ or $n + 10$, then

$$\beta(\mathcal{G}_{\setminus\{n+8\}}, 1) = \beta(\mathcal{G}_{\setminus\{n+9\}}, 1) = \beta(\mathcal{G}_{\setminus\{n+10\}}, 1) = \frac{2}{2^{n+9}} = \frac{4}{2^{n+10}},$$

so the index increases also in this case. Finally, if we delete the player $n + 11$, the quota will change to almost 2 (and since all the weights are integers, the quota is then equivalent to 2) and

$$\beta(\mathcal{G}_{\setminus\{n+11\}}, 1) = \frac{3}{2^{n+9}} = \frac{6}{2^{n+10}},$$

so the index has increased again.

Now, suppose $\xi > 0$. Then

$$\beta(\mathcal{G}, 1) = \frac{3 + 3\xi}{2^{n+10}},$$

and after removing the player $n + 11$, we have

$$\beta(\mathcal{G}_{\setminus\{n+11\}}, 1) = \frac{3}{2^{n+9}} = \frac{6}{2^{n+10}} \le \frac{3 + 3\xi}{2^{n+10}}.$$

Therefore, it is possible to nonincrease player 1's Penrose-Banzhaf power index by deleting a player when we have a yes-instance of SUBSETSUM. □

We now turn to the goal of increasing the Penrose-Banzhaf index.

Theorem 4.11 *Control by deleting players to increase a distinguished player's Penrose-Banzhaf index in a WVG with changing the quota is* DP-*hard.*

Proof Once more, we apply Lemma 2.2 with the NP-complete PARTITION problem (which we again will call A as in that lemma). Let $x_1 = (a_1, \dots, a_{n_1})$ and $x_2 = (b_1, \dots, b_{n_2})$ be two instances of PARTITION, let $a = \sum_{i=1}^{n_1} a_i$ and $b = \sum_{i=1}^{n_2} b_i$, and let ξ_j be the number of x_j's solutions for PARTITION, $j \in \{1, 2\}$.

Let $n = n_1 + n_2$

and choose $\ell \in \mathbb{N}$ so that $10^\ell > 3b$. Let

$$r = \frac{\frac{a}{2} \cdot 10^\ell + 2b + 1}{\frac{a^2}{4} \cdot 10^{2\ell} + (2b + 1)a \cdot 10^\ell + 6b^2 + 5b + 1}.$$

Consider the weighted voting game

$$\mathcal{G} = \left(1, a_1 \cdot 10^\ell, \ldots, a_{n_1} \cdot 10^\ell, 2b_1, \ldots, 2b_{n_2}, b, \frac{a^2}{4} \cdot 10^{2\ell} + 2ab \cdot 10^\ell + 6b^2 + 2b; rW_\mathcal{G}\right),$$

where $W_\mathcal{G}$ again denotes the players' total weight in $\mathcal{G}$. Let player 1 be our distinguished player and let the deletion limit be $k = 1$. The quotas before and after removing either the last or the second-to-last player are as follows:

$$\begin{aligned}
q(\mathcal{G}) &= \frac{a}{2} \cdot 10^\ell + 2b + 1,\\
q(\mathcal{G}_{\setminus\{n+3\}}) &= rW_{\mathcal{G}_{\setminus\{n+3\}}}\\
&= \frac{\left(\frac{a}{2} \cdot 10^\ell + 2b + 1\right)\left(a \cdot 10^\ell + 3b + 1\right)}{\frac{a^2}{4} \cdot 10^{2\ell} + (2b+1)\, a \cdot 10^\ell + 6b^2 + 5b + 1}\\
&= \frac{\frac{a^2}{2} \cdot 10^{2\ell} + 2ab \cdot 10^\ell + a \cdot 10^\ell + \frac{3ab}{2} \cdot 10^\ell}{\frac{a^2}{4} \cdot 10^{2\ell} + (2b+1)\, a \cdot 10^\ell + 6b^2 + 5b + 1}\\
&\quad + \frac{6b^2 + 3b + \frac{a}{2} \cdot 10^\ell + 2b + 1}{\frac{a^2}{4} \cdot 10^{2\ell} + (2b+1)\, a \cdot 10^\ell + 6b^2 + 5b + 1}\\
&= \frac{\frac{a^2}{2} \cdot 10^{2\ell} + \left(\frac{7b}{2} + \frac{3}{2}\right) a \cdot 10^\ell + 6b^2 + 5b + 1}{\frac{a^2}{4} \cdot 10^{2\ell} + (2b+1)\, a \cdot 10^\ell + 6b^2 + 5b + 1},\\
q(\mathcal{G}_{\setminus\{n+2\}}) &= q(\mathcal{G}) - rb\\
&= q(\mathcal{G}) - \frac{\frac{ab}{2} \cdot 10^\ell + \frac{b^2}{2} + b}{\frac{a^2}{4} \cdot 10^{2\ell} + (2b+1)\, a \cdot 10^\ell + 6b^2 + 5b + 1}.
\end{aligned}$$

Note that $q(\mathcal{G}_{\setminus\{n+3\}})$ is in the range $(1, 2)$ and $q(\mathcal{G}_{\setminus\{n+2\}})$ in the range $(q(\mathcal{G}) - 1, q(\mathcal{G}))$. Also, note that after removing any other player, the new quota is equivalent to $q(\mathcal{G}_{\setminus\{n+2\}})$, i.e., it is greater than $q(\mathcal{G}) - 1$ as well.

Assume that $\chi_A(x_1) \geq \chi_A(x_2)$. We now show that there exists some $i \in \{2, \ldots, n+3\}$ such that

$$\beta(\mathcal{G}, 1) - \beta(\mathcal{G}_{\setminus\{i\}}, 1) < 0 \iff x_1 \in A \wedge x_2 \notin A.$$

If $x_1 \notin A \wedge x_2 \notin A$, then for all $i \in \{2, \ldots, n+3\}$, $\beta(\mathcal{G}, 1) = \beta(\mathcal{G}_{\setminus\{i\}}, 1) = 0$, so the index does not increase.

If $x_1 \in A \wedge x_2 \notin A$, then $\beta(\mathcal{G}, 1) = \frac{\xi_1}{2^{n+2}}$ and $\beta(\mathcal{G}_{\setminus\{n+2\}}, 1) = \frac{\xi_1}{2^{n+1}}$, so the index indeed can increase in this case.

If $x_1 \in A \wedge x_2 \in A$, then $\beta(\mathcal{G}, 1) = \frac{\xi_1 + \xi_1\xi_2}{2^{n+2}}$. On the other hand, we have $\beta(\mathcal{G}_{\setminus\{n+3\}}, 1) = 0$; $\beta(\mathcal{G}_{\setminus\{n+2\}}, 1) = \frac{\xi_1}{2^{n+1}} = \frac{2\xi_1}{2^{n+2}} < \frac{\xi_1 + \xi_1\xi_2}{2^{n+2}}$ (since $\xi_2 \geq 2$); for $j \in \{n_1 + 2, \ldots, n+1\}$, $\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{\xi_1 \frac{\xi_2}{2}}{2^{n+1}} = \frac{\xi_1\xi_2}{2^{n+2}} < \beta(\mathcal{G}, 1)$, and finally, for $j \in \{2, \ldots, n_1 + 1\}$, $\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{\frac{\xi_1}{2} + \frac{\xi_1}{2}\xi_2}{2^{n+1}} = \beta(\mathcal{G}, 1)$. So, no matter which player is deleted, the index of player 1 does not increase.

By Lemma 2.2, DP-hardness of our control problem follows.

The next goal we consider is to nondecrease the Penrose-Banzhaf index.

Theorem 4.12 *Control by deleting players to nondecrease a distinguished player's Penrose-Banzhaf index in a WVG with changing the quota is* coNP-*hard.*

Proof We show coNP-hardness by means of a reduction from the complement of the NP-complete PARTITION problem. Let $(a_1, \ldots, a_n)$ be a PARTITION instance, let $\alpha = \sum_{i=1}^{n} a_i$, and let ξ denote the number of its solutions.

Construct the control problem instance consisting of a game

$$\mathcal{G} = (1, 2a_1, \ldots, 2a_n, \alpha, 6\alpha^2; 2\alpha + 1)$$

with $n + 3$ players, parameter $r = \frac{2\alpha+1}{6\alpha^2+3\alpha+1}$, the distinguished player 1, and the deletion limit $k = 1$.

The quotas after removing one player from the game are as follows: For $i \in \{2, \ldots, n+2\}$, we have

$$q(\mathcal{G}_{\setminus\{i\}}) \geq 2\alpha + 1 - r \cdot 2\alpha = 2\alpha + 1 - \frac{4\alpha^2 + 2\alpha}{6\alpha^2 + 3\alpha + 1},$$

so $q(\mathcal{G}_{\setminus\{i\}}) \in (2\alpha, 2\alpha + 1)$; and when we delete the player $n + 3$, we obtain

$$q(\mathcal{G}_{\setminus\{n+3\}}) = r(3\alpha + 1) = \frac{6\alpha^2 + 5\alpha + 1}{6\alpha^2 + 3\alpha + 1} = 1 + \frac{2\alpha}{6\alpha^2 + 3\alpha + 1},$$

so $q(\mathcal{G}_{\setminus\{n+3\}})$ is in the range $(1, 2)$.

Suppose $\xi = 0$. Then

$$\beta(\mathcal{G}, 1) = \frac{1}{2^{n+2}},$$

and after removing the player $n+2$, the index will increase (therefore, will also nondecrease).

Conversely, suppose $\xi > 0$. Then

$$\beta(\mathcal{G}, 1) = \frac{1 + \xi}{2^{n+2}}.$$

If we remove a player $i \in \{2, \ldots, n + 1\}$, the index will change to

$$\beta(\mathcal{G}_{\setminus\{i\}}, 1) = \frac{\frac{1}{2}\xi}{2^{n+1}} = \frac{\xi}{2^{n+2}} < \beta(\mathcal{G}, 1).$$

If we delete the player $n + 2$, then

$$\beta(\mathcal{G}_{\setminus\{n+2\}}, 1) = \frac{1}{2^{n+1}} = \frac{2}{2^{n+2}},$$

so the index will decrease, since $\xi \geq 2$. Finally, if we remove the player $n + 3$, the quota will change to almost 2, and there will be no coalition for which player 1 will be pivotal, so the index will decrease to 0.

Finally, we consider the goal of maintaining the Penrose-Banzhaf index.

Theorem 4.13 *Control by deleting players to maintain a distinguished player's Penrose-Banzhaf index in a WVG with changing the quota is* coNP-*hard.*

Proof We again show coNP-hardness by means of a reduction from the complement of the NP-complete PARTITION problem. Let $(a_1, \ldots, a_n)$ be a PARTITION instance with $n > 1$, let $\alpha = \sum_{i=1}^{n} a_i$, and let ξ denote the number of its solutions. Construct the control problem instance consisting of a game

$$\mathcal{G} = \left(1, 3a_1, \ldots, 3a_n, \tfrac{3}{2}\alpha, \tfrac{3}{2}\alpha, 3\alpha\left(\tfrac{3}{2}\alpha - 1\right) - 1; \tfrac{3}{2}\alpha + 1\right)$$

with $n + 4$ players, the distinguished player 1, and the deletion limit $k = 1$. Note that $\frac{3}{2}\alpha + 1 = \frac{1}{3\alpha} \sum_{i \in N} w_i$.

The quota's values in the possible new games are as follows. If we delete a player $j \in \{2, \ldots, n+1\}$, then

$$q(\mathcal{G}_{\setminus\{j\}}) = \frac{3}{2}\alpha + 1 - \frac{1}{3\alpha}3a_j > \frac{3}{2}\alpha,$$

so the new quota does not change the coalitions for which player 1 is pivotal. If we delete player $n+2$ or $n+3$, then

$$q(\mathcal{G}_{\setminus\{n+2\}}) = q(\mathcal{G}_{\setminus\{n+3\}}) = \frac{3}{2}\alpha + 1 - \frac{1}{3\alpha}\frac{3}{2}\alpha = \frac{3}{2}\alpha + \frac{1}{2},$$

and again the new quota does not change the situation of player 1. Finally, if we delete player $n+4$, then

$$q(\mathcal{G}_{\setminus\{n+4\}}) = \frac{3}{2}\alpha + 1 - \frac{1}{3\alpha}(3\alpha(\frac{3}{2}\alpha - 1) - 1) = 2 + \frac{1}{3\alpha},$$

which is equivalent to 3.

We will now show that

$$(\exists j \in \{2, \ldots, n+4\}) \left[\beta(\mathcal{G}_{\setminus\{j\}}, 1) - \beta(\mathcal{G}, 1) = 0\right] \iff \xi = 0.$$

From right to left, suppose that $\xi = 0$. Then there are only two coalitions for which player 1 is pivotal: the coalition with player $n+2$ and the coalition with player $n+3$. Therefore, $\beta(\mathcal{G}, 1) = \frac{2}{2^{n+3}} = \frac{1}{2^{n+2}}$, and if we delete one of these players, player 1's new Penrose-Banzhaf index is $\beta(\mathcal{G}_{\setminus\{n+2\}}, 1) = \beta(\mathcal{G}_{\setminus\{n+3\}}, 1) = \frac{1}{2^{n+2}} = \beta(\mathcal{G}, 1)$. So, if $\xi = 0$, it can be ensured that the Penrose-Banzhaf index of player 1 will not change even when a player is deleted.

From left to right, suppose that $\xi > 0$. Then

$$\beta(\mathcal{G}, 1) = \frac{\xi + 2}{2^{n+3}} > 0.$$

If we delete a player $j \in \{2, \ldots, n+1\}$, player 1 will be pivotal for half of the coalitions from the old game, so

$$\beta(\mathcal{G}_{\setminus\{j\}}, 1) = \frac{\frac{1}{2}\xi + 2}{2^{n+2}} = \frac{\xi + 4}{2^{n+3}} > \beta(\mathcal{G}, 1).$$

If player $n+2$ or player $n+3$ is deleted, then

$$\beta(\mathcal{G}_{\setminus\{n+2\}}, 1) = \beta(\mathcal{G}_{\setminus\{n+3\}}, 1) = \frac{\xi + 1}{2^{n+2}} = \frac{2\xi + 2}{2^{n+3}} > \beta(\mathcal{G}, 1).$$

Finally, if we delete player $n+4$, player 1 will be pivotal for the coalitions with their value equal 2 and there is no such a coalition in the new game (as in the old game) and

$$\beta(\mathcal{G}_{\setminus\{n+4\}}, 1) = 0 < \beta(\mathcal{G}, 1).$$

Summing up, there is no possibility to maintain player 1's Penrose-Banzhaf index if $\xi > 0$ and some player has to be deleted. Therefore, control by deleting players with changing the quota to increase a distinguished player's Penrose-Banzhaf index is coNP-hard. □

It remains to show our complexity results for the Shapley-Shubik index when players are deleted from a game. We start with the goals of either decreasing or nonincreasing this power index. This time, we merely obtain NP-hardness (instead of DP-hardness).

Theorem 4.14 *Control by deleting players to decrease or to nonincrease a distinguished player's Shapley-Shubik index in a weighted voting game with changing the quota is* NP-*hard.*

Proof We show NP-hardness by means of a reduction from the SubsetSum problem. Let $(a_1, \ldots, a_n; m)$ be a given instance of SubsetSum and let $\alpha = \sum_{i=1}^{n} a_i$. Let ξ be the number of SubsetSum solutions for $(a_1, \ldots, a_n; m)$. By Lemma 2.1, we can assume that each of the solutions has the same size, namely $\frac{2}{3}n$, and therefore, that $n \geq 3$.

Let $y_1, \ldots, y_{n+4}, s, \tilde{q}, x \in \mathbb{N}$ be chosen such that:

$$y_{n+4} = 1,$$
$$y_i > (2n-1)\sum_{j=i+1}^{n+4} y_j \quad \text{for each } i \in \{1, \ldots, n+3\},$$
$$10^s > (2n-1)\sum_{j=1}^{n+4} y_j,$$
$$\tilde{q} > 4\alpha \cdot 10^s + 2(2n-1)\sum_{j=1}^{n+4} y_j,$$
$$x = y_1 + \cdots + y_n.$$

Consider the following weighted voting game with changing quota:

$$\begin{aligned}\mathcal{G} = (&1, 2a_1 \cdot 10^s, \ldots, 2a_n \cdot 10^s, \tilde{q} - 2m \cdot 10^s - x, (n+5)\tilde{q}^2 + (2\alpha \cdot 10^s - 2m \cdot 10^s - x)\tilde{q},\\ &\tilde{q} - (2n-1)y_1, \underbrace{y_1, \ldots, y_1}_{2n-1}, \ldots, \tilde{q} - (2n-1)y_{n+4}, \underbrace{y_{n+4}, \ldots, y_{n+4}}_{2n-1}; \tilde{q} + 1)\end{aligned}$$

with $2n^2 + 9n + 3$ players, the parameter

$$r = \frac{\tilde{q} + 1}{(n+5)\tilde{q}^2 + (n + 5 + (2\alpha - 2m) \cdot 10^s - x)\tilde{q} + (2\alpha - 2m) \cdot 10^s - x + 1},$$

the distinguished player 1, and the deletion limit $k = 1$.

Let us calculate the new quotas: For $i \in \{2, \ldots, n+2\} \cup \{n+4, \ldots, 2n^2 + 9n + 3\}$, we have

$$\begin{aligned} q(\mathcal{G}_{\setminus\{i\}}) &\geq \tilde{q} + 1 - r\tilde{q} \\ &= \tilde{q} + 1 - \frac{(\tilde{q})^2 + \tilde{q}}{(n+5)\tilde{q}^2 + (n + 5 + (2\alpha - 2m) \cdot 10^s - x)\tilde{q} + (2\alpha - 2m) \cdot 10^s - x + 1} \\ &> \tilde{q}, \end{aligned}$$

so $q(\mathcal{G}_{\setminus\{i\}})$ is in the range $(\tilde{q}, \tilde{q} + 1)$, and

$$\begin{aligned} q(\mathcal{G}_{\setminus\{n+3\}}) &= r((n+5)\tilde{q} + 2\alpha \cdot 10^s - 2m \cdot 10^s - x + 1) \\ &= \frac{(n+5)(\tilde{q})^2 + (n + 6 + 2\alpha \cdot 10^s - 2m \cdot 10^s - x)\tilde{q} + 2\alpha \cdot 10^s - 2m \cdot 10^s - x + 1}{(n+5)\tilde{q}^2 + (n + 5 + 2\alpha \cdot 10^s - 2m \cdot 10^s - x)\tilde{q} + 2\alpha \cdot 10^s - 2m \cdot 10^s - x + 1} \\ &= 1 + \frac{\tilde{q}}{(n+5)\tilde{q}^2 + (n + 5 + (2\alpha - 2m) \cdot 10^s - x)\tilde{q} + (2\alpha - 2m) \cdot 10^s - x + 1}, \end{aligned}$$

so $q(\mathcal{G}_{\setminus\{n+3\}})$ is in the range $(1, 2)$.

We now show that the following three statements are pairwise equivalent:

1. $(a_1, \ldots, a_n; m)$ is a yes-instance of SUBSETSUM, i.e., $\xi > 0$.
2. There is a player $j > 1$ whose deletion decreases the Shapley-Shubik index of player 1: $(\exists j \in \{2, \ldots, 2n^2 + 9n + 3\}) \left[\varphi(\mathcal{G}_{\setminus\{j\}}, 1) - \varphi(\mathcal{G}, 1) < 0\right]$.
3. There is a player $j > 1$ whose deletion nonincreases the Shapley-Shubik index of player 1: $(\exists j \in \{2, \ldots, 2n^2 + 9n + 3\}) \left[\varphi(\mathcal{G}_{\setminus\{j\}}, 1) - \varphi(\mathcal{G}, 1) \leq 0\right]$.

(1) implies (2): Suppose that $\xi > 0$. Then

$$\varphi(\mathcal{G}, 1) = (n+4)\frac{(2n)!(2n^2+7n+2)!}{(2n^2+9n+3)!} + \xi(2n-1)^n \frac{(\frac{5}{3}n+1)!(2n^2+\frac{22}{3}n+1)!}{(2n^2+9n+3)!},$$

and after removing the player $n+2$, we have

$$\begin{aligned}\varphi(\mathcal{G}_{\setminus\{n+2\}}, 1) &= (n+4)\frac{(2n)!(2n^2+7n+1)!}{(2n^2+9n+2)!} \\ &= (n+4)\frac{(2n)!(2n^2+7n+2)!}{(2n^2+9n+3)!}\frac{2n^2+9n+3}{2n^2+7n+2} \\ &= (n+4)\frac{(2n)!(2n^2+7n+2)!}{(2n^2+9n+3)!}\left(1+\frac{2n+1}{2n^2+7n+2}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\varphi(\mathcal{G}, 1) - \varphi(\mathcal{G}_{\setminus\{n+2\}}, 1) \\ &= \xi(2n-1)^n \frac{(\frac{5}{3}n+1)!(2n^2+\frac{22}{3}n+1)!}{(2n^2+9n+3)!} - (n+4)\frac{(2n+1)!(2n^2+7n+1)!}{(2n^2+9n+3)!}.\end{aligned}$$

Since

$$\frac{(\frac{5}{3}n+1)!(2n^2+\frac{22}{3}n+1)!}{(2n^2+9n+3)!} > \frac{(2n+1)!(2n^2+7n+1)!}{(2n^2+9n+3)!}$$

and

$$(2n-1)^n > n+4$$

for any $n \geq 2$, it follows that

$$\varphi(\mathcal{G}, 1) - \varphi(\mathcal{G}_{\setminus\{n+2\}}, 1) > 0.$$

That is, it is possible to decrease the Shapley-Shubik index of the distinguished player by deleting a player from the game $\mathcal{G}$.

(2) implies (3): is obvious.

(3) implies (1): To prove the contrapositive, suppose that $\xi = 0$. Then

$$\varphi(\mathcal{G}, 1) = (n+4)\frac{(2n)!(2n^2+7n+2)!}{(2n^2+9n+3)!}.$$

We will show that no matter which player from $\{2, \ldots, 2n^2 + 9n + 3\}$ we delete, this will increase the Shapley-Shubik index of the distinguished player 1.

If we remove one of the players from $\{2, \ldots, n+2\}$, player 1 will remain pivotal for the same coalitions, so the Shapley-Shubik index of 1's will increase. Each player $i \in \{n+4, \ldots, 2n^2 + 9n + 3\}$ belongs to exactly one coalition for which the distinguished player 1 is pivotal and 1 will stay pivotal for the remaining coalitions. Hence, if we delete i, the index will change to

$$\begin{aligned}\varphi(\mathcal{G}_{\setminus\{i\}}, 1) &= (n+3)\frac{(2n)!(2n^2+7n+1)!}{(2n^2+9n+2)!}\\ &= (n+3)\frac{2n^2+9n+3}{2n^2+7n+2}\frac{(2n)!(2n^2+7n+2)!}{(2n^2+9n+3)!}\\ &= (n+3)\Big(1+\frac{2n+1}{2n^2+7n+2}\Big)\frac{(2n)!(2n^2+7n+2)!}{(2n^2+9n+3)!}\\ &= \Big(n+3+\frac{2n^2+7n+3}{2n^2+7n+2}\Big)\frac{(2n)!(2n^2+7n+2)!}{(2n^2+9n+3)!}\\ &> (n+4)\frac{(2n)!(2n^2+7n+2)!}{(2n^2+9n+3)!}\\ &= \varphi(\mathcal{G}, 1).\end{aligned}$$

Finally, if we remove the player $n+3$, player 1 will be pivotal only for coalitions with weight 2, so

$$\begin{aligned}\varphi(\mathcal{G}_{\setminus\{n+3\}}, 1) &= (2n-1)\frac{1!(2n^2+9n)!}{(2n^2+9n+2)!}\\ &= \frac{1}{2}(2n-1)(2n^2+9n+3)\frac{2!(2n^2+9n)!}{(2n^2+9n+3)!}\\ &> \frac{1}{2}(4n^3+16n^2-3n-3)\frac{(2n)!(2n^2+7n+2)!}{(2n^2+9n+3)!}\\ &= \Big(2n^3+8n^2-\frac{3}{2}n-\frac{3}{2}\Big)\frac{(2n)!(2n^2+7n+2)!}{(2n^2+9n+3)!},\end{aligned}$$

and because $n+4 < 2n^3+8n^2-\frac{3}{2}n-\frac{3}{2}$ for any $n \geq 1$, the Shapley-Shubik index of the distinguished player has increased also in this case. □

Next, we turn to the goals of either increasing or nondecreasing a player's Shapley-Shubik index.

Theorem 4.15 *Control by deleting players to increase or to nondecrease a distinguished player's Shapley-Shubik index in a weighted voting game with changing the quota is* NP-*hard.*

Proof We show NP-hardness by means of a reduction from the SUBSETSUM problem. Let $(a_1, \ldots, a_n; m)$ be a given instance of SUBSETSUM and let $\alpha = \sum_{i=1}^{n} a_i$. Let ξ be the number of SUBSETSUM solutions for $(a_1, \ldots, a_n; m)$. By Lemma 2.1, we can assume that each of the solutions has the same size, namely $\frac{2}{3}n$, and therefore, that $n \geq 3$.

Let $s, y \in \mathbb{N}$ be chosen such that

$$10^s > 2(n-2)y.$$

Consider the following weighted voting game with changing quota:

$$\begin{aligned}\mathcal{G} = (&1, 2a_1\cdot 10^s, \ldots, 2a_n\cdot 10^s, 8\alpha^2\cdot 10^{2s}+2\alpha\cdot 10^s, 4\alpha\cdot 10^s+4,\\ &(2\alpha-2m)\cdot 10^s, 2m\cdot 10^s-2(n-2)y, \underbrace{2y, \ldots, 2y}_{n-2}; 2\alpha\cdot 10^s+1)\end{aligned}$$

with $2n + 3$ players, the parameter $r = \frac{2\alpha \cdot 10^s + 1}{8\alpha^2 \cdot 10^{2s} + 10\alpha \cdot 10^s + 5}$, the distinguished player 1, and the deletion limit $k = 1$.

Let us calculate the new quotas: For $i \in \{2, \ldots, n+1\} \cup \{n+4, \ldots, 2n+3\}$, we have

$$\begin{aligned} q(\mathcal{G}_{\setminus\{i\}}) &\geq 2\alpha \cdot 10^s + 1 - r \cdot 2\alpha \cdot 10^s \\ &= 2\alpha \cdot 10^s + 1 - \frac{4\alpha^2 \cdot 10^{2s} + 2\alpha \cdot 10^s}{8\alpha^2 \cdot 10^{2s} + 10\alpha \cdot 10^s + 5} \\ &> 2\alpha \cdot 10^s, \end{aligned}$$

so $q(\mathcal{G}_{\setminus\{i\}})$ is in the range $(2\alpha \cdot 10^s, 2\alpha \cdot 10^s + 1)$. Next,

$$\begin{aligned} q(\mathcal{G}_{\setminus\{n+2\}}) &= r(8\alpha \cdot 10^s + 5) \\ &= \frac{16\alpha^2 \cdot 10^{2s} + 18\alpha \cdot 10^s + 5}{8\alpha^2 \cdot 10^{2s} + 10\alpha \cdot 10^s + 5} \\ &= 1 + \frac{8\alpha^2 \cdot 10^{2s} + 8\alpha \cdot 10^s}{8\alpha^2 \cdot 10^{2s} + 10\alpha \cdot 10^s + 5}, \end{aligned}$$

so $q(\mathcal{G}_{\setminus\{n+2\}})$ is in the range $(1, 2)$. Finally,

$$\begin{aligned} q(\mathcal{G}_{\setminus\{n+3\}}) &= r(8\alpha^2 \cdot 10^{2s} + 6\alpha \cdot 10^s + 1) \\ &= \frac{16\alpha^3 \cdot 10^{3s} + 20\alpha^2 \cdot 10^{2s} + 8\alpha \cdot 10^s + 1}{8\alpha^2 \cdot 10^{2s} + 10\alpha \cdot 10^s + 5} \\ &= 2\alpha \cdot 10^s - \frac{2\alpha \cdot 10^s - 1}{8\alpha^2 \cdot 10^{2s} + 10\alpha \cdot 10^s + 5}, \end{aligned}$$

so $q(\mathcal{G}_{\setminus\{n+3\}})$ is in the range $(2\alpha \cdot 10^s - 1, 2\alpha \cdot 10^s)$.

We now show that the following three statements are pairwise equivalent:

1. $(a_1, \ldots, a_n; m)$ is a yes-instance of SUBSETSUM, i.e., $\xi > 0$.
2. There is a player $j > 1$ whose deletion increases the Shapley-Shubik index of player 1: $(\exists j \in \{2, \ldots, 2n+3\})\left[\varphi(\mathcal{G}_{\setminus\{j\}}, 1) - \varphi(\mathcal{G}, 1) > 0\right]$.
3. There is a player $j > 1$ whose deletion nondecreases the Shapley-Shubik index of player 1: $(\exists j \in \{2, \ldots, 2n+3\})\left[\varphi(\mathcal{G}_{\setminus\{j\}}, 1) - \varphi(\mathcal{G}, 1) \geq 0\right]$.

(1) implies (2): Suppose that $\xi > 0$. Then

$$\varphi(\mathcal{G}, 1) = 2\frac{n!(n+2)!}{(2n+3)!} + \xi\frac{(\frac{2}{3}n+1)!(\frac{4}{3}n+1)!}{(2n+3)!},$$

and after removing some player i with weight $2y$, we have

$$\begin{aligned} \varphi(\mathcal{G}_{\setminus\{i\}}, 1) &= \frac{n!(n+1)!}{(2n+2)!} + \xi\frac{(\frac{2}{3}n+1)!(\frac{4}{3}n)!}{(2n+2)!} \\ &= \left(1 + \frac{n+1}{n+2}\right)\frac{n!(n+2)!}{(2n+3)!} + \left(1 + \frac{\frac{2}{3}n+2}{\frac{4}{3}+1}\right)\xi\frac{(\frac{2}{3}n+1)!(\frac{4}{3}n+1)!}{(2n+3)!}. \end{aligned}$$

Therefore,

$$\begin{aligned}\varphi(\mathcal{G}_{\setminus\{i\}},1)-\varphi(\mathcal{G},1) &= \xi\,\frac{(\frac{2}{3}n+2)!(\frac{4}{3}n)!}{(2n+3)!}-\frac{n!(n+1)!}{(2n+3)!}\\ &\geq \frac{(\frac{2}{3}n+2)!(\frac{4}{3}n)!}{(2n+3)!}-\frac{n!(n+1)!}{(2n+3)!}\\ &> 0\end{aligned}$$

for $n \geq 3$, so the Shapley-Shubik index of the distinguished player has increased by deleting a player from the game $\mathcal{G}$.

(2) implies (3): is obvious.

(3) implies (1): To prove the contrapositive, suppose that $\xi = 0$. Then

$$\varphi(\mathcal{G},1) = 2\frac{n!(n+2)!}{(2n+3)!}.$$

If we remove a player $i \in \{2,\dots,n+1\} \cup \{n+4,\dots,2n+3\}$, then

$$\varphi(\mathcal{G}_{\setminus\{i\}},1) = \frac{n!(n+1)!}{(2n+2)!} = \frac{2n+3}{n+2}\frac{n!(n+2)!}{(2n+3)!} = \left(1+\frac{n+1}{n+2}\right)\frac{n!(n+2)!}{(2n+3)!} < \varphi(\mathcal{G},1).$$

If we delete the player $n+2$, the Shapley-Shubik index of player 1 will decrease to 0 because 1 can be pivotal only for coalitions with total weight 2, which are impossible to form. We will get an analogous situation if we remove the player $n+3$ because the new quota will be equivalent to $2\alpha \cdot 10^{2s}$, and since all the weights (except that of player 1) are even integers, it is impossible to form a coalition containing the player 1 with an even total weight. □

Finally, we consider the goal of maintaining a player's Shapley-Shubik index.

Theorem 4.16 *Control by deleting players to maintain a distinguished player's Shapley-Shubik index in a weighted voting game with changing the quota is* coNP*-hard.*

Proof We show coNP-hardness by means of a reduction from the PARTITION problem. Let $(a_1,\dots,a_n)$ be a PARTITION instance with $n > 1$, let $\alpha = \sum_{i=1}^{n} a_i$, and let ξ denote the number of its solutions. Construct the control problem instance consisting of a game

$$\mathcal{G} = (1, 6a_1, \dots, 6a_n, 6\alpha, 9\alpha(3\alpha-1)-1; 9\alpha+1)$$

with $n+3$ players, the distinguished player 1, and the deletion limit $k = 1$. Note that $9\alpha + 1 = \frac{1}{3\alpha}\sum_{i\in N} w_i$.

The values of the quota in possible new games are as follows. If we delete a player $j \in \{2,\dots,n+1\}$, then

$$q(\mathcal{G}_{\setminus\{j\}}) = 9\alpha + 1 - \frac{1}{3\alpha}6a_j > 9\alpha - 1$$

and

$$q(\mathcal{G}_{\setminus\{j\}}) = 9\alpha + 1 - \frac{1}{3\alpha}6a_j \leq 9\alpha + 1 - \frac{2}{\alpha} < 9\alpha + 1,$$

which is equivalent to 9α. If player $n+2$ is deleted, then

$$q(\mathcal{G}_{\setminus\{n+2\}}) = 9\alpha + 1 - \frac{1}{3\alpha}6\alpha = 9\alpha - 1.$$

Finally, if we delete player $n + 3$, then

$$q(\mathcal{G}_{\setminus\{n+3\}}) = 9\alpha + 1 - \frac{1}{3\alpha}(9\alpha(3\alpha - 1) - 1)$$
$$= 9\alpha + 1 - 9\alpha + 3 + \frac{1}{3\alpha} = 4 + \frac{1}{3\alpha}$$

which is equivalent to 5.

We will show that

$$(\exists j \in \{2, \ldots, n+3\}) \left[\varphi(\mathcal{G}_{\setminus\{j\}}, 1) - \varphi(\mathcal{G}, 1) = 0\right] \iff \xi = 0.$$

From right to left, suppose that $\xi = 0$. Then $\varphi(\mathcal{G}, 1) = 0$, and it does not matter which player will be deleted, the Shapley-Shubik index of the distinguished player always remains equal to 0, so it does not change.

From left to right, we show the contrapositive. Suppose that $\xi > 0$. Then

$$\varphi(\mathcal{G}, 1) = \frac{\sum_{i=1}^{\frac{1}{2}\xi}[(\|S_i\| + 1)!(n - \|S_i\| + 1)! + (n - \|S_i\| + 1)!(\|S_i\| + 1)!]}{(n+3)!},$$

which is positive. If we delete any player $j \in \{2, \ldots, n+3\}$, then player 1's Shapley-Shubik index decreases to 0. Therefore, control by deleting players with changing the quota to maintain a distinguished player's Shapley-Shubik index is coNP-hard. □

As mentioned before, Rey and Rothe [8] presented also an upper bound of $\mathrm{NP}^{\mathrm{PP}}$ (which, recall, is the class of problems that can be solved by an NP oracle machine accessing a PP oracle set) for the computational complexity of the problems they were studying. Exactly the same argumentation[3] is also valid for weighted voting games with changing quota. Therefore, we obtain the same upper bound: All our control problems regarding weighted voting games with changing quota are contained in $\mathrm{NP}^{\mathrm{PP}}$.

5 Conclusions and future research

We have continued the work on structural control by adding or deleting players in WVGs initiated by Rey and Rothe [8]. In particular, we have solved most of their open problems and have fixed a minor flaw in the proof of a lower bound for how much the Shapley-Shubik index can change by deleting players. We have also modified their model in a natural way by making the quota of a new WVG resulting from adding or deleting players dependent on the total weight of the players, and we have initiated the complexity analysis in this new model.

Still, some problems remain open for future work. First, it would be interesting to study the goal of *nondecreasing* a distinguished player's Shapley-Shubik power index in the model of Rey and Rothe [8] (see the only question mark in Table 1). Furthermore, there is still a huge gap between the lower bounds we prove here and the upper bound of $\mathrm{NP}^{\mathrm{PP}}$ stated at various places in the paper. Can we find better upper bounds or can we raise our lower bounds, for example to PP-hardness (or, ideally, even to $\mathrm{NP}^{\mathrm{PP}}$-hardness[4])? Such an improvement of lower bounds to PP-hardness was accomplished by Rey and Rothe [8] for some of their problems, and we succeeded to do so for some of our problems in the proof of Theorem 4.7.

[3] Essentially, the result comes from the fact that computing the numerator of the Penrose-Banzhaf index is #P-parsimonious-complete and computing the numerator of the Shapley-Shubik index is #P-many-one-complete.

[4] Note that $\mathrm{NP}^{\mathrm{PP}}$ is a huge complexity class that in particular contains all of the polynomial hierarchy by Toda's result [38].

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Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Non-financial interests Author Jörg Rothe currently is on the following editorial boards of scientific journals:
• *Annals of Mathematics and Artificial Intelligence* (AMAI), Associate Editor, since 01/2020,
• *Journal of Artificial Intelligence Research* (JAIR), Associate Editor, since 09/2017, and
• *Journal of Universal Computer Science* (J.UCS), Editorial Board, since 01/2005.

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