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The price to pay for forgoing normalization in fair division of indivisible goods

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Abstract

We study the complexity of fair division of indivisible goods and consider settings where agents can have nonzero utility for the empty bundle. This is a deviation from a common normalization assumption in the literature, and we show that this inconspicuous change can lead to an increase in complexity: In particular, while an allocation maximizing social welfare by the Nash product is known to be easy to detect in the normalized setting whenever there are as many agents as there are resources, without normalization it can no longer be found in polynomial time, unless P $=$ NP. The same statement also holds for egalitarian social welfare. Moreover, we show that it is NP-complete to decide whether there is an allocation whose Nash product social welfare is above a certain threshold if the number of resources is a multiple of the number of agents. Finally, we consider elitist social welfare and prove that the increase in expressive power by allowing negative coefficients again yields NP-completeness.

Keywords Fair division · Indivisible goods · Social welfare · Computational complexity

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1 Introduction

We consider problems of social welfare optimization for allocating indivisible resources (or goods or objects or items) and study them in terms of their computational complexity. For

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an overview of the field, see the survey by Chevaleyre et al. [8] and the book chapters by Bouveret et al. [3] and by Lang and Rothe [23]. Methods of fair division can be applied, for example, in course allocation [6] and divorce settlement [5]. A prototypical fair allocation setting with indivisible goods consists of participating agents, the goods to be allocated, and the agents' preferences. We assume that agents' preferences are represented by k-additive utility functions. Thus agents assign to each subset of goods (i.e., to each *bundle*) a utility value. k-additive utility functions allow for a restricted kind of complementarity and substitutability, where $k = 1$ corresponds to no synergies between resources and larger values of k effectively enforce an upper bound on the number of resources that may have synergistic effects among themselves. An allocation, that is, a partition of the set of goods, assigns to each agent a bundle. Therefore, agents derive a certain amount of utility from an allocation. In order to aggregate the individual utility values to a common measure, social welfare functions such as the sum, minimum, or product are used. Social welfare optimization then describes the process of finding allocations that maximize this aggregate value.

We focus on a common assumption in the fair division literature, namely, that an agent that receives no resources should have utility zero. We deviate from this normalization assumption. For a fixed number of agents and resources, this allows for more opportunities to increase social welfare because agents can now forgo receiving any resources at all. While it may happen that some agent has to receive a certain resource under the normalization assumption in order to guarantee some utility level among all agents, it is now possible that a greater utility level is achievable by assigning no resources to some agents. The excess resources can then be allocated to other agents. For illustration, consider a setting with a rich agent and a poor agent and the set of goods being basic necessities of life. The rich agent distributes the utility mass differently than the poor agent. Because the rich agent has her basic needs already covered, her marginal benefit of these goods is small and the utility mass is concentrated at the empty bundle. In contrast, the poor agent's utility mass may be distributed equally over the goods. Hence, higher social welfare can be achieved by allocating the goods to the poor agent in this example. This example also highlights a use case for nonnormalized utility functions because division problems are not always solved in isolation. Instead, participants already own goods, which can have an impact on their (reported) preferences and is related to "dynamical" fair division when new goods are to be allocated to the agents (see, e.g., the work of Kash et al. [22]).

Note that performing a simple "shift" to convert nonnormalized utility functions to normalized utility functions does not capture the allocation model. In order to simulate the fact that no resources can be assigned to an agent while still realizing positive utility, additional resources have to be introduced into the original model. This is problematic for the setting where there are as many agents as resources, which we are going to consider because this is one of the few settings where polynomial-time algorithms do exist.

Our contribution Our main contribution is to show that allowing nonnormalized utility functions comes at a steep computational cost, namely, that it is unlikely that polynomial-time algorithms for maximizing social welfare exist. This is in contrast to the setting of normalized utility functions, where, under certain restrictions, such algorithms are available. More concretely, we show that dropping the normalization assumption leads to problems that are even strongly NP-complete [17]. Therefore, unless P = NP, they cannot even have a fully polynomial-time approximation scheme (FPTAS) and they cannot have pseudo-polynomial-time algorithms.

In addition, we also consider elitist social welfare that can be maximized in polynomial time for k-additive utilities whenever all coefficients in the k-additive representation are nonnegative. We show that such an algorithm cannot exist for the same problem under k-additive utility functions for $k \geq 2$ and *arbitrary* coefficients, assuming $\mathrm{P} \neq \mathrm{NP}$. This is based on a reduction that Chevaleyre et al. [9, 10] designed for utilitarian social welfare.

Related work Additive utility functions with a possibly nonzero utility value for the empty bundle are also called modular utility/valuation functions. In the context of negotiation schemes, Chevaleyre et al. [12] showed that modular valuation functions are maximal in the sense that no larger domain that includes modular valuation functions guarantees convergence in the studied negotiation setting.

Roos and Rothe [30] showed that the general problem of maximizing the Nash product (to be formally defined and denoted by $\mathbb{Q}^{+}$-NPSWO$_{k\text{-ADD}}$ in Section 2) is NP-complete. NP-completeness still holds when the given allocation setting has only two agents and normalized utility functions. Their hardness result rests on a reduction from the NP-complete problem PARTITION and is not "NP-hard in the strong sense" (unlike our result for $\mathbb{Q}^{+}$-NPSWO$^{n=m}_{k\text{-ADD}}$ in Theorem 1, where the superscript "$n = m$" indicates that there are as many resources as agents). In addition, they and, independently, Ramezani and Endriss [29] showed that this problem is NP-complete also when utilities are given in the bundle form [27, 30]. Also for other representation forms that we have not considered here, analogous results have been obtained [29] (see also, e.g., [13] for the approximability of Nash product social welfare).

Regarding the optimization problem, Nguyen et al. [28] proposed a polynomial-time algorithm that provides an allocation with maximal Nash product if both the number of agents equals the number of resources to distribute and the utility functions are normalized.

Based on the work of Irving et al. [20], Golovin [18] provided an algorithm solving the problem $\mathbb{Q}$-MAX-ESW$^{n=m}_{1\text{-ADD}}$ with normalized utility functions in polynomial time. The paper by Bansal and Sviridenko [1] provides one of the many approximability results on maximizing egalitarian social welfare, see the survey by Nguyen et al. [28] for an overview.

Our results are concerned with centralized auctions "without money" (i.e., without an optional divisible resource). Another popular model is a decentralized auction with money as proposed by Endriss et al. [16] based on the work of Sandholm [31]. Dunne et al. [15] already showed that, in such settings, the problems WELFARE-IMPROVEMENT (WI) and UTILITARIAN-SOCIAL-WELFARE-OPTIMIZATION (USWO) are NP-complete for monotonic utility functions regardless of whether they are normalized or not. Based on this, Chevaleyre et al. [11] presented a decentralized approach with an additional undirected graph (the negotiation topology) over the set of agents. They were able to show that the WI problem is also NP-complete if we restrict deals on agents which form a clique on the negotiation topology and ask if there exists an envy-free allocation. They also pointed out that all these problems remain NP-hard for nonnormalized utility functions.

Damamme et al. [14] proposed a model using ordinal single-peaked preferences, which can be interpreted as normalized utility functions by using the Borda score,[1] and rational swaps without money. The main goal is to find a sequence of rational swaps from an initial allocation to an allocation such that $sw_u(\pi)$ or $sw_e(\pi)$ exceeds a given threshold. They showed that answering this question is an NP-complete problem as well.

[1] That means $u_i(r_j) = n - k + 1$ if r_j is the k-th preferred resource for agent a_i.

2 Preliminaries

Let A denote a set of n *agents* and R a set of m indivisible and nonshareable *resources*. Each agent $a_i \in A$ is equipped with a utility function $u_i : 2^R \to \mathbb{Q}$ and $U = (u_1, \ldots, u_n)$. Then (A, R, U) is an *allocation setting*. A utility function u over resources R is *k-additive* if for every $X \subseteq R$ there is a (unique) coefficient $\alpha^X \in \mathbb{Q}$, which vanishes if $|X| > k$, such that for every $Y \subseteq R$,

$$u(Y) = \sum_{X \subseteq Y} \alpha^X.$$

In an allocation setting (A, R, U), an *allocation* π is a partition of R into $n = |A|$ (possibly empty) subsets. Then $\pi(a_i)$ denotes the bundle that agent a_i receives. Denote by $\Pi_{A,R}$ the set of all allocations for agents A and resources R.

We measure the social welfare of an allocation π using

– *utilitarian social welfare*:

$$sw_u(\pi) = \sum_{a_i \in A} u_i(\pi(a_i)),$$

– *egalitarian social welfare*:

$$sw_e(\pi) = \min_{a_i \in A} u_i(\pi(a_i)),$$

– *Nash product social welfare*:

$$sw_N(\pi) = \prod_{a_i \in A} u_i(\pi(a_i)), \text{ and}$$

– *elitist social welfare*:

$$sw_E(\pi) = \max_{a_i \in A} u_i(\pi(a_i)).$$

Utilitarian social welfare, sw_u, captures the average utility that agents receive in an allocation setting. Clearly, lopsided allocations are possible when a single agent receives all the goods. This is put to an extreme under elitist social welfare, sw_E, whose usage can be justified, e.g., in settings where the center controls all agents. At the other side of the spectrum is egalitarian social welfare, sw_e. Maximizing egalitarian social welfare corresponds to paying attention to the worst-off agent only, neglecting concerns of efficiency. Nash product social welfare, sw_N, strikes a balance between sw_u and sw_e in the sense that balanced utility values maximize sw_N and its outcomes are Pareto-efficient (see also the paper by Caragiannis et al. [7]).

Let us now define our optimization problems and their associated decision problems, starting with the most prominent one: the problem of maximizing utilitarian social welfare.

$\mathbb{Q}$-Maximum-Utilitarian-Social-Welfare$_{k\text{-Add}}$	
Input:	An allocation setting (A, R, U), where each utility function $u_i : 2^R \to \mathbb{Q}$ is represented in k-additive form.
Output:	$\max\{sw_u(\pi) \mid \pi \in \Pi_{A,R}\}$

We will also use the shorthand $\mathbb{Q}$-MAX-USW$_{k\text{-ADD}}$ for this problem. If we require in addition that the number of agents be equal to the number of resources, the resulting problem is denoted by $\mathbb{Q}$-MAX-USW$_{k\text{-ADD}}^{n=m}$; analogously, this superscript "$n = m$" indicates the same restriction for the problems defined below.

The decision problem associated with the above optimization problem $\mathbb{Q}$-MAX-USW$_{k\text{-ADD}}$ is defined as follows:

	$\mathbb{Q}$-UTILITARIAN-SOCIAL-WELFARE-OPTIMIZATION$_{k\text{-ADD}}$
Given:	An allocation setting (A, R, U), where each utility function $u_i : 2^R \to \mathbb{Q}$ is represented in k-additive form, and a number $K \in \mathbb{N}$.
Question:	Does there exist an allocation $\pi \in \Pi_{A,R}$ such that $sw_u(\pi) \geq K$?

Again, we will also use the shorthand $\mathbb{Q}$-USWO$_{k\text{-ADD}}$ for this problem. Furthermore, by replacing utilitarian social welfare by other types of social welfare, we can define the following decision and optimization problems. Here, the symbol $\mathbb{Q}^+$ denotes the set of nonnegative rational numbers.

- $\mathbb{Q}$-EGALITARIAN-SOCIAL-WELFARE-OPTIMIZATION$_{k\text{-ADD}}$ ($\mathbb{Q}$-ESWO$_{k\text{-ADD}}$) and $\mathbb{Q}$-MAX-EGALITARIAN-SOCIAL-WELFARE$_{k\text{-ADD}}$ ($\mathbb{Q}$-MAX-ESW$_{k\text{-ADD}}$),
- $\mathbb{Q}^+$-NASH-PRODUCT-SOCIAL-WELFARE-OPTIMIZATION$_{k\text{-ADD}}$ ($\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$) and $\mathbb{Q}^+$-MAX-NASH-PRODUCT-SOCIAL-WELFARE$_{k\text{-ADD}}$ ($\mathbb{Q}^+$-MAX-NPSW$_{k\text{-ADD}}$), and
- $\mathbb{Q}$-ELITIST-SOCIAL-WELFARE-OPTIMIZATION$_{k\text{-ADD}}$ ($\mathbb{Q}$-ELSWO$_{k\text{-ADD}}$) and $\mathbb{Q}$-MAX-ELITIST-SOCIAL-WELFARE$_{k\text{-ADD}}$ ($\mathbb{Q}$-MAX-ELSW$_{k\text{-ADD}}$).

We assume the reader to be familiar with the basic notions of complexity theory, such as the complexity classes P (deterministic polynomial time) and NP (nondeterministic polynomial time), polynomial-time many-one reducibility, and the notions of NP-hardness and -completeness based on this reducibility.

3 Nash product social welfare

In this section, we study the complexity of social welfare optimization by the Nash product, assuming k-additive utility functions for $k \geq 1$.

3.1 Allowing nonnormalized utility functions

We show that, assuming P $\neq$ NP, $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}^{n=m}$ is no longer solvable in polynomial time if utility functions are not required to be normalized, i.e., if $u_i(\emptyset) = \lambda_i$ with $\lambda_i \in \mathbb{Q}^+ \setminus \{0\}$ for at least one agent a_i. Concretely, we show NP-completeness of $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}^{n=m}$. An ordinary (i.e., not in the "strong sense" [17]) NP-hardness proof can be obtained based on the observation that in the construction of Roos and Rothe [30] arbitrarily many agents may be added to the given allocation setting without changing its Nash product (see the preliminary ISAIM 2018 [24] and M-PREF 2018 [25] versions of this paper for details). Here, however, we give a different proof that even establishes strong NP-hardness.

We reduce from the following NP-complete problem [21]:

EXACT COVER BY 3 SETS (X3C)

Given: A set $B = \{b_1, \ldots, b_{3m}\}$ for a natural number m and a collection $\mathcal{S} = \{S_1, \ldots, S_n\}$ of subsets $S_i \subseteq B$ with $|S_i| = 3$ for each $1 \leq i \leq n$.
Question: Does there exist an index set $I \subseteq \{1, \ldots, n\}$ with $|I| = m$ such that $B = \bigcup_{i \in I} S_i$ is a disjoint union?

The following lemma will be helpful and can be shown by an application of the inequality between arithmetic mean and geometric mean.

Lemma 1 *Let $x_1, \ldots, x_n$ be n nonnegative real numbers satisfying $\sum_{i=1}^{n} x_i \leq n$. Then*

$$\prod_{i=1}^{n} x_i \leq 1, \tag{1}$$

where equality holds in (1) if and only if $x_1 = \cdots = x_n = 1$.

Let $X = \{x_1, \ldots, x_n\}$ be a finite set and $\sigma_X(i)$ the i-th smallest element in X. A bijective function $\mu : X \to \{1, \ldots, n\}$ *maps X monotonically onto the set* $\{1, \ldots, n\}$ if $\mu(\sigma_X(k)) = k$ holds.

Theorem 1 *For each $k \geq 1$, $\mathbb{Q}^+$-NPSWO$^{n=m}_{k\text{-ADD}}$ is strongly* NP*-complete.*

Proof Membership of $\mathbb{Q}^+$-NPSWO$^{n=m}_{k\text{-ADD}}$ in NP is easy to observe. To show NP-hardness in the strong sense, we provide a polynomial reduction from X3C, which is NP-complete in the strong sense and satisfies the property that all numbers are bounded by a polynomial, to the problem $\mathbb{Q}^+$-NPSWO$^{n=m}_{1\text{-ADD}}$, i.e., for the case $k = 1$. From this, the result follows immediately for each $k > 1$ because 1-additive utility functions are k-additive (see, e.g., [27, Footnote 8]).

Let $(B, \mathcal{S})$ be an X3C instance with $B = \{b_1, \ldots, b_{3m}\}$ and $\mathcal{S} = \{S_1, \ldots, S_n\}$. Without loss of generality, assume $n \geq m$ and construct a $\mathbb{Q}^+$-NPSWO$^{n=m}_{1\text{-ADD}}$ instance $((A, R, U), K)$ with

$$\begin{aligned} A &= \left\{a_1, \ldots, a_{3m+(n-m)}\right\}, \\ R &= \left\{r_j^B \mid b_j \in B\right\} \cup \left\{r_1^N, \ldots, r_{n-m}^N\right\}, \\ K &= 1, \end{aligned}$$

and 1-additive utility functions in U given by the following coefficients:

$$\begin{aligned} \alpha_i^{\{r_j^B\}} &= \tfrac{1}{3} & &(1 \leq i \leq n \text{ and } b_j \in S_i), \\ \alpha_i^{\{r_j^N\}} &= 1 & &(1 \leq i \leq n \text{ and } 1 \leq j \leq n-m), \\ \alpha_i^{\emptyset} &= 1 & &(n+1 \leq i \leq 3m+n-m). \end{aligned}$$

All other coefficents are zero (by definition). Obviously, the number of agents is equal to the number of resources ($|A| = |R| = 3m + n - m$) and the coefficients are bounded by a constant. Resources of the form r_j^B are called base resources and resources of the form r_1^N are called dummy resources.

It remains to show that $(B, \mathcal{S})$ is a yes-instance of X3C if and only if $((A, R, U), K)$ is a yes-instance of $\mathbb{Q}^+$-NPSWO$^{n=m}_{\text{1-ADD}}$.

From left to right, suppose that $(B, \mathcal{S})$ is a yes-instance of X3C. Then there is an index set $I \subseteq \{1, \dots, n\}$ with $|I| = m$ such that $B = \bigcup_{i \in I} S_i$ is a disjoint union. Let $\bar{I} = \{1, \dots, n\} \setminus I$ and define

$$\mu : \bar{I} \to \{1, \dots, n - m\}$$

which maps every index $i \in \bar{I}$ monotonically to the set $\{1, \dots, n - m\}$. The allocation

$$\pi(a_i) = \begin{cases} \{r_p^B \mid b_p \in S_i\} & \text{if } i \in I \\ \{r_{\mu(i)}^N\} & \text{if } i \in \bar{I} \\ \emptyset & \text{if } i \in \{n+1, \dots, 3m + n - m\} \end{cases}$$

satisfies

$$\begin{aligned} sw_N(\pi) &= \left(\prod_{i \in I} \left(\sum_{b_j \in S_i} \alpha_i^{\{r_j^B\}}\right)\right) \cdot \left(\prod_{i \in \{1,\dots,n\} \setminus I} \alpha_i^{\{r_{\mu(i)}^N\}}\right) \cdot \left(\prod_{i=n+1}^{3m+(n-m)} \alpha_i^{\emptyset}\right) \\ &= \prod_{i \in I} \left(3 \cdot \frac{1}{3}\right) \cdot 1^{n-m} \cdot 1^{2m} \\ &= 1^{3m+(n-m)} \\ &\geq K. \end{aligned}$$

From right to left, suppose that there exists an allocation π with $sw_N(\pi) \geq 1$. It is easy to see that no dummy agent a_i $(n + 1 \leq i \leq 3m + n - m)$ can improve her utility value by getting nonempty bundles. Also note that the utilitarian social welfare for the first n agents is bounded above by

$$\sum_{i=1}^{n} u_i(a_i) \leq \frac{1}{3} \cdot 3m + n - m = n. \tag{2}$$

So we only have to look at allocations π restricted to agents a_i, $1 \leq i \leq n$, with $sw_N(\pi) = 1$ and thus $u_i(\pi(a_i))) = 1$, due to (2) and Lemma 1. Therefore, all resources have to be allocated to the first n agents (otherwise, one of the first n agents has utility strictly less than one).

It is easy to check that every agent must be allocated at least three base resources or one dummy resource to reach the threshold $K = 1$. The reduction requires that there are only $n-m$ dummy resources that can be assigned to the agents. So the remaining $n-(n-m) = m$ agents need to receive exactly the base resources r_j^B. Since every resource is assumed to be nonshareable, each of the $3m$ base resources must be allocated. We also know that $\alpha_i^{\{r_j^B\}}$ is set to $\frac{1}{3}$ only for three distinct r_j^B. So there must be m bundles $T_i \subseteq \{r_j^B \mid b_j \in B\}$ with $|T_i| = 3$ where each r_j^B belongs to exactly one T_i. Hence, there is an exact cover, which completes the proof. □

Example 1 Let $(B, \mathcal{S})$ with

$$\begin{aligned} B &= \{b_1, \dots, b_9\}, \\ \mathcal{S} &= \{S_1, S_2, S_3, S_4, S_5\}, \text{ and} \\ S_1 &= \{b_1, b_2, b_4\},\ S_2 = \{b_2, b_5, b_7\},\ S_3 = \{b_1, b_6, b_9\}, \\ S_4 &= \{b_3, b_4, b_8\},\ S_5 = \{b_2, b_6, b_8\} \end{aligned}$$

be a yes-instance of X3C. Construct from $(B, \mathcal{S})$ the $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$ instance $((A, R, U), K)$ according to the reduction given in the proof of Theorem 1, with lower bound $K = 1$, agents $A = \{a_1, a_2, \dots, a_{11}\}$, resources $R = \{r_1^B, \dots, r_9^B, r_1^N, r_2^N\}$, and utility functions

$$\begin{aligned} u_1(X) &= \frac{1}{3}[r_1^B] + \frac{1}{3}[r_2^B] + \frac{1}{3}[r_4^B] + 1[r_1^N] + 1[r_2^N], \\ u_2(X) &= \frac{1}{3}[r_2^B] + \frac{1}{3}[r_5^B] + \frac{1}{3}[r_7^B] + 1[r_1^N] + 1[r_2^N], \\ u_3(X) &= \frac{1}{3}[r_1^B] + \frac{1}{3}[r_6^B] + \frac{1}{3}[r_9^B] + 1[r_1^N] + 1[r_2^N], \\ u_4(X) &= \frac{1}{3}[r_3^B] + \frac{1}{3}[r_4^B] + \frac{1}{3}[r_8^B] + 1[r_1^N] + 1[r_2^N], \\ u_5(X) &= \frac{1}{3}[r_2^B] + \frac{1}{3}[r_6^B] + \frac{1}{3}[r_8^B] + 1[r_1^N] + 1[r_2^N], \\ u_i(X) &= 1[\emptyset] \end{aligned}$$

for $i \in \{6, \dots, 11\}$, where $[Y]$ is 1 if $Y \subseteq X$ and is 0 otherwise (and we omit set parentheses around singletons for convenience). An exact cover is given by $\mathcal{S}' = \{S_2, S_3, S_4\}$. So let $I = \{2, 3, 4\}$ and $\bar{I} = \{1, 5\}$, and construct $\mu : \{1, 5\} \to \{1, 2\}$ as follows:

$$\mu(1) = 1 \quad \text{and} \quad \mu(5) = 2.$$

The allocation

$$\pi(a) = \begin{cases} \{r_1^N\} & \text{if } a = a_1 \\ \{r_2^B, r_5^B, r_7^B\} & \text{if } a = a_2 \\ \{r_1^B, r_6^B, r_9^B\} & \text{if } a = a_3 \\ \{r_3^B, r_4^B, r_8^B\} & \text{if } a = a_4 \\ \{r_2^N\} & \text{if } a = a_5 \\ \emptyset & \text{if } a \in \{a_6, \dots, a_{11}\} \end{cases}$$

satisfies $sw_N(\pi) = 1$.

The hardness result of Theorem 1 depends on the fact that the number of agents with utility functions that are not normalized is a function of n. Suppose a constant c denotes the number of agents whose utility functions are not normalized. Then, for $n = m$, by exhaustive search we can find an allocation of maximum Nash product social welfare in time $\mathcal{O}(m^{3c}t)$, where t is the polynomial running time of the algorithm by Nguyen et al. [28] for maximizing the Nash product under normalized utility functions.

To sketch this proof (of upcoming Theorem 2) in some more detail, the algorithm considers all 2^c possibilities for which of the c agents are going to be ignored and therefore receive no goods. Denote by d the number of agents that are not ignored among the c agents that could be ignored. If an agent a_i is not ignored, we set $u_i(\emptyset) = 0$. The overall plan is to use the matching algorithm by neglecting the $c - d$ ignored agents. Therefore, we have to reduce the number of goods in lockstep as well in order to have an equal number of agents and goods. Since $c-d$ agents are ignored, $c-d$ goods have to be merged with some of the remaining goods to supergoods. To achieve this, there are $\binom{m}{c-d}$ ways to choose the $c-d$ goods, and for each such choice there are $(m-(c-d))^{c-d}$ ways to merge the chosen goods to supergoods. Then an agent's utility for a supergood is the utility for the empty bundle plus the utility of the bundle that contains all goods that make up that supergood. In this reduced instance, we have an equal number of agents and goods. Therefore, we can apply the algorithm by Nguyen et al. [28] for normalized utility functions. We adjust the social welfare of the output allocation by multiplying by the utility values of the ignored agents for the empty bundle and collect the resulting allocation in a list. After we have considered all 2^c possibilties, we pick an allocation of maximum welfare.

Now, let π^* be an allocation of maximum welfare according to the above procedure. Let $\widehat{\pi}$ be some allocation. Then the Nash product of π^* is no less than the Nash product of $\widehat{\pi}$. If $\widehat{\pi}$ was in the list of the above procedure, the claim holds. If $\widehat{\pi}$ was not in the list, then denote by I the set of ignored agents in $\widehat{\pi}$ (i.e., agents that receive no goods). Consider the suballocation $\widehat{\pi}_I$ that results from $\widehat{\pi}$ by removing the entries of the ignored agents in I. Some agents receive only one good, whereas other agents receive multiple goods. However, this instance was already considered in the exhaustive procedure above: There is an iteration in the above algorithm where agents in I are ignored and supergoods are formed according to the bundles in $\widehat{\pi}_I$ that contain multiple goods. Since $\widehat{\pi}$ is not in the list, the matching-based algorithm by Nguyen et al. [28] finds an allocation π'_I with greater or equal social welfare, i.e., $sw_N(\pi'_I) \geq sw_N(\widehat{\pi}_I)$. After adjusting for the ignored agents, the resulting allocation π' is added to the list. Because the set of ignored agents is the same for π'_I and $\widehat{\pi}_I$, the social welfare of $\widehat{\pi}_I$ cannot become greater than the social welfare of π'_I after the adjustment. Overall, we have $sw_N(\pi^*) \geq sw_N(\pi') \geq sw_N(\widehat{\pi})$. Hence, we have shown the following result:

Theorem 2 *For $m \geq 2$, the problem $\mathbb{Q}^+$-MAX-NPSW$^{n=m}_{1\text{-ADD}}$ restricted to instances where the number of agents with nonnormalized utility functions is bounded by some constant c can be solved in time $\mathcal{O}(m^{3c}p)$, where p is a number that is bounded above by a function that polynomially depends on the size of the input.*

To summarize the results of this section: For the hardness of $\mathbb{Q}^+$-NPSWO$^{n=m}_{1\text{-ADD}}$ to occur in Theorem 1 it is essential that the number of agents whose utility functions are nonnormalized is not bounded by a constant, as indicated by Theorem 2 for $\mathbb{Q}^+$-MAX-NPSW$^{n=m}_{1\text{-ADD}}$.

3.2 Further restrictions

In this section, we go back to the standard model again where all utility functions are normalized. However, we now focus on other restrictions on the utility functions. In particular, note that the problem $\mathbb{Q}^+$-NPSWO$^{n=m}_{k\text{-ADD}}$ is a special case of $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$, so NP-hardness of the former is immediately inherited by the latter, and this also holds for only two agents [30, Theorem 5.1]. We now consider the case where the number of resources to distribute is a multiple of the number of agents.

Theorem 3 *Fix an integer $p \geq 2$. For each $k \geq 1$, the problem $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$ restricted to instances with $|R| = p \cdot |A|$ and normalized utility functions is strongly* NP*-complete.*

Proof Membership of the problem in NP is easy to observe. To show strong NP-hardness, we present a polynomial reduction from the problem X3C with coefficients bounded by a constant to $\mathbb{Q}^+$-NPSWO$_{1\text{-ADD}}$ restricted to instances with $|R| = p \cdot |A|$ (again, this suffices to prove strong NP-hardness of $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$ for each $k \geq 1$).

Let $(B, \mathcal{S})$ be an X3C instance with $B = \{b_1, \ldots, b_{3m}\}$ and $\mathcal{S} = \{S_1, \ldots, S_n\}$. Without loss of generality, assume $n \geq m$ and construct a $\mathbb{Q}^+$-NPSWO$_{1\text{-ADD}}$ instance $((A, R, U), K)$ with

$$A = \{a_1, \ldots, a_{3m+n}\},$$
$$K = 1,$$

the resource set given by

$$R = N \cup D \cup R_1 \cup R_2$$

with

$$N = \{r_1^N, \ldots, r_{n-m}^N\}, \qquad D = \{r_1^D, \ldots, r_{n+m}^D\},$$
$$R_1 = \{r_j^{R1} \mid b_j \in B\}, \qquad R_2 = \{r_j^{R2} \mid b_j \in B\},$$

and the 1-additive utility functions in U defined by the following coefficients:

$$\alpha_i^{\{r_j^{R1}\}} = \tfrac{1}{3} \qquad (1 \leq i \leq n \text{ and } b_j \in S_i),$$
$$\alpha_i^{\{r_j^{N}\}} = 1 \qquad (1 \leq i \leq n \text{ and } 1 \leq j \leq n-m),$$
$$\alpha_i^{\{r_j^{R2}\}} = 1 \qquad (n+1 \leq i \leq n+3m \text{ and } 1 \leq j \leq 3m).$$

This utility function is also represented graphically in Fig. 1.

Obviously, the number of resources is twice the number of agents: $2|A| = |R| = 2 \cdot (3m + n)$.

It remains to show that $(B, \mathcal{S})$ is a yes-instance of X3C if and only if $((A, R, U), K)$ is a yes-instance of $\mathbb{Q}^+$-NPSWO$_{1\text{-ADD}}$.

From left to right, suppose that $(B, \mathcal{S})$ is a yes-instance of X3C. Thus there is an index set $I \subseteq \{1, \ldots, n\}$ with $|I| = m$ such that $B = \bigcup_{i \in I} S_i$ is a disjoint union. Let $\bar{I} = \{1, \ldots, n\} \setminus I$ and define

$$\mu : \bar{I} \to \{1, \ldots, n-m\}$$

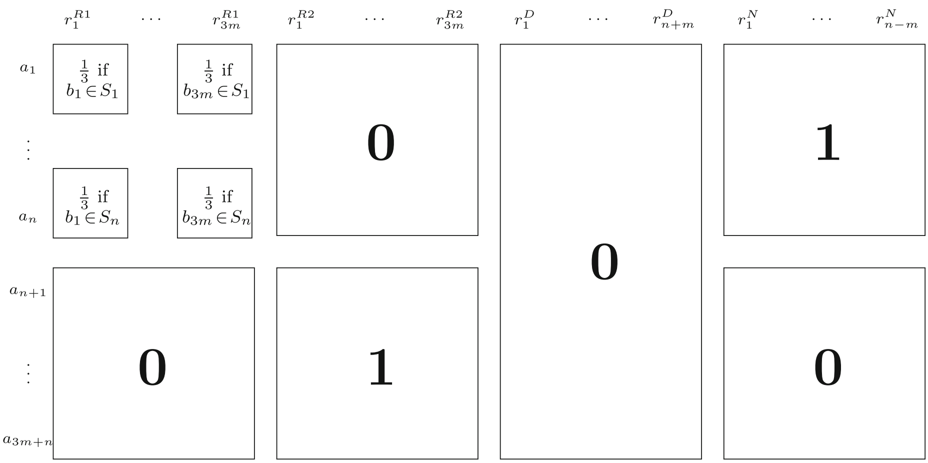

Fig. 1 Utility values of the $\mathbb{Q}^+$-NPSWO$_{1\text{-ADD}}$ instance constructed in the proof of Theorem 3

which maps every index $i \in \bar{I}$ monotonically to the set $\{1, \dots, n-m\}$. The allocation

$$\pi(a_i) = \begin{cases} \{r_p^{R1} \mid b_p \in S_i\} & \text{if } i \in I \\ \{r_{\mu(i)}^{N}\} & \text{if } i \in \{1, \dots, n\} \setminus I \\ \{r_{i-n}^{R2}\} \cup D & \text{if } i = n+1 \\ \{r_{i-n}^{R2}\} & \text{if } i \in \{n+2, \dots, n+3m\} \end{cases}$$

satisfies

$$\begin{aligned} sw_N(\pi) &= \left(\prod_{i\in I}\left(\sum_{b_j\in S_i} \alpha_i^{\{r_{b_j}^{R1}\}}\right)\right) \cdot \left(\prod_{i\in\{1,\dots,n\}\setminus I} \alpha_i^{\{r_{\mu(i)}^{N}\}}\right) \\ &\quad \cdot \left(\alpha_{n+1}^{\{r_1^{R2}\}} + \sum_{j=1}^{n+m} \alpha_{n+1}^{\{r_j^{D}\}}\right) \cdot \left(\prod_{i=n+2}^{n+3m} \alpha_i^{\{r_{i-n}^{R2}\}}\right) \\ &= 1^m \cdot 1^{n-m} \cdot (1 + (n+m)\cdot 0) \cdot 1^{n+3m-(n+2)+1} \\ &= 1^{n+3m} \\ &\geq K. \end{aligned}$$

From right to left, suppose that $(B, \mathcal{S})$ is a no-instance of X3C and assume, for a contradiction, that there were an allocation π satisfying $sw_N(\pi) \geq 1$.

We make the following observations: First, any agent $a_{n+1}, \dots, a_{n+3m}$ must be allocated exactly one resource from R_2 to get a utility value not equal to zero.[2] Second, it is irrelevant who of them gets the dummy resources D.

So the resources $T = R_1 \cup N$ remain unallocated and we only have to concentrate on the first n agents. Using

$$\sum_{i=1}^{n} u_i(a_i) \leq \frac{1}{3} \cdot 3m + (n-m) = n$$

and Lemma 1, we have $u_1(\pi(a_1)) = \cdots = u_n(\pi(a_n)) = 1$, since $sw_N(\pi) \geq 1$ is assumed.

Let $\widehat{I}$ be the largest (cardinality-wise) index subset such that $\bigcup_{i\in\widehat{I}} S_i \subsetneq B$ is a disjoint union. It is clear that $|\widehat{I}| = \ell < m$ holds, for otherwise $(B, \mathcal{S})$ would be a yes-instance of X3C.

Due to the reduction, we get $u_i(T) = 1$ if and only if $T = S_j$, $1 \leq j \leq n$ or $T = \{r_z^N\}$, $1 \leq z \leq n-m$.

Only $n - m$ agents can be allocated a resource from N. So there are

$$(3m + n) - 3m - (n - m) = m$$

agents with no resource yet. Due to $\ell < m$ there remain $m - \ell > 0$ agents who need exactly three resources to get a utility value of 1. But there are at most two resources in a bundle that are not distributed yet. Hence, $u_i(\pi(a_i)) \leq \frac{2}{3} < 1$ for these agents and we get the desired contradiction by $sw_N(\pi) < 1$. □

[2]The only resources for which the agents $A' = \{a_{n+1}, \dots, a_{n+3m}\}$ get a nonzero utility value are in R_2. Since we have normalized utility functions and the cardinality of A' is equal to $|R_2|$, every agent $a' \in A'$ must be allocated exactly one resource from R_2.

With a simple modification of the reduction from the proof of Theorem 1, we can prove strong NP-completeness of the $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$ problem: Omit all dummy agents $a_{n+1}, \ldots, a_{3m+(n-m)}$ and thus obtain an instance of $\mathbb{Q}^+$-NPSWO$_{1\text{-ADD}}$ with n agents and $3m + (n - m)$ resources. It is obvious that the resulting instance only contains normalized utility functions,[3] but $|R| = |A|$ cannot be achieved. It then follows immediately that there exists an exact cover if and only if there exists an allocation π with $sw_N(\pi) \geq 1$ in the constructed $\mathbb{Q}^+$-NPSWO$_{1\text{-ADD}}$ instance. This gives Corollary 1, which improves on an earlier result by Nguyen et al. [27] (see also [30]), who also show NP-completeness of $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$ but not *strong* NP-completeness since their reduction is from PARTITION.

Corollary 1 *For each $k \geq 1$, $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$ is strongly NP-complete.*

4 Egalitarian social welfare

In this section, we study the complexity of egalitarian social welfare optimization, again assuming k-additive utility functions for $k \geq 1$.

As we did in the previous section for the Nash product, we now investigate whether the normalization requirements for the algorithm mentioned above are necessary. We will show that without this normalization, the corresponding decision problem is strongly NP-complete.

Theorem 4 *For each $k \geq 1$, $\mathbb{Q}$-ESWO$^{n=m}_{k\text{-ADD}}$ is strongly NP-complete.*

Proof We show that the same polynomial reduction given for the Nash product social welfare in the proof of Theorem 1 works for egalitarian social welfare as well. That is, we now show that $(B, \mathcal{S})$ is a yes-instance of X3C if and only if $((A, R, U), K)$, as constructed in the proof of Theorem 1 from $(B, \mathcal{S})$, is a yes-instance of $\mathbb{Q}$-ESWO$^{n=m}_{1\text{-ADD}}$. This will give strong NP-hardness of $\mathbb{Q}$-ESWO$^{n=m}_{1\text{-ADD}}$ and, consequently, of $\mathbb{Q}$-ESWO$^{n=m}_{k\text{-ADD}}$ for each $k \geq 1$.

From left to right, suppose that $(B, \mathcal{S})$ with $B = \{b_1, \ldots, b_{3m}\}$ and $\mathcal{S} = \{S_1, \ldots, S_n\}$ is a yes-instance of X3C. Thus there is an index set $I \subseteq \{1, \ldots, n\}$ with $|I| = m$ such that $B = \bigcup_{i \in I} S_i$ is a disjoint union. Let $\bar{I} = \{1, \ldots, n\} \setminus I$ and define

$$\mu : \bar{I} \to \{1, \ldots, n - m\},$$

which maps every index $i \in \bar{I}$ monotonically to the set $\{1, \ldots, n - m\}$. The allocation

$$\pi(a_i) = \begin{cases} \{r_p^B \mid b_p \in S_i'\} & \text{if } i \in I \\ \{r^N_{\mu(i)}\} & \text{if } i \in \bar{I} \\ \emptyset & \text{if } i \in \{n+1, \ldots, 3s + (n - m)\} \end{cases}$$

satisfies

$$sw_e(\pi) = \min\left\{3 \cdot \frac{1}{3}, 1\right\} = 1 \geq K.$$

[3] In the reduction presented in the proof of Theorem 1, only the dummy agents $a_{n+1}, \ldots, a_{3m+(n-m)}$ have nonnormalized utility functions.

From right to left, suppose that there exists an allocation π with $sw_e(\pi) \geq 1$. First note that the utility of each of the agents $a_{n+1}, \dots, a_{3m+(n-m)}$ is at least one. So it is sufficient to focus on the first n agents. By using the inequality

$$\sum_{i=1}^{n} u_i(a_i) \leq \frac{1}{3} \cdot 3m + (n-m) = n,$$

we again get $u_i(a_i) = \cdots = u_n(a_n) = 1$. It is obvious that every agent must be allocated at least three normal resources or one dummy resource to reach the threshold $K = 1$. From the reduction we know that there are only $n - m$ dummy resources that can be assigned to the agents. So the remaining $n - (n - m) = m$ agents must get their utility values from the base resources r_j^B. Since every resource is assumed to be nonshareable, each of the $3m$ base resources must be allocated. We also know that $\alpha_i^{\{r_j^B\}}$ is set to $\frac{1}{3}$ only for three distinct r_j^B. So there must be m bundles $T_i \subseteq \{r_j^B \mid b_j \in B\}$ with $|T_i| = 3$, where each r_j^B belongs to exactly one T_i. Hence, there is an exact cover. □

Modifying the reduction from the proof of Theorem 4 the same way that we modified the reduction from the proof of Theorem 1 to prove Corollary 1, we obtain strong NP-completeness of $\mathbb{Q}$-ESWO$_{k\text{-ADD}}$ in Corollary 2. This again improves on an earlier result stating that $\mathbb{Q}$-ESWO$_{k\text{-ADD}}$ is NP-complete, as was mentioned (without proof) by Bouveret et al. [4] and explicitly proven by Bouveret in his thesis [2] (an implicit proof can be found already in the work of Lipton et al. [26]).

Corollary 2 *For each $k \geq 1$, $\mathbb{Q}$-ESWO$_{k\text{-ADD}}$ is strongly* NP*-complete.*

5 Elitist social welfare with normalized utility functions

Finally, we make a small observation regarding elitist social welfare with normalized utility functions.[4] Heinen et al. [19] observed that the problem $\mathbb{Q}$-ELSWO$_{1\text{-ADD}}$ (which is called n-RANK DICTATOR in their paper) can be solved in polynomial time. It is not hard to see that essentially the same argument gives the same result for $\mathbb{Q}$-ELSWO$_{k\text{-ADD}}$ for each $k \geq 2$, provided that all coefficients in the k-additive representation are nonnegative.[5] However, if negative coefficients are allowed, this decision problem turns NP-complete, which follows immediately from a known reduction due to Chevaleyre et al. [9].

Theorem 5 *For each $k \geq 2$, $\mathbb{Q}$-ELSWO$_{k\text{-ADD}}$ with arbitrary coefficients is* NP*-complete.*

Proof It is obvious that $\mathbb{Q}$-ELSWO$_{k\text{-ADD}}$ is in NP for each $k \geq 2$: Nondeterministically, choose an allocation π and verify whether $\max\{u_i(\pi(a_i)) \mid a_i \in A\} \geq K$.

[4]This section is a bit special in two regards. First, the elitist social welfare is not as well-motivated as any of the other social welfare measures; one might argue that it is orthogonal to the “social” in *social welfare* and to the “fair” in *fair division*, and thus is more of theoretical interest. Second, unlike in the previous sections, we do not consider *non*normalized utility functions here. However, we mention that our approach to proving Theorem 5 would also work for *non*normalized utility functions whenever $k = 1$ or ($k \geq 2$ and all coefficients are positive).

[5]In particular, this assumption ensures that every agent realizes the highest utility by receiving all resources.

Table 1 2-additive terms for u_1 assuming $i \neq j$

Clause	2-additive term
$(x_i \vee x_i)$	$1[x_i]$
$(\neg x_i \vee \neg x_i)$	$1 - [x_i]$
$(x_i \vee x_j)$	$[x_i] + [x_j] - [x_i][x_j]$
$(x_i \vee \neg x_j)$	$[x_i] + (1 - [x_j]) - [x_i] \cdot (1 - [x_j])$
$(\neg x_i \vee \neg x_j)$	$(1 - [x_i]) + (1 - [x_j]) - (1 - [x_i]) \cdot (1 - [x_j])$

To prove NP-hardness of $\mathbb{Q}$-ELSWO$_{2\text{-ADD}}$, we make use of a reduction due to Chevaleyre et al. [10] who showed NP-hardness of $\mathbb{Q}$-USWO$_{2\text{-ADD}}$ by a reduction from the well-known NP-complete problem MAXIMUM-2-SATISFIABILITY (for short, MAX-2-SAT), which is defined as follows:

MAXIMUM-2-SATISFIABILITY	
Given:	A boolean formula φ in conjunctive normal form, where each clause has exactly two literals, and a nonnegative integer K.
Question:	Does there exist a truth assignment simultaneously satisfying at least K clauses of φ?

Consider the reduction from the proof of [10, Proposition 8], which reduces the problem MAX-2-SAT to $\mathbb{Q}$-USWO$_{2\text{-ADD}}$. That means that a MAX-2-SAT instance (φ, K) is mapped to a $\mathbb{Q}$-USWO$_{2\text{-ADD}}$ instance $((A, R, U), K)$ consisting of one resource for each variable of φ, two agents, a_1 and a_2, with utilities $u_2 \equiv 0$ and u_1 as shown in Table 1 such that when there are T satisfied clauses, there are exactly T additive terms equal to 1. (Recall the notation "$[x_i]$" used in Table 1 from Example 1.)

Note that utility function u_1 can also have negative coefficients. It holds that $((A, R, U), K)$ is a yes-instance of $\mathbb{Q}$-USWO$_{2\text{-ADD}}$ exactly if there exists an allocation π with $sw_u(\pi) = u_1(\pi(a_1)) + u_2(\pi(a_2)) = u_1(\pi(a_1)) \geq K$, if and only if there exists an allocation π with $sw_E(\pi) = \max\{u_1(\pi(a_1)), u_2(\pi(a_2))\} = \max\{T, 0\} \geq K$, which in turn is equivalent to $((A, R, U), K)$ being a yes-instance of $\mathbb{Q}$-ELSWO$_{2\text{-ADD}}$. Hence, MAX-2-SAT reduces to $\mathbb{Q}$-ELSWO$_{2\text{-ADD}}$ in polynomial time.

The NP-hardness claim for $\mathbb{Q}$-ELSWO$_{k\text{-ADD}}$, $k > 2$, follows immediately. □

6 Conclusions

We have studied the implications of the normalization assumption in fair division of indivisible goods. For the common notions of egalitarian and Nash product social welfare, we have shown that this assumption is crucial to have polynomial-time algorithms in certain settings. The key idea of the NP-hardness proofs for nonnormalized utility functions is that dummy agents can be inserted easily to ensure the cardinality constraint. For $n = m$, the results also suggest that there is no general (for a superconstant number of agents with nonnormalized utility functions) and efficient transformation to simulate allocation settings with nonnormalized utility functions using normalized utility functions only, as this would imply P = NP. This is interesting because assigning nonzero utility to the empty bundle corresponds to merely having a positive base level of happiness. Regarding elitist social welfare, note that the reduction in Theorem 5 can also produce nonnormalized utility functions.

In the future, it might be worthwhile to study the effect of the normalization assumption in settings apart from fair division such as (cooperative) game theory.

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References

1. Bansal, N., Sviridenko, M.: The Santa Claus problem. In: Proceedings of the 38th ACM Symposium on Theory of Computing, pp. 31–40. ACM (2006)
2. Bouveret, S.: Fair allocation of indivisible items: Modeling, computational complexity and algorithmics. Ph.D. Thesis, Institut Supérieur De L'Aéronautique Et De l'Espace, Toulouse, France (2007)
3. Bouveret, S., Chevaleyre, Y., Maudet, N.: Handbook of computational social choice, chap. 12. In: Brandt, F., Conitzer, V., Endriss, U., Lang, J., Procaccia, A. (eds.), pp. 284–310. Cambridge University Press, Cambridge (2016)
4. Bouveret, S., Lemaître, M., Fargier, H., Lang, J.: Allocation of indivisible goods: A general model and some complexity results (extended abstract). In: Proceedings of the 4th international joint conference on autonomous agents and multiagent systems, pp. 1309–1310. ACM Press (2005)
5. Brams, S., Taylor, A.: Fair division: From cake-cutting to dispute resolution. Cambridge University Press, Cambridge (1996)
6. Budish, E.: The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. Journal of Political Economy **119**(6), 1061–1103 (2011)
7. Caragiannis, I., Kurokawa, D., Moulin, H., Procaccia, A., Shah, N., Wang, J.: The unreasonable fairness of maximum Nash welfare. In: Proceedings of the 17th ACM conference on economics and computation, pp. 305–322. ACM (2016)
8. Chevaleyre, Y., Dunne, P., Endriss, U., Lang, J., Lemaître, M., Maudet, N., Padget, J., Phelps, S., Rodríguez-Aguilar, J., Sousa, P.: Issues in multiagent resource allocation. Informatica **30**(1), 3–31 (2006)
9. Chevaleyre, Y., Endriss, U., Estivie, S., Maudet, N.: Multiagent resource allocation with k-additive utility functions. In: Proceedings of the DIMACS-LAMSADE workshop on computer science and decision theory, Annales du LAMSADE, vol. 3, pp. 83–100 (2004)
10. Chevaleyre, Y., Endriss, U., Estivie, S., Maudet, N.: Multiagent resource allocation in k-additive domains: Preference representation and complexity. Ann. Oper. Res. **163**(1), 49–62 (2008)
11. Chevaleyre, Y., Endriss, U., Maudet, N.: Allocating goods on a graph to eliminate envy. In: Proceedings of the 22nd AAAI conference on artificial intelligence, pp. 700–705. AAAI Press (2007)
12. Chevaleyre, Y., Endriss, U., Maudet, N.: Simple negotiation schemes for agents with simple preferences: sufficiency, necessity and maximality. Auton. Agent. Multi-Agent Syst. **20**(2), 234–259 (2010)
13. Cole, R., Devanur, N., Gkatzelis, V., Jain, K., Mai, T., Vazirani, V., Yazdanbod, S.: Convex program duality, fisher markets, and Nash social welfare. In: Proceedings of the 18th ACM conference on economics and computation, pp. 459–460. ACM (2017)
14. Damamme, A., Beynier, A., Chevaleyre, Y., Maudet, N.: The power of swap deals in distributed resource allocation. In: Proceedings of the 14th international conference on autonomous agents and multiagent systems, pp. 625–633. IFAAMAS (2015)
15. Dunne, P., Wooldridge, M., Laurence, M.: The complexity of contract negotiation. Artif. Intell. **164**(1–2), 23–46 (2005)
16. Endriss, U., Maudet, N., Sadri, F., Toni, F.: Negotiating socially optimal allocations of resources. J. Artif. Intell. Res. **25**, 315–348 (2006)
17. Garey, M., Johnson, D.: Computers and Intractability: A Guide to the Theory of NP-completeness. W. H Freeman and Company (1979)
18. Golovin, D.: Max-min fair allocation of indivisible goods. Tech. Rep. CMU-CS-05-144, School of Computer Science Carnegie Mellon University (2005)
19. Heinen, T., Nguyen, N., Rothe, J.: Fairness and rank-weighted utilitarianism in resource allocation. In: Proceedings of the 4th international conference on algorithmic decision theory, pp. 521–536. Springer-Verlag Lecture Notes in Artificial Intelligence #9346 (2015)
20. Irving, R., Leather, P., Gusfield, D.: An efficient algorithm for the "optimal" stable marriage. Journal of the ACM **34**(3), 532–543 (1987)
21. Karp, R.: Reducibility among Combinatorial Problems. In: Miller, R., Thatcher, J. (eds.) Complexity of computer computations, pp. 85–103. Plenum Press (1972)

22. Kash, I., Procaccia, A., Shah, N.: No agent left behind: Dynamic fair division of multiple resources. J. Artif. Intell. Res. **51**, 579–603 (2014)
23. Lang, J., Rothe, J.: Fair division of indivisible goods. In: Rothe, J. (ed.) Economics and computation. An introduction to algorithmic game theory, computational social choice, and fair division, springer texts in business and economics, Chap. 8. Springer-Verlag (2015)
24. Lange, P., Nguyen, N., Rothe, J.: The price to pay for forgoing normalization in fair division of indivisible goods. In: Nonarchival website proceedings of the 15th International Symposium on Artificial Intelligence and Mathematics. http://isaim2018.cs.virginia.edu/papers/ISAIM2018_Lange_etal.pdf (2018)
25. Lange, P., Nguyen, N., Rothe, J.: The price to pay for forgoing normalization in fair division of indivisible goods. In: Nonarchival website proceedings of the 11th Multidisciplinary Workshop on Advances in Preference Handling. http://www.mpref-2018.preflib.org/wp-content/uploads/2017/12/paper_10.pdf (2018)
26. Lipton, R., Markakis, E., Mossel, E., Saberi, A.: On approximately fair allocations of indivisible goods. In: Proceedings of the 5th ACM conference on electronic commerce, pp. 125–131. ACM Press (2004)
27. Nguyen, N., Nguyen, T., Roos, M., Rothe, J.: Computational complexity and approximability of social welfare optimization in multiagent resource allocation. Journal of Autonomous Agents and Multi-Agent Systems **28**(2), 256–289 (2014)
28. Nguyen, T., Roos, M., Rothe, J.: A survey of approximability and inapproximability results for social welfare optimization in multiagent resource allocation. Ann. Math. Artif. Intell. **68**(1–3), 65–90 (2013)
29. Ramezani, S., Endriss, U.: Nash social welfare in multiagent resource allocation. In: Agent-mediated electronic commerce. Designing trading strategies and mechanisms for electronic markets, pp. 117–131. Springer-Verlag Lecture Notes in Business Information Processing #79 (2010)
30. Roos, M., Rothe, J.: Complexity of social welfare optimization in multiagent resource allocation. In: Proceedings of the 9th international conference on autonomous agents and multiagent systems, pp. 641–648. IFAAMAS (2010)
31. Sandholm, T.: Contract types for satisficing task allocation. In: Proceedings of the AAAI spring symposium: Satisficing models, pp. 23–25 (1998)