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Properties of skeptical c-inference for conditional knowledge bases and its realization as a constraint satisfaction problem

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Abstract While the axiomatic system P is an important standard for plausible, nonmonotonic inferences from conditional knowledge bases, it is known to be too weak to solve benchmark problems like Irrelevance or Subclass Inheritance. Ordinal conditional functions provide a semantic base for system P and have often been used to design stronger inference relations, like Pearl's system Z, or c-representations. While each c-representation shows excellent inference properties and handles particularly Irrelevance and Subclass Inheritance properly, it is still an open problem which c-representation is the best. In this paper, we consider the skeptical inference relation, called *c-inference*, that is obtained by taking all c-representations of a given knowledge base into account. We study properties of c-inference and show in particular that it preserves the properties of solving Irrelevance and Subclass Inheritance. Based on a characterization of c-representations as solutions of a Constraint Satisfaction Problem (CSP), we also model skeptical c-inference as a CSP and prove soundness and completeness of the modelling, ensuring that constraint solvers can be used for implementing c-inference.

Keywords Conditional · Conditional knowledge base · System P · System Z · C-representation · C-inference · Constraint satisfaction problem

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1 Introduction

Sets of conditionals of the form *If A then normally B* can serve as a knowledge base to a reasoning agent. While such knowledge bases may contain all relevant rules for an agent, they usually do not contain enough information to represent all plausible beliefs that a reasoning agent, operating based on this knowledge, should have. Thus, for a reasoning agent it is essential to extend a knowledge base to what is called a complete *epistemic state*, containing all beliefs necessary to answer arbitrary questions [21]. There are many ways to represent the epistemic state of an agent obtained from a set of qualitative conditionals, e.g. using probabilities [1], ϵ-semantics [31], big-stepped probabilities [3, 33], the system-of-spheres semantics of Lewis [26], possibility theory [3, 17], or ordinal conditional functions [34, 35]; for a formal comparison of different semantics of conditional knowledge and the formal interrelationships and possible translations among them see e.g. [9].

Determining inductive inferences based on knowledge bases of conditional rules is an important task in nonmonotonic reasoning. Here, calculi like Adams' system P [1], the rational closure of the knowledge base [27], probabilistic approaches like p-entailment [18], reasoning under maximum entropy [30], or possibilistic inference methods [3, 17] have been developed, as well as the inductive methods of Pearl's system Z [20, 32] or c-representations [22, 23]. The latter two rely on Spohn's ordinal conditional functions [34, 35] for calculating inferences which means that the underlying preferential model [28] always is set up upon a total preorder on the set of possible worlds. In this article, we define a novel inductive inference relation, called *c-inference,* as a skeptical inference over the (infinitely many) c-representations of a knowledge base. We show that this inference relation, even if set up upon a partial ordering of the worlds, exceeds system P and handles important benchmarks of plausible reasoning, like the Drowning Problem or Irrelevance, properly. We model the c-representations of a knowledge base $\mathcal{R}$ as a constraint satisfaction problem (CSP), denoted by $CR(\mathcal{R})$ (cf. [10]), and prove correctness and completeness of this modelling. Since $CR(\mathcal{R})$ is solvable if and only if $\mathcal{R}$ is consistent, this CSP modelling additionally yields an alternative to the tolerance partitioning algorithm [20, 32] for checking the consistency of $\mathcal{R}$. We also show how c-inference can be realized by a CSP and prove that every c-inference can be reduced to the solvability of a constraint satisfaction problem. Hence, constraint solvers can be used for checking the consistency of a knowledge base $\mathcal{R}$, for computing c-representations for $\mathcal{R}$, and for implementing c-inference in the context of $\mathcal{R}$.

This article revises and largely extends the conference paper [5]: We investigate skeptical c-inference with respect to further general axioms put forward for nonmonotonic inference from conditional knowledge bases representing sets of default rules. The constraint satisfaction problems for c-representations and for c-inference are sharpened to CSPs over finite domains, and we investigate the influence of reducing the size of the finite domains, thereby simplifying the CSPs. We demonstrate, that in general, while still preserving soundness and completeness of the modellings, there are different minimal upper bounds for the CSPs modelling c-representations (up to equivalences) and modelling c-inference, respectively.

The rest of this paper is organized as follows: In Section 2 we recall the basics of conditionals, ordinal conditional functions, plausible inference, system P, system Z and c-representations as far as needed for the formal background of this paper. In Section 3, we prove that the CSP $CR(\mathcal{R})$ for c-representations of a knowledge base $\mathcal{R}$ is a correct and complete modelling of c-representations. Section 4 defines c-inference as skeptical inference relation over all c-representations of a knowledge base. We prove that c-inference not only satisfies but exceeds system P, study c-inference with respect to general axioms for nonmonotonic inference, and show that c-inference handles selected benchmarks properly. A characterization of c-inference as a CSP along with a correctness and completeness theorem is given in Section 5. In Section 6, CSPs over finite domains for c-representations and for c-inference are developed. In Section 7 we give a brief overview of a system implementing skeptical c-inference, along with other nonmonotonic inference relations. In Section 8, we conclude and point out further work.

2 Conditionals, OCF, and plausible inference

Let $\Sigma = \{v_1, ..., v_m\}$ be a finite propositional alphabet. From Σ we obtain the propositional language $\mathfrak{L}$ as the set of formulas of Σ closed under negation $\neg$, conjunction $\wedge$, and disjunction $\vee$, as usual; for formulas $A, B \in \mathfrak{L}$, $A \Rightarrow B$ denotes the material implication and stands for $\neg A \vee B$. For shorter formulas, we abbreviate conjunction by juxtaposition (i.e., AB stands for $A \wedge B$), and negation by overlining (i.e., $\overline{A}$ is equivalent to $\neg A$). A *literal* is a propositional variable v_i or a negated propositional variable $\overline{v_i}$. A conjunction that mentions every variable in Σ, is called a complete conjunction over Σ. Let Ω denote the set of possible worlds over $\mathfrak{L}$; Ω will be taken here simply as the set of all propositional interpretations over $\mathfrak{L}$ and can be identified with the set of all complete conjunctions over Σ. For $\omega \in \Omega$, $\omega \models A$ means that the propositional formula $A \in \mathfrak{L}$ holds in the possible world ω.

A *conditional* $(B|A)$ with $A, B \in \mathfrak{L}$ encodes the defeasible rule "if A then normally B" and is a trivalent logical entity with the evaluation [14, 22]

$$(B \mid A)(\omega) = \begin{cases} \textit{true} & \text{iff} \quad \omega \models AB \quad \text{(verification)}, \\ \textit{false} & \text{iff} \quad \omega \models A\overline{B} \quad \text{(falsification)}, \\ \textit{undefined} & \text{iff} \quad \omega \models \overline{A} \quad \text{(not applicable)}. \end{cases}$$

A *knowledge base* $\mathcal{R} = \{(B_1|A_1), ..., (B_n|A_n)\}$ is a finite set of conditionals. A conditional $(B \mid A)$ is *tolerated* by a set of conditionals $\mathcal{R}$ if there is a world $\omega \in \Omega$ such that $\omega \models AB$ and $\omega \models \bigwedge_{i=1}^{n}(A_i \Rightarrow B_i)$, i.e., if ω verifies $(B \mid A)$ and does not falsify any conditional in $\mathcal{R}$.

An *Ordinal Conditional Function* (OCF, ranking function) [34, 35] is a function $\kappa : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ that assigns to each world $\omega \in \Omega$ an implausibility rank $\kappa(\omega)$, that is, the higher $\kappa(\omega)$, the more surprising ω is. OCFs have to satisfy the normalization condition that there has to be a world that is maximally plausible, i.e., the preimage of 0 cannot be empty, formally $\kappa^{-1}(0) \neq \emptyset$. The rank of a formula A is defined to be the rank of the least surprising world that satisfies A, formally

$$\kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\}. \tag{1}$$

The set of models of tautologies is the complete set of possible worlds, therefore the normalization condition directly gives us $\kappa(\top) = 0$. In accordance with general order-theoretical conventions, we set $\kappa(\bot) = \infty$.

An OCF κ *accepts* a conditional $(B \mid A)$ (denoted by $\kappa \models (B \mid A)$) if the verification of the conditional is less surprising than its falsification, i.e., if $\kappa(AB) < \kappa(A\overline{B})$. This can also be understood as a nonmonotonic inference relation between the premise A and the conclusion B: We say that A *κ-entails* B (written $A \mid\!\sim^{\kappa} B$) if κ accepts the conditional $(B \mid A)$, formally

$$\kappa \models (B \mid A) \text{ iff } \kappa(AB) < \kappa(A\overline{B}) \text{ iff } A \mid\!\sim^{\kappa} B. \tag{2}$$

Note that $\kappa(AB) < \kappa(A\overline{B})$ is equivalent to $\kappa(AB) - \kappa(A) > 0$, giving us

$$\kappa \models (B \mid A) \text{ iff } \kappa(AB) - \kappa(A) > 0. \tag{3}$$

The acceptance relation in (2) is extended as usual to a set $\mathcal{R}$ of conditionals by defining $\kappa \models \mathcal{R}$ if $\kappa \models (B \mid A)$ for all $(B \mid A) \in \mathcal{R}$. This is synonymous to saying that κ is *admissible* with respect to $\mathcal{R}$ [20]. An OCF that is admissible with respect to $\mathcal{R}$ is called a *ranking model* of $\mathcal{R}$.

A knowledge base $\mathcal{R}$ is *consistent* if there exists a ranking model of $\mathcal{R}$. Such an OCF can be found if and only if in every non-empty subset $\mathcal{R}' \subseteq \mathcal{R}$ there is a conditional $(B|A) \in \mathcal{R}'$ that is tolerated by $\mathcal{R}'$. This condition is equivalent to the existence of an ordered partitioning $(\mathcal{R}_0, ..., \mathcal{R}_m)$ of $\mathcal{R}$ with the property that for every $0 \leqslant i \leqslant m$ every conditional $(B|A) \in \mathcal{R}_i$ is tolerated by $\bigcup_{j=i}^{m} \mathcal{R}_j$ [20, 32].

Example 1 ($\mathcal{R}_{bird}$) We illustrate the definitions and propositions in this article with the well-known penguin example. Here, the variables in the alphabet $\Sigma = \{p, b, f\}$ indicate whether something is a bird (b) or not ($\overline{b}$), can fly (f) or not ($\overline{f}$) and whether something is a penguin (p) or not ($\overline{p}$) which results in the possible worlds $\Omega = \{pbf, pb\overline{f}, p\overline{b}f, p\overline{b}\overline{f}, \overline{p}bf, \overline{p}b\overline{f}, \overline{p}\overline{b}f, \overline{p}\overline{b}\overline{f}\}$. The knowledge base $\mathcal{R}_{bird} = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ consists of the four conditionals:

δ_1 : $(f \mid b)$ "If something is a bird, it usually can fly."
δ_2 : $(\overline{f} \mid p)$ "If something is a penguin, it usually cannot fly."
δ_3 : $(\overline{f} \mid pb)$ "If something is a penguin bird, it usually cannot fly."
δ_4 : $(b \mid p)$ "If something is a penguin, it usually is a bird."

This knowledge base is consistent: For $\mathcal{R}_0 = \{(f \mid b)\}$ and $\mathcal{R}_1 = \mathcal{R}_{bird} \backslash \mathcal{R}_0$ we have the ordered partitioning $(\mathcal{R}_0, \mathcal{R}_1)$ such that every conditional in $\mathcal{R}_0$ is tolerated by $\mathcal{R}_0 \cup \mathcal{R}_1 = \mathcal{R}_{bird}$ and every conditional in $\mathcal{R}_1$ is tolerated by $\mathcal{R}_1$. For instance, $(f \mid b)$ is tolerated by $\mathcal{R}_{bird}$ since there is, for example, the world $\overline{p}bf$ with $\overline{p}bf \models bf$ as well as $\overline{p}bf \models (p \Rightarrow \overline{f}) \wedge (pb \Rightarrow \overline{f}) \wedge (p \Rightarrow b)$. Table 1 shows an OCF κ that is a ranking model for $\mathcal{R}_{bird}$; for instance we have $\kappa \models (f \mid b)$ since $\kappa(bf) = \min\{\kappa(pbf), \kappa(\overline{p}bf)\} = \min\{2, 0\} = 0$ and $\kappa(b\overline{f}) = \min\{\kappa(pb\overline{f}), \kappa(\overline{p}b\overline{f})\} = \min\{1, 1\} = 1$ and therefore $\kappa(bf) < \kappa(b\overline{f})$.

The following *p-entailment* is an established inference in the area of ranking functions.

Definition 1 (p-entailment [20]) Let $\mathcal{R}$ be a conditional knowledge base and let A, B be formulas. A *p-entails* B in the context of $\mathcal{R}$, written $A \mid\!\sim^{p}_{\mathcal{R}} B$, if $A \mid\!\sim^{\kappa} B$ for all $\kappa \models \mathcal{R}$.

P-entailment can be characterized as follows:

Table 1 Ranking model κ for the knowledge base $\mathcal{R}_{bird}$ in Example 1

ω	pbf	$pb\overline{f}$	$p\overline{b}f$	$p\overline{b}\overline{f}$	$\overline{p}bf$	$\overline{p}b\overline{f}$	$\overline{p}\overline{b}f$	$\overline{p}\overline{b}\overline{f}$
$\kappa(\omega)$	2	1	2	2	0	1	0	0

Proposition 1 ([16, 19, 20]) *Let $\mathcal{R}$ be a conditional knowledge base and let A, B be formulas. A p-entails B in the context of a knowledge base $\mathcal{R}$, if and only if $\mathcal{R} \cup \{(\overline{B} \mid A)\}$ is inconsistent.*

Example 2 ($\mathcal{R}_{bird}$, cont'd) We illustrate p-entailment with the running example. Here the knowledge base $\mathcal{R}_{bird}$ p-entails, for instance, that non-flying penguins are birds, formally, $p\overline{f} \mathrel{|\!\sim}^{p}_{\mathcal{R}} b$: Using Prop. 1, we observe that $\mathcal{R}_{bird} \cup \{(\overline{b} \mid p\overline{f})\}$ is inconsistent because every world ω that verifies the conditional $(\overline{b} \mid p\overline{f})$, i.e., $\omega \models p\overline{b}\,\overline{f}$, violates $(b \mid p)$, and every world ω that verifies $(b \mid p)$, i.e., $\omega \models pb$, violates $(\overline{b} \mid p\overline{f})$. Therefore, the conditional $(\overline{b} \mid p\overline{f})$ is neither tolerated by $\mathcal{R}_{bird}$ nor does it tolerate $\mathcal{R}_{bird}$ and hence $\mathcal{R}_{bird} \cup \{(\overline{b} \mid p\overline{f})\}$ is inconsistent. Hence by Definition 1 we obtain $p\overline{f} \mathrel{|\!\sim}^{p}_{\mathcal{R}} b$.

Nonmonotonic inference relations are usually evaluated by means of properties. In particular, the axiom system P [1] provides an important standard for plausible, nonmonotonic inferences. With $\mathrel{|\!\sim}$ being a generic nonmonotonic inference operator and A, B, C being formulas in $\mathfrak{L}$, the six properties of system P are defined as follows:

(REF)	Reflexivity	for all $A \in \mathfrak{L}$ it holds that $A \mathrel{|\!\sim} A$
(LLE)	Left Logical Equivalence	$A \equiv B$ and $B \mathrel{|\!\sim} C$ imply $A \mathrel{|\!\sim} C$
(RW)	Right weakening	$B \models C$ and $A \mathrel{|\!\sim} B$ imply $A \mathrel{|\!\sim} C$
(CM)	Cautious Monotony	$A \mathrel{|\!\sim} B$ and $A \mathrel{|\!\sim} C$ imply $AB \mathrel{|\!\sim} C$
(CUT)		$A \mathrel{|\!\sim} B$ and $AB \mathrel{|\!\sim} C$ imply $A \mathrel{|\!\sim} C$
(OR)		$A \mathrel{|\!\sim} C$ and $B \mathrel{|\!\sim} C$ imply $(A \vee B) \mathrel{|\!\sim} C$

We refer to Dubois and Prade [15] for the relation between p-entailment and system P:

Proposition 2 ([15]) *Let A, B be formulas and let $\mathcal{R}$ be a conditional knowledge base. Then B follows from A in the context of $\mathcal{R}$ with the rules of system P if and only if A p-entails B in the context of $\mathcal{R}$.*

So, given a knowledge base $\mathcal{R}$, system P inference is the same as p-entailment.

Two inference relations which are defined by specific OCFs obtained inductively from a knowledge base $\mathcal{R}$ have received some attention: system Z and c-representations, or the induced inference relations, respectively, both show excellent inference properties. We recall both approaches briefly.

System Z [32] is based upon the ranking function κ^Z, which is the unique Pareto-minimal OCF that accepts $\mathcal{R}$. The system is set up by forming an ordered partition $(\mathcal{R}_0, ..., \mathcal{R}_m)$ of $\mathcal{R}$, where each $\mathcal{R}_i$ is the (with respect to set inclusion) maximal subset of $\bigcup_{j=i}^{m} \mathcal{R}_j$ that is tolerated by $\bigcup_{j=i}^{m} \mathcal{R}_j$. This partitioning is unique due to the maximality. The resulting OCF κ^Z is defined by assigning to each world ω a rank of 1 plus the maximal index $1 \leqslant i \leqslant m$ of the partition that contains conditionals falsified by ω or 0 if ω does not falsify any conditional in $\mathcal{R}$. Formally, for all $(B \mid A) \in \mathcal{R}$ and for $Z(B \mid A) = i$ if and only if $(B \mid A) \in \mathcal{R}_i$, the OCF κ^Z is given by

$$\kappa^Z(\omega) = \begin{cases} 0 & \text{if } \omega \text{ does not falsify any conditional in } \mathcal{R}, \\ \max\{Z(B \mid A) | (B \mid A) \in \mathcal{R}, \omega \models A\overline{B}\} + 1 & \text{otherwise.} \end{cases}$$

Other than system Z, the approach of c-representations does not use the most severe falsification of a conditional, but assigns an individual impact to each conditional and generates the world ranks as a sum of impacts of falsified conditionals.

Definition 2 (c-representation [22, 23]) Let $\mathcal{R}$ be a knowledge base. A *c-representation* of $\mathcal{R}$ is a ranking function κ constructed from integer impacts $\eta_i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ assigned to each conditional $(B_i \mid A_i)$ such that κ accepts $\mathcal{R}$ and is given by the following equation:

$$\kappa(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \eta_i \tag{4}$$

The relationship between these two approaches of inductively generating or finding a ranking model for a consistent knowledge base is well-researched. For instance, it has been shown that the two approaches differ in their inferences and that, in general, neither is contained in the other [24, 36].

Examples of system Z and c-representations are given in the following sections.

3 Soundness and completeness of a CSP modeling of c-representations

In [10], a modeling of c-representations as solutions of a constraint satisfaction problem is proposed and employed for computing c-representations using constraint logic programming. In this section, we first recall this modeling, and then prove its soundness and completeness.

Definition 3 (***CR***$(\mathcal{R})$ [10]) Let $\mathcal{R} = \{(B_1 \mid A_1), \ldots, (B_n \mid A_n)\}$. The constraint satisfaction problem for c-representations of $\mathcal{R}$, denoted by $CR(\mathcal{R})$, on the constraint variables $\{\eta_1, \ldots, \eta_n\}$ ranging over $\mathbb{N}$ is given by the conjunction of the constraints, for all $i \in \{1, \ldots, n\}$:

$$\eta_i \geqslant 0 \tag{5}$$

$$\eta_i > \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j \tag{6}$$

A solution of $CR(\mathcal{R})$ is an n-tuple $(\eta_1, \ldots, \eta_n)$ of natural numbers. For a constraint satisfaction problem *CSP*, the set of solutions is denoted by *Sol*(*CSP*). Thus, with $Sol(CR(\mathcal{R}))$ we denote the set of all solutions of $CR(\mathcal{R})$.

Example 3 ($\mathcal{R}_{bird}$, cont'd) The verification/falsification behaviour of the conditionals in $\mathcal{R}_{bird}$ from Example 1 is given in Table 2. Based on these evaluations, the constraints in $CR(\mathcal{R}_{bird})$ according to (6) are

$$\eta_1 > \min\{\eta_2 + \eta_3, 0\} - \min\{0, 0\} = 0 \tag{7}$$

$$\eta_2 > \min\{\eta_1, \eta_4\} - \min\{\eta_3, \eta_4\} \tag{8}$$

$$\eta_3 > \eta_1 - \eta_2 \tag{9}$$

$$\eta_4 > \min\{\eta_2 + \eta_3, \eta_1\} - \min\{\eta_2, 0\} = \min\{\eta_2 + \eta_3, \eta_1\} \tag{10}$$

Table 2 Verification / falsification behavior of $\mathcal{R}_{bird}$: (+) indicates verification, (−) falsification and an empty cell non-applicability

	pbf	$pb\overline{f}$	$p\overline{b}f$	$p\overline{b}\overline{f}$	$\overline{p}bf$	$\overline{p}b\overline{f}$	$\overline{p}\overline{b}f$	$\overline{p}\overline{b}\overline{f}$
$\delta_1 = (f\|b)$	+	−			+	−		
$\delta_2 = (\overline{f}\|p)$	−	+	−	+				
$\delta_3 = (\overline{f}\|pb)$	−	+						
$\delta_4 = (b\|p)$	+	+	−	−				

because $\eta_i \geqslant 0$ for all $1 \leqslant i \leqslant n$. The inequality (9) is equivalent to $\eta_2 + \eta_3 > \eta_1$, which together with (10) gives us $\eta_4 > \eta_1$, and we finally obtain

$$\begin{aligned} \eta_1 &> 0 \\ \eta_2 &> \eta_1 - \min\{\eta_3, \eta_4\} \\ \eta_3 &> \eta_1 - \eta_2 \\ \eta_4 &> \eta_1. \end{aligned} \tag{11}$$

Three possible solutions for this system (11) that also satisfy the constraint (5) are $\vec{\eta}^{(1)} = (1, 1, 1, 2)$, $\vec{\eta}^{(2)} = (1, 2, 0, 2)$ and $\vec{\eta}^{(3)} = (1, 0, 3, 2)$.

Proposition 3 (Soundness of $\boldsymbol{CR(\mathcal{R})}$) *For* $\mathcal{R} = \{(B_1 \mid A_1), \dots, (B_n \mid A_n)\}$ *let* $\vec{\eta} = (\eta_1, \dots, \eta_n) \in Sol(CR(\mathcal{R}))$. *Then the function* κ *defined by the equation system given by* (4) *is a c-representation that accepts* $\mathcal{R}$.

Proof To show that the modeling given by the constraint satisfaction problem $CR(\mathcal{R})$ is sound, we have to show that every solution of the constraints given by Eqs. (5) and (6) is a c-representation of $\mathcal{R}$. We will use the techniques for showing c-representation properties given in [22] to show that κ accepts $\mathcal{R}$, i.e., for all $(B_i|A_i) \in \mathcal{R}$

$$\kappa(A_i B_i) < \kappa(A_i \overline{B}_i). \tag{12}$$

Using the definition of ranks of formulas for each $1 \leqslant i \leqslant n$, (12) gives us

$$\min_{\omega \models A_i B_i} \{\kappa(\omega)\} < \min_{\omega \models A_i \overline{B}_i} \{\kappa(\omega)\}. \tag{13}$$

We now use (4) and get

$$\min_{\omega \models A_i B_i} \Bigg\{ \sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j \overline{B_j}}} \eta_i \Bigg\} < \min_{\omega \models A_i \overline{B}_i} \Bigg\{ \underbrace{\sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j \overline{B_j}}} \eta_i}_{(a)} \Bigg\}. \tag{14}$$

In each line i, η_i is a summand of (a) and hence this summand can be extracted from the minimum, yielding

$$\min_{\omega \models A_i B_i} \Bigg\{ \sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j \overline{B_j}}} \eta_i \Bigg\} < \min_{\omega \models A_i \overline{B}_i} \Bigg\{ \sum_{\substack{1 \leqslant j \leqslant n; i \neq j \\ \omega \models A_j \overline{B_j}}} \eta_i \Bigg\} + \eta_i. \tag{15}$$

We rearrange the inequality in (15) and get

$$\eta_i > \min_{\omega \models A_i B_i} \Big\{ \underbrace{\sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j \overline{B_j}}}}_{(b)} \eta_i \Big\} - \min_{\omega \models A_i \overline{B}_i} \Big\{ \sum_{\substack{1 \leqslant j \leqslant n; i \neq j \\ \omega \models A_j \overline{B_j}}} \eta_i \Big\}. \tag{16}$$

In the first minimum in (16), $(B_i | A_i)$ is never falsified, so i can be removed from the range of (b), which gives us

$$\eta_i > \min_{\omega \models A_i B_i} \Big\{ \sum_{\substack{1 \leqslant j \leqslant n; i \neq j \\ \omega \models A_j \overline{B_j}}} \eta_i \Big\} - \min_{\omega \models A_i \overline{B}_i} \Big\{ \sum_{\substack{1 \leqslant j \leqslant n; i \neq j \\ \omega \models A_j \overline{B_j}}} \eta_i \Big\} \tag{17}$$

which is (6). Therefore, every solution of $CR(\mathcal{R})$ ensures that the resulting κ defined by (4) in Definition 2 is a c-representation that accepts $\mathcal{R}$. □

Definition 4 ($\kappa_{\vec{\eta}}$, $\mathcal{O}(CR(\mathcal{R}))$) For $\vec{\eta} \in Sol(CR(\mathcal{R}))$ and κ as in (4), κ is the *OCF induced by* $\vec{\eta}$ and is denoted by $\kappa_{\vec{\eta}}$. The set of all OCFs induced by the solutions of $CR(\mathcal{R})$ is denoted by $\mathcal{O}(CR(\mathcal{R})) = \{\kappa_{\vec{\eta}} \mid \vec{\eta} \in Sol(CR(\mathcal{R}))\}$.

Example 4 ($\mathcal{R}_{bird}$, cont'd) Using (4) with the solutions for the CSP calculated in Example 3 gives us the OCFs shown in Table 3. All of these OCFs accept the knowlege base $\mathcal{R}_{bird}$; for $\vec{\eta}^{(1)} = (1, 1, 1, 2)$ we have, for instance,

$$\begin{aligned}
\kappa_{\vec{\eta}^{(1)}}(bf) &= \min\{\kappa_{\vec{\eta}^{(1)}}(pbf), \kappa_{\vec{\eta}^{(1)}}(\overline{p}bf)\} = 0 \\
\kappa_{\vec{\eta}^{(1)}}(b) &= \min\{\kappa_{\vec{\eta}^{(1)}}(pbf), \kappa_{\vec{\eta}^{(1)}}(pb\overline{f}), \kappa_{\vec{\eta}^{(1)}}(\overline{p}bf), \kappa_{\vec{\eta}^{(1)}}(\overline{p}b\overline{f})\} = 0 \\
\kappa_{\vec{\eta}^{(1)}}(b\overline{f}) &= \min\{\kappa_{\vec{\eta}^{(1)}}(pb\overline{f}), \kappa_{\vec{\eta}^{(1)}}(\overline{p}b\overline{f})\} = 1
\end{aligned}$$

and hence

$$\kappa_{\vec{\eta}^{(1)}}(f|b) = \kappa_{\vec{\eta}^{(1)}}(bf) - \kappa_{\vec{\eta}^{(1)}}(b) = 0 < 1 = \kappa_{\vec{\eta}^{(1)}}(b\overline{f}) - \kappa_{\vec{\eta}^{(1)}}(b) = \kappa_{\vec{\eta}^{(1)}}(\overline{f}|b).$$

The other ranks for the verification and falsification, respectively, of the conditionals with respect to these ranking functions are given in Table 4 from which we can see that each of the three induced OCFs accepts $\mathcal{R}_{bird}$.

Proposition 4 (Completeness of $\boldsymbol{CR(\mathcal{R})}$) *Let κ be a c-representation for a knowledge base $\mathcal{R} = \{(B_1 \mid A_1), \ldots, (B_n \mid A_n)\}$, i.e., $\kappa \models \mathcal{R}$. Then there is a vector $\vec{\eta} \in Sol(CR(\mathcal{R}))$ such that $\kappa = \kappa_{\vec{\eta}}$.*

Table 3 Induced ranking functions for the solution vectors $\vec{\eta}^{(1)} = (1, 1, 1, 2)$, $\vec{\eta}^{(2)} = (1, 2, 0, 2)$, and $\vec{\eta}^{(3)} = (1, 0, 3, 2)$ in the penguin example $\mathcal{R}_{bird}$ (cf. Examples 3 and 4)

ω	pbf	$pb\overline{f}$	$p\overline{b}f$	$p\overline{b}\,\overline{f}$	$\overline{p}bf$	$\overline{p}b\overline{f}$	$\overline{p}\overline{b}f$	$\overline{p}\overline{b}\,\overline{f}$
$\kappa_{\vec{\eta}^{(1)}}(\omega)$	2	1	3	2	0	1	0	0
$\kappa_{\vec{\eta}^{(2)}}(\omega)$	2	1	4	2	0	1	0	0
$\kappa_{\vec{\eta}^{(3)}}(\omega)$	3	1	2	2	0	1	0	0

Table 4 Acceptance of conditionals in the penguin example $\mathcal{R}_{bird}$ by the OCFs in Table 3. Note that $\kappa(B|A) = \kappa(AB) - \kappa(A)$ for all $A, B \in \mathfrak{L}$

	$\kappa_{\vec{\eta}^{(1)}}$				$\kappa_{\vec{\eta}^{(2)}}$				$\kappa_{\vec{\eta}^{(3)}}$			
	verif.		falsif.	accpt?	verif.		falsif.	accpt?	verif.		falsif.	accpt?
$(f \mid b)$	0	<	1	✓	0	<	1	✓	0	<	1	✓
$(\overline{f} \mid p)$	0	<	1	✓	0	<	1	✓	0	<	1	✓
$(\overline{f} \mid pb)$	0	<	1	✓	0	<	1	✓	0	<	2	✓
$(b \mid p)$	2	<	1	✓	0	<	1	✓	0	<	1	✓

Proof A ranking function is a c-representation for $\mathcal{R}$ if and only if it is composed by (4) and accepts $\mathcal{R}$. For the proof of Proposition 3 we have shown that these two conditions are equivalent to the impacts being chosen to satisfy (5) and (6). Therefore, for every c-representation κ for $\mathcal{R}$ there is a vector $\vec{\eta} \in Sol(CR(\mathcal{R}))$ such that $\kappa = \kappa_{\vec{\eta}}$, as proposed. □

It has been shown that there is a c-representation for a knowledge base $\mathcal{R}$ if and only if $\mathcal{R}$ is consistent [22, 23]. The soundness and completeness results in Propositions 3 and 4 give us that $CR(\mathcal{R})$ is solvable if and only if there is a c-representation for $\mathcal{R}$. This gives us an additional criterion for the consistency of a knowledge base which we formalize as follows.

Corollary 1 (Consistency Criterion) *A knowledge base $\mathcal{R}$ is consistent if and only if the constraint satisfaction problem $CR(\mathcal{R})$ has a solution.*

Applying a constraint satisfaction solver, Corollary 1 gives us an implementable alternative to the tolerance test algorithm in [32].

4 Skeptical inference based on c-representations

Equation (2) defines an inference relation $\mid\!\sim^{\kappa}$ based on a single OCF κ. For a given knowledge base $\mathcal{R}$ and two formulas A, B we will now introduce a novel skeptical inference relation based on all c-representations.

Definition 5 (c-inference, $\mid\!\sim^{c}_{\mathcal{R}}$) Let $\mathcal{R}$ be a knowledge base and let A, B be formulas. B is *a (skeptical) c-inference from A in the context of $\mathcal{R}$*, denoted by $A \mid\!\sim^{c}_{\mathcal{R}} B$, if $A \mid\!\sim^{\kappa} B$ holds for all c-representations κ for $\mathcal{R}$.

We will show that skeptical c-inference is different from p-entailment, which is equivalent to the skeptical inference relation obtained by considering all OCFs that accept $\mathcal{R}$, and that it is able to preserve high-quality inference properties that inference based on single c-representations has.

Example 5 ($\mid\!\sim^{c}_{\mathcal{R}}$) Consider the three OCFs $\kappa_{\vec{\eta}^{(1)}}$, $\kappa_{\vec{\eta}^{(2)}}$, and $\kappa_{\vec{\eta}^{(3)}}$ from Table 3 calculated in Example 4 that are induced by the solutions $\vec{\eta}^{(1)}$, $\vec{\eta}^{(2)}$, and $\vec{\eta}^{(3)}$ of $CR(\mathcal{R}_{bird})$ given in Example 3. In Table 5, their acceptance properties with respect to some conditionals that are not contained in $\mathcal{R}_{bird}$ are given. From the acceptance properties in Table 5, we conclude that b is not a c-inference of pf in the context of $\mathcal{R}_{bird}$, denoted by $pf \not\mid\!\sim^{c}_{\mathcal{R}_{bird}} b$, since we find that for all c-representations $\kappa_{\vec{\eta}}$ of $\mathcal{R}_{bird}$ the ranks of the respective worlds

Table 5 Acceptance properties for the ranking functions $\kappa_{\vec{\eta}^{(1)}}, \kappa_{\vec{\eta}^{(2)}}, \kappa_{\vec{\eta}^{(3)}}$ (Table 3) that accept $\mathcal{R}_{bird}$ with respect to three conditionals not contained in $\mathcal{R}_{bird}$

	$\kappa_{\vec{\eta}^{(1)}}$				$\kappa_{\vec{\eta}^{(2)}}$				$\kappa_{\vec{\eta}^{(3)}}$			
	verif.		falsif.	accpt.	verif.		falsif.	accpt.	verif.		falsif.	accpt.
$(b \mid pf)$	0	<	1	✓	0	<	2	✓	1	>	0	↯
$(\overline{b} \mid pf)$	1	>	0	↯	2	>	0	↯	0	<	1	✓
$(b\overline{f} \mid p)$	0	<	1	✓	0	<	1	✓	0	<	1	✓

are $\kappa_{\vec{\eta}}(pbf) = \eta_2 + \eta_3$ and $\kappa_{\vec{\eta}}(p\overline{b}f) = \eta_2 + \eta_4$, which with the solutions of Example 3 result in the following concrete ranks and relationships:

$$\kappa_{\vec{\eta}^{(1)}}(pbf) = 2 < 3 = \kappa_{\vec{\eta}^{(1)}}(p\overline{b}f)$$
$$\kappa_{\vec{\eta}^{(2)}}(pbf) = 2 < 4 = \kappa_{\vec{\eta}^{(2)}}(p\overline{b}f)$$
$$\kappa_{\vec{\eta}^{(3)}}(pbf) = 3 < 2 = \kappa_{\vec{\eta}^{(3)}}(p\overline{b}f)$$

So there are c-representations of $\mathcal{R}_{bird}$, for instance $\kappa_{\vec{\eta}^{(3)}}$, with the property that $\kappa_{\vec{\eta}^{(3)}} \not\models (b \mid pf)$. Likewise there are c-representations of $\mathcal{R}_{bird}$, for instance $\kappa_{\vec{\eta}^{(1)}}$, with the property that $\kappa_{\vec{\eta}^{(2)}} \not\models (\overline{b} \mid pf)$, from which we obtain $pf \not\mathrel{|\!\sim}^{c}_{\mathcal{R}_{bird}} \overline{b}$. Thus, by c-inference we can neither infer that flying penguins are birds nor can we infer that flying penguins are not birds.

On the other hand, c-inference allows the plausible conclusion that flying birds are not penguins, i.e., we have $bf \mathrel{|\!\sim}^{c}_{\mathcal{R}_{bird}} \overline{p}$. For a first illustration, to see that the conditional $(\overline{p} \mid bf)$ is accepted by the three OCFs given in Example 4, observe that $\kappa_{\vec{\eta}^{(1)}} \models (\overline{p} \mid bf)$ since $\kappa_{\vec{\eta}^{(1)}}(\overline{p}bf) = 0 < 2 = \kappa_{\vec{\eta}^{(1)}}(pbf)$, $\kappa_{\vec{\eta}^{(2)}} \models (\overline{p} \mid bf)$ since $\kappa_{\vec{\eta}^{(2)}}(\overline{p}bf) = 0 < 2 = \kappa_{\vec{\eta}^{(2)}}(pbf)$, and $\kappa_{\vec{\eta}^{(3)}} \models (\overline{p} \mid bf)$ since $\kappa_{\vec{\eta}^{(3)}}(\overline{p}bf) = 0 < 3 = \kappa_{\vec{\eta}^{(3)}}(pbf)$. More generally, the conditional $(\overline{p} \mid bf)$ is accepted by all c-representations of $\mathcal{R}_{bird}$ since we have $\kappa_{\vec{\eta}}(\overline{p}bf) = 0$ and $\kappa_{\vec{\eta}}(pbf) = \eta_2 + \eta_3$, for every solution $\vec{\eta}$ of the CSP because of (4) and Table 2. From the system of inequalities (11) in Example 3 we obtain that $\eta_2 + \eta_3 > \eta_1 > 0$. Therefore $\kappa_{\vec{\eta}}(\overline{p}bf) < \kappa_{\vec{\eta}}(pbf)$ which implies $\kappa_{\vec{\eta}} \models (\overline{p} \mid bf)$ for all solutions $\vec{\eta}$, by which we obtain $bf \mathrel{|\!\sim}^{c}_{\mathcal{R}_{bird}} \overline{p}$ using Definition 5.

Another c-inference of $\mathcal{R}_{bird}$ is that non-flying penguins are birds, i.e., we have $p\overline{f} \mathrel{|\!\sim}^{c}_{\mathcal{R}_{bird}} b$: According to (4) and Table 2 we have $\kappa(pb\overline{f}) = \eta_1$ and $\kappa(p\overline{b}\,\overline{f}) = \eta_4$. The conditions (11) in Example 3 give us that $\eta_4 > \eta_1$ and therefore we have $p\overline{f} \mathrel{|\!\sim}^{\kappa} b$ by (2) for every c-repesentation κ of $\mathcal{R}_{bird}$ and hence $p\overline{f} \mathrel{|\!\sim}^{c}_{\mathcal{R}_{bird}} b$.

Thus, overall from this example we obtain that for $\mathcal{R}_{bird}$ we have, for instance, $p\overline{f} \mathrel{|\!\sim}^{c}_{\mathcal{R}_{bird}} b$ but neither $pf \not\mathrel{|\!\sim}^{c}_{\mathcal{R}_{bird}} b$ nor $pf \not\mathrel{|\!\sim}^{c}_{\mathcal{R}_{bird}} \overline{b}$.

Comparing Definition 5 to Definition 1, we find that c-inference is defined in full analogy to p-entailment but with the set of OCF that accept $\mathcal{R}$ being restricted to c-representations of $\mathcal{R}$. An obvious question is what the exact relationship between c-inference and p-entailment is, and which features of c-representations still hold for c-inference. First, we show that c-inference satisfies system P but allows for additional inferences.

Proposition 5 *Let $\mathcal{R}$ be a knowledge base and let A, B be formulas. If B can be inferred from A in the context of $\mathcal{R}$ using system P, then it can also be c-inferred from A in the context of $\mathcal{R}$.*

Proof By Definition 1, B can be p-entailed from A in the context of $\mathcal{R}$ if and only if the conditional $(B \mid A)$ is accepted by every OCF that accepts $\mathcal{R}$. Naturally, if $(B \mid A)$ is accepted by all OCFs that accept $\mathcal{R}$, then $(B \mid A)$ is also accepted by every subset of all OCFs that accept $\mathcal{R}$. Since every c-representation accepts $\mathcal{R}$, we obtain that if B is p-entailed from A given $\mathcal{R}$, then it is also c-inferred. Since p-entailment is equivalent to system P inference (cf. Proposition 2) we conclude that every system P inference from $\mathcal{R}$ can also be drawn using c-inference on $\mathcal{R}$. □

To show that c-inference allows for inferences beyond system P we consider the following example.

Example 6 ($\mathcal{R}'_{bird}$) We use the knowledge base

$$\mathcal{R}'_{bird} = \{\delta_1 : (f \mid b),\ \delta_4 : (b \mid p)\}$$

which is a proper subset of $\mathcal{R}_{bird}$ from Example 1. For each impact vector $\vec{\eta} = (\eta_1, \eta_4)$ for $\mathcal{R}'_{bird}$, we obtain the inequalities $\eta_1 > 0$ and $\eta_4 > 0$ by the verification/falsification behavior from Table 6, implying

$$\kappa_{\vec{\eta}}(pf) = \min\{\kappa_{\vec{\eta}}(pbf), \kappa_{\vec{\eta}}(p\overline{b}f)\} = \min\{0, \eta_4\} = 0$$
$$\kappa_{\vec{\eta}}(p\overline{f}) = \min\{\kappa_{\vec{\eta}}(pb\overline{f}), \kappa_{\vec{\eta}}(p\overline{b}\,\overline{f})\} = \min\{\eta_1, \eta_4\} > 0$$

and hence $\kappa_{\vec{\eta}}(pf) < \kappa_{\vec{\eta}}(p\overline{f})$ for the OCF induced by $\vec{\eta}$. Thus, $\kappa(pf) < \kappa(p\overline{f})$ for every c-representation κ of $\mathcal{R}'_{bird}$, giving us $p \mathrel{|\!\sim}^{c}_{\mathcal{R}'_{bird}} f$. Note that this inference is reasonable with respect to $\mathcal{R}'_{bird}$, since $\mathcal{R}'_{bird}$ does not contain any information that can inhibit this chaining of rules.

Proposition 6 *There are knowledge bases $\mathcal{R}$ and propositions A, B such that B is c-entailed, but not p-entailed, from A in the context of $\mathcal{R}$.*

Proof $\mathcal{R}'_{bird}$ from Example 6 is an example for such a knowledge base. Here, we have $p \mathrel{|\!\sim}^{c}_{\mathcal{R}'_{bird}} f$. From Proposition 1 we obtain that if we had $p \mathrel{|\!\sim}^{p}_{\mathcal{R}'_{bird}} f$, then $\mathcal{R}'_{bird} \cup \{(\overline{f} \mid p)\}$ would be inconsistent. This is not the case since $\mathcal{R}'_{bird} \cup \{(\overline{f} \mid p)\}$ is consistent (e.g., with the tolerance partitioning $(\{(f \mid b)\}, \{(\overline{f} \mid p), (b \mid p)\}))$, which gives us that $(f \mid p)$ is c-entailed, but not p-entailed from $\mathcal{R}'_{bird}$. □

From these two propositions we conclude:

Corollary 2 *Every system P entailment of a knowledge base $\mathcal{R}$ is also a c-inference of $\mathcal{R}$; the converse is not true in general.*

Table 6 Verification/falsification behavior and abstract weights for $\mathcal{R}'_{bird}$ from Example 6 and the OCF $\kappa_{(\eta_1,\eta_4)}$ induced by an impact vector (η_1, η_4)

	pbf	$pb\overline{f}$	$p\overline{b}f$	$p\overline{b}\,\overline{f}$	$\overline{p}bf$	$\overline{p}b\overline{f}$	$\overline{p}\,\overline{b}f$	$\overline{p}\,\overline{b}\,\overline{f}$
$\delta_1 = (f \mid b)$	+	−			+	−		
$\delta_4 = (b \mid p)$	+	+	−	−				
$\kappa_{(\eta_1,\eta_4)}(\omega)$	0	η_1	η_4	η_4	0	η_1	0	0

Since Corollary 2 gives us that c-inference includes and extends system P inferences, c-inference satisfies all properties included in system P. We here recall some major properties that can be derived from system P and hence also hold for c-inference ($A, B, C \in \mathfrak{L}$):

(SCL) Supraclassicality $A \models B$ implies $A \mathrel{|\!\!\sim} B$ [28]

states that the set of nonmonotonic inferences from a formula includes the classical consequences of this formula; (SCL) follows from (REF) and (RW).

(DED) Deduction $AB \mathrel{|\!\!\sim} C$ implies $A \mathrel{|\!\!\sim} (B \Rightarrow C)$ [12]

introduces the so-called "tough half of the deduction theorem" to nonmonotonic inference relations; (DED) follows from (LLE), (OR), (SCL) and (RW).

(AND) $A \mathrel{|\!\!\sim} B$ and $A \mathrel{|\!\!\sim} C$ implies $A \mathrel{|\!\!\sim} BC$ [25]

states that if from a formula ϕ two formulas can be inferred, the conjunction of both formulas can also be inferred; (AND) is a consequence of (CM), (CUT) and (SCL).

If two conclusions of the same premise can be joined conjunctively using (AND), one might ask whether this is also valid for two premises with the same conclusion. This property is formalized by the following property which, however, cannot be derived in system P:

(CI) Conjunctive Insistence $A \mathrel{|\!\!\sim} C$ and $B \mathrel{|\!\!\sim} C$ implies $AB \mathrel{|\!\!\sim} C$ [11]

Conjunctive insistence states that (like with (AND) in the conclusion) the conjunction of two formulas that allow for an identical inference still allows for this inference.

The following proposition shows that also c-inference does not satisfy (CI) in general:

Proposition 7 *There are knowledge bases such that c-inference violates (CI).*

We give an example for such a knowledge base.

Example 7 We use the knowledge base

$$\mathcal{R} = \{\delta_1 : (a\overline{b} \vee \overline{a}b \vee \overline{c} \mid \top),\ \delta_2 : (c \mid \top)\}$$

with the verification/falsification behavior shown in Table 7. The system of inequalities (4) for this knowledge base reduces to $\eta_1 > 0$ and $\eta_2 > 0$, and so for any solution vector $\vec{\eta}$, the ranks $\kappa_{\vec{\eta}}(\omega)$ of the worlds are calculated as given in the last line in Table 7. Regardless of the concrete values for η_1 and η_2 this gives us the following relations for $\kappa = \kappa_{\vec{\eta}}$:

$$\kappa(a\overline{b}c) = 0 < \eta_1 = \kappa(abc) \tag{18}$$

$$\kappa(a\overline{b}c) = 0 < \eta_2 = \kappa(a\overline{b}\overline{c}) \tag{19}$$

$$\kappa(a\overline{b}c) = 0 < \eta_2 = \kappa(ab\overline{c}) \tag{20}$$

Table 7 Verification/falsification and (generic) c-representation for $\mathcal{R}$ in Example 7

ω	abc	$ab\overline{c}$	$a\overline{b}c$	$a\overline{b}\overline{c}$	$\overline{a}bc$	$\overline{a}b\overline{c}$	$\overline{a}\overline{b}c$	$\overline{a}\overline{b}\overline{c}$
Verifies	δ_2	δ_1	δ_1, δ_2	δ_1	δ_1, δ_2	δ_1	δ_2	δ_1
Falsifies	δ_1	δ_2	—	δ_2	—	δ_2	δ_1	δ_2
$\kappa_{\vec{\eta}}(\omega)$	η_1	η_2	0	η_2	0	η_2	η_1	η_2

Therefore

$$\kappa(a\overline{b}c) = \underbrace{\min\{\kappa(abc), \kappa(a\overline{b}c)\} < \min\{\kappa(ab\overline{c}), \kappa(a\overline{b}\overline{c})\}}_{(21.a)} \tag{21}$$

for all c-representations κ of $\mathcal{R}$, where by the definition of the rank of formulas in (1), (21.a) gives us $\kappa(ac) < \kappa(a\overline{c})$.

Hence, with the definition of κ-entailment in (2), $a \mathrel{|\!\!\sim}^{\kappa} c$ for all c-representations κ of $\mathcal{R}$ and therefore $a \mathrel{|\!\!\sim}^{c}_{\mathcal{R}} c$. Likewise, from Table 7 we obtain the following relations:

$$\kappa(\overline{a}bc) = 0 < \eta_1 = \kappa(abc) \tag{22}$$

$$\kappa(\overline{a}bc) = 0 < \eta_2 = \kappa(\overline{a}b\overline{c}) \tag{23}$$

$$\kappa(\overline{a}bc) = 0 < \eta_2 = \kappa(ab\overline{c}) \tag{24}$$

Therefore

$$\kappa(\overline{a}bc) = \underbrace{\min\{\kappa(abc), \kappa(\overline{a}bc)\} < \min\{\kappa(ab\overline{c}), \kappa(\overline{a}b\overline{c})\}}_{(25.b)} \tag{25}$$

and with the same justification as above, (25.b) gives us $b \mathrel{|\!\!\sim}^{c}_{\mathcal{R}} c$. From Table 7 we further obtain $\kappa(abc) = \eta_1$ and $\kappa(ab\overline{c}) = \eta_2$. The CSP $CR(\mathcal{R})$ does not determine a relation between η_1 and η_2, so solutions with $\eta_1 > \eta_2$, $\eta_1 = \eta_2$ or $\eta_1 < \eta_2$ are valid solutions and therefore there are c-representations for $\mathcal{R}$ such that $\kappa(abc) \not< \kappa(ab\overline{c})$, giving us $ab \not\mathrel{|\!\!\sim}^{c}_{\mathcal{R}} c$ in contrast to the property (CI).

We have shown that c-inference exceeds system P. In the following we examine benchmarks for plausible inference relations, namely Subclass Inheritance, Irrelevance, and Rule Chaining, which we illustrate using the following modification of the running example.

Example 8 ($\mathcal{R}^*_{bird}$) We extend the alphabet $\Sigma = \{p, b, f\}$ of our running example knowledge base $\mathcal{R}_{bird}$ from Example 1 with the variable w for *having wings* (w) or not ($\overline{w}$), the variable a for *being airborne*(a) or not ($\overline{a}$), and the variable r for *being red* (r) or not ($\overline{r}$) to obtain the alphabet $\Sigma^* = \{p, b, f, w, a, r\}$. We use the knowledge base

$$\mathcal{R}^*_{bird} = \left\{\delta_1 : (f \mid b),\ \delta_2 : (\overline{f} \mid p),\ \delta_4 : (b \mid p),\ \delta_5 : (w \mid b),\ \delta_6 : (a \mid f)\right\}$$

where the conditional $\delta_5 = (w \mid b)$ encodes the rule that birds usually have wings, and the conditional $\delta_6 = (a \mid f)$ encodes the rule that flying things are usually airborne; the other three conditionals $\delta_1, \delta_2, \delta_4$ are the same as in $\mathcal{R}_{bird}$. The verification / falsification behavior of the worlds for the knowledge base $\mathcal{R}^*_{bird}$ is given in Table 8. For each impact vector $\vec{\eta} = (\eta_1, \eta_2, \eta_4, \eta_5, \eta_6)$ for $\mathcal{R}^*_{bird}$, the constraints defined by (5) and (6) give us the following system of inequations:

$$\eta_1 > 0 \tag{26}$$

$$\eta_2 > \min\{\eta_1, \eta_4\} \tag{27}$$

$$\eta_4 > \min\{\eta_1, \eta_2\} \tag{28}$$

$$\eta_5 > 0 \tag{29}$$

$$\eta_6 > 0. \tag{30}$$

If we assume $\eta_1 \geqslant \eta_2$ then (28) would give us $\eta_4 > \eta_2$ which would imply that $\eta_1 < \eta_4$ by (27). But then, (27) would also require $\eta_1 < \eta_2$ in contradiction to the assumption. Therefore, we conclude $\eta_1 \ngeqslant \eta_2$ and hence $\eta_1 < \eta_2$, which gives us the inequalities

$$\begin{aligned} \eta_1 &> 0 \\ \eta_2 &> \eta_1 \\ \eta_4 &> \eta_1 \\ \eta_5 &> 0 \\ \eta_6 &> 0. \end{aligned} \tag{31}$$

An inference relation suffers from the *Drowning Problem* [2, 32] if it does not allow to infer properties of a superclass for a subclass that is exceptional with respect to another property because the respective conditional is "drowned" by others. E.g., penguins are exceptional birds with respect to flying but not with respect to having wings. So we would reasonably expect that penguins have wings. However, system Z is known to suffer from the Drowning Problem, as the following example shows.

Example 9 (Drowning Problem) System Z partitions the knowledge base $\mathcal{R}^*_{bird}$ of Example 8 into $(\mathcal{R}_0 = \{\delta_1, \delta_5, \delta_6\}, \mathcal{R}_1 = \{\delta_2, \delta_4\})$ which results in the ranking function κ^Z given in Table 8 (rightmost columns). Here we have $\kappa^Z(pw) = 1 = \kappa^Z(p\overline{w})$ and therefore we cannot infer whether penguins have wings: Every world ω with $\omega \models p$ falsifies a conditional (cf. Table 8), and the minimal rank of every world satisfying p is 1. Since the conditional $(w \mid b)$ is in $\mathcal{R}_0$ and system Z always takes this maximal index of the partitions containing conditionals falsified by ω, this conditional will never contribute to the rank of any such world ω; its effect is "drowned" in the effects of the other conditionals.

The Drowning Problem distinguishes between inference relations that allow for Subclass Inheritance only for non-exceptional subclasses (like System Z inference) and inference relations that allow for subclass inheritance for exceptional subclasses (like inference with minimal c-representations, cf. [24, 36]). Here we show that this property is preserved by c-inference, the skeptical inference over all c-representations.

Observation 1 *Skeptical c-inference does not suffer from the Drowning Problem in Example 8.*

Proof From the observations for c-representations of $\mathcal{R}^*_{bird}$ in Example 8 together with Definition 2 we obtain, for each impact vector $\vec{\eta} = (\eta_1, \eta_2, \eta_4, \eta_5, \eta_6)$ for $\mathcal{R}^*_{bird}$ and the correspondingly induced OCF $\kappa_{\vec{\eta}}$,

$$\begin{aligned} \kappa_{\vec{\eta}}(pw) &= \min\{\eta_2, \eta_2 + \eta_6, \eta_1, \eta_2 + \eta_4, \eta_2 + \eta_4 + \eta_6, \eta_4\} \\ &= \eta_1 \end{aligned}$$

according to (31) and

$$\kappa_{\vec{\eta}}(p\overline{w}) = \min\{\underbrace{\eta_2 + \eta_5}_{>\eta_1}, \underbrace{\eta_2 + \eta_5 + \eta_6}_{>\eta_1}, \underbrace{\eta_1 + \eta_5}_{>\eta_1}, \underbrace{\eta_2 + \eta_4}_{>\eta_1}, \underbrace{\eta_2 + \eta_4 + \eta_6}_{>\eta_1}, \underbrace{\eta_4}_{>\eta_1}\}.$$

This implies $\kappa_{\vec{\eta}}(pw) < \kappa_{\vec{\eta}}(p\overline{w})$ for all impact vectors $\vec{\eta}$ of $\mathcal{R}^*_{bird}$, and therefore $\kappa(pw) < \kappa(p\overline{w})$ for all c-representations of $\mathcal{R}^*_{bird}$ and hence $p \mid\!\sim^c_{\mathcal{R}^*_{bird}} w$. That is, in

Table 8 Verification / falsification behavior of the worlds for the knowledge base $\mathcal{R}^*_{bird}$ (Example 8) and ranking function κ^Z obtained from $\mathcal{R}^*_{bird}$ using System Z

ω	Verifies	Falsifies	κ^Z	ω	Verifies	Falsifies	κ^Z
$pbfwar$	$\delta_1, \delta_4, \delta_5, \delta_6$	δ_2	2	$\overline{p}bfwar$	$\delta_1, \delta_5, \delta_6$	—	0
$pbfwa\overline{r}$	$\delta_1, \delta_4, \delta_5, \delta_6$	δ_2	2	$\overline{p}bfwa\overline{r}$	$\delta_1, \delta_5, \delta_6$	—	0
$pbfw\overline{a}r$	$\delta_1, \delta_4, \delta_5$	δ_2, δ_6	2	$\overline{p}bfw\overline{a}r$	δ_1, δ_5	δ_6	1
$pbfw\overline{a}\overline{r}$	$\delta_1, \delta_4, \delta_5$	δ_2, δ_6	2	$\overline{p}bfw\overline{a}\overline{r}$	δ_1, δ_5	δ_6	1
$pbf\overline{w}ar$	$\delta_1, \delta_4, \delta_6$	δ_2, δ_5	2	$\overline{p}bf\overline{w}ar$	δ_1, δ_6	δ_5	1
$pbf\overline{w}a\overline{r}$	$\delta_1, \delta_4, \delta_6$	δ_2, δ_5	2	$\overline{p}bf\overline{w}a\overline{r}$	δ_1, δ_6	δ_5	1
$pbf\overline{w}\overline{a}r$	δ_1, δ_4	$\delta_2, \delta_5, \delta_6$	2	$\overline{p}bf\overline{w}\overline{a}r$	δ_1	δ_5, δ_6	1
$pbf\overline{w}\overline{a}\overline{r}$	δ_1, δ_4	$\delta_2, \delta_5, \delta_6$	2	$\overline{p}bf\overline{w}\overline{a}\overline{r}$	δ_1	δ_5, δ_6	1
$pb\overline{f}war$	$\delta_2, \delta_4, \delta_5$	δ_1	1	$\overline{p}b\overline{f}war$	δ_5	δ_1	1
$pb\overline{f}wa\overline{r}$	$\delta_2, \delta_4, \delta_5$	δ_1	1	$\overline{p}b\overline{f}wa\overline{r}$	δ_5	δ_1	1
$pb\overline{f}w\overline{a}r$	$\delta_2, \delta_4, \delta_5$	δ_1	1	$\overline{p}b\overline{f}w\overline{a}r$	δ_5	δ_1	1
$pb\overline{f}w\overline{a}\overline{r}$	$\delta_2, \delta_4, \delta_5$	δ_1	1	$\overline{p}b\overline{f}w\overline{a}\overline{r}$	δ_5	δ_1	1
$pb\overline{f}\overline{w}ar$	δ_2, δ_4	δ_1, δ_5	1	$\overline{p}b\overline{f}\overline{w}ar$	—	δ_1, δ_5	1
$pb\overline{f}\overline{w}a\overline{r}$	δ_2, δ_4	δ_1, δ_5	1	$\overline{p}b\overline{f}\overline{w}a\overline{r}$	—	δ_1, δ_5	1
$pb\overline{f}\overline{w}\overline{a}r$	δ_2, δ_4	δ_1, δ_5	1	$\overline{p}b\overline{f}\overline{w}\overline{a}r$	—	δ_1, δ_5	1
$pb\overline{f}\overline{w}\overline{a}\overline{r}$	δ_2, δ_4	δ_1, δ_5	1	$\overline{p}b\overline{f}\overline{w}\overline{a}\overline{r}$	—	δ_1, δ_5	1
$p\overline{b}fwar$	δ_6	δ_2, δ_4	2	$\overline{p}\overline{b}fwar$	δ_6	—	0
$p\overline{b}fwa\overline{r}$	δ_6	δ_2, δ_4	2	$\overline{p}\overline{b}fwa\overline{r}$	δ_6	—	0
$p\overline{b}fw\overline{a}r$	—	$\delta_2, \delta_4, \delta_6$	2	$\overline{p}\overline{b}fw\overline{a}r$	—	δ_6	1
$p\overline{b}fw\overline{a}\overline{r}$	—	$\delta_2, \delta_4, \delta_6$	2	$\overline{p}\overline{b}fw\overline{a}\overline{r}$	—	δ_6	1
$p\overline{b}f\overline{w}ar$	δ_6	δ_2, δ_4	2	$\overline{p}\overline{b}f\overline{w}ar$	δ_6	—	0
$p\overline{b}f\overline{w}a\overline{r}$	δ_6	δ_2, δ_4	2	$\overline{p}\overline{b}f\overline{w}a\overline{r}$	δ_6	—	0
$p\overline{b}f\overline{w}\overline{a}r$	—	$\delta_2, \delta_4, \delta_6$	2	$\overline{p}\overline{b}f\overline{w}\overline{a}r$	—	δ_6	1
$p\overline{b}f\overline{w}\overline{a}\overline{r}$	—	$\delta_2, \delta_4, \delta_6$	2	$\overline{p}\overline{b}f\overline{w}\overline{a}\overline{r}$	—	δ_6	1
$p\overline{b}\overline{f}war$	δ_2	δ_4	2	$\overline{p}\overline{b}\overline{f}war$	—	—	0
$p\overline{b}\overline{f}wa\overline{r}$	δ_2	δ_4	2	$\overline{p}\overline{b}\overline{f}wa\overline{r}$	—	—	0
$p\overline{b}\overline{f}w\overline{a}r$	δ_2	δ_4	2	$\overline{p}\overline{b}\overline{f}w\overline{a}r$	—	—	0
$p\overline{b}\overline{f}w\overline{a}\overline{r}$	δ_2	δ_4	2	$\overline{p}\overline{b}\overline{f}w\overline{a}\overline{r}$	—	—	0
$p\overline{b}\overline{f}\overline{w}ar$	δ_2	δ_4	2	$\overline{p}\overline{b}\overline{f}\overline{w}ar$	—	—	0
$p\overline{b}\overline{f}\overline{w}a\overline{r}$	δ_2	δ_4	2	$\overline{p}\overline{b}\overline{f}\overline{w}a\overline{r}$	—	—	0
$p\overline{b}\overline{f}\overline{w}\overline{a}r$	δ_2	δ_4	2	$\overline{p}\overline{b}\overline{f}\overline{w}\overline{a}r$	—	—	0
$p\overline{b}\overline{f}\overline{w}\overline{a}\overline{r}$	δ_2	δ_4	2	$\overline{p}\overline{b}\overline{f}\overline{w}\overline{a}\overline{r}$	—	—	0

difference to system Z (see Example 9), using c-inference we can infer that penguins have wings in the context of $\mathcal{R}^*_{bird}$, even if they are exceptional birds with respect to flying. □

It is straightforward to explain more generally why c-inference does not suffer from a Drowning Problem. C-inference is the skeptical inference of all c-representations of a knowledge base $\mathcal{R}$. These OCFs are set up such that every rank of every world takes the impact of every single conditional into account independently, i.e., a world that falsifies a conditional is usually less plausible than a world that, ceteris paribus, does not falsify this conditional. If we presuppose that all η_i are strictly positive, then we can definitely exclude the Drowning Problem; this means that in c-representations with strictly positive impacts, no conditional can simply "drown" in a set of others.

But even if, as observed above, c-inference does not suffer from the Drowning Problem in the usual examples, in general, this skeptical inference relation does not allow for rational transitive inferences.

(RT) Rational Transitivity $A \mid\!\sim B$, $B \mid\!\sim C$ and $A \mid\!\not\sim \overline{C}$ implies $A \mid\!\sim C$ [13]

Proposition 8 *There are knowledge bases such that c-inference violates (RT).*

We give an example for such a knowledge base.

Example 10 We illustrate that there are knowledge bases such that c-inference violates (RT) with the knowledge base

$$\mathcal{R} = \{\delta_1 : (\overline{a}|\top),\ \delta_2 : (b|a),\ \delta_3 : (\overline{b} \vee c|\overline{a})\}$$

whose verification/falsification behavior is shown in Table 9. The constraint satisfcation problem $CR(\mathcal{R})$ for $\mathcal{R}$ reduces to $\eta_1 > 0$, $\eta_2 > 0$, and $\eta_3 > 0$, with no further constraints on the impacts. So for all c-representations κ of $\mathcal{R}$ we have the inferences $a \mid\!\sim^c_{\mathcal{R}} b$ since $\kappa(ab) = \eta_1 < \eta_1 + \eta_2 = \kappa(a\overline{b})$, $b \mid\!\sim^c_{\mathcal{R}} c$ since $\kappa(bc) = 0 < \min\{\eta_1, \eta_3\} = \kappa(b\overline{c})$, and $a \mid\!\not\sim^c_{\mathcal{R}} \overline{c}$ because $\kappa(a\overline{c}) = \eta_1 = \kappa(ac)$. Because of this equality we also have $a \mid\!\not\sim^c_{\mathcal{R}} c$ in contrast to (RT).

Overall from Observation 1 and Proposition 8 we obtain that using c-inference, it is possible to connect explicitly stated knowledge, but not any possible inferences in a (rational) transitive way.

Another benchmark for plausible reasoning is *Irrelevance*. It is safe to assume that a variable is not relevant for an inference based on a knowledge base if the variable does not appear in any conditional of the knowledge base.

Proposition 9 (c-inference and Irrelevance) *Variables that do not appear in the knowledge base do not change the outcome of the inferences drawn with c-inference.*

Proof Let Σ be a propositional alphabet and $d \in \Sigma$, and let $\mathcal{R}$ be a conditional knowledge base where there is no conditional $(B_i \mid A_i) \in \mathcal{R}$ such that either d or $\overline{d}$ appears in the conjunction $A_i B_i$. Let ω, ω' be a pair of worlds such that $\omega = \mathbf{o} \wedge d$ and $\omega' = \mathbf{o} \wedge \overline{d}$. Since neither d nor $\overline{d}$ is a member of any conjunction $A_i B_i$ of the conditionals $(B_i \mid A_i) \in \mathcal{R}$, the sets of

Table 9 Verification/falsification and (generic) c-representation for $\mathcal{R}$ in Example 10

ω	abc	$ab\overline{c}$	$a\overline{b}c$	$a\overline{b}\overline{c}$	$\overline{a}bc$	$\overline{a}b\overline{c}$	$\overline{a}\overline{b}c$	$\overline{a}\overline{b}\overline{c}$
Verifies	δ_2	δ_2	—	—	δ_1, δ_3	δ_1	δ_1, δ_3	δ_1, δ_3
Falsifies	δ_1	δ_1	δ_1, δ_2	δ_1, δ_2	—	δ_3	—	—
$\kappa_{\vec{\eta}}(\omega)$	η_1	η_1	$\eta_1 + \eta_2$	$\eta_1 + \eta_2$	0	η_3	0	0

conditionals falsified by ω and by ω', respectively, are identical. By Definition 2 this means that $\kappa(\omega) = \kappa(\omega')$. This implies that for every two formulas A, B, which are composed from the language of the alphabet $\Sigma \setminus \{d\}$, and for every configuration $\dot{d}$ of d, the conjunction AB (respectively $A\overline{B}$) falsifies a conditional $(B_i \mid A_i)$ if and only if $AB\dot{d}$ (respectively $A\overline{B}\dot{d}$) falsifies the conditional, and therefore $\kappa(AB) = \min\{\kappa(ABd), \kappa(AB\overline{d})\} = \kappa(AB\dot{d})$ and also $\kappa(A\overline{B}) = \min\{\kappa(A\overline{B}d), \kappa(A\overline{B}\,\overline{d})\} = \kappa(A\overline{B}\dot{d})$. This means for all c-representations κ of $\mathcal{R}$, if $\kappa(AB) < \kappa(A\overline{B})$, then also $\kappa(AB\dot{d}) < \kappa(A\overline{B}\dot{d})$. Thus, if $A \mid\!\sim^{c}_{\mathcal{R}} B$, then also $A\dot{d} \mid\!\sim^{c}_{\mathcal{R}} B$. □

We illustrate the behavior of $\mid\!\sim^{c}_{\mathcal{R}}$ regarding variables that are not relevant using the knowledge base $\mathcal{R}^{*}_{bird}$ from Example 8:

Example 11 (c-inference and Irrelevance) Table 8 gives us that the behavior of all worlds ω for $\mathcal{R}^{*}_{bird}$ with $\omega \models r$ is, ceteris paribus, identical to the behavior of all worlds ω with $\omega \models \overline{r}$. Thus, we conclude directly that for all fixed configurations $\dot{p}, \dot{b}, \dot{f}, \dot{w}, \dot{a}$ of $\{p, b, f, w, a\}$, we have $\kappa(\dot{p}\dot{b}\dot{f}\dot{w}\dot{a}r) = \kappa(\dot{p}\dot{b}\dot{f}\dot{w}\dot{a}\overline{r})$. This means that, for instance, since in the context of $\mathcal{R}^{*}_{bird}$ we can infer that birds can fly ($b \mid\!\sim^{c}_{\mathcal{R}^{*}_{bird}} f$), we can also infer that red birds can fly ($br \mid\!\sim^{c}_{\mathcal{R}^{*}_{bird}} f$).

Combining the conditionals in a knowledge base by Rule Chaining is a natural element of plausible reasoning and is, e.g., the base of syllogisms. However, we know that transitivity is not a general inference rule in nonmonotonic logics. But we would expect that chaining rules yields plausible inferences as long as there is no reason to believe the opposite.

Example 12 (c-inference and chaining rules) We use again the knowledge base $\mathcal{R}^{*}_{bird}$ from Example 8. Given that we have $(f \mid b)$ and $(a \mid f)$ in the knowledge base, and no interference between b and a , we would expect that chaining these rules is reasonable and that we can infer that birds are usually airborne. With Table 8 and (31) in Example 8 we obtain that $\kappa(b\overline{a}) = \min\{\eta_1, \eta_6\} > 0 = \kappa(ba)$ and hence $b \mid\!\sim^{c}_{\mathcal{R}^{*}_{bird}} a$, as supposed.

Note that Example 12 illustrates that c-inference does not rely on a total, but a partial ordering of the impacts imposed by Eqs. (5) and (6) and therefore, via Definition 2, a partial ordering of the worlds for drawing inferences. In the example, nothing can be derived about the concrete values of η_1 and η_6 except that they are positive (see (31)). This is sufficient to guarantee the considered skeptical inference.

5 Characterizing c-inference by a CSP

For a given OCF κ, the relation $\kappa \models (B \mid A)$ can be checked by determining $\kappa(AB)$ and $\kappa(A\overline{B})$ according to (2). For checking the skeptical c-inference relation $A \mid\!\sim^{c}_{\mathcal{R}} B$ the countably infinite set of all c-representations for $\mathcal{R}$ has to be taken into account. In the previous section, we showed that such a c-inference can not be reduced to the inconsistency of $\mathcal{R} \cup \{(\overline{B} \mid A)\}$ which is a consequence of Proposition 6. In the following, we will show that the relation $\mid\!\sim^{c}_{\mathcal{R}}$ can be characterized by a constraint satisfaction problem, implying that $\mid\!\sim^{c}_{\mathcal{R}}$ can be computed using a constraint-based approach.

Definition 6 (**$CR_{\mathcal{R}}(B \mid A)$, $\neg CR_{\mathcal{R}}(B \mid A)$**) Let $\mathcal{R} = \{(B_1 \mid A_1), \ldots, (B_n \mid A_n)\}$ and $(B \mid A)$ be a conditional. The *acceptance constraint for* $(B \mid A)$ *with respect to* $\mathcal{R}$, denoted by $CR_{\mathcal{R}}(B \mid A)$, is the following constraint:

$$\min_{\omega \models AB} \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \eta_i \; < \; \min_{\omega \models A\overline{B}} \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \eta_i \tag{32}$$

Likewise, $\neg CR_{\mathcal{R}}(B \mid A)$ denotes the negation of (32), i.e., it denotes the constraint:

$$\min_{\omega \models AB} \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \eta_i \; \geqslant \; \min_{\omega \models A\overline{B}} \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \eta_i \tag{33}$$

Note that $CR_{\mathcal{R}}(B \mid A)$ is a constraint on the constraint variables $\eta_1, \ldots, \eta_n$ which are used in the CSP $CR(\mathcal{R})$, but it does not introduce any new variables not already occurring in $CR(\mathcal{R})$; this observation also holds for the constraint $\neg CR_{\mathcal{R}}(B \mid A)$.

The following proposition shows that the skeptical c-inference relation $\mid\!\sim^{c}_{\mathcal{R}}$ can be modeled by a CSP.

Proposition 10 (c-inference as a CSP) *Let* $\mathcal{R} = \{(B_1 \mid A_1), \ldots, (B_n \mid A_n)\}$ *be a consistent knowledge base and* A, B *formulas.*

Then the following holds:

$$A \mid\!\sim^{c}_{\mathcal{R}} B \;\; \textit{iff} \;\; CR(\mathcal{R}) \cup \{\neg CR_{\mathcal{R}}(B \mid A)\} \textit{ has no solution.} \tag{34}$$

Proof Assume that $A \mid\!\sim^{c}_{\mathcal{R}} B$ holds, i.e., $\kappa \models (B \mid A)$ holds for all c-representations κ for $\mathcal{R}$. If $CR(\mathcal{R}) \cup \{\neg CR_{\mathcal{R}}(B \mid A)\}$ were solvable with a solution $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ then $\kappa_{\vec{\eta}} \models \mathcal{R}$ according to Prop. 3 where

$$\kappa_{\vec{\eta}}(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \eta_i \tag{35}$$

due to (4). Furthermore, since $\vec{\eta}$ also solves $\neg CR_{\mathcal{R}}(B \mid A)$, (33) holds. Applying (35) to (33) yields

$$\min_{\omega \models AB} \kappa_{\vec{\eta}}(\omega) \; \geqslant \; \min_{\omega \models A\overline{B}} \kappa_{\vec{\eta}}(\omega) \tag{36}$$

and further applying (1) to (36) yields

$$\kappa_{\vec{\eta}}(AB) \geqslant \kappa_{\vec{\eta}}(A\overline{B}). \tag{37}$$

Using (2), this implies

$$\kappa_{\vec{\eta}} \not\models (B \mid A), \tag{38}$$

contradicting the assumption $A \mid\!\sim^{c}_{\mathcal{R}} B$. Thus, $CR(\mathcal{R}) \cup \{\neg CR_{\mathcal{R}}(B \mid A)\}$ does not have a solution.

For the other direction, we use contraposition and assume that $A \not\mid\!\sim^{c}_{\mathcal{R}} B$ holds. Therefore, since $\mathcal{R}$ is consistent, there is a c-representation κ with $\kappa \models \mathcal{R}$ and $\kappa \not\models (B \mid A)$. According to Prop. 4, there is a solution $\vec{\eta} = (\eta_1, \ldots, \eta_n) \in Sol(CR(\mathcal{R}))$ such that $\kappa = \kappa_{\vec{\eta}}$. From $\kappa_{\vec{\eta}} \not\models (B \mid A)$ we get the following:

$$\kappa_{\vec{\eta}}(AB) \geqslant \kappa_{\vec{\eta}}(A\overline{B}) \tag{39}$$

Applying (1) to (39) yields

$$\min\{\kappa_{\vec{\eta}}(\omega) \mid \omega \models AB\} \; \geqslant \; \min\{\kappa_{\vec{\eta}}(\omega) \mid \omega \models A\overline{B}\} \tag{40}$$

and further applying (4) to (40) yields

$$\min\{\sum_{\substack{1\leqslant i\leqslant n\\ \omega\models A_i\overline{B}_i}} \eta_i \mid \omega \models AB\} \geqslant \min\{\sum_{\substack{1\leqslant i\leqslant n\\ \omega\models A_i\overline{B}_i}} \eta_i \mid \omega \models A\overline{B}\} \tag{41}$$

which is equivalent to (33). Thus, $CR(\mathcal{R}) \cup \{\neg CR_{\mathcal{R}}(B \mid A)\}$ has a solution, completing the proof. □

In Section 4 we already discussed that c-inference and p-entailment are defined in analogy, but using the set of all c-representations of a knowledge base $\mathcal{R}$ rather than the set of all OCFs that accept $\mathcal{R}$ when defining inference leads to the differences shown above. While our CSP modeling of the inference closely resembles the characterization of p-entailment given in Proposition 1, there is a major difference: while the characterization in Proposition 1 tests whether an augmented knowledge base is consistent, the characterization in Proposition 10 tests for the solvability of an augmentation of the CSP specifying the c-representations of the knowledge base. If we compare both approaches, Corollary 1 gives us that $\mathcal{R} \cup (\overline{B} \mid A)$ is consistent if and only if $CR(\mathcal{R} \cup (\overline{B} \mid A))$ has a solution, hence the nonexistence of a solution of $CR(\mathcal{R} \cup (\overline{B} \mid A))$ is, by Proposition 1, equivalent to the question whether the entailment $A \mathrel{|\!\!\sim}^{p}_{\mathcal{R}} B$ holds.

Since we have shown that c-inference $A \mathrel{|\!\!\sim}^{c}_{\mathcal{R}} B$ is characterized by $CR(\mathcal{R}) \cup \{\neg CR_{\mathcal{R}}(B \mid A)\}$ not being solvable in Proposition 10, and that c-inference exceeds system P inference in Corollary 2, we conclude:

Corollary 3 *Let $\mathcal{R}$ be a conditional knowledge base and let A, B be formulas. If $CR(\mathcal{R} \cup (\overline{B} \mid A))$ does not have a solution, then $CR(\mathcal{R}) \cup \{\neg CR_{\mathcal{R}}(B \mid A)\}$ does not have a solution; the converse is not true in general.*

6 Using constraints over finite domains

In this section, we will sharpen the CSPs for c-representations and for c-inference to CSPs over finite domains. Note that if the knowledge base $\mathcal{R}$ is consistent, there are in general infinitely many solution vectors in $Sol(CR(\mathcal{R}))$, inducing an infinite set of c-representations accepting $\mathcal{R}$. Therefore, our finite domains approach will represent c-representations up to inferential equivalence. In order to make this precise, we first introduce the following notion.

Definition 7 (self-fulfilling) A conditional $(B \mid A)$ with $A \models B$ is called a *self-fulfilling* conditional.

Thus, a conditional is self-fulfilling if it can not be falsified by any world. Examples of self-fulfilling conditionals are $(a \vee b \mid a)$, $(b \mid ab)$, and $(\top \mid a)$, but also the self-fulfilling conditionals $(a \mid \bot)$ and $(\top \mid \bot)$ that can be neither falsified nor verified by any world. The next proposition elaborates how self-fulfilling conditionals in a consistent knowledge base $\mathcal{R}$ influence the set of solution vectors in $Sol(CR(\mathcal{R}))$ and the set of c-representations being a model of $\mathcal{R}$. Recall that $\mathcal{O}(CR(\mathcal{R}))$ is the set of OCFs induced by solutions of $Sol(CR(\mathcal{R}))$.

Proposition 11 *Let $\mathcal{R} = \{(B_1 \mid A_1), \ldots, (B_n \mid A_n)\}$ be a consistent knowledge base, and let κ_{uni} be the uniform OCF with $\kappa_{\mathrm{uni}}(\omega) = 0$ for all $\omega \in \Omega$.*

1. *If $\mathcal{R}$ is empty, then:*

$$Sol(CR(\mathcal{R})) = \{()\} \tag{42}$$

$$\mathcal{O}(CR(\mathcal{R})) = \{\kappa_{\mathrm{uni}}\} \tag{43}$$

2. *If $\mathcal{R}$ is non-empty and all conditionals in $\mathcal{R}$ are self-fulfilling, then:*

$$|Sol(CR(\mathcal{R}))| = \infty \tag{44}$$

$$\mathcal{O}(CR(\mathcal{R})) = \{\kappa_{\mathrm{uni}}\} \tag{45}$$

3. *If $\mathcal{R}$ is non-empty and $\mathcal{R}$ contains at least one conditional that is not self-fulfilling, then:*

$$|Sol(CR(\mathcal{R}))| = \infty \tag{46}$$

$$|\mathcal{O}(CR(\mathcal{R}))| = \infty \tag{47}$$

4. *If $(B_i \mid A_i) \in \mathcal{R}$ is self-fulfilling, then:*

$$\mathcal{O}(CR(\mathcal{R})) = \mathcal{O}(CR(\mathcal{R} \setminus \{(B_i \mid A_i))\}) \tag{48}$$

Proof If $\mathcal{R}$ is empty, then (42) holds trivially as the empty vector is the only solution of $CR(\mathcal{R})$, and (43) holds since $\kappa_{()} = \kappa_{\mathrm{uni}}$ due to (4).

For the rest of the proof, we will use the following observation:

$$\text{If } \vec{\eta} = (\eta_1, \ldots, \eta_n) \in Sol(CR(\mathcal{R})) \text{ and } k \in \mathbb{N}, k \geqslant 1,$$
$$\text{then } k \cdot \vec{\eta} = (k \cdot \eta_1, \ldots, k \cdot \eta_n) \in Sol(CR(\mathcal{R})). \tag{49}$$

This observation holds since the inequations of $CR(\mathcal{R})$ satisfied by $\vec{\eta}$ are still satisfied when they are all multiplied by k.

Thus, for non-empty $\mathcal{R}$, Eqs. (44) and (46) hold provided that $Sol(CR(\mathcal{R}))$ is not just a singleton set containing the vector containing only 0, i.e., provided that $Sol(CR(\mathcal{R})) \neq \{\vec{\eta}^{\,0}\}$ where $\vec{\eta}^{\,0} = (0, \ldots, 0)$. So let us assume that $\vec{\eta}^{\,0}$ was the only element in $Sol(CR(\mathcal{R}))$. We have $\kappa_{\vec{\eta}^{\,0}} = \kappa_{\mathrm{uni}}$ according to (4). Together with the acceptance condition $\kappa_{\vec{\eta}^{\,0}}(A_i B_i) < \kappa_{\vec{\eta}^{\,0}}(A_i \overline{B_i})$ for the conditional $(B_i \mid A_i)$, this implies $\kappa_{\vec{\eta}^{\,0}}(A_i B_i) = 0$ and $\kappa_{\vec{\eta}^{\,0}}(A_i \overline{B_i}) = \infty$. Thus, $A_i B_i$ is satisfiable, while $A_i \overline{B_i}$ is unsatisfiable and therefore $A_i \models B_i$ so that $(B_i \mid A_i)$ is self-fulfilling. For every self-fulfilling conditional $(B_i \mid A_i) \in \mathcal{R}$, the inequation (6) in $CR(\mathcal{R})$ reduces to

$$\eta_i > \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j \;-\; \min_{\emptyset} 0 \tag{50}$$

since there is no ω with $\omega \models A_i \overline{B_i}$. Because $A_i B_i$ is satisfiable and every $\eta_j \in \mathbb{N}$, the first minimization term in (50) yields a finite number, and since for the second minimization term we have $\min_{\emptyset} 0 = \infty$, (50) further reduces to

$$\eta_i > -\infty \tag{51}$$

which in turn can be removed from $CR(\mathcal{R})$ since it is implied by (5). Therefore, since all conditionals in $\mathcal{R}$ are self-fulfilling if $Sol(CR(\mathcal{R})) = \{\vec{\eta}^{\,0}\}$, $CR(\mathcal{R})$ is equivalent to the simple set of equations

$$\eta_i \geqslant 0 \tag{52}$$

for $i \in \{1, \ldots, n\}$. Now for any $i \in \{1, \ldots, n\}$, the vector $(0, \ldots, 1, \ldots, 0)$ obtained from $\vec{\eta}^{\,0}$ by replacing the i-th component by 1, is another trivial solution of $CR(\mathcal{R})$ as given by (52), contradicting the assumption that $\vec{\eta}^{\,0}$ is the only element in $Sol(CR(\mathcal{R}))$ and thus completing the proof of (44) and (46).

The derivations above show that if all conditionals in $\mathcal{R}$ are self-fulfilling, $CR(\mathcal{R})$ is equivalent to the constraints given by (52); hence any vector $\vec{\eta} = (t_1, \ldots, t_n) \in \mathbb{N}^n$ is a solution of $CR(\mathcal{R})$. For any such $\vec{\eta}$, we get $\kappa_{\vec{\eta}} = \kappa_{\text{uni}}$ due to (4) since no world falsifies any conditional in $\mathcal{R}$, implying (45).

For showing (47), let $(B_i \mid A_i) \in \mathcal{R}$ be a not self-fulfilling conditional. Then there is a world ω falsifying $(B_i \mid A_i)$, and for every $\vec{\eta} \in Sol(CR(\mathcal{R}))$, we must have $\kappa_{\vec{\eta}}(\omega) > 0$. For every $k \in \mathbb{N}$, $k > 1$, the observation (49) implies $\kappa_{k \cdot \vec{\eta}}(\omega) = k \cdot \kappa_{\vec{\eta}}(\omega) \neq \kappa_{\vec{\eta}}(\omega)$. Thus, there are infinitely many different OCFs in $\mathcal{O}(CR(\mathcal{R}))$, implying (47).

For showing (48), let $(B_i \mid A_i) \in \mathcal{R}$ be a self-fulfilling conditional. As shown above, the constraint for η_i stemming from (6) in $CR(\mathcal{R})$ reduces to simply $\eta_i \geqslant 0$. Since there is no world falsifying $(B_i \mid A_i)$, η_i does not occur in the constraints (6) for any of the other η_j with $i \neq j$, nor in the equation (4) for any ω. Hence, for every $\vec{\eta} = (\eta_1, \ldots, \eta_n) \in Sol(CR(\mathcal{R}))$, we have $\vec{\eta}\,' = (\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_n) \in Sol(\mathcal{R} \setminus \{(B_i \mid A_i)\})$ and $\kappa_{\vec{\eta}}(\omega) = \kappa_{\vec{\eta}'}(\omega)$ for every $\omega \in \Omega$, implying (48) and completing the proof. □

OCFs like $\kappa_{\vec{\eta}}$ and $\kappa_{k \cdot \vec{\eta}}$ as used in the proof of Proposition 11 are examples of ranking functions that have identical inference behaviour.

Definition 8 ($\equiv_{\mid\sim}$) Two ranking functions κ, κ' are *inferentially equivalent*, denoted by $\kappa \equiv_{\mid\sim} \kappa'$ if for all $(B \mid A)$ it is the case that $\kappa \models (B \mid A)$ if and only if $\kappa' \models (B \mid A)$.

For instance, we have $\kappa_{\vec{\eta}} \equiv_{\mid\sim} \kappa_{k \cdot \vec{\eta}}$; in general, two ranking functions are inferentially equivalent if and only if they induce the same total preorder on worlds.

Proposition 12 ($\equiv_{\mid\sim}$) *For ranking functions κ and κ', we have $\kappa \equiv_{\mid\sim} \kappa'$ if and only if for all $\omega_1, \omega_2 \in \Omega$ it is the case that $\kappa(\omega_1) \leqslant \kappa(\omega_2)$ iff $\kappa'(\omega_1) \leqslant \kappa'(\omega_2)$.*

Proof Assume $\kappa(\omega_1) \leqslant \kappa(\omega_2)$ iff $\kappa'(\omega_1) \leqslant \kappa'(\omega_2)$.

$$\begin{aligned}
&\kappa \models (B \mid A) \\
&\text{iff } \kappa(AB) < \kappa(A\overline{B}) \\
&\text{iff } \min\{\kappa(\omega) \mid \omega \models AB\} \leqslant \min\{\kappa(\omega) \mid \omega \models A\overline{B}\} \\
&\text{iff } \min\{\kappa'(\omega) \mid \omega \models AB\} \leqslant \min\{\kappa'(\omega) \mid \omega \models A\overline{B}\} \\
&\text{iff } \kappa'(AB) < \kappa'(A\overline{B}) \\
&\text{iff } \kappa' \models (B \mid A)
\end{aligned}$$

To show the other direction assume $\kappa \equiv_{\mid\!\sim} \kappa'$ and let $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$. The following derivation exploits the fact that ω is logically equivalent to $(\omega \vee \omega') \wedge \omega$ for any two worlds ω and ω'.

$$\begin{aligned}
&\kappa(\omega_1) < \kappa(\omega_2) \\
\text{iff } &\kappa((\omega_1 \vee \omega_2) \wedge \omega_1) < \kappa((\omega_1 \vee \omega_2) \wedge \overline{\omega_1}) \\
\text{iff } &\kappa \models (\omega_1 \mid \omega_1 \vee \omega_2) \\
\text{iff } &\kappa' \models (\omega_1 \mid \omega_1 \vee \omega_2) \\
\text{iff } &\kappa'((\omega_1 \vee \omega_2) \wedge \omega_1) < \kappa'((\omega_1 \vee \omega_2) \wedge \overline{\omega_1}) \\
\text{iff } &\kappa'(\omega_1) < \kappa'(\omega_2)
\end{aligned}$$

□

Example 13 ($\mathcal{R}_{bird}$, cont'd) Recall the knowledge base $\mathcal{R}_{bird}$ from Example 1. The two impact vectors $\vec{\eta}^{(1)} = (1, 1, 1, 2)$ and $\vec{\eta}^{(2)} = (1, 2, 0, 2)$ given in Example 3 induce different ranking functions (given in Table 3) that are inferentially equivalent, i.e. $\kappa_{\vec{\eta}^{(1)}} \equiv_{\mid\!\sim} \kappa_{\vec{\eta}^{(2)}}$.

It is obvious that $\equiv_{\mid\!\sim}$ is an equivalence relation on OCFs. For any set $\mathcal{O}$ of OCFs, the set of equivalence classes induced by $\mathcal{O}$ will be denoted by $\mathcal{O}_{/\equiv_{\mid\!\sim}}$. Although there are in general both infinitely many solutions to $CR(\mathcal{R})$ and infinitely many ranking functions induced by them, the following observation trivially holds since the number of worlds is finite.

Proposition 13 *For any knowledge base $\mathcal{R}$, the set of equivalence classes $\mathcal{O}(CR(\mathcal{R}))_{/\equiv_{\mid\!\sim}}$ is finite.*

While skeptical c-inference in the context of $\mathcal{R}$ is defined taking all c-representations accepting $\mathcal{R}$ into account, we will now sharpen $CR(\mathcal{R})$ by introducing an upper bound for the impact values, yielding a finite domain (FD) constraint system.

Definition 9 (**$CR^u(\mathcal{R})$**) Let $\mathcal{R} = \{(B_1 \mid A_1), \ldots, (B_n \mid A_n)\}$ and $u \in \mathbb{N}$. The finite domain constraint satisfaction problem $CR^u(\mathcal{R})$ on the constraint variables $\{\eta_1, \ldots, \eta_n\}$ ranging over $\mathbb{N}$ is given by the conjunction of the constraints, for all $i \in \{1, \ldots, n\}$:

$$\eta_i \geqslant 0 \tag{53}$$

$$\eta_i > \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j \tag{54}$$

$$\eta_i \leqslant u \tag{55}$$

The notations and constructions introduced for $CR(\mathcal{R})$ will tacitly be used also for $CR^u(\mathcal{R})$, e.g., $Sol(CR^u(\mathcal{R}))$ and $\mathcal{O}(CR^u(\mathcal{R}))$. In particular, we also define c-inference with respect to a maximal impact value that can be viewed as a kind of resource-bounded inference operation.

Definition 10 (c-inference under maximal impact value, $\mid\!\sim^{c,u}_{\mathcal{R}}$) Let $\mathcal{R}$ be a knowledge base, $u \in \mathbb{N}$, and let A, B be formulas. B is *a (skeptical) c-inference from A in the context of $\mathcal{R}$ under maximal impact value u*, denoted by $A \mid\!\sim^{c,u}_{\mathcal{R}} B$, if $A \mid\!\sim^{\kappa} B$ holds for all c-representations κ with $\kappa \in \mathcal{O}(CR^u(\mathcal{R}))$.

Proposition 14 ($\mid\!\sim^{c,u}_{\mathcal{R}}$) *Let $\mathcal{R}$ be a knowledge base, $u, u' \in \mathbb{N}$, and let A, B be formulas. Then*

$$A \mid\!\sim^{c}_{\mathcal{R}} B \quad \textit{implies} \quad A \mid\!\sim^{c,u}_{\mathcal{R}} B$$

and for $u' \geqslant u$

$$A \mid\!\sim^{c,u'}_{\mathcal{R}} B \quad \textit{implies} \quad A \mid\!\sim^{c,u}_{\mathcal{R}} B.$$

Proof It suffices to show that for the respective set of c-representations to be taken into account, the subset relationships

$$\mathcal{O}(CR^{u}(\mathcal{R})) \;\subseteq\; \mathcal{O}(CR^{u'}(\mathcal{R})) \;\subseteq\; \mathcal{O}(CR(\mathcal{R}))$$

hold which is the case due to the definitions of $CR(\mathcal{R})$ and $CR^{u}(\mathcal{R})$. □

Thus, Proposition 14 shows that $\mid\!\sim^{c,u}_{\mathcal{R}}$ approximates c-inference $\mid\!\sim^{c}_{\mathcal{R}}$. The following definition introduces a criterion for a maximal impact value ensuring that $\mid\!\sim^{c,u}_{\mathcal{R}}$ fully realizes skeptical c-inference. For an OCF κ, the definition uses the total preorder $\preccurlyeq_{\kappa}$ on worlds given by

$$\omega_1 \preccurlyeq_{\kappa} \omega_2 \quad \text{iff} \quad \kappa(\omega_1) \leqslant \kappa(\omega_2).$$

Definition 11 (regular, minimally regular) For a knowledge base $\mathcal{R}$ let $\hat{u} \in \mathbb{N}$ be the smallest number such that

$$\left|\{\preccurlyeq_{\kappa} \mid \kappa \in \mathcal{O}(CR^{\hat{u}}(\mathcal{R}))\}\right| \;=\; |\{\preccurlyeq_{\kappa} \mid \kappa \in \mathcal{O}(CR(\mathcal{R}))\}|\,. \tag{56}$$

Then $CR^{u}(\mathcal{R})$ is called *regular* if $u \geqslant \hat{u}$, and $CR^{\hat{u}}(\mathcal{R})$ is *minimally regular*; we also say that u is *regular for* $\mathcal{R}$ and $\hat{u}$ is *minimally regular for* $\mathcal{R}$.

Every regular $CR^{u}(\mathcal{R})$ is a sound and complete modelling of the set of all c-representations for $\mathcal{R}$ in the following sense:

Proposition 15 (Soundness and Completeness of $CR^{u}(\mathcal{R})$) *Let $\mathcal{R}$ be a knowledge base and $CR^{u}(\mathcal{R})$ be a regular FD constraint system for $\mathcal{R}$. Then:*

– *(Soundness) For every vector $\vec{\eta} \in Sol(CR^{u}(\mathcal{R}))$, the induced OCF $\kappa_{\vec{\eta}}$ is a c-representation κ for $\mathcal{R}$ (i.e., $\kappa_{\vec{\eta}} \models \mathcal{R}$).*
– *(Completeness) For every c-representation κ for $\mathcal{R}$ (i.e., $\kappa \models \mathcal{R}$), there is a vector $\vec{\eta}\,' \in Sol(CR^{u}(\mathcal{R}))$ such that $\kappa \equiv_{\mid\sim} \kappa_{\vec{\eta}\,'}$.*

Proof Soundness of $CR^{u}(\mathcal{R})$ immediately follows from the soundness of $CR(\mathcal{R})$ (Prop. 3) since $Sol(CR^{u}(\mathcal{R})) \subseteq Sol(CR(\mathcal{R}))$.

For showing completeness of $CR^{u}(\mathcal{R})$, let κ be a c-representation accepting $\mathcal{R}$. Since $Sol(CR^{u}(\mathcal{R})) \subseteq Sol(CR(\mathcal{R}))$, regularity of $CR^{u}(\mathcal{R})$ and completeness of $CR(\mathcal{R})$ imply that, since the inference relations induced by an OCF κ depend solely on $\preccurlyeq_{\kappa}$, the respective equivalence classes $\mathcal{O}(CR^{u}(\mathcal{R}))_{/\equiv_{\mid\sim}}$ and $\mathcal{O}(CR(\mathcal{R}))_{/\equiv_{\mid\sim}}$ can be enumerated such that $\mathcal{O}(CR^{u}(\mathcal{R}))_{/\equiv_{\mid\sim}} = \{K^{u}_{1}, \ldots, K^{u}_{r}\}$, $\mathcal{O}(CR(\mathcal{R}))_{/\equiv_{\mid\sim}} = \{K_1, \ldots, K_r\}$, with $K^{u}_{i} \subseteq K_i$ for

$$r = \left|\{\preccurlyeq_{\kappa} \mid \kappa \in \mathcal{O}(CR^{\hat{u}}(\mathcal{R}))\}\right| = |\{\preccurlyeq_{\kappa} \mid \kappa \in \mathcal{O}(CR(\mathcal{R}))\}|$$

and $i = 1, \dots, r$. Furthermore, completeness of $CR(\mathcal{R})$ ensures that there is $\vec{\eta} \in Sol(CR(\mathcal{R}))$ such that $\kappa_{\vec{\eta}} = \kappa$. Hence, there is $p \in \{1, \dots, r\}$ such that $\kappa_{\vec{\eta}} \in K_p$. Observing that $K_p^u \subseteq K_p$, we can choose $\vec{\eta}\,' \in Sol(CR^u(\mathcal{R}))$ such that $\kappa_{\vec{\eta}\,'} \in K_p^u$ and thus $\kappa_{\vec{\eta}\,'} \equiv_{\mid\sim} \kappa_{\vec{\eta}} \equiv_{\mid\sim} \kappa$, completing the proof. □

An immediate consequence of Proposition 15 is that for every regular $CR^u(\mathcal{R})$, the inference relation $\mid\sim_{\mathcal{R}}^{c,u}$ coincides with c-inference $\mid\sim_{\mathcal{R}}^{c}$ over all c-representations of $\mathcal{R}$.

Proposition 16 *Let $\mathcal{R}$ be a knowledge base, $CR^u(\mathcal{R})$ regular, and A, B be formulas. Then*

$$A \mid\sim_{\mathcal{R}}^{c} B \quad \textit{iff} \quad A \mid\sim_{\mathcal{R}}^{c,u} B.$$

Proof The $\Rightarrow$ direction is an immediate consequence of the subset relationship $Sol(CR^u(\mathcal{R})) \subseteq Sol(CR(\mathcal{R}))$. Regularity of $CR^u(\mathcal{R})$ ensures the $\Leftarrow$ direction since for each equivalence class K_i from the proof of Proposition 15, there is at least one solution of $CR^u(\mathcal{R})$ inducing an OCF belonging to the corresponding equivalence class K_i^u. □

In Section 5, we showed how c-inference can be modelled by a constraint satisfaction problem. This modelling directly transfers to the finite domains case so that c-inference is precisely captured by a constraint satisfaction problem over finite domains.

Proposition 17 (c-inference as a FD CSP) *Let $\mathcal{R}$ be a consistent knowledge base, $CR^u(\mathcal{R})$ regular, and A, B formulas. Then the following holds:*

$$A \mid\sim_{\mathcal{R}}^{c} B \;\; \textit{iff} \;\; CR^u(\mathcal{R}) \cup \{\neg CR_{\mathcal{R}}(B \mid A)\} \textit{ does not have a solution.} \tag{57}$$

Proof We can reuse the derivations used in the proof of Proposition 10 by employing soundness and completeness of $CR^u(\mathcal{R})$ (Proposition 15) instead of the soundness and completeness of $CR(\mathcal{R})$ (Propositions 3 and 4). □

Note that since $\neg CR_{\mathcal{R}}(B \mid A)$ does not introduce any variables not already in the CSP $CR^u(\mathcal{R})$ over finite domains, also $CR^u(\mathcal{R}) \cup \{\neg CR_{\mathcal{R}}(B \mid A)\}$ is a CSP over finite domains. Thus, with a regular $CR^u(\mathcal{R})$ for a knowledge base $\mathcal{R}$, we can immediately exploit the techniques developed for finite domain constraint solvers, as they are available, e.g., in constraint logic programming, for computing the inference relation $A \mid\sim_{\mathcal{R}}^{c} B$ induced by c-representations (cf. [4, 10]).

According to Proposition 15, a regular $CR^u(\mathcal{R})$ captures every c-representation of $\mathcal{R}$ up to $\equiv_{\mid\sim}$ equivalence. When we are not interested in capturing all c-representations of $\mathcal{R}$ (up to $\equiv_{\mid\sim}$ equivalence), but aim at capturing c-inference instead, we can specify a maximal impact value from this perspective in order to obtain a CSP over finite domains.

Definition 12 (sufficient $CR^u(\mathcal{R})$) Let $\mathcal{R}$ be a knowledge base and let $u \in \mathbb{N}$. Then $CR^u(\mathcal{R})$ is called *sufficient* if for all formulas A, B we have

$$A \mid\sim_{\mathcal{R}}^{c} B \quad \text{iff} \quad A \mid\sim_{\mathcal{R}}^{c,u} B.$$

If $CR^u(\mathcal{R})$ is sufficient, we will also call *u sufficient for $\mathcal{R}$*.

The next proposition states the relationship between a regular and a sufficient $CR^u(\mathcal{R})$.

Proposition 18 (regular vs. sufficient) *Let $\mathcal{R}$ be a consistent knowledge base and $u \in \mathbb{N}$. If $CR^u(\mathcal{R})$ is regular, then it is also sufficient; the converse is not true in general.*

Proof If $CR^u(\mathcal{R})$ is regular, it is also sufficient according to Proposition 16.

For the other direction, it suffices to give a $CR^u(\mathcal{R})$ that is sufficient, but not regular. Consider the alphabet $\Sigma = \{a, b\}$ and the knowledge base $\mathcal{R} = \{(a \mid \top), (b \mid \top)\}$. The CSP $CR^2(\mathcal{R})$ is regular and has four solutions:

$$\vec{\eta}^{(1)} = (1, 1) \qquad \vec{\eta}^{(2)} = (1, 2) \qquad \vec{\eta}^{(3)} = (2, 1) \qquad \vec{\eta}^{(4)} = (2, 2)$$

Note that $\vec{\eta}^{(4)}$ is a multiple of $\vec{\eta}^{(1)}$; hence, the inference relations associated with the ranking functions induced by these two solutions are inferentially equivalent. The induced ranking functions for the four solutions of $CR^2(\mathcal{R})$ are given in Table 10. Obviously the CSP $CR^1(\mathcal{R})$ only has $\vec{\eta}^{(1)}$ as a solution. Every inference that requires a different ranking of the worlds $a\overline{b}$ and $\overline{a}b$ only holds when considering either $\kappa_{\vec{\eta}^{(2)}}$ or $\kappa_{\vec{\eta}^{(3)}}$, but not when taking $\kappa_{\vec{\eta}^{(1)}}$ into account. Therefore these inferences do not hold under skeptical inference. On the other hand, every inference that holds in $\kappa_{\vec{\eta}^{(1)}}$ does also hold in $\kappa_{\vec{\eta}^{(2)}}$ and $\kappa_{\vec{\eta}^{(3)}}$. Therefore $CR^1(\mathcal{R})$ is sufficient but not regular. □

A corresponding observation is that a sufficient $CR^u(\mathcal{R})$ is sound in the sense of Proposition 15, but it is not complete since there might be a c-representation of $\mathcal{R}$ that is not captured by any solution of $CR^u(\mathcal{R})$.

Corollary 4 *Let $\mathcal{R}$ be a knowledge base and $CR^u(\mathcal{R})$ sufficient. Then the following holds:*

- *(Soundness) For every vector $\vec{\eta} \in Sol(CR^u(\mathcal{R}))$, the induced OCF $\kappa_{\vec{\eta}}$ is a c-representation κ for $\mathcal{R}$ (i.e., $\kappa_{\vec{\eta}} \models \mathcal{R}$).*
- *(Non-Completeness) In general, there might be a c-representation κ for $\mathcal{R}$ such that for all $\vec{\eta}' \in Sol(CR^u(\mathcal{R}))$ it is the case that $\kappa \not\equiv_{\mid\sim} \kappa_{\vec{\eta}'}$.*

Proof As in Proposition 15, soundness of $CR^u(\mathcal{R})$ immediately follows from the soundness of $CR(\mathcal{R})$ (Prop. 3) since $Sol(CR^u(\mathcal{R})) \subseteq Sol(CR(\mathcal{R}))$.

For showing that $CR^u(\mathcal{R})$ is not complete in general, consider again the knowledge base $\mathcal{R} = \{(a \mid \top), (b \mid \top)\}$. The only solution to $CR^1(\mathcal{R})$, $\vec{\eta}^{(1)}$ induces a ranking function that is not inferentially equivalent to $\kappa_{\vec{\eta}^{(2)}}$ (see again Table 10). □

Note that the missing completeness of a sufficient $CR^u(\mathcal{R})$ does not prevent the exact modeling of c-inference by the solutions of $CR^u(\mathcal{R})$, as was already pointed out in the proof of Proposition 18.

Table 10 Ranking functions for $R = \{(a \mid \top), (b \mid \top)\}$ used in the proof of Proposition 18

ω	ab	$a\overline{b}$	$\overline{a}b$	$\overline{a}\overline{b}$
$\kappa_{\vec{\eta}^{(1)}}(\omega)$	0	1	1	2
$\kappa_{\vec{\eta}^{(2)}}(\omega)$	0	2	1	3
$\kappa_{\vec{\eta}^{(3)}}(\omega)$	0	1	2	3
$\kappa_{\vec{\eta}^{(4)}}(\omega)$	0	2	2	4

Example 14 (minimally regular) We expand the knowledge base $\mathcal{R}$ from the proof of Proposition 18 systematically with one additional atom c and a conditional $(c \mid \top)$ to

$$\mathcal{R}' = \{(a \mid \top), (b \mid \top), (c \mid \top)\}$$

over the alphabet $\Sigma = \{a, b, c\}$. Then $CR^4(\mathcal{R})$ is minimally regular, producing 31 inferentially different c-representations. If we add just an additional atom d to Σ, the number of possible worlds doubles, but the number of inferentially different c-representations stays at 31, because there is no conditional causing a preference of a world $\dot{a}\dot{b}\dot{c}d$ over $\dot{a}\dot{b}\dot{c}\overline{d}$ or vice versa. Extending the knowledge base to

$$\mathcal{R}'' = \{(a \mid \top), (b \mid \top), (c \mid \top), (d \mid \top)\}$$

now causes worlds ω with $\omega \models d$ to be preferred over worlds ω' with $\omega' \models \overline{d}$. $CR^{10}(\mathcal{R})$ is now minimally regular, producing 1519 inferentially different c-representations.

Thus, while $u = 4$ is minimally regular for $\mathcal{R}'$, adding the conditional $(d \mid \top)$ causes $u = 10$ to be minimally regular for $\mathcal{R}''$. This effect is particularly strong here because there are no interactions between the conditionals in $\mathcal{R}'$ or in $\mathcal{R}''$, leading to various degrees of freedom when ordering the possible worlds while still accepting every conditional in the knowledge base. If we introduce some limited interactions between conditionals by replacing $(d \mid \top)$ in $\mathcal{R}''$ with the conditional $(d \mid a)$, creating

$$\mathcal{R}''' = \{(a \mid \top), (b \mid \top), (c \mid \top), (d \mid a)\},$$

the number of inferentially different c-representations is reduced from 1519 to 961. However, $u = 10$ remains minimally regular for $\mathcal{R}'''$.

For a given knowledge base $\mathcal{R}$ it is still an open problem how to approximate or to directly determine minimal values u such that $CR^u(\mathcal{R})$ is regular or sufficient, respectively. All examples we have investigated so far suggest that u corresponding to the number of conditionals in $\mathcal{R}$, i.e. $u = |\mathcal{R}|$, ensures that $CR^u(\mathcal{R})$ is sufficient; if this observation holds generally it would imply that $\mid\!\sim^{c,u}_{\mathcal{R}}$ with $u = |\mathcal{R}|$ precisely models c-inference $\mid\!\sim^{c}_{\mathcal{R}}$. This observation is also supported by the investigation on minimally sufficient bounds carried out in [7, 8], where for a sequence of simple knowledge bases $\mathcal{R}_i$, generalizing the knowledge bases $\mathcal{R}'$ and $\mathcal{R}''$ from Example 14 and containing only conditional facts, $u = |\mathcal{R}_i| - 1$ is formally proven to be sufficient.

7 Implementation

In [4] the reasoning platform `InfOCF` is presented. `InfOCF` provides implementations of OCF-based inference relations for propositional conditional knowledge bases. The user can load knowledge bases from files, and several sets of OCFs (e.g., sets of minimal c-representations [4]) can be calculated. Using these sets of models, the user can perform skeptical, credulous, and weakly-skeptical [6] c-inference. For comparison, `InfOCF` also implements system P [1] and system Z [20, 32] inference.

For experiments regarding regular and sufficient bounds for finite domain c-inference, `InfOCF` allows for the specification of a maximal impact value that is taken into account in solutions of the CSP generated for a loaded knowledge base.

Figure 1 shows the options for inference as well as some results of answered queries. The option for the maximal impact labeled "Automatic" induces a heuristic calculation that estimates the maximal impact needed for a regular CSP. This is done by calculating the

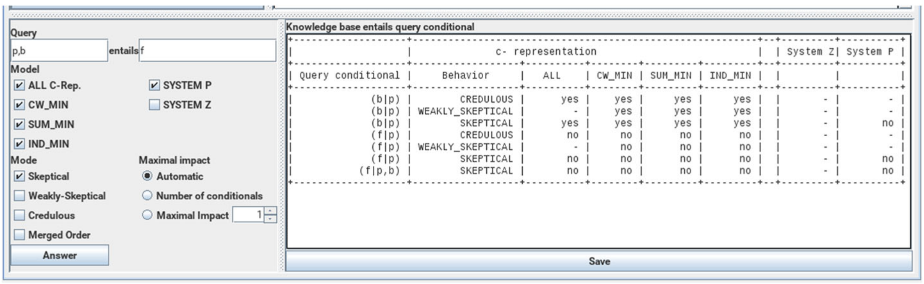

Fig. 1 Part of the user interface of `InfOCF` implementing various OCF-based inference relations. The queries on the right are answerd with respect to the knowledge base $\mathcal{R}_{bird}$ from Example 1. The options selected on the left were selected for the last query. Note that system Z inference was not performed because it was not selected and system P inference is inherently skeptical, hence it was only performed when skeptical was selected

induced partial orders of all OCF solutions to a CSP with the number of conditionals as the maximal impact as a starting point. The maximal impact is then increased systematically until no new partial orders are obtained from the new solutions. The option labeled "Number of Conditionals" sets the maximal impact to the number of conditionals in the currently loaded knowledge base. In all our experiments, we were able to show this value to be sufficient, by calculating the acceptance of all syntactically different conditionals under skeptical inference [7]. The third option allows for experimentation with a selected maximal impact.

While the user interface of `InfOCF` is implemented in Java, the core component modelling c-representations and c-inference is implemented using constraint logic programming. For any knowledge base $\mathcal{R}$ and any query conditional $(B \mid A)$, the constraint systems $CR^u(\mathcal{R})$ (cf. (53)–(54)) and $CR^u(\mathcal{R}) \cup \{\neg CR_{\mathcal{R}}(B \mid A)\}$ (cf. (57)) are transformed directly into a high-level representation in SICStus Prolog[1] using the SICStus Prolog library clp(fd) [29] for constraint logic programming over finite domains; for details of the Prolog code we refer to [4].

8 Conclusions and future work

We introduced the novel inference relation *c-inference* as the skeptical inference over all c-representations of a given conditional knowledge base $\mathcal{R}$. We proved that c-inference exceeds the skeptical inference of all OCFs that accept $\mathcal{R}$, the latter being equivalent to Adams' system P. In particular, we showed that c-inference shares important benchmark properties with inference based on single c-representations, e.g., subclass inheritance for exceptional subclasses (the "Drowning Problem") and Irrelevance, and also allows for Rule Chaining in a rational way. This is all the more remarkable because c-inference is based on a partial preorder on worlds.

Based on a CSP modelling of c-representations, we also characterized c-inference as a constraint satisfaction problem and proved its soundness and completeness, implying that skeptical inference over infinitely many c-representations can be implemented by using a constraint solver. We discussed sharpening the CSPs to finite domains, leading to CSPs that

[1] http://www.sics.se/isl/sicstuswww/site/index.html

might be easier to solve. Since there is a c-representation for a knowledge base if and only if the knowledge base is consistent, the CSP modeling also provides an alternative consistency test apart from the tolerance test [32]. Implementing the calculation of c-representations using a constraint solver has been demonstrated successfully in [10], and in [4] an extension of this implementation to c-inference is proposed. Our current work includes extending this implementation, further evaluating it empirically, and investigating the complexity of this approach. Another aspect of further work concerns the open problems of approximating, determining and comparing minimal upper bounds for the finite domains in the used CSPs.

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