Ann Math Artif Intell (2017) 79:267–289
DOI 10.1007/s10472-016-9522-x

Submodularity and its application to some global constraints

D. Magos[1] · **I. Mourtos**[2]

Published online: 26 August 2016

Abstract Submodularity defines a general framework rich of important theoretical properties while accommodating numerous applications. Although the notion has been present in the literature of Combinatorial Optimization for several decades, it has been overlooked in the analysis of global constraints. The current work illustrates the potential of submodularity as a powerful tool for such an analysis. In particular, we show that the cumulative constraint, when all tasks are identical, has the *submodular/supermodular representation property,* i.e., it can be represented by a submodular/supermodular system of linear inequalities. Motivated by that representation, we show that the system of any two (global) constraints not necessarily of the same type, each bearing the above-mentioned property, has an integral relaxation given by the conjunction of the linear inequalities representing each individual constraint. This result is obtained through the use of the celebrated polymatroid intersection theorem.

Keywords Submodular system · G-polymatroid · Global constraint · Cumulative

Mathematics Subject Classification 2010 90C10 · 90C90 · 68T99

This research has been co-financed by the European Union (European Social Fund ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: Thales. Investing in knowledge society through the European Social Fund.

✉ D. Magos
dmagos@teiath.gr

I. Mourtos
mourtos@aueb.gr

[1] Department of Informatics, Technological Educational Institute of Athens, Ag. Spyridonos Str., 12210, Egaleo, Greece

[2] Department of Management Science & Technology, Athens University of Economics and Business, 76 Patission Ave., 104 34, Athens, Greece

1 Introduction

Global constraints offer declarative simplicity in modeling complex real-life applications often with the use of discrete variables. Usually, the resulting model is communicated to a software platform for resolution. Such solvers use a variety of methods that have been developed mainly within the field of Constraint Programming (CP). As these models are typically not easy to resolve, many computational environments also take advantage of Mathematical Programming (MP) techniques. In this way, these solvers implement a unified resolution framework that has been advocated and described in various works (see [6, 9–11, 25] and references contained therein.)

The power of such a framework is greatly enhanced by the ability to represent a global constraint by linear inequalities that describe the convex hull of points satisfying that constraint. If such a representation is known, then resolution can be accomplished by Linear Programming, especially when an optimal solution is sought. Surveys on the linearization of several global constraints can be found in [8–10, 18]. Work, where some (or all) of the facets of the convex hulls associated with global constraints are established, includes [22] *(alldifferent),* [23] (*cardinality rules*), [12] (the *cumulative* constraint), [24] (the *sum* constraint), [13] (the *circuit* constraint). Related research on *binary* constraint satisfaction problems appears in [2, 21]. Furthermore, the linear representation of various systems of alldifferent constraints has been studied [1, 3, 15, 16].

The current work falls within such a framework. It utilizes the powerful theory of Polymatroids and Submodularity, developed in the field of MP (particularly Combinatorial Optimization) in order to establish results analogous to the ones presented above. Although that theory has been motivated by a wide range of applications, its contribution to the domain of global constraints is rather limited. To the best of our knowledge, the paper by D. Magos is the only relevant work where it is shown that the linear representation of the alldifferent constraint has the submodular/supermodular representation property [14]. In the current work, we derive the convex hull of the cumulative constraint in the case of identical tasks P_S and derive its dimension. We show that its defining inequalities form a submodular/supermodular system and establish that P_S is a *generalized polymatroid.* The direct consequence is that if the functions providing the right-hand side of these inequalities are integral, P_S is integral too. We also use results from the theory of submodular systems to identify the facets of P_S thus deriving its minimal description when it is of full dimension. Subsequently, we provide an extension to the celebrated Polymatroid Intersection Theorem [20, Corollary 46.1a] which is directly applicable to any system of two global constraints each bearing the submodular/supermodular property: an integral relaxation of such a system is given by the conjunction of the linear inequalities representing each of the global constraints individually.

2 Background

2.1 Polytopes and integrality

We review some basic definitions from MP theory [19]. Let $n, m \in \mathbb{Z}_+$. A *polyhedron* $P \subseteq \mathbb{R}^n$ is the set of points that satisfy a finite number of linear inequalities, that is $P = \{x \in \mathbb{R}^n : Ax \geq b\}$, where (A, b) is an $m \times (n+1)$ matrix. We say that the system of inequalities $Ax \geq b$ is *defining* for the polyhedron P. A bounded polyhedron is called a *polytope.* Given a polytope P, a point $x \in P$ is an extreme point of P if there do not exist $x^1, x^2 \in P$,

$x^1 \neq x^2$ such that $x = 0.5x^1 + 0.5x^2$. A polytope has a finite number of extreme points and every other point of that polytope can be expressed as a *convex combination* of its extreme points. Hence, a polytope can be described either by its extreme points or by its defining linear inequalities. Thus, if C is the set of extreme points of P and conv$\{C\}$ denotes the smallest convex region including the points of C, then $\{x \in \mathbb{R}^n : Ax \geq b\} = P = \text{conv}\{C\}$. The system $Ax \geq b$ is called a *linear description* of P. The polytope P is called *integral* if all its extreme points are integral. Given two polytopes $\bar{P}$ and $\hat{P}$ such that $\bar{P} \subseteq \hat{P}$, if the polytope $\hat{P}$ is integral then it is called *an integral relaxation* of $\bar{P}$. The following definition establishes the notion of *total dual integrality.*

Definition 1 ([5]) A system of linear inequalities $Ax \geq b$ is called *totally dual integral* (TDI), if for all integral w such that $z = \min\{wx : Ax \geq b\}$ is finite, the dual $\max\{yb : yA = w, y \in \mathbb{R}^m\}$ has an integral optimal solution.

2.2 Submodularity

The following definition can be found in any standard textbook on polyhedral combinatorics (e.g. [20]).

Definition 2 Given a set S, a function $g : 2^S \rightarrow \mathbb{R}$ is called submodular if

$$g(A \cup B) + g(A \cap B) \leq g(A) + g(B), \forall A, B \subseteq S. \tag{1}$$

An equivalent definition can be given with respect to the non-increasing first differences.

Lemma 3 ([17]) Given a set S, a function $g : 2^S \rightarrow \mathbb{R}$ is called submodular iff

$$g(B \cup \{j\}) - g(B) \leq g(A \cup \{j\}) - g(A), \forall A \subset B \subset S, j \in S \setminus B. \tag{2}$$

Proof (Necessity) Let $A \subset B \subset S$ and $j \in S \setminus B$. Then (1) for sets C, D where $C = B, D = A \cup \{j\}$ yields

$$g(C \cup D) + g(C \cap D) \leq g(C) + g(D) \Rightarrow$$
$$g(B \cup (A \cup \{j\})) + g(B \cap (A \cup \{j\})) \leq g(B) + g(A \cup \{j\}).$$

Because $B \cup (A \cup \{j\}) = B \cup \{j\}$ and $B \cap (A \cup \{j\}) = A$, the above implies

$$g(B \cup \{j\}) - g(B) \leq g(A \cup \{j\}) - g(A).$$

(Sufficiency) Again let A, B, j defined as previously. It is easy to see that (2) implies

$$g(B \cup C) - g(B) \leq g(A \cup C) - g(A), \tag{3}$$

where $C \cap B = \emptyset$.

For any $X, Y \subseteq S$, the sets $X \cap Y, X \setminus Y, Y$ qualify for the roles of A, C, B, respectively, in (3). Substitution yields

$$g(Y \cup (X \setminus Y)) - g(Y) \leq g((X \cap Y) \cup (X \setminus Y)) - g(X \cap Y).$$

Because $Y \cup (X \setminus Y) = Y \cup X$ and $(X \cap Y) \cup (X \setminus Y) = X$, the above yields

$$g(Y \cup X) + g(X \cap Y) \leq g(X) + g(Y)$$

which is exactly (1) for $X, Y \subseteq S$. □

g is called *supermodular* if '$\leq$' is replaced by '$\geq$' in the above definitions. Finally, if (1), (2) hold as equalities g is called *modular*. Clearly, the notions of super- submodularity are symmetric; it is easy to see that if g is submodular, $-g$ is supermodular and vice versa.

In his seminal paper [4], Edmonds initiated a systematic study of submodular functions. Among the main contributions of that work is the definition of a polytope with the use of a submodular function g. That is, a *submodular system* (S, g) is associated with the polytope

$$P(S, g) = \{x \in \mathbb{R}^S : x \geq 0, x(U) \leq g(U), \forall U \subseteq S\},$$

where $x(U) = \sum_{u \in U} x_u$. P is called a *polymatroid* while dropping the non-negative constraints yields an *extended polymatroid* denoted as $EP(S, g)$ (c.f. [20, p.767]). An important property of the inequality system defining $P(S, g)$ is that it is totally dual integral (TDI) [4]. The same applies for the system defining $EP(S, f)$ [20, Corollary 44.3c]. The key implication of TDIness is that if g is integer-valued, then $P(S, g)$ (and thus $EP(S, g)$) is integral [20, Corollary 44.3d].

A most celebrated result is that the above property carries over to the intersection of two (extended) polymatroids. That is, given two submodular functions g_0, g_1 defined on the same set S, the inequality system $x \geq 0, x(U) \leq g_i(U), \forall U \subseteq S, i = 0, 1$ is TDI [4]. Note that this system defines the polytope $P(S, g_0) \cap P(S, g_1)$, i.e., the intersection of the polymatroids $P(S, g_0)$ and $P(S, g_1)$. An analogous result holds for $EP(S, g_0) \cap EP(S, g_1)$ [20, Theorem 46.1]. The consequence is known as *the polymatroid intersection theorem* [20, Corollary 46.1a] given next.

Theorem 4 *The intersection of two integral (extended) polymatroids is integral.*

Structures that are symmetrical to the ones presented above are defined if instead of a submodular function g, we consider a supermodular function f defined on a set S. Thus a *contrapolymatroid* is the polytope

$$\tilde{P}(S, f) = \{x \in \mathbb{R}^S : x \geq 0, x(U) \geq f(U), \forall U \subseteq S\},$$

In an analogous manner, an extended contrapolymatroid is defined. A theorem analogous to Theorem 4 holds for contrapolymatroids as well [20, Corollary 46.1d]. The intersection of an integral (extended) polymatroid and an integral (extended) contrapolymatroid is also integral [20, Theorem 46.2, Corollary 46.2a].

An even more general model is that of a *generalized polymatroid (g-polymatroid)* [7]. Two set functions f, g defined on a set S are called *compliant* if

$$g(A) - f(B) \geq g(A \setminus B) - f(B \setminus A), \forall A, B \subseteq S. \quad (4)$$

The pair (f, g) is called *strong* if the two functions are compliant and f is supermodular whereas g is submodular. For a strong pair (f, g), the polytope

$$Q(S, f, g) = \{x \in \mathbb{R}^S : f(U) \leq x(U) \leq g(U), \forall U \subseteq S\}$$

is a g-polymatroid. It is known that the system defining a g-polymatroid is TDI [7, Proposition 2.12]. Also the linear system defining the intersection of two g-polymatroids $Q(S, f_0, g_0) \cap Q(S, f_1, g_1)$, where (f_0, g_0), (f_1, g_1), are strong pairs is TDI [7, Proposition 4.1].

3 The Cumulative constraint with identical tasks

The resource-constrained scheduling problem (RCSP) involves scheduling a set of tasks $N = \{1, \dots, n\}$ on a renewable resource with limited capacity C. Each task $j \in N$ has an earliest release date r_j^{lb}, a latest release date r_j^{ub}, a due date d_j and consumes c_j units of the resource while running, non-preemptively, for p_j time units. Given a time horizon of T time units, we are interested in determining the start time s_j of each task $j \in N$ such that

$$\sum_{j \in N_t} c_j \qquad \leq C, \qquad \forall\, t \in \{0, \dots, T\}, \tag{5}$$

$$r_j^{\text{lb}} \leq s_j \;\leq r_j^{\text{ub}} = d_j - p_j,\; \forall\, j \in N, \tag{6}$$

where $N_t = \{j \in N : s_j \leq t < s_j + p_j\}$ is the set of tasks running at time t. We refer to a vector $s \in \mathbb{R}^n$ satisfying (5)–(6) as a *schedule*. Such a system is modelled by the so-called cumulative constraint [9, Section 4.16]

$$\text{cumulative}(s|p, c, C). \tag{7}$$

whose set of solutions are all schedules s.

We focus on the case of identical tasks, i.e., $p_j = p_0$, $c_j = c_0$, $r_j^{\text{lb}} = r_0$, and $r_j^{\text{ub}} = r_1$, for all $j \in N$. To be more descriptive, in this case, we write (7) as

$$\text{cumulative}(s_j : j \in N|p_0, c_0, C). \tag{8}$$

Hence, the resource can process at most $\delta = \lfloor C/c_0 \rfloor$ tasks at a time. Table 1 summarizes the notation introduced above.

Given $K \subseteq N$, observe that $p_0 \cdot \lceil |K| / \delta \rceil$ is the minimum number of time periods needed to process a set of K tasks. We define

$$\rho(K) = \left\lceil \frac{|K|}{\delta} \right\rceil - 1.$$

Remark 5 The function $\rho : 2^N \to \mathbb{Z}_+$ is non-decreasing on the cardinality of the subsets of N.

We define,

$$f(r_0, K) \;= |K|r_0 + \rho(K) \cdot p_0 \cdot \left(|K| - \tfrac{\delta}{2}(\rho(K) + 1)\right), \tag{9}$$

$$g(r_1, K) \;= |K|r_1 - \rho(K) \cdot p_0 \cdot \left(|K| - \tfrac{\delta}{2}(\rho(K) + 1)\right). \tag{10}$$

Table 1 Notation associated with the Cumulative constraint on identical tasks

N : set of tasks to be run non-preemptively
C : amount of a renewable resource
s_j : starting time of task $j \in N$
p_0 : processing time of a task
c_0 : quantity of resource consumed by a task
r_0 : earliest release date of a task
r_1 : latest release date of a task
$\delta = \lfloor C/c_0 \rfloor$: maximum number of tasks processed simultaneously

We begin by showing how (9) and (10) relate to feasible schedules. Hereafter, let $s(K) = \sum_{j \in K} s_j$, for $K \subseteq N$.

Proposition 6 For $K \subseteq N$, $f(r_0, K)$ and $g(r_1, K)$ are the minimum and maximum value of $s(K)$ in any feasible schedule s, respectively.

Proof We show the result for $f(r_0, K)$ (the proof for $g(r_1, K)$ is similar). The value of $s(K)$ is minimized when the tasks in K are processed as early as possible. Recall that, by definition, $\rho(K)$ is the number of periods of length p_0 needed to process $|K|$ tasks, minus one. Therefore, the first $\rho(K)$ periods of the schedule will be full, that is, there will be δ tasks running. The tasks that run during the first period have $s_j = r_0$, the tasks that run during the second period have $s_j = r_0 + p_0$, and so on, until (and including) the $\rho(K)$-th period. Therefore, the sum of the start times of these first $\delta \cdot \rho(K)$ tasks is equal to $\sum_{i=0}^{\rho(K)-1}(r_0 + ip_0)$ multiplied by δ. The remaining $|K| - \delta\rho(K)$ tasks will have $s_j = r_0 + p_0 \cdot \rho(K)$. Summing these start times with the former expression, we obtain

$$f(r_0, K) = \sum_{i=0}^{\rho(K)-1}(r_0 + ip_0)\delta + (r_0 + \rho(K)p_0)\left(|K| - \delta\rho(K)\right),$$

which reduces to (9) after some algebraic manipulations. □

Example 1 Consider a set of $N = \{1, 2, \ldots, 7\}$ tasks. Suppose that each task runs non-preemptively, consumes two units of a resource and requires three periods of processing time. Assume that the amount of the renewable resource available is limited to 5 units. Furthermore, the processing cannot start earlier than day number 2 and later than day number 14. The parameters of the problem are $C = 5$, $c_0 = 2$, $p_0 = 3$, $r_0 = 2$, $r_1 = 14$. The number of tasks running at each time period cannot be more than $\delta = \lfloor 5/2 \rfloor = 2$. The minimum number of days needed for all tasks to be processed is $p_0 \cdot \lceil |N| / \delta \rceil = 3 \cdot \lceil 7/2 \rceil = 12$. Also, $\rho(N) = 3$. Plausible schedules include $\hat{s}, \tilde{s}, \bar{s}$ where

$$\hat{s}_1 = 3, \hat{s}_2 = 4, \hat{s}_3 = 7, \hat{s}_4 = 7, \hat{s}_5 = 10, \hat{s}_6 = 11, \hat{s}_7 = 14,$$
$$\tilde{s}_1 = 2, \tilde{s}_2 = 2, \tilde{s}_3 = 5, \tilde{s}_4 = 5, \tilde{s}_5 = 8, \tilde{s}_6 = 8, \tilde{s}_7 = 11,$$
$$\bar{s}_1 = 5, \bar{s}_2 = 8, \bar{s}_3 = 8, \bar{s}_4 = 11, \bar{s}_5 = 11, \bar{s}_6 = 14, \bar{s}_7 = 14.$$

Observe that $\tilde{s}(N) = f(2, 7) = 41$, $\hat{s}(N) = 56$ and $\bar{s}(N) = g(14, 7) = 71$.

Intuitively, because tasks have common characteristics, the sum of starting times of a set K of tasks starting as early (or as late) as possible cannot be less than the corresponding sum of a set of K' of tasks where $\left|K'\right| \leq |K|$. This is formalized as a remark.

Remark 7 The functions f, g are non-decreasing on the cardinality of the subsets of N.

The preceding analysis implies that any schedule satisfies

$$g(r_1, K) \geq s(K) \geq f(r_0, K), \forall K \subseteq N. \tag{11}$$

In fact, (11) yields the linear representation of the cumulative constraint for identical tasks as shown next.

Theorem 8 *The convex hull of schedules is*

$$P_S = \{s \in \mathbb{R}^N : s \text{ satisfies } (11)\} \tag{12}$$

Proof Clearly $P_S \neq \emptyset$ if and only if $r_1 \geq r_0 + p_0 \cdot \rho(N)$. Let $\alpha s \leq \beta$ be an inequality that is valid for P_S and $P^{(\alpha,\beta)} = \{s \in P_S : \alpha s = \beta\}$ the face it induces. Either (i) at least one coefficient of $\alpha s \leq \beta$ is positive or (ii) all such coefficients are non-positive (thus the inequality $-\alpha s \geq -\beta$ has only non-negative coefficients). We prove that $P^{(\alpha,\beta)}$ is included in a face defined by an inequality $s(K) \leq g(r_1, K)$ (if (i) holds) or an inequality $s(K) \geq f(r_0, K)$ (if (ii) holds).

Case 8.1 $\alpha_k > 0$ for some $k \in N$

Let $\alpha_{\max} = \max\{\alpha_k : k \in N\}$. By hypothesis $\alpha_{\max} > 0$. Define $K = \{k \in N : \alpha_k = \alpha_{\max}\}$, i.e. K is the set indexing the maximum coefficients in the left-hand side of $\alpha s \leq \beta$ (and also $\alpha_k > 0$ for all $k \in K$). We prove that each vector $s \in P^{(\alpha,\beta)}$ that defines a schedule satisfies $s(K) = g(r_1, K)$.

Thus, assume to the contrary a schedule $s' \in P^{(\alpha,\beta)}$ such that $s'(K) < g(r_1, K)$. The implication here is that, in the schedule s', not all tasks indexed by K start 'as late as possible'. Thus consider the task indexed by K with the earliest start time, i.e. $k_0 = \arg\min\{s'_k : k \in K\}$. Since $s'(K) < g(r_1, K)$, task k_0 starts no later than $r_1 - (\rho(K) + 1) \cdot p_0$, i.e. $s'_{k_0} \leq r_1 - (\rho(K) + 1) \cdot p_0$. Moreover, in the schedule s', either there are δ tasks processed in each period $t = r_1 - (\rho(K) - 1) \cdot p_0, \ldots, r_1$ (and the remaining tasks processed in $r_1 - \rho(K) \cdot p_0$) or there is a period $t \in \{r_1 - \rho(K) \cdot p_0, \ldots, r_1 - p_0\}$ in which fewer than δ tasks are processed (or fewer than w tasks are processed in r_1).

In the former case (all periods after $r_1 - \rho(K) \cdot p_0$ are 'full'), $s'(K) < g(r_1, K)$ implies that there is a task not indexed by K which starts no earlier than $r_1 - \rho(K) \cdot p_0$, i.e. there is $k_1 \in N \backslash K$ such that $s'_{k_1} \geq r_1 - \rho(K) \cdot p_0$. Recalling that $s'_{k_0} \leq r_1 - (\rho(K) + 1) \cdot p_0$ yields that $s'_{k_0} < s'_{k_1}$ although $\alpha_{k_0} > \alpha_{k_1}$ (follows from $k_0 \in K$ and $k_1 \in N \backslash K$). Construct the point $\bar{s}$ as $\bar{s}_{k_0} = s'_{k_1}, \bar{s}_{k_1} = s'_{k_0}$ and $\bar{s}_k = s'_k$ for all $k \in N \backslash \{k_0, k_1\}$. Observe that, although $\bar{s}$ is a schedule of P_S,

$$\begin{aligned} \alpha\bar{s} - \alpha s' &= \alpha_{k_0}\bar{s}_{k_0} + \alpha_{k_1}\bar{s}_{i_1} - \alpha_{k_0}s'_{k_0} - \alpha_{k_1}s'_{k_1} = \\ &= (\alpha_{k_0} - \alpha_{k_1}) \cdot (s'_{k_1} - s'_{k_0}) > 0 \end{aligned}$$

or $\alpha\bar{s} > \alpha s' = \beta$. Hence inequality $\alpha s \leq \beta$ is not valid for P_S, since violated by $\bar{s}$, i.e. a contradiction to our assumption that $s'(K) < g(r_1, K)$.

In the latter case (there is a period after $r_1 - \rho(K) \cdot p_0$ that is not 'full'), let $t_0 > r_1 - (\rho(K) + 1) \cdot p_0$ be a period in which an additional task can be processed. The schedule $\bar{s} \in P_S$ derived as $\bar{s}_{k_0} = t_0, \bar{s}_k = s'_k$ for all $k \in N \backslash \{k_0, k_1\}$ task k_0 is processed in period t_0 and all other tasks are processed as before (i.e., as in the schedule s). Clearly, $\bar{s}_{k_0} > s'_{k_0}$ and $\alpha_{k_0} > 0$ by hypothesis. But then,

$$\alpha\bar{s} - \alpha s' = \alpha_{k_0} \cdot (\bar{s}_{k_0} - s'_{k_0}) > 0$$

implies that $\alpha s \leq \beta$ is not valid for P_S, since violated by $\bar{s}$, thus contradicting again that $s'(K) < g(r_1, K)$.

It follows that $s(K) = g(r_1, K)$ for all schedules $s \in P^{(\alpha,\beta)}$. Since each face of P_S is the convex hull of the schedules belonging to it, $P^{(\alpha,\beta)} \subseteq \{s \in P_S : s(K) = g(r_1, K)\}$.

Therefore, the only inequalities of the form $\alpha s \leq \beta$ (with a positive left-hand side coefficient) required for the description of P_S are $s(K) \leq g(r_1, K)$, $K \subseteq N$.

Case 8.2 $\alpha_k \leq 0$ for all $k \in N$

Let $\alpha_{\min} = \min\{\alpha_k : k \in N\}$. Unless $\alpha s \leq \beta$ is an all-zeros inequality (thus being trivially redundant) $\alpha_{\min} < 0$. Define $K = \{k \in N : \alpha_k = \alpha_{\min}\}$, i.e. K is the set indexing the minimum coefficients in $\alpha s \leq \beta$ and $\alpha_k < 0$ for all $k \in K$. We prove that $s(K) = f(r_0, K)$ for any schedule $s \in P^{(\alpha,\beta)}$.

Assuming to the contrary a schedule $s' \in P^{(\alpha,\beta)}$ such that $s'(K) > f(r_0, K)$ implies that not all tasks indexed by K start 'as early as possible' in s'. Thus consider the task indexed by K with the latest start time, i.e. $k_0 = \arg\max\{s'_k : k \in K\}$. Since $s'(K) > f(r_0, K)$, task k_0 starts no earlier than $r_0 + (\rho(K)+1) \cdot p_0$, i.e. $s'_{k_0} \leq r_0 + (\rho(K)+1) \cdot p_0$. Moreover, either there are δ tasks processed in each period $t = r_0, r_0 + p_0, \ldots, r_0 + \rho(K) \cdot p_0$ or there is a period $t \in \{r_0, r_0 + p_0, \ldots, r_0 + \rho(K) \cdot p_0\}$ in which fewer than δ tasks are processed.

In the former case (all periods $r_0, r_0+p_0, \ldots, r_0+\rho(K)\cdot p_0$ are 'full'), $s'(K) > f(r_0, K)$ implies that there is $k_1 \in N \backslash K$ such that $s'_{k_1} \leq r_0 + \rho(K) \cdot p_0$. Thus $s'_{k_0} > s'_{k_1}$ although $\alpha_{k_0} < \alpha_{k_1}$ (follows from the definition of K). Construct the point $\bar{s}$ as $\bar{s}_{k_0} = s'_{k_1}$, $\bar{s}_{k_1} = s'_{k_0}$ and $\bar{s}_k = s'_k$ for all $k \in N \backslash \{k_0, k_1\}$. Observe that, although $\bar{s}$ is a schedule of P_S, $\alpha\bar{s} - \alpha s' = (\alpha_{k_0} - \alpha_{k_1}) \cdot (s'_{k_1} - s'_{k_0}) > 0$ or $\alpha\bar{s} > \alpha s' = \beta$, i.e. a contradiction to the assumption that $s'(K) > f(r_0, K)$.

In the latter case let $t_0 \leq r_0 + \rho(K) \cdot p_0$ be a period in which an additional task can be processed. The schedule $\bar{s} \in P_S$ derived as $\bar{s}_{k_0} = t_0$, $\bar{s}_k = s'_k$ for all $k \in N \backslash \{k_0, k_1\}$, task k_0 is processed in period t_0 and all other tasks are processed as in the schedule s. Clearly, $\bar{s}_{k_0} < s'_{k_0}$ and $\alpha_{k_0} < 0$ by hypothesis. But then, $\alpha\bar{s} - \alpha s' = \alpha_{k_0} \cdot (\bar{s}_{k_0} - s'_{k_0}) > 0$ yields again a contradiction to $s'(K) > f(r_0, K)$.

It follows that $s(K) = f(r_0, K)$ for all extreme points $s \in P^{(\alpha,\beta)}$, thus $P^{(\alpha,\beta)} \subseteq \{s \in P_S : s(K) = f(r_0, K)\}$. Therefore, the inequalities $s(K) = f(r_0, K)$, $K \subseteq N$, are the only ones required (among the inequalities of the form $\alpha s \leq \beta$ without a positive coefficient) for the description of P_S. □

It is easy to see that P_S is empty if $r_1 < r_0 + \rho(N) \cdot p_0$. In this case, there are not enough time periods for all the tasks in N to start. That is, $f(r_0, N) > g(r_1, N)$. A more general result can be derived from the following two lemmas.

Lemma 9 $f(r_0, N) = g(r_1, N)$ if $r_1 = r_0 + \rho(N) \cdot p_0$ and δ divides $|N|$.

Proof Considering (9) and (10) for N and equating them yields

$$\begin{aligned} g(r_1, N) &= f(r_0, N) \Rightarrow \\ &|N|\, r_1 - p_0 \cdot \rho(N) \left(|N| - \frac{\delta}{2}(\rho(N)+1) \right) \\ &= |N|\, r_0 + p_0 \cdot \rho(N) \left(|N| - \frac{\delta}{2}(\rho(N)+1) \right) \Rightarrow \\ &|N|\,(2 \cdot p_0 \cdot \rho(N) - (r_1 - r_0)) = \delta \cdot p_0 \cdot \rho(N) \cdot (\rho(N)+1) \end{aligned}$$

Setting in the above equation $r_1 - r_0 = p_0 \cdot \rho(N)$ and dividing both parts by $p_0 \cdot \rho(N)$, we get

$$\frac{|N|}{\delta} = \left\lceil \frac{|N|}{\delta} \right\rceil$$

which implies that δ divides $|N|$. □

Lemma 10 If $P_S \neq \emptyset$ then there does not exist $K \subset N$ such that $f(r_0, K) = g(r_1, K)$, for any r_0, r_1.

Proof Since $P_S \neq \emptyset$, it can only be that $r_1 \geq r_0 + \rho(N) \cdot p_0$. Now assume there is $K \subset N$ such that $f(r_0, K) = g(r_1, K)$. That yields

$$r_1 - r_0 = \rho(K) \cdot p_0 \cdot \mu, \tag{13}$$

$$\mu = \left(2 - \frac{\lceil |K|/\delta \rceil}{|K|/\delta}\right). \tag{14}$$

Observe that $\mu \leq 1$. If $r_1 > r_0 + \rho(N) \cdot p_0$ (implying $f(r_0, N) < g(r_1, N)$) then (13) yields $\rho(K) > \rho(N)$ contradicting Remark 5 as $|K| < |N|$.

Next assume that $r_1 = r_0 + \rho(N) \cdot p_0$ but δ delta does not divide $|N|$ (implying again that $f(r_0, N) < g(r_1, N)$). Then, we have that

$$\rho(N) = \rho(K) \cdot \mu. \tag{15}$$

For this to hold μ must be equal to one implying that δ divides $|K|$. But then as δ does not divide $|N|$ and $|N| > |K|$, it follows that $\rho(N) > \rho(K)$.

Finally, if $f(r_0, N) = g(r_1, N)$ then by Lemma 9, δ divides $|N|$ and thus $\mu < 1$ if δ does not divide $|K|$. In this case (15) cannot hold given that $\rho(N) \geq \rho(K)$. On the other hand, if δ divides both $|K|$ and $|N|$ then $\mu = 1$ but $\rho(N) > \rho(K)$. □

Corollary 11

$$\dim P_S = \begin{cases} -1, & r_1 < r_0 + \rho(N) \cdot p_0, \\ |N| - 1, & r_1 = r_0 + \rho(N) \cdot p_0, \delta \text{ divides } |N|, \\ |N|, & \textit{otherwise.} \end{cases}$$

Proof By Lemmas 9, 10, when $P_S \neq \emptyset$, the system (11) may include up to one equality; that is $g(r_1, N) = f(r_0, N) = s(N)$ when $r_1 = r_0 + \rho(N) \cdot p_0$ and δ divides $|N|$. □

Next, we show how submodularity enters the picture.

Proposition 12 Functions $f(r_0, K)$ and $g(r_1, K)$ are supermodular and submodular on 2^N, respectively.

Proof To establish the supermodularity of f it suffices to show (2) where f replaces g and '$\geq$' '$\leq$'. That is, we must prove

$$f(r_0, K \cup \{j\}) - f(r_0, K) \geq f(r_0, \bar{K} \cup \{j\}) - f(r_0, \bar{K}), \tag{16}$$

for any $\bar{K} \subset K \subset N$ and $j \in N \backslash K$. By the definition of $f(r_0, K)$, the difference $f(r_0, \bar{K} \cup \{j\}) - f(r_0, \bar{K})$ is solely determined by the size of the sets $\bar{K} \cup \{j\}$ and $\bar{K}$. Moreover, since all tasks are identical and $f(r_0, K)$ represents the minimum sum of start times in a (feasible) schedule, $f(r_0, \bar{K} \cup \{j\}) - f(r_0, \bar{K})$ equals the start time of the last task among the $|\bar{K} \cup \{j\}|$ tasks. But the start time of that task (irrespective of whether that task is j) equals $r_0 + \rho(\bar{K} \cup \{j\}) \cdot p_0 = f(r_0, \bar{K} \cup \{j\}) - f(r_0, \bar{K})$. Similarly, $f(r_0, K \cup \{j\}) - f(r_0, K) = r_0 + \rho(K \cup \{j\}) \cdot p_0$. Hence, showing (16) is equivalent to showing that $\rho(\bar{K} \cup \{j\}) \leq \rho(K \cup \{j\})$. The latter follows from the fact that $\left(\bar{K} \cup \{j\}\right) \subset (K \cup \{j\})$, and by noticing that $\rho(K)$ is non-decreasing with respect to $|K|$. The submodularity of $g(r_1, K)$ is shown in an analogous manner. □

The following statement which is proven in the Appendix reveals a fundamental property of P_S.

Theorem 13 *P_S is a generalized polymatroid.*

Corollary 14 *The polytope P_S is integral if r_0, r_1 and p_0 are integers.*

Proof As P_S is a generalized polymatroid, the system of inequalities (11) is TDI. Given that that r_0, r_1 and p_0 are integers, the functions $f(r_0, K)$, $g(r_1, K)$ yield integer values for any $K \subseteq N$. □

Also, the polytopes defined from P_S for $r_0 = -\infty$ and $r_1 = \infty$ form an extended polymatroid and contrapolymatroid, respectively, thus constituting integral relaxations of P_S, when the conditions of the above theorem hold. Submodularity can also be used to identify the facets of P_S when it is of full dimension. The following definition is adapted from [7][Chapter VI, page 548]

Definition 15 Let h denote any of the functions f, g, while $\hat{h}$ the other one. Consider a set $K \subseteq N$.

- A set $N_1 \subset K$ is called an *inner* (h, K)*-separator* if
$$h(N_1) + h(K \setminus N_1) = h(K).$$
- A set $N_1 \subseteq N \setminus K$ is called an *outer* (h, K)*-separator* if
$$h(N_1 \cup K) - \hat{h}(N_1) = h(K).$$

[7][Corollary 1.2, page 548] yields that $s(K) \geq f(r_0, K)$ and $s(K) \leq g(r_1, K)$ define facets of the full-dimensional P_S iff there is no inner (h, K)-separator and no outer (h, K)-separator where h is defined as above. In the following statements, we identify the sets $K \subset N$ for which this true.

Proposition 16 For $|K| = 1$ or $|K| > \delta$, there is no inner (f, K)-separator. The same applies for g.

Proof We show the result for f. By definition there can be no inner separator if $|K| = 1$.

Next, let $|K| > \delta$. Since $\emptyset \subset N_1 \subset K$ and function ρ is non-decreasing, $\rho(K) \geq \rho(N_1)$ and $\rho(K) \geq \rho(K \backslash N_1)$. We show that at least one of these two inequalities is strict, i.e., $\rho(K) > \rho(N_1)$ or $\rho(K) > \rho(K \backslash N_1)$. Assuming the contrary and adding the two equalities, we get

$$2\rho(K) = \rho(N_1) + \rho(K \backslash N_1). \tag{17}$$

Next, considering Lemma 24, (see Appendix), with $n = 2, a_0 = |K|, a_1 = |N_1|, a_2 = |K \setminus N_1|$ and $b = \delta$, we obtain

$$\left\lceil \frac{|K|}{\delta} \right\rceil + \theta_{\delta,|K|} = \left\lceil \frac{|N_1|}{\delta} \right\rceil + \left\lceil \frac{|K \setminus N_1|}{\delta} \right\rceil,$$

where $\theta_{\delta,|K|} \in \{0, 1\}$. Moving $\theta_{\delta,|K|}$ to the right-hand side and subtracting one from both sides, yields

$$\rho(K) \geq \rho(N_1) + \rho(K \setminus N_1).$$

Substituting the right-hand side from (17), we arrive at $\rho(K) \geq 2\rho(K)$ which is a contradiction because $\rho(K) \geq 1$ (i.e., $\rho(K) \neq 0$); this stems from the hypothesis that $|K| > \delta$.

Recall that $f(r_0, K)$ denotes the minimum sum of start times of $|K|$ tasks in a feasible schedule, i.e. δ tasks start at each period $t = r_0, \ldots, r_0 + p_0 \cdot (\rho(K) - 1)$ and $|K| - \rho(K) \cdot \delta$ tasks start at period $r_0 + p_0 \cdot \rho(K)$. Let $\{s_k, k \in K\}$ denote the start times of such a schedule sorted in increasing order, i.e. $s_1 \leq s_2 \leq \cdots \leq s_{|K|}$ and $s(K) = f(r_0, K)$.

Similarly, $f(r_0, N_1)$ and $f(r_0, K \setminus N_1)$ represent the minimum sum of start times for two schedules comprising $|N_1|$ and $|K| - |N_1|$ tasks respectively; that is, δ tasks start at each period $t = r_0, \ldots r_0 + p_0 \cdot (\rho(N_1) - 1)$, $|N_1| - p_0 \cdot \rho(N_1) \cdot \delta$ tasks start at period $r_0 + p_0 \cdot \rho(N_1)$, another δ tasks start at each period $t = r_0, \ldots, p_0 \cdot (\rho(K \setminus N_1) - 1)$ and $|K \setminus N_1| - \rho(K \setminus N_1) \cdot \delta$ tasks start at period $\rho(K \setminus N_1)$. Let $\{s'_k, k \in K\}$ denote the start times of all $|K|$ tasks in these two schedules in increasing order, i.e. $s'_1 \leq s'_2 \leq \cdots \leq s'_{|K|}$ and $s'(K) = f(r_0, N_1) + f(r_0, K \setminus N_1)$. It is not difficult to see that, since all tasks are identical and $\rho(K) > \rho(N_1)$ or $\rho(K) > \rho(K \setminus N_1)$, $s'_k \leq s_k$ for all $k \in K$ while $s'_{k^*} < s_{k^*}$ for some $k^* \in K$ (i.e. there is at least one task starting 'earlier' in the second case). Therefore, $f(r_0, N_1) + f(r_0, K \setminus N_1) < f(r_0, K)$. The proof for $g(r_1, K)$ follows along similar lines. □

Proposition 17 If P_S is full-dimensional then there is no outer (g, K)-separator. The same holds for function f.

Proof Suppose there exists $N_1 \subseteq N \setminus K$ such that it is an outer (g, K)-separator. That implies

$$g(r_1, N_1 \cup K) = f(r_0, N_1) + g(r_1, K). \tag{18}$$

Recall that the first period when tasks start in a feasible schedule s such that $s(N_1 \cup K) = g(r_1, N_1 \cup K)$ is $r_1 - p_0 \cdot \rho(N_1 \cup K)$. The only way for (18) to hold is for a schedule s' such that $s'(N_1) = f(r_0, N_1)$ to have the same first period that tasks start, i.e.,

$$r_0 = r_1 - p_0 \cdot \rho(N_1 \cup K) \Rightarrow r_1 - r_0 = p_0 \cdot \rho(N_1 \cup K). \tag{19}$$

For P_S to be full-dimensional we need $r_1 - r_0 \geq p_0 \cdot \rho(N)$. That together with (19) imply that $\rho(N_1 \cup K) \geq \rho(N)$ which can hold only as equality because $N_1 \cup K \subseteq N$, i.e.,

$$\rho(N_1 \cup K) = \rho(N) \Rightarrow \left\lceil \frac{N}{\delta} \right\rceil = \left\lceil \frac{N_1 \cup K}{\delta} \right\rceil \tag{20}$$

For (18) to hold as equality according to schedule s exactly δ tasks must start at period $r_1 - p_0 \cdot \rho(N_1 \cup K)$. It follows that

$$|N_1 \cup K| - \delta \cdot \rho(N_1 \cup K) = \delta \Rightarrow \rho(N_1 \cup K) = \frac{|N_1 \cup K|}{\delta} - 1 \tag{21}$$

(21) implies that δ divides $|N_1 \cup K|$. Then (20) yields

$$\left\lceil \frac{N}{\delta} \right\rceil = \frac{|N_1 \cup K|}{\delta}$$

which, given that $N_1 \cup K \subseteq N$, implies $N_1 \cup K = N$. But then δ divides N and because $r_1 - r_0 = p_0 \cdot \rho(N)$ the polytope P_S is not full-dimensional by Corollary 11 (contradiction). The proof for f follows in an analogous manner. □

The two propositions above imply the following theorem.

Theorem 18 *If P_S is full dimensional then (9) and (10)*

- *define facets for $|K| > \delta$ or $|K| = 1$ and*
- *are redundant for $2 \leq |K| \leq \delta$.*

The efficacy of the theory of submodularity will become even more apparent in the next section.

4 The polytope of two submodular systems

Prompted by the paradigm of the single cumulative constraint analyzed above, we take the next step to consider the more general situation of a system of two such constraints. Therefore, suppose that there are two sets of tasks, not necessarily distinct. Each set is to be processed by a different machine with the restriction that common tasks, between the two sets, must be started in both machines at the same time period. The tasks of each set are treated identically by the machine they are submitted to; they all have the same processing time and they all require the same amount of renewable resource offered by the machine. Also all tasks of both sets have the same earliest release date r_0 and latest released date r_1. If the tasks are indexed by the sets N_0 and N_1, respectively, a schedule $s \in \mathbb{R}^{N_0 \cup N_1}$ must satisfy

$$\text{cumulative}(s_j : j \in N_0 | p_0, c_0, C_0), \tag{22}$$

$$\text{cumulative}(s_j : j \in N_1 | p_1, c_1, C_1), \tag{23}$$

$$r_0 \leq s_j \leq r_1, \forall j \in N_0 \cup N_1, \tag{24}$$

where c_i is the amount of resource required by each task of the set N_i when processed by the machine with resource capacity C_i, for $i = 0, 1$. Similarly, all tasks of the set N_i have the same processing time p_i.

We seek for a linear relaxation of such a system. Clearly, the theory presented in Section 2 does not address this situation. All intersection theorems require a common ground set S, whereas in the example above we have submodular/supermodular systems defined on different ground sets; for constraint (22) the corresponding system is (N_0, f_0, g_0) while for (23) it is (N_1, f_1, g_1). Moreover, observe that $f_0 \neq f_1$ as these functions are instantiations of (9) for different processing times and consumption rates in the two cumulative constraints (22), (23). The same applies for g_0, g_1.

Motivated by the above situation, we extend the theory presented in Section 2 for the case of two submodular/supermodular systems defined on two distinct ground sets in terms of different submodular/supermodular functions. In the following proofs, the functions under examination are not assumed of a particular form other than being submodular (supermodular) and mapping the empty set to zero. It follows that the results obtained are not pertinent solely to the system described by (22)-(24). Rather, they address a wide spectrum of models composed of structures having the submodular/supermodular representation property. Examples of such models include the intersection of two alldifferent constraints [1], two polymatroids associated with matroids on distinct ground sets, etc.

Consider two submodular functions h_0, h_1 defined on the sets S_0, S_1, respectively, such that $h_0(\emptyset) = h_1(\emptyset) = 0$. Hereafter all additions on indices are taken mod 2.

Proposition 19 The function q_i defined on the set $S = S_0 \cup S_1$ as

$$q_i(U) = h_i(U \cap S_i) + h_{i+1}(U \setminus S_i), U \subseteq S$$

is submodular, for $i = 0, 1$.

Proof Let $i = 0$; we show that function q_0 is submodular, i.e., we show that

$$q_0(T \cap U) + q_0(T \cup U) \leq q_0(T) + q_0(U), \tag{25}$$

for all $T, U \subseteq S_0 \cup S_1$.

Inequality (1) written for h_0 and $A = T \cap S_0$, $B = U \cap S_0$ (notice that $A, B \subseteq S_0$) yields

$$h_0\left((T \cap S_0) \cap (U \cap S_0)\right) + h_0\left((T \cap S_0) \cup (U \cap S_0)\right) \leq h_0\left(T \cap S_0\right) + h_0\left(U \cap S_0\right).$$

Since

$$(T \cap S_0) \cap (U \cap S_0) = (T \cap U) \cap S_0,$$

and

$$(T \cap S_0) \cup (U \cap S_0) = (T \cup U) \cap S_0,$$

the above expression becomes

$$h_0((T \cap U) \cap S_0) + h_0((T \cup U) \cap S_0) \leq h_0\left(T \cap S_0\right) + h_0\left(U \cap S_0\right). \tag{26}$$

In an analogous manner, (1) written for h_1 and $A = T \setminus S_0$, $B = U \setminus S_0$ (notice that $A, B \subseteq S_1$) yields

$$h_1\left((T \setminus S_0) \cap (U \setminus S_0)\right) + h_1\left((T \setminus S_0) \cup (U \setminus S_0)\right) \leq h_1\left(T \setminus S_0\right) + h_1\left(U \setminus S_0\right).$$

After observing that

$$(T \setminus S_0) \cap (U \setminus S_0) = (T \cap U) \setminus S_0,$$

and

$$(T \setminus S_0) \cup (U \setminus S_0) = (T \cup U) \setminus S_0,$$

the above inequality becomes

$$h_1\left((T \cap U) \setminus S_0\right) + h_1\left((T \cup U) \setminus S_0\right) \leq h_1\left(T \setminus S_0\right) + h_1\left(U \setminus S_0\right).$$

Adding (26) to (27) yields

$$\begin{aligned}[h_0((T \cap U) \cap S_0) + h_1((T \cap U) \setminus S_0)] + [h_0((T \cup U) \cap S_0) + h_1((T \cup U) \setminus S_0)] \\ \leq [h_0\left(T \cap S_0\right) + h_1\left(T \setminus S_0\right)] + [h_1\left(U \setminus S_0\right) + h_0\left(U \cap S_0\right)]\end{aligned}$$

which is exactly (25). □

Theorem 20 *For $i = 0, 1$, let h_i be a submodular function defined on the set S_i such that $h_i(\emptyset) = 0$. The system*

$$x(U) \leq h_i(U), \forall U \subseteq S_i, i = 0, 1 \tag{27}$$

is TDI.

Proof A direct consequence of Proposition 19 is that the system

$$x(U) \leq q_0(U), \forall U \subseteq S_0 \cup S_1, \tag{28}$$
$$x(U) \leq q_1(U), \forall U \subseteq S_0 \cup S_1, \tag{29}$$

is defining for $EP(S_0 \cup S_1, q_0) \cap EP(S_0 \cup S_1, q_1)$ and therefore is TDI, by [20, Theorem 46.1].

Notice however that, for $i = 0, 1$, inequality $x(U) \leq q_i(U)$ is the sum of the inequalities

$$x(U \cap S_i) \leq h_i(U \cap S_i) = q_i(U \cap S_i),$$
$$x(U \setminus S_i) \leq h_{i+1}(U \setminus S_i) = q_{i+1}(U \setminus S_i),$$

unless $U \subseteq S_i$. (Note that the equalities in the above expressions hold because $h_i(\emptyset) = 0, i = 0, 1$.) Thus, (28) and (29) after removing redundant inequalities, become

$$x(U) \leq q_0(U), \forall U \subseteq S_0,$$
$$x(U) \leq q_1(U), \forall U \subseteq S_1.$$

Observe that the removal performed does not affect TDIness [19, p. 322 (41)].

Last, notice that, for $U \subseteq S_i$, $q_i(U) = h_i(U)$ since $h_i(\emptyset) = 0$. Therefore the above system is equivalent to

$$x(U) \leq h_0(U), \forall U \subseteq S_0, \tag{30}$$
$$x(U) \leq h_1(U), \forall U \subseteq S_1. \tag{31}$$

□

It is straightforward that Theorem 20 extends to the case where $x_s \geq 0$, for all $s \in S_0 \cup S_1$; that is, the system including (30), (31) and $x_s \geq 0$, for all $s \in S_0 \cup S_1$, is also TDI. This implies a generalization of the polymatroid intersection theorem, i.e., Theorem 4 becomes a special case defined for $S_0 = S_1$. Analogous results hold for supermodular systems as the proofs of Lemma 19 and Theorem 20 carry out in an analogous fashion when supermodular functions take the place of submodular ones.

Our results extend to the case of g-polymatroids. That is, let (v_i, h_i) be a strong pair defined on the ground set S_i, for $i = 0, 1$. The functions

$$q_i(U) = h_i(U \cap S_i) + h_{i+1}(U \setminus S_i), U \subseteq S_0 \cup S_1, \tag{32}$$
$$r_i(U) = v_i(U \cap S_i) + v_{i+1}(U \setminus S_i), U \subseteq S_0 \cup S_1, \tag{33}$$

are submodular and supermodular, respectively, and the system $v_i(U) \leq x(U) \leq h_i(U), \forall U \subseteq S_i, i = 0, 1$, is equivalent to $r_i(U) \leq x(U) \leq q_i(U), \forall U \subseteq S_0 \cup S_1, i = 0, 1$. It remains to show the following.

Lemma 21 The functions r_i, q_i are compliant for $i = 0, 1$.

Proof Let $i = 0$; (4) for r_0, q_0 becomes

$$q_0(A) - r_0(B) \geq q_0(A \setminus B) - r_0(B \setminus A), \forall A, B \subseteq S_0 \cup S_1.$$

Substituting from (32), (33) yields

$$\begin{aligned} & h_0(A \cap S_0) + h_1(A \setminus S_0) - (v_0(B \cap S_0) + v_1(B \setminus S_0)) \\ & \geq h_0((A \setminus B) \cap S_0) + h_1((A \setminus B) \setminus S_0) \\ & -(v_0(B \setminus A) \cap S_0) + v_1((B \setminus A) \setminus S_0)) \end{aligned} \tag{34}$$

Since v_0, h_0 are compliant,

$$\begin{aligned} & h_0(A \cap S_0) - v_0(B \cap S_0) \\ & \geq h_0((A \cap S_0) \setminus (B \cap S_0)) - v_0((B \cap S_0) \setminus (A \cap S_0)) \\ & = h_0((A \setminus B) \cap S_0) - v_0((B \setminus A) \cap S_0). \end{aligned} \tag{35}$$

Since v_1, h_1 are compliant,

$$\begin{aligned} & h_1(A \setminus S_0) - v_1(B \setminus S_0) \\ & \geq h_1((A \setminus S_0) \setminus (B \setminus S_0)) - v_1((B \setminus S_0) \setminus (A \setminus S_0)) \\ & = h_1((A \setminus B) \setminus S_0) - v_1((B \setminus A) \setminus S_0). \end{aligned} \tag{36}$$

Adding (35) to (36) yields (34). □

The following statement is now straightforward.

Theorem 22 *For $i = 0, 1$, let (v_i, h_i) be a strong pair defined on the set S_i such that $h_i(\emptyset) = v_i(\emptyset) = 0$. The system*

$$v_i(U) \leq x(U) \leq h_i(U), \forall U \subseteq S_i, i = 0, 1,$$

is TDI.

The above theorem, in conjunction with Theorem 8, implies that a integer relaxation of the convex hull of the points satisfying the two cumulative constraints (22),...,(24) is given by (11) written for N_0, N_1.

5 Conclusions

It is usually the case that a model includes more than one global constraint. In terms of the line of research adopted here, this amounts to deriving linear representations of systems of global constraints. This is a daunting task given that even a single global constraint might imply the solution of a problem which is inherently difficult ($\mathcal{NP}$-hard). Even if one succeeds in analyzing such a system, the results will be ad-hoc; they will apply to that specific configuration only (i.e., the system consisting of the specific constraints). The current work exploits the notion of submodularity to provide a first step towards a more general framework. First, it is shown that the cumulative constraint on identical tasks has *the submodular/supermodular representation property;* it is linearly represented by a submodular/supermodular system. Motivated by this result and the corresponding result for the all-different constraint [14], our work establishes an integral relaxation of a system of two (global) constraints when each of these has the submodular/supermodular representation property; the constraints of the system must not be necessarily of the same type (e.g., two cumulative constraints). Once the submodular/supermodular representation property holds for each constraint individually and the defining functions are integral, we immediately obtain an integral relaxation of the associated polytope. Thus, the current work provides a strong motivation for more global constraints to be examined under the prism of submodularity.

Appendix

First, we state two auxiliary results.

Remark 23 For any a and $b \in \mathbb{Z}_+$, it is true that

$$\left\lceil \frac{a}{b} \right\rceil = \frac{a + \epsilon_{b,a}(b - \upsilon_{b,a})}{b}, \tag{37}$$

where $\upsilon_{b,a} = a \bmod b$, and $\epsilon_{b,a} = 1$ if $\upsilon_{b,a} \neq 0$, and $\epsilon_{b,a} = 0$ otherwise.

Lemma 24 Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}_+$ such that $a_1 + \ldots + a_n = a_0$. Then, for any $b \in \mathbb{Z}_+$,

$$\left\lceil \frac{a_1}{b} \right\rceil + \ldots + \left\lceil \frac{a_n}{b} \right\rceil = \left\lceil \frac{a_0}{b} \right\rceil + \theta_{b,a_0},$$

where $0 \leq \theta_{b,a_0} \leq n - \left\lceil \frac{n}{b} \right\rceil$.

Proof First observe that θ_{b,a_0} is integer. Solving, in the above expression, with respect to θ_{a_0} while substituting terms from the equation of Remark 1, we get

$$\begin{aligned} \theta_{b,a_0} &= \sum_{i=1,\ldots n} \frac{a_i + \epsilon_{b,a_i}(b - \upsilon_{b,a_i})}{b} - \frac{a_0 + \epsilon_{b,a_0}(b - \upsilon_{b,a_0})}{b} \\ &= \frac{1}{b}\Big(\sum_{i=1,\ldots n} \epsilon_{b,a_i}(b - \upsilon_{b,a_i}) - \epsilon_{b,a_0}(b - \upsilon_{b,a_0})\Big). \end{aligned}$$

The maximum value of the right-hand side is attained if $\epsilon_{b,a_i} = 1$, $\upsilon_{b,a_i} = 1$ and $\epsilon_{b,a_0} = 0$. Therefore

$$\theta_{b,a_0} \leq n\frac{b-1}{b} = n - \frac{n}{b}$$

yielding $\theta_{b,a_0} \leq n - \left\lceil \frac{n}{b} \right\rceil$ since θ_{b,a_0} is integer. □

To show Theorem 13, we observe that, for any $X, Y \subseteq N$,

$$|Y| = |Y \setminus X| + |Y \cap X|, \tag{38}$$

$$|X| = |X \setminus Y| + |Y \cap X|, \tag{39}$$

$$|Y \cup X| = |Y \setminus X| + |Y \cap X| + |X \setminus Y|. \tag{40}$$

The above equalities in conjunction with Lemma 24 yield

$$\theta_{|Y|} = \left\lceil \frac{|Y \setminus X|}{\delta} \right\rceil + \left\lceil \frac{|Y \cap X|}{\delta} \right\rceil - \left\lceil \frac{|Y|}{\delta} \right\rceil, \tag{41}$$

$$\theta_{|X|} = \left\lceil \frac{|X \setminus Y|}{\delta} \right\rceil + \left\lceil \frac{|Y \cap X|}{\delta} \right\rceil - \left\lceil \frac{|X|}{\delta} \right\rceil, \tag{42}$$

$$\theta_{|Y \cup X|} = \left\lceil \frac{|Y \setminus X|}{\delta} \right\rceil + \left\lceil \frac{|Y \cap X|}{\delta} \right\rceil + \left\lceil \frac{|X \setminus Y|}{\delta} \right\rceil - \left\lceil \frac{|Y \cup X|}{\delta} \right\rceil, \tag{43}$$

where

$$\theta_{|Y|}, \theta_{|X|} \leq \quad 1, \text{ if } \delta \geq 2, \tag{44}$$

$$\theta_{|Y \cup X|} \leq \begin{cases} 2, & \text{if } \delta \geq 3, \\ 1, & \text{if } \delta = 2. \end{cases} \tag{45}$$

We can drop the ceiling operator from (41), (42), (43) by substituting terms from the equation of Remark 1. To simplify notation we will drop the δ subscript from ϵ and υ. This convention will be used throughout. Then canceling out equivalent terms from (38), (39), (40) yields respectively,

$$\theta_{|Y|} = \frac{\epsilon_{|Y \setminus X|}(\delta - \upsilon_{|Y \setminus X|}) + \epsilon_{|Y \cap X|}(\delta - \upsilon_{|Y \cap X|}) - \epsilon_{|Y|}(\delta - \upsilon_{|Y|})}{\delta}, \tag{46}$$

$$\theta_{|X|} = \frac{\epsilon_{|X \setminus Y|}(\delta - \upsilon_{|X \setminus Y|}) + \epsilon_{|Y \cap X|}(\delta - \upsilon_{|Y \cap X|}) - \epsilon_{|X|}(\delta - \upsilon_{|X|})}{\delta}, \tag{47}$$

$$\theta_{|Y \cup X|} = \frac{\epsilon_{|Y \setminus X|}(\delta - \upsilon_{|Y \setminus X|}) + \epsilon_{|Y \cap X|}(\delta - \upsilon_{|Y \cap X|})}{\delta} + \frac{\epsilon_{|X \setminus Y|}(\delta - \upsilon_{|X \setminus Y|}) - \epsilon_{|Y \cup X|}(\delta - \upsilon_{|Y \cup X|})}{\delta}. \tag{48}$$

We are now ready to prove Theorem 13.

Proof For $r_1 \geq r_0 + p_0\rho(N)$ and any $X, Y \subseteq N$ with $X \setminus Y \neq \emptyset$ and $Y \setminus X \neq \emptyset$ we must show that f, g are compliant, i.,e.,

$$F = g(Y) - g(Y \setminus X) - (f(X) - f(X \setminus Y)) \geq 0. \tag{49}$$

By the definition of g

$$g(Y) - g(Y \setminus X) = (|Y| - |Y \setminus X|)r_1 - p_0\rho(Y)(|Y| - \frac{\delta}{2}(\rho(Y) + 1)) + p_0\rho(Y \setminus X)(|Y \setminus X| - \frac{\delta}{2}(\rho(Y \setminus X) + 1)),$$

while by the definition of f

$$f(X) - f(X \setminus Y) = (|X| - |X \setminus Y|)r_0 + p_0\rho(X)(|X| - \frac{\delta}{2}(\rho(X) + 1)) - p_0\rho(X \setminus Y)(|X \setminus Y| - \frac{\delta}{2}(\rho(X \setminus Y) + 1)).$$

Putting it all in (49) and observing from (38), (39) that

$$|X \cap Y| = |Y| - |Y \setminus X| = |X| - |X \setminus Y| \tag{50}$$

yields

$$\begin{aligned} F = {} & |X \cap Y|\,(r_1 - r_0) \\ & - p_0(\rho(Y)\,|Y| + \rho(X)\,|X| - \rho(Y \setminus X)\,|Y \setminus X| - \rho(X \setminus Y)\,|X \setminus Y|) \\ & + p_0\frac{\delta}{2}((\rho(Y) - \rho(Y \setminus X))(\rho(Y) + \rho(Y \setminus X) + 1) \\ & + (\rho(X) - \rho(X \setminus Y))(\rho(X) + \rho(X \setminus Y) + 1)). \end{aligned}$$

Substituting $|Y|$ and $|X|$ from (38) and (39) respectively, we get

$$\begin{aligned} F &= |X \cap Y|\,(r_1 - r_0) \\ &- p_0((\rho(Y) + \rho(X))\,|X \cap Y| \\ &+ (\rho(Y) - \rho(Y \setminus X))\,|Y \setminus X| + (\rho(X) - \rho(X \setminus Y))\,|X \setminus Y|) \\ &+ p_0 \frac{\delta}{2}((\rho(Y) - \rho(Y \setminus X))(\rho(Y) + \rho(Y \setminus X) + 1) \\ &+ (\rho(X) - \rho(X \setminus Y))(\rho(X) + \rho(X \setminus Y) + 1)), \end{aligned}$$

yielding

$$\begin{aligned} F &= |X \cap Y|\,(r_1 - r_0) \\ &- p_0((\rho(Y) + \rho(X))\,|X \cap Y|) \\ &+ p_0((\rho(Y) - \rho(Y \setminus X))(\frac{\delta}{2}(\rho(Y) + \rho(Y \setminus X) + 1) - |Y \setminus X|) \\ &+ (\rho(X) - \rho(X \setminus Y))(\frac{\delta}{2}(\rho(X) + \rho(X \setminus Y) + 1) - |X \setminus Y|)). \end{aligned} \tag{51}$$

Because

$$\begin{aligned} |Y \setminus X| &= \frac{|Y \setminus X|}{2} + \frac{|Y \setminus X|}{2} \\ &= \frac{|Y \setminus X|}{2} + \frac{|Y|}{2} - \frac{|X \cap Y|}{2}, \end{aligned}$$

(due to (38)), we have that

$$\begin{aligned} &\tfrac{\delta}{2}\ (\rho(Y) + \rho(Y \setminus X) + 1) - |Y \setminus X| \\ &= \frac{1}{2}(\delta \left\lceil \frac{|Y|}{\delta} \right\rceil - |Y|) + \frac{1}{2}(\delta \left\lceil \frac{|Y \setminus X|}{\delta} \right\rceil - |Y \setminus X|) + \frac{|X \cap Y|}{2} - \frac{\delta}{2}, \end{aligned}$$

while substituting terms in brackets from the equation of Remark 23, we obtain

$$\begin{aligned} &\tfrac{\delta}{2}\ (\rho(Y) + \rho(Y \setminus X) + 1) - |Y \setminus X| \\ &= \frac{1}{2}((|X \cap Y| - \delta) + \delta(\epsilon_{|Y|} + \epsilon_{|Y \setminus X|}) - \epsilon_{|Y|}\upsilon_{|Y|} - \epsilon_{|Y \setminus X|}\upsilon_{|Y \setminus X|}) \end{aligned} \tag{52}$$

In an analogous manner, we obtain

$$\begin{aligned} &\tfrac{\delta}{2}\ (\rho(X) + \rho(X \setminus Y) + 1) - |X \setminus Y| \\ &= \frac{1}{2}((|X \cap Y| - \delta) + \delta(\epsilon_{|X|} + \epsilon_{|X \setminus Y|}) - \epsilon_{|X|}\upsilon_{|X|} - \epsilon_{|X \setminus Y|}\upsilon_{|X \setminus Y|}). \end{aligned} \tag{53}$$

Plugging (52), (53) in (51) and performing further substitutions from the equation of Remark 1, (38)–(43) and (48), we derive

$$F = \big(|X \cap Y|\,(r_1 - r_0) - |X \cap Y|\,p_0 \rho(Y \cup X)\big) + F' p_0,$$

where

$$\begin{aligned} F' &= \epsilon_{|Y\cap X|}\frac{\delta-\upsilon_{|Y\cap X|}}{\delta}(\epsilon_{|Y\setminus X|}(\delta-\upsilon_{|Y\setminus X|}) \\ &+ \epsilon_{|X\setminus Y|}(\delta-\upsilon_{|X\setminus Y|})+\epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|})-\delta) \\ &+ \frac{|X\cap Y|}{\delta}\epsilon_{|Y\cup X|}(\delta-\upsilon_{|Y\cup X|})+\frac{\delta}{2}(\theta_{|Y|}(\theta_{|Y|}+1)+\theta_{|X|}(\theta_{|X|}+1)) \\ &- \theta_{|Y|}(\epsilon_{|Y\setminus X|}(\delta-\upsilon_{|Y\setminus X|})+\epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|})) \\ &- \theta_{|X|}(\epsilon_{|X\setminus Y|}(\delta-\upsilon_{|X\setminus Y|})+\epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|})) \end{aligned} \tag{54}$$

Notice that the first bracket evaluates to a non-negative quantity for $r_1 - r_0 \geq p_0\rho(N)$, since $\rho(N) \geq \rho(Y \cup X)$. Thus, it remains to show that F' also evaluates to a non-negative quantity.

Case 1 $\theta_{|X|} = \theta_{|Y|} = \theta_z$, where $\theta_z \in \{0, 1\}$.

$$\begin{aligned} F' &= (\epsilon_{|Y\cap X|}\frac{\delta-\upsilon_{|Y\cap X|}}{\delta}-\theta_z)(\epsilon_{|Y\setminus X|}(\delta-\upsilon_{|Y\setminus X|}) \\ &+ \epsilon_{|X\setminus Y|}(\delta-\upsilon_{|X\setminus Y|})+\epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|})) \\ &- \delta\epsilon_{|Y\cap X|}\frac{\delta-\upsilon_{|Y\cap X|}}{\delta}+\frac{|X\cap Y|}{\delta}\epsilon_{|Y\cup X|}(\delta-\upsilon_{|Y\cup X|}) \\ &+ \delta\theta_z(\theta_z+1)-\theta_z\epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|}). \end{aligned} \tag{55}$$

Subcase 1.1 $\theta_z = 0$.

(48) implies that

$$\begin{aligned} &\delta\theta_{|Y\cup X|}+\epsilon_{|Y\cup X|}(\delta-\upsilon_{|Y\cup X|}) \\ &= \epsilon_{|Y\setminus X|}(\delta-\upsilon_{|Y\setminus X|})+\epsilon_{|X\setminus Y|}(\delta-\upsilon_{|X\setminus Y|})+\epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|}) \end{aligned}$$

and thus

$$\begin{aligned} F' &= \epsilon_{|Y\cap X|}\frac{\delta-\upsilon_{|Y\cap X|}}{\delta}(\delta\theta_{|Y\cup X|}+\epsilon_{|Y\cup X|}(\delta-\upsilon_{|Y\cup X|})-\delta) \\ &+ \frac{|X\cap Y|}{\delta}\epsilon_{|Y\cup X|}(\delta-\upsilon_{|Y\cup X|}) \\ &= \frac{\epsilon_{|Y\cup X|}(\delta-\upsilon_{|Y\cup X|})}{\delta}(|X\cap Y|+\epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|})) \\ &+ \epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|})(\theta_{|Y\cup X|}-1). \end{aligned} \tag{56}$$

If $\epsilon_{|Y\cap X|} = 0$ then (56) implies $F' \geq 0$. Otherwise, we will show that $F' \geq 0$. Hence, let $\epsilon_{|Y\cap X|} = 1$. If $\epsilon_{|Y\cup X|} = 0$ then $\theta_{|Y\cup X|}$ must be greater than or equal to one (implying $F' \geq 0$) since if $\theta_{|Y\cup X|} = 0$ (48) yields

$$\epsilon_{|Y\setminus X|}(\delta-\upsilon_{|Y\setminus X|})+(\delta-\upsilon_{|Y\cap X|})+\epsilon_{|X\setminus Y|}(\delta-\upsilon_{|X\setminus Y|})=0$$

which cannot be true as $(\delta - \upsilon_{|Y\cap X|}) \geq 1$ and the remaining terms of the right-hand side are nonnegative. If $\epsilon_{|Y\cup X|} = 1$ then (56) becomes

$$\begin{aligned} F' &= (\delta-\upsilon_{|Y\cup X|})(\frac{|X\cap Y|-\upsilon_{|Y\cap X|}}{\delta}+1) \\ &+ (\delta-\upsilon_{|Y\cap X|})(\theta_{|Y\cup X|}-1). \end{aligned}$$

Clearly $F' \geq 0$ if $\theta_{|Y \cup X|} \geq 1$. Thus assume $\theta_{|Y \cup X|} = 0$ and because $\frac{|X \cap Y| - \upsilon_{|Y \cap X|}}{\delta} \geq 0$

$$F' \geq (\delta - \upsilon_{|Y \cup X|}) - (\delta - \upsilon_{|Y \cap X|}).$$

Observe that if $\theta_{|Y \cup X|} = 0$ then (48) yields

$$(\delta - \upsilon_{|Y \cup X|}) = (\delta - \upsilon_{|Y \cap X|}) + \epsilon_{|X \setminus Y|}(\delta - \upsilon_{|X \setminus Y|}) + \epsilon_{|Y \setminus X|}(\delta - \upsilon_{|Y \setminus X|})$$

and because the two last terms of the right-hand side are nonnegative, we have that

$$(\delta - \upsilon_{|Y \cup X|}) \geq (\delta - \upsilon_{|Y \cap X|})$$

implying that $F' \geq 0$.

Subcase 1.2 $\theta_z = 1$.

(55) yields

$$\begin{aligned} F' &= \delta\theta_{|Y \cup X|}(\epsilon_{|Y \cap X|}\frac{\delta - \upsilon_{|Y \cap X|}}{\delta} - 1) \\ &+ 2(\delta - \epsilon_{|Y \cap X|}(\delta - \upsilon_{|Y \cap X|})) \\ &+ \frac{\epsilon_{|Y \cup X|}(\delta - \upsilon_{|Y \cup X|})}{\delta}(|X \cap Y| + \epsilon_{|Y \cap X|}(\delta - \upsilon_{|Y \cap X|}) - \delta). \end{aligned} \tag{57}$$

If $\epsilon_{|Y \cap X|} = 0$ then (57) becomes

$$F' = (2 - \theta_{|Y \cup X|})\delta + \epsilon_{|Y \cup X|}(\delta - \upsilon_{|Y \cup X|})(\frac{|X \cap Y|}{\delta} - 1). \tag{58}$$

In this case, $\delta \mid |X \cap Y|$ implying that $\frac{|X \cap Y|}{\delta} = k$, where $k \geq 0$ and integer. For $k \geq 1$, the above implies that $F' \geq 0$. Next assume that $k = 0$ implying that $|X \cap Y| = 0$. Then (40) becomes $|Y \cup X| = |Y \setminus X| + |X \setminus Y|$, (43) $\left\lceil \frac{|Y \setminus X|}{\delta} \right\rceil + \left\lceil \frac{|X \setminus Y|}{\delta} \right\rceil = \left\lceil \frac{|Y \cup X|}{\delta} \right\rceil + \theta_{|Y \cup X|}$. In this case, Lemma 24 implies $\theta_{|Y \cup X|} \leq 1$. Thus (58) yields

$$F' \geq \delta - \epsilon_{|Y \cup X|}(\delta - \upsilon_{|Y \cup X|}) \geq 1,$$

since $\delta - 1 \geq \upsilon_{|Y \cup X|} \geq 1$.

If $\epsilon_{|Y \cap X|} = 1$ then (57) becomes

$$\begin{aligned} F' = &\frac{\epsilon_{|Y \cup X|}(\delta - \upsilon_{|Y \cup X|})}{\delta}(|X \cap Y| - \upsilon_{|Y \cap X|}) \\ &-\theta_{|Y \cup X|}\upsilon_{|Y \cap X|} + 2\upsilon_{|Y \cap X|} \end{aligned}$$

which implies that $F' \geq 0$ since $|X \cap Y| - \upsilon_{|Y \cap X|} \geq 0$ and $\theta_{|Y \cup X|} \leq 2$.

Case 2 $\theta_{|Y|} = 1 - \theta_{|X|}$

Without loss of generality assume that $\theta_{|X|} = 0$ yielding $\theta_{|Y|} = 1$. (46) leads to

$$\delta + \epsilon_{|Y|}(\delta - \upsilon_{|Y|}) = \epsilon_{|Y \setminus X|}(\delta - \upsilon_{|Y \setminus X|}) + \epsilon_{|Y \cap X|}(\delta - \upsilon_{|Y \cap X|}).$$

That implies

$$\epsilon_{|Y \setminus X|} = \epsilon_{|Y \cap X|} = 1 \tag{59}$$

because the left-hand side is greater than or equal to δ whereas each of the terms of the right-hand side evaluates to at most $\delta - 1$. Thus (46) yields

$$\epsilon_{|Y|}(\delta - \upsilon_{|Y|}) = \delta - \upsilon_{|Y \setminus X|} - \upsilon_{|Y \cap X|} \geq 0 \tag{60}$$

since the left-hand side is non-negative.

Setting $\theta_{|X|} = 0$ and $\epsilon_{|Y\cap X|} = 1$ in (47) yields

$$\delta - \upsilon_{|Y\cap X|} + \epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) = \epsilon_{|X|}(\delta - \upsilon_{|X|})$$

implying that

$$\epsilon_{|X|} = 1 \tag{61}$$

since $\delta - \upsilon_{|Y\setminus X|} + \epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) \geq 1$. (47) yields

$$\epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) = \upsilon_{|Y\cap X|} - \upsilon_{|X|} \geq 0 \tag{62}$$

because the left-hand side is non-negative.

(54) becomes

$$\begin{aligned} F' &= -\frac{\upsilon_{|Y\cap X|}}{\delta}(\delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|}) \\ &\quad \frac{\delta - \upsilon_{|Y\cap X|}}{\delta}\epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) \\ &\quad + \frac{|X\cap Y|}{\delta}\epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|}). \end{aligned} \tag{63}$$

(48) yields

$$\epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|}) = (\delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|}) + (\upsilon_{|Y\cap X|} - \upsilon_{|X|}) + \delta(1 - \theta_{|Y\cup X|}).$$

Substituting in (63), we obtain

$$\begin{aligned} F' &= -\upsilon_{|Y\cap X|}(\delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|}) \\ &\quad \frac{\delta - \upsilon_{|Y\cap X|}}{\delta}\epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) \\ &+ \frac{|X\cap Y|}{\delta}((\delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|}) + (\upsilon_{|Y\cap X|} - \upsilon_{|X|}) + \delta(1 - \theta_{|Y\cup X|})) \\ &= (\delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|})(\frac{|X\cap Y| - \upsilon_{|Y\cap X|}}{\delta}) \\ &+ \frac{\delta - \upsilon_{|Y\cap X|}}{\delta}\epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) \\ &+ \frac{|X\cap Y|}{\delta}(\upsilon_{|Y\cap X|} - \upsilon_{|X|}) + |X\cap Y|\,(1 - \theta_{|Y\cup X|}). \end{aligned}$$

Because of $\delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|} \geq 0$ (by (60)), $\frac{|X\cap Y| - \upsilon_{|Y\cap X|}}{\delta} \geq 0$ and $\upsilon_{|Y\cap X|} - \upsilon_{|X|} \geq 0$ (by (62)), we have that $F' \geq 0$ if $\theta_{|Y\cup X|} \leq 1$. Assume that $\theta_{|Y\cup X|} = 2$. It is easy to see that (48) and (47) yield

$$\delta\theta_{|Y\cup X|} = \delta\theta_{|X|} + \epsilon_{|X|}(\delta - \upsilon_{|X|}) + \epsilon_{|Y\setminus X|}(\delta - \upsilon_{|Y\setminus X|}) - \epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|}).$$

In this case $\theta_{|X|} = 0$, $\epsilon_{|X|} = 1$ (by (61)), $\epsilon_{|Y\setminus X|} = 1$ (by (59)) and $\theta_{|Y\cup X|} = 2$ yielding

$$\begin{aligned} 2\delta &= 2\delta - \upsilon_{|X|} - \upsilon_{|Y\setminus X|} - \epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|}) \Rightarrow \\ \epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|}) &= -(\upsilon_{|X|} + \upsilon_{|Y\setminus X|}) \end{aligned}$$

leading to a contradiction since the left-hand side is strictly non-negative and the right-hand side is negative ((61) and (59) imply that $\upsilon_{|X|}, \upsilon_{|Y\setminus X|} \geq 1$). Thus it can only be $\theta_{|Y\cup X|} \leq 1$ and therefore $F' \geq 0$. □

Compliance with Ethical Standards

Conflict of interests The authors declare that they have no conflict of interest

Funding This research has been co-financed by the European Union (European Social Fund ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: Thales. Investing in knowledge society through the European Social Fund.

References

1. Appa, G., Magos, D., Mourtos, I.: On the system of two all_different predicates. Inf. Process. Lett. **94**, 99–105 (2005)
2. Aron, I.D., Leventhal, D.H., Sellmann, M.: A totally unimodular description of the consistent value polytope for binary constraint programming. In: Beck, J.C., Smith, B.M. (eds.) Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems, Third International Conference, CPAIOR 2006, Cork, Ireland, May 31 - June 2, 2006, Proceedings, volume 3990 of Lecture Notes in Computer Science, p. 2006
3. Bergman, D., Hooker, J.N.: Graph coloring inequalities from all-different systems. Constraints **19**(4), 404–433 (2014)
4. Edmonds, J.: Submodular functions, matroids and certain polyhedra. In: Guy, R., Hanani, H., Sauer, N., Schonheim, J. (eds.) Combinatorial Structures and their Applications, pp. 69–87. Gordon and Beach, New York (1970)
5. Edmonds, J., Giles, R.: A min-max relation for submodular functions in graphs. In: Hammer, P., Johnson, E., Korte, B.H., Newhauser, G. (eds.) Studies in Integer Programming (Workshop on Integer Programming, Bonn 1975), pp. 185–204. North-Holland, Amsterdam (1977)
6. Focacci, F., Lodi, A., Milano, M.: Mathematical programming techniques in constraint programming A short overview. J. Heuristics **8**(1), 7–17 (2002)
7. Frank, A., Tardos, E.: Generalized polymatroids. Math. Program. **42**, 489–563 (1988)
8. Hooker, J.: Logic-based Methods for Optimization: Combining Optimization and Constraint Satisfaction. Wiley-Interscience series in Discrete Mathematics and Optimization. John Wiley & Sons (2000)
9. Hooker, J.: Integrated methods for optimization, volume 100 of International series in Operations Research & Management Science. Springer, New York (2007)
10. Hooker, J.N.: A search-infer-and-relax framework for integrating solution methods. In: Barták, R., Milano, M. (eds.) Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems, Second International Conference, CPAIOR 2005, Prague, Czech Republic, May 30 - June 1, 2005 Proceedings, volume 3524 of Lecture Notes in Computer Science, p. 2005
11. Hooker, J.N., Lama, M.A.O.: Mixed logical-linear programming. Discret. Appl. Math. **96-97**, 395–442 (1999)
12. Hooker, J.N., Yan, H.: A relaxation of the cumulative constraint. In: Hentenryck, P.V. (ed.) Principles and Practice of Constraint Programming - CP 2002, 8th International Conference, CP 2002, Ithaca, NY, USA, September 9-13, 2002, Proceedings, volume 2470 of Lecture Notes in Computer Science, pp. 686–690. Springer (2002)
13. Kaya, L.G., Hooker, J.N.: Domain reduction for the circuit constraint. In: van Beek, P. (ed.) Principles and Practice of Constraint Programming - CP 2005, 11th International Conference, CP 2005, Sitges, Spain, October 1-5, 2005, Proceedings, volume 3709 of Lecture Notes in Computer Science, p. 846. Springer (2005)
14. Magos, D.: The constraint of difference and total dual integrality. In: Ketikidis, P.H., Margaritis, K.G., Vlahavas, I.P., Chatzigeorgiou, A., Eleftherakis, G., Stamelos, I. (eds.) Proceedings of the 17th Panhellenic Conference on Informatics, PCI Thessaloniki, Greece - September 19 - 21, 2013, pp. 188–194, ACM (2013)
15. Magos, D., Mourtos, I.: On the facial structure of the alldifferent system. SIAM J. Discrete Math. **25**(1), 130–158 (2011)
16. Magos, D., Mourtos, I., Appa, G.: A polyhedral approach to the alldifferent system. Math Program. **132**(1-2), 209–260 (2012)
17. Queyranne, M.: An introduction to set functions and optimization. Slides of IMA Seminar. Institute of Mathematics and its Applications, University of Minnesota Available online at https://www.ima.umn.edu/optimization/seminar/queyranne.pdf (2002)
18. Refalo, P.: Linear formulation of constraint programming models and hybrid solvers. In: Dechter, R. (ed.) Principles and Practice of Constraint Programming - CP 2000, 6th International Conference, Singapore,

September 18-21, 2000, Proceedings, volume 1894 of Lecture Notes in Computer Science, pp. 369–383. Springer (2000)
19. Schrijver, A.: Theory of linear and integer programming. John Willey & Sons, New York (1986)
20. Schrijver, A.: Combinatorial Optimization, polyhedra and efficiency. Springer, Berlin (2004)
21. Sellmann, M., Mercier, L., Leventhal, D.: The linear programming polytope of binary constraint problems with bounded tree-width. In: Hentenryck, P.V., Wolsey, L. (eds.) Integration of AI and OR techniques in Constraint Programming for Combinatorial Optimization problems. 4th International Conference CPAIOR 2007, volume 4510 of Lecture Notes in Computer Science, pp. 275–287. Springer (2007)
22. Williams, H.P., Yan, H.: Representations of the all_different predicate of constraint satisfaction in integer programming. INFORMS J. on Computing **13**(2), 96–103 (2001)
23. Yan, H., Hooker, J.: Tight representation of logical constraints as cardinality rules. Math. Program. **85**, 363–377 (1999)
24. Yunes, T.H.: On the sum constraint: Relaxation and applications. In: Proceedings of the 8th International Conference on Principles and Practice of Constraint Programming, CP '02, pp. 80–92. Springer-Verlag (2002)
25. Yunes, T.H., Aron, I.D., Hooker, J.N.: An integrated solver for optimization problems. Oper. Res. **58**(2), 342–356 (2010)