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A new modal logic for reasoning about space: spatial propositional neighborhood logic

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Abstract It is widely accepted that spatial reasoning plays a central role in artificial intelligence, for it has a wide variety of potential applications, e.g., in robotics, geographical information systems, and medical analysis and diagnosis. While spatial reasoning has been extensively studied at the algebraic level, modal logics for spatial reasoning have received less attention in the literature. In this paper we propose a new modal logic, called spatial propositional neighborhood logic (SpPNL for short) for spatial reasoning through directional relations. We study the expressive power of SpPNL, we show that it is able to express meaningful spatial statements, we prove a representation theorem for abstract spatial frames, and we devise a (non-terminating) sound and complete tableaux-based deduction system for it. Finally, we compare SpPNL with the well-known algebraic spatial reasoning system called rectangle algebra.

Keywords Qualitative spatial logic · Deduction systems based on tableaux

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1 Introduction

The principal goal of qualitative spatial representation and reasoning is to capture the common-sense knowledge about space and provide a calculus to handle with spatial information without recursing to a often intractable (or unavailable) quantitative

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model. Although a qualitative formalism may provide only approximate solutions, it copes with the indeterminacy of spatial data and allows inferences based on incomplete spatial knowledge.

As for other qualitative reasoning formalisms (e.g., temporal reasoning), spatial reasoning can be viewed under three different, somehow complementary, points of view. We may distinguish between the *algebraic* level, that is, purely existential theories formulated as constraint satisfaction systems over jointly exhaustive and mutually disjoint set of topological, directional, or combined relations; the *first-order* level, that is, first-order theories of topological, directional, or combined relations; and the *modal logic* level, where a (usually propositional) modal language is interpreted over opportune Kripke structures representing space. The latter approach, though computationally less efficient than the algebraic one, it is often very expressive and allows one to formalize a wide variety of natural language expressions. Spatial reasoning can be also classified in *topological*-based and *directional*-based (in which we are interested here), depending on the type of relations considered. Topological relations can be defined between objects (viewed as set of points) without referring to their shape or their mutual position, while directional-based spatial reasoning (in this paper, we fix our attention on two-dimensional space) is closely related with the shape of the considered object, and the reference system becomes important for the choice of the set of relations. Recent work has been focused on algebraic systems for mixed relations. For a comprehensive, rather recent survey on the various formalisms (topological, directional, and combined constraint systems and relations) see, e.g., [10].

According to [15], we distinguish three different types of reference systems, namely *intrinsic* ones, which depends on some inherent properties of a certain object which serves as relatum, *relative* ones, which rely on a viewpoint that is distinct from relatum or the object to be localized, and *absolute* ones, which, impose a fixed and immutable orientation (e.g., defined by gravity or some other physical property). From now on, we focus our attention on a relative reference system.

In this work we present a new modal logic, called spatial propositional neighborhood logic (SpPNL for short), for reasoning about two-dimensional space by means of directional relations. Regions are approximated by their minimum bounding box, and four modal operators allow one to move along the x- and the y-axis. We present a first-order theory for abstract spatial frames, i.e., we prove a representation theorem, devise a non-terminating sound and complete external tableaux-based deduction system for SpPNL, and, finally, we study the expressive power of SpPNL by showing, on the one hand, that it is able to express useful spatial properties and, on the other hand, by comparing SpPNL with rectangle algebra. It is worth noticing that comparing our approach with the previous (modal) ones presents some difficulties, basically because most of the previous work is based on topological relations instead of directional relations. Nevertheless, the expressive power of SpPNL can be compared with Lutz and Wolter's modal logic of topological relations [16] over a region structure in the Euclidean space $\mathbb{D} \times \mathbb{D}$, where regions are limited to rectangles (see next section); to some extent, SpPNL can be considered as a fragment of it. Preliminary results on SpPNL have been presented in [18, 21, 22].

The present work is organized as follows. First, we briefly review the state-of-the-art on modal logics for spatial reasoning, comparing in some way our approach to the previous ones, and in Section 3 we formally present syntax and semantics of

SpPNL. In Section 4 we provide a representation theorem for abstract spatial frames, while in Section 5 we show some simple examples of application of SpPNL and study the expressive power of SpPNL by means of a comparison with rectangle algebra. Finally, in Section 6 we devise a non-terminating tableaux-based deduction system for it, before concluding.

2 Modal logics for spatial reasoning

In this section we briefly review the literature on modal logics for spatial reasoning, and we compare the approach of SpPNL to spatial reasoning with the one of previous work.

In the context of modal logics for spatial reasoning, we mention Bennett's work [6, 7], later extended by Bennett himself, Cohn, Wolter and Zakharyaschev in [5]. In [6], Bennett proposes to interpret regions as subsets of a given topological space, and shows how it is possible to exploit both the classical propositional calculus and the intuitionistic propositional calculus, together with certain meta-level constraint concerning entailments between formulas, for reasoning about space with topological relations. In such a way, a spatial topological constraints problem can be solved by checking the satisfiability of a logical formula. In [7] Bennett extends his approach by the use of modal languages. Bennett takes into consideration the modal logic S4, and interprets the modal operator in a topological sense, as the interior operator of a given topology. Moreover, in the same work, a modal convex-hull operator is defined and studied, by translating a first-order axiomatization into a modal schemata. In [5], the authors consider a multi-modal system for spatio-temporal reasoning, based on Bennett's previous work. Further research on this issue can be found in [24], where Nutt gives a rigorous foundation of the translation of topological relations into modal logic, introducing generalized topological relations. It is worth to point out that Bennett, Cohn, Wolter, Zakharyaschev, and Nutt's results basically exploit the finite model property and decidability of the classical propositional logic, the modal logic S4, and of some of their extensions. For a recent investigation concerning the major mathematical theories of space from a modal standpoint, see [2].

Unlike Bennett, Cohn, Wolter and Zakharyaschev's work, an important attempt to exploit the whole expressive power of modal logic for reasoning about space (instead of using it for constraint solving) is that of Lutz and Wolter's modal logic for topological relations [16]. Lutz and Wolter present a new propositional modal logic, where propositional variables are interpreted in the regions of topological space, and references to other regions are enabled by modal operators interpreted as topological relations.

There are many possible choices for the set of relations. For example, the set RCC8 [11, 25] contains the relations *equal* (*eq*), *disconnected* (*di*), *externally connected* (*ec*), *tangential proper part* (*tpp*), *inverse of tangential proper part* (*tppi*), *non-tangential proper part* (*ntpp*), *inverse of non-tangential proper part* (*ntppi*), and *partially overlap* (*po*). Among other possibilities, we mention a refinement of RCC8 into 23 relations, and the set RCC5, obtained from RCC8 by keeping the relations *eq* and *po*, but coarsening the relations *tpp* and *ntpp* into a new relation (*proper part*), the relations *tppi* and *ntppi* into a new relation (*inverse of proper part*), and the relations *ec* and *dc* into a new relation *disconnected* (see e.g. [10, 12] for a detailed discussion). Regions are defined as non-empty regular closed subsets of a topological space with no further assumption and Lutz and Wolter analyze the computational properties and the expressive power of the modal logic for different interpretations. They consider the Euclidean space $\mathbb{R}^n$ ($n > 1$), where $\mathbb{R}$ is the set of real numbers, with different topologies, among others: (1) the set of all non-empty closed subsets of topological space; (2) the set of all (hyper)-rectangles; (3) substructures of the above region structures. The modal logic for topological relations presents a bad computational behavior, and it turns out to be generally undecidable. For the matter of a comparison with the present work, Lutz and Wolter have shown that the satisfiability problem for the modal logic of RCC8 relations when the set of basic regions is exactly the set of all (hyper)-rectangles on $\mathbb{R}^2$ is not even recursively enumerable, which means that it is not even possible to devise a semi-decidability method for it.

As for directional relations we mention Venema's compass logic introduced in [26] and further studied in [19]. Compass logic features four modal operators, namely $\Diamond$, $\Diamond$, $\Diamond$, and $\Diamond$, and propositional variables are interpreted as points in the Euclidean two-dimensional space. The modalities are interpreted as the natural *north*, *south*, *east*, and *west* relations between two given points. For example, given a point with coordinates (d_x, d_y) such that p holds on it, one is able to reach a point with coordinates (d'_x, d_y), where $d_x < d'_x$, such that q holds on it, by the formula $p \wedge \Diamond q$. In [19], Marx and Reynolds show that compass logic is undecidable even in the class of all two-dimensional frames. Moreover, Güsgen [13], and Mukerjee and Joe [17] introduced rectangle algebra (RA), which has later been studied by Balbiani, Condotta, and Del Cerro [3, 4]. RA allows one to express any relation between two rectangles in an Euclidean space $\mathbb{D} \times \mathbb{D}$. To our knowledge, the set of all 169 relations between any two rectangles has not been studied at the modal logic level. Nevertheless, it easy to see that the natural propositional modal logic based on RA is not recursively enumerable at least when interpreted in the same classes of frames as Lutz and Wolter's modal logic of topological relations. Indeed, by a straightforward translation it is possible to express the RCC8 relations in RA, which means that the modal logic of topological relations in the topological space of all rectangles on some Euclidean space $\mathbb{D} \times \mathbb{D}$ is a fragment of the modal logic based on RA.

3 Syntax and semantics of SpPNL

Spatial propositional neighborhood logic can be considered as the natural two-dimensional extension of an interval-based temporal logic called propositional neighborhood logic (PNL) [14]. The language for PNL contains a set of propositional variables $\mathcal{AP}$, the propositional logical connectives $\neg$ and $\vee$, and the modalities $\langle A \rangle$ and $\langle \overline{A} \rangle$, the dual operators of which will be denoted by $[A]$ and $[\overline{A}]$, respectively. The remaining classical propositional connectives can be considered as abbreviations. Formulas are recursively defined as follows:

$$\phi = p \mid \neg\phi \mid \phi \vee \psi \mid \langle A \rangle\phi \mid \langle \overline{A} \rangle\phi.$$

The semantics of PNL is given in terms of linear-time models, over which are defined intervals of the type $[d, d']$, and the modalities $\langle A \rangle$ and $\langle \overline{A} \rangle$ correspond to Allen's relations *met by* and *meets*, respectively (see [1]). PNL has been deeply studied (see, e.g, [14, 20]); as we will see, we will be able to adapt some of the results concerning PNL to the spatial case.

The language for SpPNL consists of a set of propositional variables $\mathcal{AP}$, the logical connectives $\neg$ and $\vee$, and the modalities $\langle \mathrm{E} \rangle$, $\langle \mathrm{W} \rangle$, $\langle \mathrm{N} \rangle$, $\langle \mathrm{S} \rangle$. The other logical connectives, as well as the logical constants $\top$ and $\bot$, can be defined in the usual way. SpPNL well formed *formulas*, denoted by ϕ, ψ, ..., are recursively defined as follows (where $p \in \mathcal{AP}$):

$$\phi = p \mid \neg\phi \mid \phi \vee \psi \mid \langle \mathrm{E} \rangle\phi \mid \langle \mathrm{W} \rangle\phi \mid \langle \mathrm{N} \rangle\phi \mid \langle \mathrm{S} \rangle\phi.$$

Given any two linearly ordered sets $\mathbb{H} = \langle H, < \rangle$ and $\mathbb{V} = \langle V, < \rangle$, we call *spatial frame* the structure $\mathbb{F} = (\mathbb{H} \times \mathbb{V})$, and we denote by $\mathbb{O}(\mathbb{F})$ the set of all *objects* (rectangles), that is, $\mathbb{O}(\mathbb{F}) = \{\langle (h, v), (h', v') \rangle \mid h < h', v < v', h, h' \in \mathbb{H}\, v, v' \in \mathbb{V}\}$. The semantics of SpPNL is given in terms of *spatial models* of the type $M = \langle \mathbb{F}, \mathbb{O}(\mathbb{F}), \mathcal{V} \rangle$, where $\mathbb{F}$ is a spatial frame, and $\mathcal{V} : \mathbb{O}(\mathbb{F}) \mapsto 2^{\mathcal{AP}}$ is a *spatial valuation function*. The *truth* relation for a well formed SpPNL-formula ϕ in a model M and an object $\langle (h, v), (h', v') \rangle$ is given by the following clauses:

- $M, \langle (h, v), (h', v') \rangle \Vdash p$ if and only if $p \in \mathcal{V}(\langle (h, v), (h', v') \rangle)$, for any $p \in \mathcal{AP}$;
- $M, \langle (h, v), (h', v') \rangle \Vdash \neg\phi$ if and only if it is not the case that $M, \langle (h, v), (h', v') \rangle \Vdash \phi$;
- $M, \langle (h, v), (h', v') \rangle \Vdash \phi \vee \psi$ if and only if $M, \langle (h, v), (h', v') \rangle \Vdash \phi$ or $M, \langle (h, v), (h', v') \rangle \Vdash \psi$;
- $M, \langle (h, v), (h', v') \rangle \Vdash \langle \mathrm{E} \rangle\phi$ if only if there exists $h'' \in \mathbb{H}$ such that $h' < h''$, and $M, \langle (h', v), (h'', v') \rangle \Vdash \phi$;
- $M, \langle (h, v), (h', v') \rangle \Vdash \langle \mathrm{W} \rangle\phi$ if only if there exists $h'' \in \mathbb{H}$ such that $h'' < h$, and $M, \langle (h'', v), (h, v') \rangle \Vdash \phi$;
- $M, \langle (h, v), (h', v') \rangle \Vdash \langle \mathrm{N} \rangle\phi$ if only if there exists $v'' \in \mathbb{V}$ such that $v' < v''$, and $M, \langle (h, v'), (h', v'') \rangle \Vdash \phi$;
- $M, \langle (h, v), (h', v') \rangle \Vdash \langle \mathrm{S} \rangle\phi$ if only if there exists $v'' \in \mathbb{V}$ such that $v'' < v$, and $M, \langle (h, v''), (h', v) \rangle \Vdash \phi$;

As usual, we denote by [X] the dual operator of the modality $\langle \mathrm{X} \rangle$, where $\langle \mathrm{X} \rangle \in \{\langle \mathrm{E} \rangle, \langle \mathrm{W} \rangle, \langle \mathrm{N} \rangle, \langle \mathrm{S} \rangle\}$, and by $M \Vdash \phi$ the fact that ϕ is *valid* on M.

In order to give a first idea of the expressive power of SpPNL, we list hereby some simple valid formulas:

1. $p \rightarrow [E]\langle \mathrm{W} \rangle p$ (i.e., if p holds in the current rectangle, then no matter how we go on some rectangle to the east of the current one, we are always able to go 'back' to p);
2. $(\langle \mathrm{W} \rangle\langle \mathrm{W} \rangle\top \wedge \langle \mathrm{E} \rangle\langle \mathrm{W} \rangle p) \rightarrow p \vee \langle \mathrm{W} \rangle\langle \mathrm{E} \rangle\langle \mathrm{E} \rangle p \vee \langle \mathrm{W} \rangle\langle \mathrm{W} \rangle\langle \mathrm{E} \rangle p$ (i.e., the horizontal domain is linearly ordered);
3. $\langle \mathrm{N} \rangle\langle \mathrm{E} \rangle p \rightarrow \langle \mathrm{E} \rangle\langle \mathrm{N} \rangle p$ (i.e., the relations N and E are commutative).

In the rest of this section, we prove that the satisfiability problem for SpPNL is undecidable. To this end, we prove that the satisfiability problem for compass logic [26], which has been shown to be undecidable [19], can be polynomially reduced to the one for SpPNL. As recalled in the previous section, compass logic features four modal operators, namely $\Diamond_{\rightarrow}$, $\Diamond_{\leftarrow}$, $\Diamond_{\uparrow}$, and $\Diamond_{\downarrow}$, and propositional variables are interpreted as points in the Euclidean two-dimensional space. Well formed formulas, here denoted by $f, g, \ldots$, can be obtained by the following abstract syntax:

$$f = p \mid \neg f \mid f \vee g \mid \overset{\rightarrow}{\Diamond} f \mid \overset{\leftarrow}{\Diamond} f \mid \overset{\uparrow}{\Diamond} f \mid \overset{\downarrow}{\Diamond} f.$$

The modalities are interpreted as the natural *north*, *south*, *east*, and *west* relations between two given points. For example, given a point with coordinates (d_x, d_y) such that p holds on it, one is able to reach a point with coordinates (d'_x, d_y), where $d_x < d'_x$, such that q holds on it, by the formula $p \wedge \overset{\rightarrow}{\Diamond} q$. Usually, compass logic is interpreted on the Euclidean space $\mathbb{D} \times \mathbb{D}$, where $\mathbb{D} = \langle D, < \rangle$ is any linearly ordered set; nevertheless, in the context of the following proof, for both SpPNL and compass logic it does not matter whether or not the spatial frame is built from the Cartesian product of two identical sets or not. Thus, for reasons of simplicity, in the rest of this section, we will work under the hypothesis that both logics are interpreted over an Euclidean space of the type $\mathbb{D} \times \mathbb{D}$, and we will denote points by the symbols $d_x, d_y, d'_x, \ldots$ in both cases.

Consider the following translation τ from compass logic formulas to SpPNL-formulas:

- $\tau(p) = p$;
- $\tau(\neg f) = \neg \tau(f)$;
- $\tau(f \vee g) = \tau(f) \vee \tau(g)$;
- $\tau(\overset{\uparrow}{\Diamond} f) = \langle \mathrm{N} \rangle \tau(f)$;
- $\tau(\overset{\rightarrow}{\Diamond} f) = \langle \mathrm{E} \rangle \tau(f)$;
- $\tau(\overset{\downarrow}{\Diamond} f) = [N] \langle \mathrm{S} \rangle \langle \mathrm{S} \rangle \tau(f)$;
- $\tau(\overset{\leftarrow}{\Diamond} f) = [E] \langle \mathrm{W} \rangle \langle \mathrm{W} \rangle \tau(f)$.

Now we prove that for any compass logic formula f, we have that f is satisfiable if and only if the SpPNL-formula $\tau(f)$ is satisfiable. We consider a particular class of SpPNL-models. Given a finite set of propositional variables AP and any SpPNL-model $M = \langle \mathbb{F}, \mathbb{O}(\mathbb{F}), \mathcal{V} \rangle$, where $\mathbb{F}$ is a spatial frame of the type $\mathbb{D} \times \mathbb{D}$; then we say that M is *upright local* (with respect to AP) if, for every propositional letter $p \in AP$ and every object $\langle (d'_x, d'_y), (d_x, d_y) \rangle$, we have that $p \in \mathcal{V}(\langle (d'_x, d'_y), (d_x, d_y) \rangle)$ if and only if $p \in \mathcal{V}(\langle (d''_x, d''_y), (d_x, d_y) \rangle)$ for all $d''_x < d_x$ and $d''_y < d_y$. This class of SpPNL-models features a uniform valuation of the propositional letters in AP over all objects having the same upright corner. Such a upright locality constraint is definable in the language of SpPNL by the following formula:

$$\begin{aligned} \Lambda(AP) = \textstyle\bigwedge_{p \in AP'} \big(& p \leftrightarrow ([E]\langle \mathrm{S} \rangle [N] \overline{p}_x \wedge [N] \langle \mathrm{E} \rangle [W] \overline{p}_y) \\ & \wedge \neg p \leftrightarrow ([E]\langle \mathrm{S} \rangle [N] \neg \overline{p}_x \wedge [N] \langle \mathrm{E} \rangle [W] \neg \overline{p}_y) \end{aligned}$$

where, for every $p \in AP$, we extended the language with new propositional variables $\overline{p}_x$ and $\overline{p}_y$. The length of $\Lambda(AP)$ is polynomial in $|AP|$.

Lemma 1 *Given a SpPNL-model M and a finite set of propositional variables AP', if $M, \Vdash \Lambda(AP)$, then M is upright local with respect to AP.*

Let $AP|_f$ be the set of all propositional variables occurring in f. By using a technique similar to the one which can be found in [8], it is possible to prove that $\tau(f)$ is satisfiable over upright local SpPNL-models if and only if f is satisfiable, by providing a suitable translation between compass logic models and upright local

SpPNL-models, and vice versa (notice that the universal operator is definable in SpPNL, as we will see in the Section 5).

Lemma 2 *If f is any compass logic formula, then f is satisfiable if and only if $\tau(f)$ is satisfiable in the class of all upright local SpPNL-models.*

Theorem 3 *The satisfiability problem for SpPNL is not decidable.*

4 Representation theorem for spatial frames

In this section we consider the problem of finding a sound and complete first-order representation for spatial frames. As we have recalled in the Introduction, in the literature of spatial reasoning some attention has been given to (existential) theories such as rectangle algebra and region connection calculus; nevertheless, for some reason, no representation theorems have been shown for spatial frames (based on directional relations) so far. Some results on this topic can be found in [16], and in [3].

Let us start with some useful definitions.

Definition 4 An *abstract spatial frame* (*ASF*) is a triple $ASF = \langle U, E, N\rangle$, where U is any non-empty set, and $E, N \subseteq U \times U$.

Abstract spatial frames are first-order structures built up from a non-empty universe U and two binary relations E (*east*) and N (*north*). The main problem is now to provide opportune first-order conditions on E and N in order to make an abstract spatial frame isomorphic to a (concrete) spatial frame defined as in the previous section. Elements of U will be called (abstract) *objects*. Intuitively, E (resp., N) correspond to the RA-relation (mi, e) [resp., (e, mi)]. As observed in [3], these two relations must be sufficient to express any other RA-relation.

Definition 5 Let $ASF = \langle U, E, N\rangle$ be an abstract spatial frame. Then, the relation $W \subseteq U \times U$ is defined as follows: $\forall x, y(xWy \leftrightarrow yEx)$, and the relation $S \subseteq U \times U$ is defined as follows: $\forall x, y(xSy \leftrightarrow yNx)$.

Now, let $R_1 = EW \cup EEW \cup EWW$, $R_2 = SN \cup SNN \cup SSN$, and consider the following first-order conditions:

- Same objects have the same endpoints:

  A1) $\forall x, y(\exists z(xEz \wedge zWy) \rightarrow \forall z(xEz \rightarrow zWy))$;
  A2) $\forall x, y(\exists z(xWz \wedge zEy) \rightarrow \forall z(xWz \rightarrow zEy))$;
  A3) $\forall x, y(\exists z(xNz \wedge zSy) \rightarrow \forall z(xNz \rightarrow zSy))$;
  A4) $\forall x, y(\exists z(xSz \wedge zNy) \rightarrow \forall z(xSz \rightarrow zNy))$;

- Abstract spatial frames are plane:

  B1) $\forall x, y, z[xEy \wedge xEz \rightarrow y = z \vee \exists w(\forall k(zEk \rightarrow yEw \wedge wEk) \wedge \forall k(kWw \rightarrow kWz \wedge wWy)) \vee \exists w(\forall k(yEk \rightarrow zEw \wedge wEk) \wedge \forall k(kWw \rightarrow kWy \wedge wEz))]$;

B2) $\forall x, y, z[xWy \wedge xWz \rightarrow y = z \vee \exists w(\forall k(zWk \rightarrow yWw \wedge wWk) \wedge \forall k(kEw \rightarrow kEz \wedge wEy)) \vee \exists w(\forall k(yWk \rightarrow zWw \wedge wWk) \wedge \forall k(kEw \rightarrow kEy \wedge wEz))]$;
B3) $\forall x, y, z[xNy \wedge xNz \rightarrow y = z \vee \exists w(\forall k(zNk \rightarrow yNw \wedge wNk) \wedge \forall k(kSw \rightarrow kSz \wedge wSy)) \vee \exists w(\forall k(yNk \rightarrow zNw \wedge wNk) \wedge \forall k(kSw \rightarrow kSy \wedge wNz))]$;
B4) $\forall x, y, z[xSy \wedge xSz \rightarrow y = z \vee \exists w(\forall k(zSk \rightarrow ySw \wedge wSk) \wedge \forall k(kNw \rightarrow kNz \wedge wNy)) \vee \exists w(\forall k(ySk \rightarrow zSw \wedge wSk) \wedge \forall k(kNw \rightarrow kNy \wedge wSz))]$;

- Relations are pseudo-transitive:

C1) $\forall x, y, z, w(xEy \wedge yEz \wedge zEw \rightarrow \exists k(xEk \wedge kEw))$;
C2) $\forall x, y, z, w(xWy \wedge yWz \wedge zWw \rightarrow \exists k(xWk \wedge kWw))$;
C3) $\forall x, y, z, w(xNy \wedge yNz \wedge zNw \rightarrow \exists k(xNk \wedge kNw))$;
C4) $\forall x, y, z, w(xSy \wedge ySz \wedge zSw \rightarrow \exists k(xSk \wedge kSw))$;

- Abstract objects have non-zero area:

D1) $\forall x, y, z, w(xEy \wedge yWz \wedge zWw \rightarrow x \neq w)$;
D2) $\forall x, y, z, w(xNy \wedge ySz \wedge zSw \rightarrow x \neq w)$;
D3) $\forall x, y, z, w(xWy \wedge yEz \wedge zEw \rightarrow x \neq w)$;
D4) $\forall x, y, z, w(xSy \wedge yNz \wedge zNw \rightarrow x \neq w)$;

- Abstract spatial frames are normal:

E1) $\forall x, y(\forall z(zEx \leftrightarrow zEy) \wedge \forall z(zWx \leftrightarrow zWy) \wedge \forall z(zNx \leftrightarrow zNy) \wedge \forall z(zSx \leftrightarrow zSy)) \rightarrow x = y$;

- Abstract spatial frames are standard (there are no 'holes'):

F1) $\forall x, y, z(xEy \wedge yNz \rightarrow \exists k(xNk \wedge kEz))$;
F2) $\forall x, y, z(xWy \wedge ySz \rightarrow \exists k(xWk \wedge kSz))$;

- Abstract spatial frames are connected:

G1) $EW(R_2) \cup EEW(R_2) \cup EWW(R_2)$ and $WE(R_2) \cup WEE(R_2) \cup WWE(R_2)$ are the universal relation on U;
G2) $SN(R_1) \cup SNN(R_1) \cup SSN(R_1)$ and $NS(R_1) \cup NSS(R_1) \cup NNS(R_1)$ are the universal relation on U.

Theorem 6 (Representation theorem) *Every abstract spatial frame such that conditions from A to E are respected is isomorphic to a (concrete) spatial frame.*

Proof Let $ASF = \langle U, E, N \rangle$ any ASF such that it respects all given conditions. We construct an underlying point-based spatial frame $\mathbb{F} = (\mathbb{H} \times \mathbb{V})$ and the set of all objects on that frame.

First, we have to define the endpoints of abstract objects. Let u be any abstract object for the universe U. We identify the minimum horizontal coordinate as follows:

$$\mathrm{hmin}(u) = \{x \in U \mid xWEu\},$$

and the minimum vertical coordinate as follows:

$$\mathrm{vmin}(u) = \{x \in U \mid xSNu\}.$$

Similarly, we can define functions hmax(u) and vmax(u). Now, we have to relate somehow horizontal and vertical coordinates. In order to do that, we first observe that the relations EW, WE, NS, and SW are equivalence relations in U (by conditions A). This means that $P_{\text{hmin}} = \{[u]_{\text{hmin}(u)} \mid u \in U\}$, $P_{\text{hmax}} = \{[u]_{\text{hmax}(u)} \mid u \in U\}$, and their vertical counterparts are partitions of U. Now, it is not difficult to see that the functions λ_h and λ_v, defined as follows:

$$\lambda\left([u]_{\text{hmax}(u)}\right) = [x]_{\text{hmin}(x)}$$

where uEx, and

$$\lambda\left([u]_{\text{vmax}(u)}\right) = [x]_{\text{vmin}(x)}$$

(where uNx), are isomorphisms. Thus we restrict our attention to the set of minimum coordinates (both in the horizontal and the vertical sense). We now define the sets $H = P_{\text{hmin}}$ (and we denote its elements by $h, h', \ldots$) and $V = P_{\text{vmin}}$ $(v, v', \ldots)$, and, for each one of them, a relation $<$ as follows:

$$h < h' \leftrightarrow hWEEh',$$

and

$$v < v' \leftrightarrow vSNNv'.$$

The relations $<$ are total ordering relations; we now restrict our attention on the relation $<$ defined in the set H, since the considerations for V are completely analogous. In order to show that $<$ totally orders H we have to prove that:

1. $<$ is irreflexive. Suppose that $h < h$ for some $h \in H$; this means that there is some $u \in U$ such that $[u]_{\text{hmin}(u)} WEE [u]_{\text{hmin}(u)}$, that is, $uWEEu$, which is in contradiction with conditions D;
2. $<$ is transitive. Suppose that $h < h'$ and $h' < h''$ for some $h, h', h'' \in H$; this means that for some $u, v, w \in U$ it holds that $[u]_{\text{hmin}[u]} WEE [v]_{\text{hmin}[v]}$, and that $[v]_{\text{hmin}[u]} WEE [w]_{\text{hmin}[v]}$, that is, $uWEEv$ and $vWEEw$; by conditions A, we have that $EWE \subseteq E$, and, thus, it holds that $uWEEEw$; by conditions C, it results that $uWEEw$, that is, $[u]_{\text{hmin}(u)} WEE [w]_{\text{hmin}(w)}$, and, finally, $h < h''$;
3. $<$ is linear. Suppose that $h < h'$ and that $h < h''$ for some $h, h', h'' \in H$; this means that for some $u, v, w \in U$ it holds that $[u]_{\text{hmin}[u]} WEE [v]_{\text{hmin}[v]}$, and that $[u]_{\text{hmin}[u]} WEE [w]_{\text{hmin}[v]}$, that is, $uWEEv$ and $uWEEw$; now, by conditions B, for some $t \in U$ it must be the case that uWt and that there are $t', t'' \in U$ with tEt', tEt'', $t'Ev$, and $t''Ew$; by conditions B, we have three possibilities: (1) $t' = t''$, which implies, by conditions A, that $u = w$, and, thus, that hmin(v) = hmin(w), that is, $h' = h''$; (2) there exists some abstract object z such that $t'Ez$ and that z for each abstract object k, $zEk \leftrightarrow t''Ek$, that is, zEw, which implies that vEt', $t'Ez$, and zEw, i.e. $vWEEw$, or, in other words, hmin(v) $<$ hmin(w) by definition; (3) similarly to (2), but exchanging the roles of t' and t'';
4. $<$ is total. Directly from conditions G.

Finally, let $\mathbb{H} = \langle H, <\rangle$ and $\mathbb{V} = \langle V, <\rangle$, and let $\mathbb{O}(\mathbb{F}) = \{\langle (h, v), (h', v')\rangle \mid h < h'$ $v < v', h, h' \in \mathbb{H}$ $v, v' \in \mathbb{V}\}$. We have to show that $\mathbb{O}(\mathbb{F})$ is isomorphic to U. Consider the mapping $\mu : U \mapsto \mathbb{O}(\mathbb{F})$ defined as

$$\mu(u) = \left\langle \left([u]_{\text{hmin}(u)}, [u]_{\text{vmin}(u)}\right), \left(\lambda\left([u]_{\text{hmax}(u)}\right), \lambda\left([u]_{\text{vmax}(u)}\right)\right)\right\rangle.$$

Clearly $[u]_{\mathrm{hmin}(u)} < [u]_{\mathrm{hmax}(u)}$, and $[u]_{\mathrm{hmin}(u)} < [u]_{\mathrm{hmax}(u)}$, which means that $\langle([u]_{\mathrm{hmin}(u)}, [u]_{\mathrm{vmin}(u)}), (\lambda([u]_{\mathrm{hmax}(u)}), \lambda([u]_{\mathrm{vmax}(u)}))\rangle$ is a well defined rectangle in $\mathbb{O}(\mathbb{F})$. We have to show that μ is an isomorphism. (1): μ is injective. Let $\mu(u) = \mu(v)$; now, $\langle([u]_{\mathrm{hmin}(u)}, [u]_{\mathrm{vmin}(u)}), (\lambda([u]_{\mathrm{hmax}(u)}), \lambda([u]_{\mathrm{vmax}(u)}))\rangle$ must be equal to $\langle([v]_{\mathrm{hmin}(v)}, [v]_{\mathrm{vmin}(v)}), (\lambda([v]_{\mathrm{hmax}(v)}), \lambda([v]_{\mathrm{vmax}(v)}))\rangle$ (component by component). By definition, for all t, t' such that uEt and vEt' (resp., W, N, S), we have that $[t]_{\mathrm{hmin}[t]} = [t']_{\mathrm{hmin}[t']}$. By conditions A, we have that u and v 'see' the same abstract objects on each of the four directions, and, by conditions E, $u = v$. (2): μ is surjective. Let $h, h' \in H$, $v, v' \in V$, and let $\langle(h, v), (h', v')\rangle \in \mathbb{O}(\mathbb{F})$, and we have to show that there exists some $u \in U$ such that $\mu(u) = \langle(h, v), (h', v')\rangle$. If $\langle(h, v), (h', v')\rangle \in \mathbb{O}(\mathbb{F})$, then $h < h'$ and $v < v'$, and there exist $t, s, w, z \in U$ such that, $[t]_{\mathrm{hmin}(t)} = h$,$[s]_{\mathrm{hmin}(s)} = h'$,$[w]_{\mathrm{vmin}(w)} = v$,$[z]_{\mathrm{vmin}(z)} = v'$. By definition, we have that $tWEEs$ and $wSNNz$, which implies the existence of some $t', w' \in U$ such that $tWEt'$, $t'Es$, $wSNw'$, and $w'Nz$. Now, from the existence of t', w', we can show that it is possible to go from t' to w' (and the other way around) through opportune elements of U (by exploiting conditions G, that is, the connectedness of the abstract spatial frame), and to deduce the existence of a certain object $u \in U$ such that $[u]_{\mathrm{hmin}(u)} = h$,$[u]_{\mathrm{hmax}(u)} = h'$, $[u]_{\mathrm{vmin}(u)} = v$, $[v]_{\mathrm{vmax}(u)} = v'$ (by conditions F), which is exactly the abstract object we are searching for (3): μ respects the relations. This is immediate by the definitions. □

5 Simple applications and expressive power of SpPNL

As we will see below, in SpPNL only 25 out of 169 (see Fig. 1) possible basic RA-relations are directly expressible. Anyway, very natural relations such as *southeast* or *northwest* can be easily expressed; for example, we can define the modal operator for *southeast* as follows:

$$\begin{aligned}\langle \mathrm{SE}\rangle\phi = {} & \langle \mathrm{E}\rangle\langle \mathrm{S}\rangle\phi \vee \langle \mathrm{E}\rangle\langle \mathrm{S}\rangle\langle \mathrm{S}\rangle\phi \vee \langle \mathrm{E}\rangle\langle \mathrm{E}\rangle\langle \mathrm{S}\rangle\phi \\ & \vee\langle \mathrm{E}\rangle\langle \mathrm{E}\rangle\langle \mathrm{S}\rangle\langle \mathrm{S}\rangle\phi.\end{aligned}$$

Notice that the above definition captures any region to the south-east of the current one, no matter if their MBB meet (on either of the two axes) or not. Also, in SpPNL it is possible to express 2 out of the 8 RCC8 topological relations (namely *disconnected* and *equal*) in the topological space of all rectangles.

As another example, we can translate in SpPNL a natural language statement borrowed from the geographical context such as: *suppose that at the southeast of the current region there exists a region containing water* (w) *at the northeast of which there are no trees* (t) *at all; so we can deduce that there exists at least one region at the east of the current one* (*with no side in common with it*) *with no trees.* Such a statement can be expressed by means of the following (valid) formula:

$$\langle \mathrm{S}\rangle\langle \mathrm{E}\rangle(w \wedge \langle \mathrm{E}\rangle\top \wedge \neg\langle \mathrm{N}\rangle\langle \mathrm{E}\rangle\, t) \rightarrow \langle \mathrm{E}\rangle\langle \mathrm{E}\rangle\neg t.$$

Now we focus our attention on the so-called rectangle algebra (RA). The considered objects in RA are rectangles whose sides are parallel to the axes of some orthogonal basis in a bidimensional Euclidean space $\mathbb{D} \times \mathbb{D}$, where $\mathbb{D} = \langle D, <\rangle$ is a

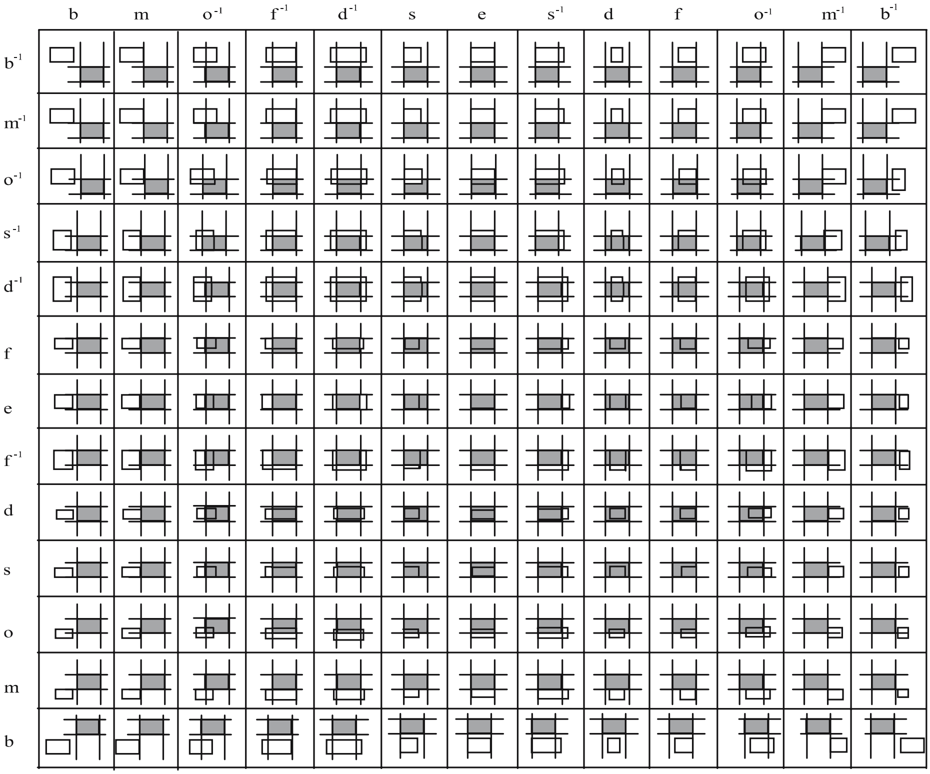


Fig. 1 The basic relations between two rectangles

linearly ordered set.[1] Since we are going to compare SpPNL with RA, for simplicity of notation we consider that SpPNL is interpreted over a spatial frame $\mathbb{D} \times \mathbb{D}$ generated by any linearly ordered set $\langle D, < \rangle$, and we denote points by $d_x, d'_x, \ldots$ when they belong to the x-axis, and by $d_y, d'_y \ldots$ for the y-axis. A *basic RA-relation* (or, simply, a *basic relation*) between two rectangles O_1 and O_2 is a pair $R = (r_i, r_j)$, where r_i and r_j (called *components*) are Allen's interval relations. As standard, we denote by r^{-1} the inverse of a basic Allen's relation r. As for example, if the rectangle O_1 is entirely included into the rectangle O_2, and no side of O_1 touches any of the sides of O_2, then the relations between O_1 and O_2 is (d, d), where d represents Allen's relation *during*. In this way, there are $13^2 = 169$ possible basic relations between any two given rectangles, as shown in Fig. 1. For RA, the basic operations of *inverse*, *composition*, and *intersection* are defined. A *relation* $\hat{R}$ in RA is a set $\{R_1, R_2, \ldots, R_n\}$, where, for each i, R_i is a basic RA-relation. The set of basic

[1] It is worth noticing that original results concerning rectangle algebra, such as consistency checking of a network of constraint, have been given for a spatial frame of the type $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers; nevertheless, RA-relations and the problem of consistency of a network of constraint can be defined for any spatial frame.

RA-relations is denoted by $\mathcal{B}$; thus, any RA-relation belongs to the set $2^{\mathcal{B}}$, which means that there are 2^{169} possible relations.

Definition 7 Let $\mathbb{O}(\mathbb{F})$ be the set of all rectangles defined in the Euclidean space $\mathbb{F} = \mathbb{D} \times \mathbb{D}$. We call RA-constraint the expression $(O_i \; \hat{R} \; O_j)$, where $O_i, O_j \in \mathbb{O}(\mathbb{F})$, and $\hat{R}$ is any RA-relation. A RA-*network* N is a pair $N = \langle \mathcal{O}, \mathcal{R} \rangle$ where $\mathcal{O}$ is a set of variables which take values over $\mathbb{O}(\mathbb{F})$ and $\mathcal{R}$ is a set of binary RA-constraints between elements of $\mathcal{O}$.

Given any RA-network N, the main problem is to know whether it is or not consistent.

Definition 8 A RA-network $\langle \mathcal{O}, \mathcal{R} \rangle$ is said to be *consistent* if and only if there exists a concrete instance of the whole set $\mathcal{O}$ such that it respects any constraint in $\mathcal{R}$.

In simple words, an RA-network is said to be consistent if and only if the spatial information represented by N is coherent. Generally speaking, the consistency problem is NP-complete [17]; Balbiani, del Cerro, and Condotta [3, 4] study tractable sub-fragments of RA.

Now, we will show how it is possible to check the consistency of any given RA-network N by checking the satisfiability of a SpPNL-formula $\phi(N)$; to this end, we first show how it is possible to express any (basic and non-basic) relation in SpPNL. Consider the following shorthand:

$$hor(\phi) = [\mathrm{W}][\mathrm{W}][\mathrm{E}]\phi \wedge [\mathrm{W}][\mathrm{E}][\mathrm{E}]\phi \wedge [\mathrm{E}][\mathrm{E}][\mathrm{W}]\phi \wedge [\mathrm{E}][\mathrm{W}][\mathrm{W}]\phi.$$

The operator $hor(\phi)$ states that the formula ϕ is satisfied by any rectangle $\langle (h, v), (h', v') \rangle$ such that v, v' are the same as the current rectangle. Similarly, an operator $ver(\phi)$ can be defined. This means that in SpPNL it is possible to express the *difference* operator:

$$[\neq](\phi) = hor(ver(\phi) \wedge \phi),$$

and, thus, to simulate the *universal modality* and *nominals*:

$$u(\phi) = \phi \wedge [\neq]\phi \;\text{ and }\; n(p) = p \wedge [\neq](\neg p),$$

where $n(p)$ states that p holds in the current rectangle and nowhere else. In the following, we will use simulation of nominals in the following in order to translate basic RA-relations into SpPNL-formulas. For a given set of propositional letters $\mathcal{AP}$, and set of object variables $O_1, O_2, \ldots$, we define the set of propositional letters $\mathcal{AP}'$ as an extension of $\mathcal{AP}$ which contains at least a new propositional letter $p(O_i)$ for any object variable O_i. Now, for a generic RA-constraint $(O_i \; \hat{R} \; O_j)$, where O_i, O_j are object variables, and $\hat{R}$ is a RA-relation, we want to write a formula $\phi_{\hat{R}}(O_i, O_j)$ such that it respects the RA-constraint $(O_i \; \hat{R} \; O_j)$.

Definition 9 Given any RA-constraint $(O_i \; \hat{R} \; O_j)$, we say that the SpPNL-formula $\phi_{\hat{R}}(O_i, O_j)$ *respects* $(O_i \; \hat{R} \; O_j)$ if and only if for any spatial model $M = \langle \mathbb{F}, \mathbb{O}(\mathbb{F}), \mathcal{V} \rangle$, we have that $M, \langle (d_x, d_y), (d'_x, d'_y) \rangle \Vdash \phi_{\hat{R}}(O_i, O_j)$ if and only if, if it is the case that

$p(O_i) \in V(\langle (d_x, d_y), (d'_x, d'_y)\rangle)$, then we have that $p(O_j) \in V(\langle (d''_x, d''_y), (d'''_x, d'''_y)\rangle)$ for some rectangle $\langle (d''_x, d''_y), (d'''_x, d'''_y)\rangle$ such that $\langle (d_x, d_y), (d'_x, d'_y)\rangle \ \hat{R} \ \langle (d''_x, d''_y), (d'''_x, d'''_y)\rangle$.

To begin with, we consider basic relations. It turns out that 169 different formulas are needed in order to translate all the basic RA-relations. We can divide such relations into three groups, as follows.

Direct relations These are the 25 basic relations that can be directly expressed in SpPNL (see Fig. 2). For example, we have that the basic constraint $(O_1 \ (e, e) \ O_2)$ can be expressed by the formula $p(O_1) \rightarrow p(O_2)$. As another example, the basic constraint $(O_1 \ (b, b) \ O_2)$ can be expressed by the formula $p(O_1) \rightarrow \langle \mathrm{E}\rangle\langle \mathrm{E}\rangle\langle \mathrm{N}\rangle\langle \mathrm{N}\rangle p(O_2)$.

Lemma 10 *For any direct basic* RA-*constraint* $(O_i \ R \ O_j)$, *there exists a* SpPNL-*formula* $\phi_R(O_i, O_j)$ *that respects* $(O_i \ R \ O_j)$.

Partially indirect relations A partially indirect RA-relation $R = (r_i, r_j)$ is any basic RA-relation such that exactly one of its components can be directly expressed in SpPNL. Focusing the attention onto a single axis, there are five Allen's relations

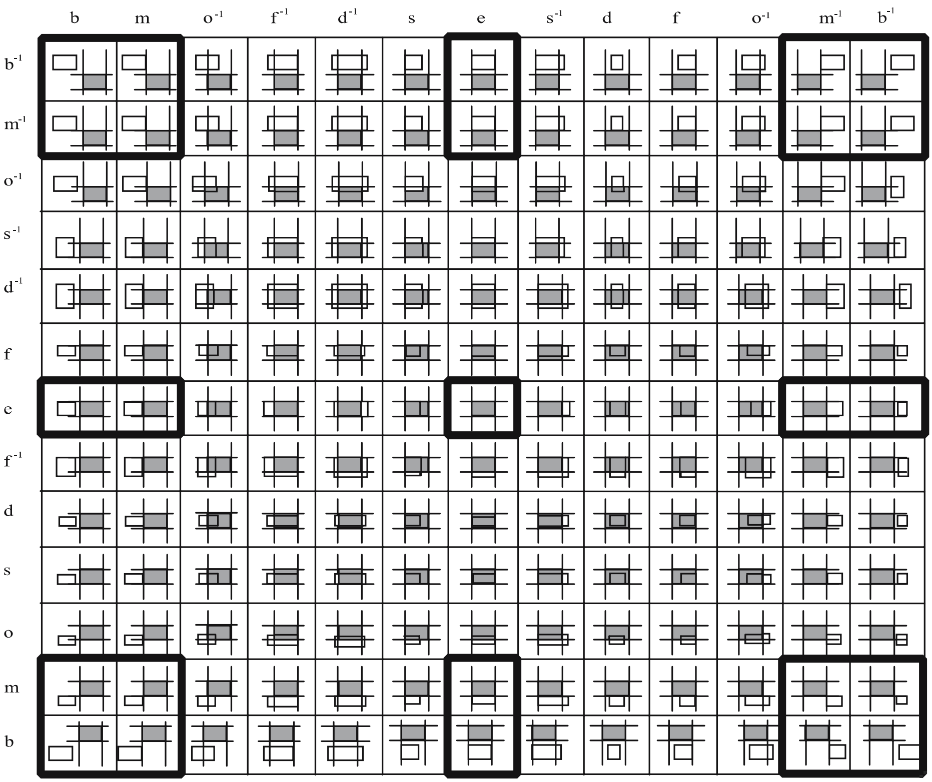
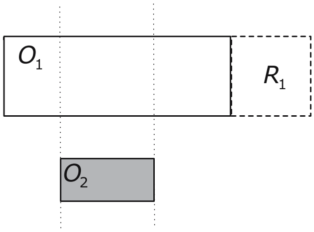


Fig. 2 The basic relations between two rectangles directly expressible in SpPNL

expressible in SpPNL, namely $\{e, m, b, m^{-1}, b^{-1}\}$. This means that there are 80 partially indirect RA-relations: if $R = (r_i, r_j)$ is a partially indirect relations, then r_i can be chosen in a set of five Allen's relations, and r_i in the remaining set of eight Allen's relations, or the other way around; for example, the relation (d^{-1}, b^{-1}) is partially indirect. It turns out that, if $(O_i\ R\ O_j)$ is partially indirect, using at most two (simulation of) nominals it is possible to write a SpPNL-formula $\phi_R(O_i, O_j)$ respecting R. Consider for example the relation $O_1\ (d^{-1}, b^{-1})\ O_2$, depicted hereafter:



The propositional variable denoted by R_1 represents a nominal that can be used in order to express the relation (d^{-1}, b^{-1}). Consider the formula $\phi_{(d^{-1},b^{-1})}(O_1,\ O_2) = p(O_1) \rightarrow \langle E\rangle\ n(p_{R_1}) \wedge \langle W\rangle\ \langle E\rangle\langle E\rangle\ (\langle E\rangle\langle E\rangle\ n(p_{R_1}) \wedge\ \langle S\rangle\langle S\rangle\ (p(O_2)))$, where $p(O_1)$ and $p(O_2)$ are propositional variables representing objects, and p_{R_1} is a propositional variable used here to simulate a nominal.

Proposition 11 *The formula* $\phi_{(d^{-1},b^{-1})}(O_1, O_2)$ *respects the* RA-*constraint* $(O_1\ (d^{-1}, b^{-1})\ O_2)$.

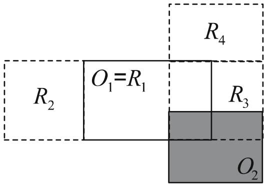
Proof Suppose that there exists a spatial model M such that $M, \langle (d_x, d_y), (d'_x, d'_y)\rangle \Vdash \phi_{(d^{-1},b^{-1})}(O_1, O_2)$. This means that $M, \langle (d_x, d_y), (d'_x,\ d'_y)\rangle \Vdash p(O_1) \rightarrow \langle E\rangle\ n(p_{R_1}) \wedge \langle W\rangle\langle E\rangle\ \langle E\rangle\ (\langle E\rangle\langle E\rangle\ n(p_{R_1}) \wedge \langle S\rangle\langle S\rangle\ (p(O_2)))$. Thus, suppose that $p(O_1) \in V(\langle (d_x, d_y), (d'_x, d'_y)\rangle)$. In this case, we have that at some object $\langle (d'_x, d_y),\ (\hat{d}_x,\ d'_y)\rangle$ such that $d'_x < \hat{d}_x$ it holds p_{R_1}, and nowhere else. So, since at the object $\langle (d_x, d_y), (d'_x, d'_y)\rangle$ it holds $\langle W\rangle\langle E\rangle\ \langle E\rangle\ (\langle E\rangle\langle E\rangle\ n(p_{R_1}) \wedge \langle S\rangle\langle S\rangle\ (p(O_2)))$, the only possible way to place $p(O_2)$ is over an object $\langle (d''_x, d''_y), (d'''_x,\ d'''_y)\rangle$ such that $d_x < d''_x$, $d'''_x < d'_x$, and $d'_y < d_y$. Thus, $\langle (d''_x, d''_y), (d'''_x, d'''_y)\rangle$ is in the relation (d^{-1}, b^{-1}) with $\langle (d_x, d_y), (d'_x, d'_y)\rangle$. $\square$

The proof of the following lemma would require 80 different cases, but it goes exactly as in the above proof.

Lemma 12 *For any partially indirect basic* RA-*constraint* $(O_i\ R\ O_j)$, *there exists a* SpPNL-*formula* $\phi_R(O_i, O_j)$ *that respects* $(O_i\ R\ O_j)$.

Indirect relations An indirect RA-relation $R = (r_i, r_j)$ is any basic RA-relation such that none of its components can be directly expressed in SpPNL. As we have seen above, 8 out of 13 Allen's relations (involving the projections of two objects on a single axis) cannot be directly expressed SpPNL. Such relations are those belonging to the set $I - \{e, m, b, m^{-1}, b^{-1}\}$, where I is the set of all Allen's relations. This means that there are 64 indirect RA-relations: if $R = (r_i, r_j)$ is a partially indirect relations, then both r_i and r_j can be chosen in a set of 8 Allen's relations;

as for example, the relation (o, o^{-1}) is indirect. It turns out that, if $(O_i\ R\ O_j)$ is indirect, by using four (simulation of) nominals it is possible to write a SpPNL-formula $\phi_R(O_i, O_j)$ respecting $(O_i\ R\ O_j)$. Consider for example the constraint $(O_1\ (o, o^{-1})\ O_2)$, depicted hereafter:



The propositional variables denoted by R_1, R_2, R_3 and R_4 represent propositional letters that can be used as simulation of nominals in order to express the relation (o, o^{-1}). Consider the formula $\phi_{(o,o^{-1})}(O_1, O_2) = p(O_1) \rightarrow n(p_{R_1}) \wedge \langle \mathrm{W}\rangle n(p_{R_2}) \wedge \langle \mathrm{E}\rangle\ \langle \mathrm{E}\rangle\ \langle \mathrm{W}\rangle([W]\neg p_{R_1}\ \wedge\ [W]\ [W]\neg p_{R_1} \wedge \langle \mathrm{W}\rangle\ \langle \mathrm{W}\rangle\ p_{R_2}\ \wedge\ (n(p_{R_3}) \wedge \langle \mathrm{N}\rangle n(p_{R_4}) \wedge \langle \mathrm{S}\rangle\ \langle \mathrm{S}\rangle\ \langle \mathrm{N}\rangle\ (p(O_2) \wedge [N]\ \neg p_{R_3} \wedge [N][N]\neg p_{R_3} \wedge \langle \mathrm{N}\rangle\ \langle \mathrm{N}\rangle\ p_{R_4})))$.

Proposition 13 *The formula $\phi_{(o,o^{-1})}(O_1, O_2)$ respects the* RA-*constraint $(O_1\ (o, o^{-1})\ O_2)$.*

Proof Suppose that there exists a spatial model M such that $M, \langle(d_x, d_y), (d'_x, d'_y)\rangle \Vdash \phi_{(o,o^{-1})}(O_1, O_2)$, and $M, \langle(d_x, d_y), (d'_x, d'_y)\rangle \Vdash p(O_1)$. This means that the same rectangle $\langle(d_x, d_y), (d'_x, d'_y)\rangle$ makes true p_{R_1}, and p_{R_1} is false everywhere else. Moreover, by $\langle \mathrm{W}\rangle n(p_{R_2})$, we have that at some rectangle $\langle(d''_x, d_y), (d_x, d'_y)\rangle$ with $d''_x < d_x$ the propositional letter p_{R_2} is true, and nowhere else. Now, the only way to place p_{R_3} is at some rectangle $\langle(\overline{d_x}, \overline{d_y}), (\overline{d'_x}, \overline{d'_y})\rangle$ such that its x-projection $[\overline{d_x}, \overline{d'_x}]$ overlaps the segment $[d_x, d'_x]$. In this way, p_{R_3} is true at $\langle(\overline{d_x}, \overline{d_y}), (\overline{d'_x}, \overline{d'_y})\rangle$ and nowhere else, and p_{R_4} is true at some rectangle $\langle(\overline{d_x}, d'_y), (\overline{d'_x}, d''_y)\rangle$, with $d'_y < d''_y$, and nowhere else. By using the same strategy as above, $p(O_2)$ must be placed at some rectangle $\langle(\overline{d_x}, \hat{d}_y), (\overline{d'_x}, \hat{d}'_y)\rangle$ such that its y-projection $[\hat{d}_y, \hat{d}'_y]$ is overlapped by $[d_y, d'_y]$. □

Again, by exploring 64 different cases it is possible to prove the following lemma.

Lemma 14 *For any indirect basic* RA-*constraint $(O_i\ R\ O_j)$, there exists a* SpPNL-*formula $\phi_R(O_i, O_j)$ that respects $(O_i\ R\ O_j)$.*

Now we consider a generic RA-constraint of the type $(O_i\ \hat{R}\ O_j)$, where $\hat{R} = \{R_1, \ldots, R_n\}$. Such a constraint is interpreted as a logical disjunction: $(O_i\ \hat{R}\ O_j)$ holds if and only if it holds $\bigvee_{i=1}^{n}((O_i\ R_i\ O_j))$. Thus, we have the following results.

Lemma 15 *Given any* RA-*constraint of the type $(O_i\ \hat{R}\ O_j)$, where $\hat{R} = \{R_1, \ldots, R_n\}$, the* SpPNL-*formula $\phi_{\hat{R}}(O_i, O_j) = \bigvee_{i=1}^{n}(\phi_{R_i}(O_i, O_j))$ respects $(O_i\ \hat{R}\ O_j)$.*

Finally, we have to prove the main result, that is, that for any RA-network $N = \{\hat{R}_1, \ldots, \hat{R}_n\}$ there exists a SpPNL-formula which is satisfiable if and only if N is consistent. Consider any RA-network N, and let $\{O_1, \ldots, O_k\}$ be the set of all objects involved in N. Now, for a given constraint of the type $(O_i\ \hat{R}_p\ O_j)$, where $\hat{R}_p = \{R_1, \ldots, R_n\}$ and $1 \leq p \leq n$, with a little abuse of notation, let us denote by $\phi_{\hat{R}}(O_i, O_j, \mathfrak{N}_p)$ the SpPNL-formula respecting $\hat{R}_p$ such that $\mathfrak{N}_p$ is the set of all propositional variables p_{R_l} used to simulate nominals in $\phi_{\hat{R}}(O_i, O_j)$. Consider the following formula:

$$\phi(N) = \bigwedge_{i=1}^{k}(e(n(p(O_i)))) \wedge \bigwedge_{i=1}^{n}(u(\phi_{\hat{R}_i}(O_l, O_m, \mathfrak{N}_i))),$$

where $e(\phi)$ is the *existential* operator (which is definable in SpPNL, as a consequence of the definability of the universal one), for each i, $p(O_i)$ is a propositional letter, for all $1 \leq i, j \leq n$, if $i \neq j$ then $\mathfrak{N}_i \cap \mathfrak{N}_j = \emptyset$, and, for all $1 \leq i \leq k$, $\mathfrak{N}_i \cap \{p(O_1), \ldots, p(O_k)\} = \emptyset$.

Theorem 16 *Let N be any* RA-*network. Then N is consistent if and only if the* SpPNL-*formula $\phi(N)$ is satisfiable.*

6 A tableau-based method for SpPNL

In this section we devise a tableau-based method for SpPNL; in this section, all considered formulas will be in *negated normal form*. We first introduce some basic terminology. A *finite tree* is a finite directed connected graph in which every node, apart from one (the *root*), has exactly one incoming edge. A *successor* of a node n is a node n' such that there is an edge from n to n'. A *leaf* is a node with no successors; a *path* is a sequence of nodes $n_0, \ldots, n_k$ such that, for all $i = 0, \ldots, k-1$, n_{i+1} is a successor of n_i; a *branch* is a path from the root to a leaf. The *height* of a node n is the maximum length (number of edges) of a path from n to a leaf. If n, n' belong to the same branch and the height of n is less than (resp. less than or equal to) the height of n', we write $n \prec n'$ (resp. $n \preceq n'$).

Definition 17 If $\mathbb{C}_h = \langle C_h, <\rangle$ and $\mathbb{C}_v = \langle C_v, <\rangle$ are finite linearly ordered sets, a *labeled formula*, with label in $\mathbb{C} = (\mathbb{C}_h \times \mathbb{C}_v)$, is a pair $(\phi, \langle (h_i, v_j), (h_k, v_l)\rangle)$, where $\phi \in$ SpPNL and $\langle (h_i, v_j), (h_k, v_l)\rangle \in \mathbb{O}(\mathbb{C})$. For a node n in a tree $\mathcal{T}$, the *decoration* $\nu(n)$ is a triple $((\phi, \langle (h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}, u_n)$, where $(\phi, \langle (h_i, v_j), (h_k, v_l)\rangle)$ is a labeled formula, with label in $\mathbb{C}$, and u_n is a *local flag function* which associates the values 0 or 1 with every branch B in $\mathcal{T}$ containing n.

The value 0 for a node n with respect to a branch B means that n can be expanded on B.

Definition 18 A *decorated tree* is a tree in which every node has a decoration $\nu(n)$.

For every decorated tree, we also use a *global flag function u* acting on pairs (*node, branch through that node*), and defined as $u(n, B) = u_n(B)$. For any branch B in a decorated tree, we denote by $\mathbb{C}_B$ the spatial frame belonging to the decoration of the leaf of B (and, similarly, we can refer to $\mathbb{C}_{h_B}$ and $\mathbb{C}_{v_B}$), and for any node

n in a decorated tree, we denote by $\Phi(n)$ the formula in its decoration. If B is a branch, then $B \cdot n$ denotes the result of the expansion of B with the node n. Similarly, $B \cdot n_1 \mid \ldots \mid n_k$ denotes the result of the expansion of B with k immediate successor nodes (producing k branches extending B).

Definition 19 Given a decorated tree $\mathcal{T}$, a branch B in $\mathcal{T}$, and a node $n \in B$ such that $\nu(n) = ((\phi, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}, u)$, with $u(n, B) = 0$, the *branch-expansion rule* for B and n is defined as follows (in all the considered cases, $u(n', B') = 0$ for all new pairs (n', B') of nodes and branches):

1. If $\phi = \neg\neg\psi$, then expand the branch to $B \cdot n_0$, with $\nu(n_0) = ((\psi, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}_B, u)$;
2. If $\phi = \psi \wedge \theta$, then expand the branch to $B \cdot n_0 \cdot n_1$, with $\nu(n_0) = ((\psi, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}_B, u)$ and $\nu(n_1) = ((\theta, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}_B, u)$;
3. If $\phi = \neg(\psi \wedge \theta)$, then expand the branch to $B \cdot n_0 \mid n_1$, with $\nu(n_0) = ((\neg\psi, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}_B, u)$ and $\nu(n_1) = ((\neg\theta, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}_B, u)$;
4. If $\phi = [\mathrm{E}]\psi$ and there exists $h_o \in \mathbb{C}_{h_B}$, such that $h_k < h_o$ that h_o has not been used yet to expand the node n on B, then take the least such h_o and expand the branch to $B \cdot n_0$, with $\nu(n_0) = ((\psi, \langle(h_k, v_j), (h_o, v_l)\rangle), \mathbb{C}_B, u)$;
5. If $\phi = [\mathrm{W}]\psi$ and there exists $h_o \in \mathbb{C}_{h_B}$, such that $h_o < h_i$ that h_o has not been used yet to expand the node n on B, then take the least such h_o and expand the branch to $B \cdot n_0$, with $\nu(n_0) = ((\psi, \langle(h_o, v_j), (h_i, v_l)\rangle), \mathbb{C}_B, u)$;
6. If $\phi = [\mathrm{N}]\psi$ and there exists $v_o \in \mathbb{C}_{v_B}$, such that $v_l < v_o$ that v_o has not been used yet to expand the node n on B, then take the least such v_o and expand the branch to $B \cdot n_0$, with $\nu(n_0) = ((\psi, \langle(h_i, v_l), (h_k, v_o)\rangle), \mathbb{C}_B, u)$;
7. If $\phi = [\mathrm{S}]\psi$ and there exists $v_o \in \mathbb{C}_{h_B}$, such that $v_o < v_j$ that v_o has not been used yet to expand the node n on B, then take the least such v_o and expand the branch to $B \cdot n_0$, with $\nu(n_0) = ((\psi, \langle(h_i, v_o), (h_k, v_j)\rangle), \mathbb{C}_B, u)$;
8. If $\phi = \langle\mathrm{E}\rangle\psi$, then, if h_{k+m} is the last element of $\mathbb{C}_{h_B}$, expand the branch to $B \cdot n_{k+1}|\ldots|n_{k+m}|n'_{k+1}|\ldots|n'_{k+m+1}$, where:
   a. For all $h_z \in \mathbb{C}_{h_B}$ such that $h_k < h_z$, $(k+1 \leq z \leq k+m)$, $\nu(n_z) = ((\psi, \langle(h_k, v_j), (h_z, v_l)\rangle), \mathbb{C}_B, u)$;
   b. For all $k \leq z \leq k+m$, let $\mathbb{C}'_{h_B}$ be the linear ordering obtained by inserting a new element h_w right after h_z, and $\nu(n'_z) = ((\psi, \langle(h_k, v_j), (h_w, v_l)\rangle), \mathbb{C}'_{h_B}, u)$;
9. If $\phi = \langle\mathrm{W}\rangle\psi$, then, if h_0 is the first element of $\mathbb{C}_{h_B}$, expand the branch to $B \cdot n_0|\ldots|n_{i-1}|n'_0|\ldots|n'_i$, where:
   a. For all $h_z \in \mathbb{C}_{h_B}$ such that $h_z < h_i$, $(0 \leq z < i)$, $\nu(n_z) = ((\psi, \langle(h_z, v_j), (h_i, v_l)\rangle), \mathbb{C}_B, u)$;
   b. For all $0 \leq z \leq i$, let $\mathbb{C}'_{h_B}$ be the linear ordering obtained by inserting a new element h_w right before h_z, and $\nu(n'_z) = ((\psi, \langle(h_w, v_j), (h_i, v_l)\rangle), \mathbb{C}'_{h_B}, u)$;
10. If $\phi = \langle\mathrm{N}\rangle\psi$, then, if v_{l+m} is the last element of $\mathbb{C}_{v_B}$, expand the branch to $B \cdot n_{l+1}|\ldots|n_{l+m}|n'_{l+1}|\ldots|n'_{l+m+1}$, where:
    a. For all $v_z \in \mathbb{C}_{v_B}$ such that $v_l < v_z$, $(l+1 \leq z \leq l+m)$, $\nu(n_z) = ((\psi, \langle(h_i, v_l), (h_k, v_z)\rangle), \mathbb{C}_B, u)$;
    b. For all $l \leq z \leq l+m$, let $\mathbb{C}'_{v_B}$ be the linear ordering obtained by inserting a new element v_w right after v_z, and $\nu(n'_z) = ((\psi, \langle(h_i, v_l), (h_k, v_w)\rangle), \mathbb{C}'_{v_B}, u)$;

11. If $\phi = \langle \mathrm{S} \rangle \psi$, then, if v_0 is the first element of $\mathbb{C}_{v_B}$, expand the branch to $B \cdot n_0 | \ldots | n_{j-1} | n'_0 | \ldots | n'_j$, where:
    a. For all $v_z \in \mathbb{C}_{v_B}$ such that $v_z < v_j$, $(0 \leq z < j)$, $\nu(v_z) = ((\psi, \langle (h_i, v_z), (h_k, v_j) \rangle), \mathbb{C}_B, u)$;
    b. For all $0 \leq z \leq j$, let $\mathbb{C}'_{h_B}$ be the linear ordering obtained by inserting a new element v_w right before v_z, and $\nu(n'_z) = ((\psi, \langle (h_i, v_w), (h_k, v_j) \rangle), \mathbb{C}'_{h_B}, u)$;

Finally, for any node m $(\neq n)$ in B and any branch B' extending B, let $u(m, B') = u(m, B)$, and for any branch B' extending B, $u(n, B') = 1$, unless $\phi = [\mathrm{X}]\psi$ (in such case $u(n, B') = 0$).

Now, we define the notions of open and closed branch. We say that a node n in a decorated tree $\mathcal{T}$ is *available on a branch* B if and only if $u(n, B) = 0$. The branch-expansion rule is *applicable* to a node n on a branch B if the node is available on B and the application of the rule generates at least one successor node with a new labeled formula. This second condition is needed to avoid looping of the application of the rule on universal modalities.

Definition 20 A branch B is *closed* if and only if there are two nodes $n, n' \in B$ such that $\nu(n) = ((\phi, \langle (h_i, v_j), (h_k, v_l) \rangle), \mathbb{C}, u)$ and $\nu(n') = ((\neg\phi, \langle (h_i, v_j), (h_k, v_l) \rangle), \mathbb{C}, u)$ for some formula ϕ, otherwise it is *open*.

Moreover, we define a *branch-expansion strategy* for a branch B in a decorated tree $\mathcal{T}$, as follows: (1) apply the branch-expansion rule to a branch B only if it is open, and (2) if B is open, apply the branch-expansion rule to the first available node one encounters moving from the root to the leaf of B to which the branch-expansion rule is applicable (if any). Clearly, an *initial tableau* for a given formula $\phi \in$ SpPNL is the decorated tree $\mathcal{T}$ with only one node *root* such that $\nu(root) = ((\phi, \langle (h_0, v_0), (h_1, v_1) \rangle), \mathbb{C}, 0)$, where $C_h = \{h_0, h_1\}$ $(h_0 < h_1)$, $C_v = \{v_0, v_1\}$ $(v_0 < v_1)$. Finally, we define a tableau for SpPNL as follows.

Definition 21 A *tableau* for a given formula $\phi \in$ SpPNL is any finite decorated tree isomorphic to a finite decorated tree $\mathcal{T}$ obtained by expanding the initial tableau for ϕ through successive applications of the branch-expansion strategy to the existing branches.

As in the classical case, a tableau for SpPNL is *closed* if and only if every branch in it is closed, otherwise it is *open*.

In Fig. 3 we show an example of a tableau for the negation of the Axiom 5, which, clearly, results in a tree such that all its branches are closed. In Table 1 the spatial domains used in the tableau are explained.

6.1 Soundness and completeness

Definition 22 Given a set S of labeled formulas with labels in $\mathbb{C}$, we say that S is *satisfiable over* $\mathbb{C}$ if there exists a spatial model $M = \langle \mathbb{F}, \mathbb{O}(\mathbb{F}), \mathcal{V} \rangle$, such that $\mathbb{F}$ is an extension of $\mathbb{C}$ and $M, \langle (h_i, v_j), (h_k, v_l) \rangle \Vdash \psi$ for all $(\psi, \langle (h_i, v_j), (h_k, v_l) \rangle) \in S$.

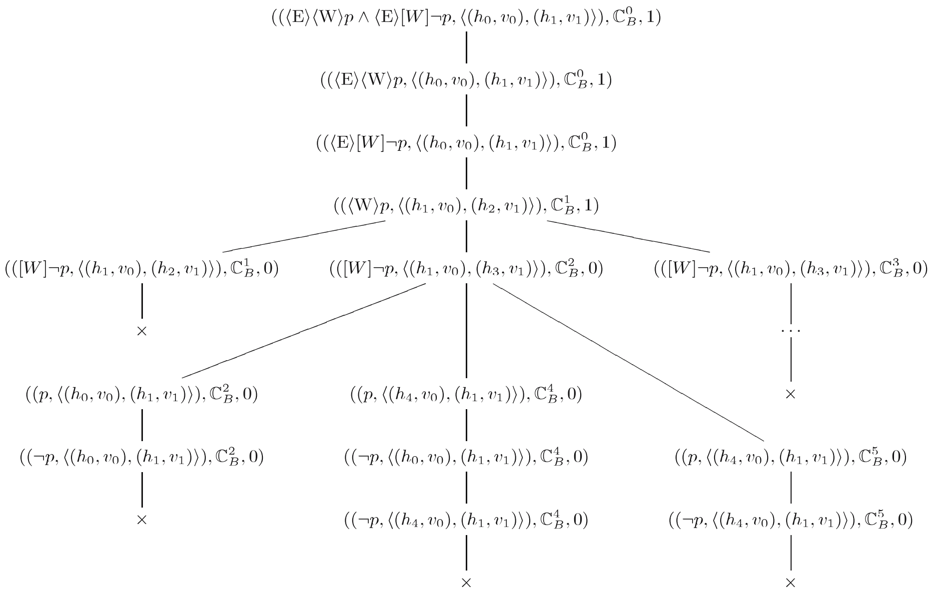


Fig. 3 A closed tableau for the formula $\langle E\rangle\langle W\rangle p \wedge \langle E\rangle[W]\neg p$

If S contains only one labeled formula, the notion of satisfiability of a (labeled) formula over $\mathbb{C}$ is equivalent to the notion of satisfiability given in Section 3.

Theorem 23 (Soundness) *If $\phi \in SpPNL$ and a tableau $\mathcal{T}$ for ϕ is closed, then ϕ is not satisfiable.*

Proof We will prove by induction on the height h of a node n in the tableau $\mathcal{T}$ that if every branch including n is closed, then the set $S(n)$ of all labeled formulas in the decorations of the nodes between n and the root is not satisfiable over $\mathbb{C}$, where $\mathbb{C}$ is the spatial frame in the decoration of n. If $h = 0$, then n is a leaf and the unique branch B containing n is closed. This means that $S(n)$ contains both the labeled formulas $(\psi, \langle(h_i, v_j), (h_k, v_l)\rangle)$ and $(\neg\psi, \langle(h_i, v_j), (h_k, v_l)\rangle)$ for some formula ψ. Take any model $M = \langle \mathbb{F}, \mathbb{O}(\mathbb{F}), \mathcal{V}\rangle$, such that $\mathbb{F}$ is an extension of $\mathbb{C}$. Clearly

Table 1 Spatial frames for the tableau in Fig. 3

i	$\mathbb{C}_{h_B}^i$	$\mathbb{C}_{v_B}^i$
0	$\{h_0, h_1\}(h_0 < h_1)$	$\{v_0, v_1\}(v_0 < v_1)$
1	$\{h_0, h_1, h_2\}(h_0 < h_1 < h_2)$	$\{v_0, v_1\}(v_0 < v_1)$
2	$\{h_0, h_1, h_2, h_3\}(h_0 < h_1 < h_3 < h_2)$	$\{v_0, v_1\}(v_0 < v_1)$
3	$\{h_0, h_1, h_2, h_3\}(h_0 < h_1 < h_2 < h_3)$	$\{v_0, v_1\}(v_0 < v_1)$
4	$\{h_0, h_1, h_2, h_3, h_4\}(h_4 < h_0 < h_1 < h_3 < h_2)$	$\{v_0, v_1\}(v_0 < v_1)$
5	$\{h_0, h_1, h_2, h_3, h_4\}(h_0 < h_4 < h_1 < h_3 < h_2)$	$\{v_0, v_1\}(v_0 < v_1)$

$M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \psi$ if and only if $M, \langle(h_i, v_j), (h_k, v_l)\rangle \nVdash \neg\psi$. Hence, $S(n)$ is notover $\mathbb{C}$. Otherwise, suppose $h > 0$. Then either n has been generated as one of the successors, but not the last one, when applying the branch-expansion rule in the case $\wedge$, or branch-expansion rule has been applied to some labeled formula $(\psi, \langle(h_i, v_j), (h_k, v_l)\rangle) \in S(n) - \{\Phi(n)\}$ to extend the branch at n. We deal with the latter case, for he former can be dealt with in the same way. Let $\mathbb{C}$ be the interval structure from the decoration of n. Notice that every branch passing through any successor of n must be closed, so the inductive hypothesis applies to all successors of n. We consider the possible cases for the branch-expansion rule applied at n, dealing with the most interesting ones only:

- Let $\psi = \xi_0 \wedge \xi_1$. Then there are two nodes $n_0, n_1 \in B$ such that $\nu(n_0) = ((\xi_0, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}, u)$, $\nu(n_1) = ((\xi_1, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}, u)$, and, without loss of generality, n_0 is the successor of n and n_1 is the successor of n_0. Since every branch containing n is closed, then every branch containing n_1 is closed. By the inductive hypothesis, $S(n_1)$ is not satisfiable over $\mathbb{C}$ since $n_1 \prec n$. Since every model over $\mathbb{C}$ satisfying $S(n)$ must, in particular, satisfy $(\xi_0 \wedge \xi_1, \langle(h_i, v_j), (h_k, v_l)\rangle)$, and hence $(\xi_0, \langle(h_i, v_j), (h_k, v_l)\rangle)$ and $(\xi_1, \langle(h_i, v_j), (h_k, v_l)\rangle)$, it follows that $S(n)$, $S(n_0)$, and $S(n_1)$ are equi-satisfiable over $\mathbb{C}$. Therefore, $S(n)$ is not satisfiable over $\mathbb{C}$;
- Let $\psi = [\mathrm{N}]\xi$. Suppose by contradiction that $S(n)$ is satisfiable over $\mathbb{C}$. Then, since $([\mathrm{N}], \langle(h_i, v_j), (h_k, v_l)\rangle) \in S(n)$, there is a model $M = \langle\mathbb{F}, \mathbb{O}(\mathbb{F}), \mathcal{V}\rangle$ such that $\mathbb{F}$ is an extension of $\mathbb{C}$ and $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash [\mathrm{N}]\xi$. So, for every $v_m \in \mathbb{C}_v$ such that $v_l < v_m$, we have that $M, \langle(h_i, v_l), (h_k, v_m)\rangle \Vdash \xi$. By construction, the immediate successor of n is n_1 such that, for an element v_o with $v_l < v_o$, $(\xi, \langle(h_i, v_l), (h_k, v_m)\rangle)$ is in the decoration of n_0. By inductive hypothesis, since $n_1 \prec n$, $S(n_1)$ and is not satisfiable over $\mathbb{C}$. Thus, such a model M cannot exist, and $S(n)$ is not satisfiable over $\mathbb{C}$;
- Let $\psi = \langle\mathrm{N}\rangle\xi$. Assuming by contradiction that $S(n)$ is satisfiable over $\mathbb{C}$, there is a model $M = \langle\mathbb{F}, \mathbb{O}(\mathbb{F}), \mathcal{V}\rangle$ such that $\mathbb{F}$ is an extension of $\mathbb{C}$ and $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \theta$ for all $(\theta, \langle(h_i, v_j), (h_k, v_l)\rangle) \in S(n)$. In particular, $M, \langle(h_i, v_l), (h_k, v_m)\rangle \Vdash \xi$ for some v_m such that $v_l < v_m$. Consider two cases:

  1. If $v_m \in \mathbb{C}_v$, then $v_m = v_o$ for some $v_l < v_o$. But among the successors of n there is a node n_o where $\nu(n_o) = ((\xi, \langle(h_i, v_l), (h_k, v_o)\rangle), \mathbb{C}, u)$, and since $n_o \prec n$, by the inductive hypothesis $S(n_o) = S(n) \cup\{(\xi_0, \langle(h_i, v_l), (h_k, v_o)\rangle)\}$ is not satisfiable over $\mathbb{C}$, which is a contradiction, and $S(n)$ is not satisfiable over $\mathbb{C}$;
  2. If $v_m \notin \mathbb{C}_v$, then there is an o such that $l < o$ and $v_l < v_o$. Hence, there is a successor n_o of n such that $\nu(n_o) = ((\xi, \langle(h_i, v_l), (h_k, v_o)\rangle), \mathbb{C}_v\cup \{v_o\}, u)$, and since $n_o \prec n$, by the inductive hypothesis $S(n_o) = S(n) \cup\{(\xi, \langle(h_i, v_l), (h_k, v_o)\rangle)\}$ is not satisfiable over $\mathbb{C}'$ (obtained by adding v_o to $\mathbb{C}_v$) which, again, is a contradiction, and $S(n)$ is not satisfiable over $\mathbb{C}$;

- The remaining cases can be dealt with in a similar way. □

Definition 24 If $\mathcal{T}_0$ is the initial tableau for a given SpPNL-formula ϕ, the *limit tableau* $\overline{\mathcal{T}}$ for ϕ is the (possibly infinite) decorated tree obtained as follows. First, for all i, $\mathcal{T}_{i+1}$ is the tableau obtained by the simultaneous application of the branch-expansion strategy to every branch in $\mathcal{T}_i$. Then, we ignore all flags from the decorations of the nodes in every $\mathcal{T}_i$. Thus, we obtain a chain by inclusion of decorated trees: $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \ldots$, and we define $\overline{\mathcal{T}} = \bigcup_{i=0}^{\infty} \mathcal{T}_i$.

Notice that the chain above may stabilize at some $\mathcal{T}_i$ if it closes, or if the branch-expansion rule is not applicable to any of its branches. If $\overline{\mathcal{T}}$ is a limit tableau, we associate with each branch B in $\overline{\mathcal{T}}$ the spatial frame $\mathbb{C}_B = \bigcup_{i=0}^{\infty} \mathbb{C}_{B_i}$, where, for all i, $\mathbb{C}_{B_i}$ is the spatial frame from the decoration of the leaf of the (sub-)branch B_i of B in $\mathcal{T}_i$. The definitions of closed and open branches readily apply to $\overline{\mathcal{T}}$.

Definition 25 A branch in a (limit) tableau is *saturated* if there are no nodes on that branch to which the branch-expansion rule is applicable on the branch. A (limit) tableau is *saturated* if every open branch in it is saturated.

Now we will show that the set of all labeled formulas on an open branch in a limit tableau has the saturation properties of a Hintikka set in first-order logic.

Lemma 26 (saturation lemma) *Every limit tableau is saturated.*

Proof Given a node n in a limit tableau $\overline{\mathcal{T}}$, we denote by $k(n)$ the distance (number of edges) between n and the root of $\overline{\mathcal{T}}$. Now, given a branch B in $\overline{\mathcal{T}}$, we will prove by induction on $k(n)$ that after every step of the expansion of that branch at which the branch-expansion rule becomes applicable to n (because n has just been introduced, or because a new point has been introduced in the spatial frame on B) that rule is subsequently applied on B to that node. Suppose the inductive hypothesis holds for all nodes with distance to the root less than l. Let $k(n) = l$ and the branch-expansion rule has become applicable to n. If there are no nodes between the root (incl. the root) and n (excl. n) to which the branch-expansion rule is applicable at that moment, the next application of the branch-expansion rule on B is to n. Otherwise, consider the closest-to-n node n^* between the root and n to which the branch-expansion rule is applicable or will become applicable on B at least once thereafter. (Such a node exists because there are only finitely many nodes between n and the root.) Since $k(n^*) < k(n)$, by the inductive hypothesis the branch-expansion rule has been subsequently applied to n^*. Then the next application of the branch-expansion rule on B must have been to n and that completes the induction. Now, assuming that a branch in a limit tableau is not saturated, consider the closest-to-the-root node n on that branch B to which the branch-expansion rule is applicable on that branch. If $\Phi(n)$ is not a universal modality, then the branch-expansion rule has become applicable to n at the step when n is introduced, and by the claim above, it has been subsequently applied, at which moment the node has become unavailable thereafter, which contradicts the assumption. Suppose that $\Phi(n) = [\mathrm{N}]\psi$. Then an application of the rule on B would create a successor with label $(\psi, \langle (h_i, v_l), (h_k, v_m) \rangle)$ on B. But v_j, v_l, v_m have already been introduced at some (finite) step of the construction of B and at the first step when the three of them, as well as n, have appeared on the branch, the branch-expansion rule has become applicable to n, hence is has been subsequently applied on B and that application must have introduced the label $(\psi, \langle(h_i, v_l), (h_k, v_m)\rangle)$, which, again, contradicts the assumption. The same holds for the remaining universal modalities. $\square$

As a corollary of the previous result, we have that if ϕ be a SpPNL-formula and $\overline{\mathcal{T}}$ is the limit tableau for ϕ, then for every open branch B in $\overline{\mathcal{T}}$, the following closure properties hold:

(1) For a node $n \in B$ such that $\nu(n) = ((\neg\neg\psi, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}, u)$, there is a node $n_0 \in B$ such that $\nu(n_0) = ((\psi, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}, u_0)$;
(2) For a node $n \in B$ such that $\nu(n) = ((\psi_0 \wedge \psi_1, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}, u)$, there is a node $n_0 \in B$ such that $\nu(n_0) = ((\psi_0, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}, u_0)$ and a node $n_1 \in B$ such that $\nu(n_1) = ((\psi_1, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}, u_1)$;
(3) For a node $n \in B$ such that $\nu(n) = ((\neg(\psi_0 \wedge \psi_1), \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}, u)$, there is a node $n_0 \in B$ such that $\nu(n_0) = ((\neg\psi_0, \langle(h_i, v_j), (h_k, v_l)\rangle), \mathbb{C}, u_0)$ or a node $n_1 \in B$ such that $\nu(n_1) = ((\neg\psi_1, \langle(h_i, v_j), (h_k, v_l)\rangle, \mathbb{C}, u_1)$;
(4) For a node $n \in B$ such that $\nu(n) = ((\langle N\rangle\psi, \langle(h_i, v_j), (h_k, v_l)\rangle, \mathbb{C}, u)$, then, for some $v_m \in \mathbb{C}_v$ such that $v_l < v_m$ there is a node $n_0 \in B$ such that $\nu(n_0) = ((\psi_0, \langle(h_i, v_l), (h_k, v_m)\rangle), \mathbb{C}', u')$ (and similarly for the other existential modalities);
(5) For a node $n \in B$ such that $\nu(n) = ([N]\psi, \langle(h_i, v_j), (h_k, v_l)\rangle, \mathbb{C}, u)$, $v_m \in \mathbb{C}_v$ such that $v_l < v_m$ there is a node $n_0 \in B$ such that $\nu(n_0) = ((\psi, \langle(h_i, v_l), (h_k, v_m)\rangle), \mathbb{C}', u')$ (and similarly for the other universal modalities).

Lemma 27 *If the limit tableau for some formula $\phi \in SpPNL$ is closed, then some finite tableau for ϕ is closed.*

Proof Suppose the limit tableau for ϕ is closed. Then every branch closes at some finite step of the construction and then remains finite. Since the branch-expansion rule always produces finitely many successors, every finite tableau is finitely branching, and hence so is the limit tableau. Then, by König's lemma, the limit tableau, being a finitely branching tree with no infinite branches, must be finite, hence its construction stabilizes at some finite stage. At that stage a closed tableau for ϕ is constructed. $\square$

Theorem 28 (Completeness) *Let $\phi \in SpPNL$ be a valid formula. Then there is a closed tableau for $\neg\phi$.*

Proof We will show that the limit tableau $\overline{\mathcal{T}}$ for $\neg\phi$ is closed, whence the claim follows by the previous lemma. By contraposition, suppose that $\overline{\mathcal{T}}$ has an open branch B. Let $\mathbb{C}_B$ be the spatial frame associated with B and $S(B)$ be the set of all labeled formulas on B. Consider the spatial model built on it, where for every object $\langle(h_i, v_j), (h_k, v_l)\rangle$ and $p \in \mathcal{AP}$, $p \in V(\langle(h_i, v_j), (h_k, v_l)\rangle)$ iff $(p, \langle(h_i, v_j), (h_k, v_l)\rangle) \in \Phi(B)$. We show by induction on ψ that, for every $(\psi, \langle(h_i, v_j), (h_k, v_l)\rangle) \in S(B)$, $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \psi$. We proceed on the complexity of ψ:

- Let $\psi = p$ or $\psi = \neg p$ where $p \in \mathcal{AP}$. Then the claim follows by definition, because if $(\neg p, \langle(h_i, v_j), (h_k, v_l)\rangle) \in S(B)$ then $(p, \langle(h_i, v_j), (h_k, v_l)\rangle) \notin S(B)$ since B is open;

- Let $\psi = \neg\neg\xi$. Then by the consequences of the saturation lemma, $(\xi, \langle(h_i, v_j), (h_k, v_l)\rangle) \in S(B)$, and by inductive hypothesis $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \xi$. So $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \psi$;
- Let $\psi = \xi_0 \wedge \xi_1$. Then by the saturation lemma, $(\xi_0, \langle(h_i, v_j), (h_k, v_l)\rangle) \in S(B)$ and $(\xi_1, \langle(h_i, v_j), (h_k, v_l)\rangle) \in S(B)$. By inductive hypothesis, $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \xi_0$ and $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \xi_1$, so $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \psi$;
- Let $\psi = \neg(\xi_0 \wedge \xi_1)$. Then by the saturation lemma, $(\neg\xi_0, \langle(h_i, v_j), (h_k, v_l)\rangle) \in S(B)$ or $(\neg\xi_1, \langle(h_i, v_j), (h_k, v_l)\rangle) \in S(B)$. By inductive hypothesis $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \neg\xi_0$ or $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \neg\xi_1$, so $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \psi$;
- Let $\psi = \langle \mathrm{N} \rangle \xi$. Then by the saturation lemma, $(\xi_0, \langle(h_i, v_l), (h_k, v_m)\rangle) \in S(B)$ and for some $v_m \in \mathbb{C}_{v_B}$ such that $v_l < v_m$. Thus, by inductive hypothesis, $M, \langle(h_i, v_l), (h_k, v_m)\rangle \Vdash \xi$, and thus $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \psi$ (and similarly for the other existential modalities);
- Let $\psi = [\mathrm{N}]\xi$. Then by the saturation lemma, for all $v_m \in \mathbb{C}_{v_B}$ such that $v_l < v_m$, $(\xi, \langle(h_i, v_l), (h_k, v_m)\rangle) \in S(B)$. Hence, for any such v_m, by the inductive hypothesis $M, \langle(h_i, v_l), (h_k, v_m)\rangle \Vdash \xi$. Thus, $M, \langle(h_i, v_j), (h_k, v_l)\rangle \Vdash \psi$.

This completes the induction. In particular, we obtain that $\neg\phi$ is satisfied in M, which is in contradiction with the assumption that ϕ is valid. □

7 Conclusions

In this paper we considered a new modal logic for qualitative spatial reasoning by means of directional relations and approximation of objects with minimum bounding boxes. SpPNL can be viewed as the natural bi-dimensional extension of the interval-based temporal logic PNL [14], and it has been shown to be quite useful for the formalization of natural spatial expressions. We presented a representation theorem and devised a tableau-based proof system for it. Moreover, we showed that SpPNL, despite its simplicity, is powerful enough to solve any rectangle algebra constraint network. As a future work, we plan to generalize SpPNL and related results to the case of n dimensions ($n > 2$), and to test an implementation of the tableaux method with real examples in the context of a project financed by the Spanish Ministry of Education. From a theoretical point of view, the results presented in this paper and in [18, 21, 22] leave as open, among others, the following interesting questions:

Question 1 Is it possible to find out some kind of syntactically defined sub-logic of SpPNL whose satisfiability/validity problem is decidable?

Question 2 There are cases in interval-based temporal logic of decidable fragments of undecidable logics have been obtained by interpreting formulas over non-standard frames presenting some kind of ‘hole’ (see [23]). It is possible to find a similar result for SpPNL?

Question 3 Very recently, an interesting functional completeness result has been found for PNL [9]; is it possible to extend such a result for some suitable fragment of first-order logic? and, if it is not the case, for which fragment of SpPNL is that possible?

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