DOI 10.1007/s10469-023-09713-2
Algebra and Logic, Vol. 61, No. 6, January, 2023
(Russian Original Vol. 61, No. 6, November-December, 2022)

A CLASS OF GENERALIZED DERIVATIONS

A. S. Zakharov*

UDC 512.55

Keywords: *differential algebra, ternary derivation, generalized derivation, Novikov–Poisson algebra, Jordan superalgebra.*

We consider a class of generalized derivations that arise in connection with the problem of adjoining unity to an algebra with generalized derivation, and of searching envelopes for Novikov–Poisson algebras. Conditions for the existence of the localization of an algebra with ternary derivation are specified, as well as conditions under which given an algebra with ternary derivation, we can construct a Novikov–Poisson algebra and a Jordan superalgebra. Finally, we show how the simplicity of an algebra with Brešar generalized derivation is connected with simplicity of the appropriate Novikov algebra.

INTRODUCTION

The emergence of ternary derivations goes back to the Jacobson triality principle (see [1]). The term 'ternary derivations' arose in [2] and has a direct connection with ternary automorphisms. Ternary derivations generalize ordinary derivations, δ-derivations introduced in [3], generalized derivations presented in [4], and generalized derivations such as in [5].

Differential algebras are connected with Novikov algebras, which appeared in [6] and [7]. Zelmanov [8] gave a description of finite-dimensional Novikov algebras over an algebraically closed field of characteristic zero. Fillipov [9] constructed examples of simple nonassociative finite-dimensional algebras over a field of positive characteristic and of infinite-dimensional simple Novikov algebras over a field of characteristic zero. Osborn [10-12] examined simple Novikov algebras over a field of prime characteristic. X. Xu [13] continued this work and furnished a

*Supported by Russian Science Foundation, project No. 21-11-00286.

Novosibirsk State Technical University. Novosibirsk State University, Novosibirsk, Russia; antzakh@gmail.com. Translated from *Algebra i Logika*, Vol. 61, No. 6, pp. 687-705, November-December, 2022. Russian DOI: 10.33048/alglog.2022.61.602. Original article submitted July 24, 2022; accepted October 13, 2023.

description of simple finite-dimensional Novikov algebras over a field of prime characteristic larger than 2. Novikov–Poisson algebras were introduced in [14] to study the tensor theory of Novikov algebras. It turned out that all known Novikov and Novikov–Poisson algebras are obtained from commutative associative derivation algebras. A basis for a free Novikov algebra was found in [15]. In [16] it was shown that every Novikov algebra embeds in a strict Novikov superalgebra of vector type. For Novikov–Poisson algebras, however, it was proved in [17] that there exists an s-identity for strict Novikov–Poisson superalgebras of vector type.

The relationship between differential algebras, Novikov–Poisson algebras, and Jordan superalgebras was studied in [18]. Every associative commutative algebra complies with a Novikov–Poisson algebra. If an associative commutative part of a Novikov–Poisson algebra contains unity, then the converse is also true. The commutator with respect to the Novikov operation is a Jordan bracket. In [19] this property was proved without imposing the condition of being unital for an associative commutative part of a Novikov–Poisson algebra. The question of whether Novikov–Poisson algebras embed in Novikov–Poisson algebras of vector type was considered in [20]. Not all Novikov–Poisson algebras embed in strict Novikov–Poisson algebras of vector type; therefore, it is natural to raise the issue of searching the universal envelope of Novikov–Poisson algebras in the class of associative commutative algebras with a generalized or ternary derivation.

1. TERNARY DERIVATIONS OF SPECIAL KIND

Let $\mathbb{F}$ be a field and A an $\mathbb{F}$-algebra. A *derivation* is a linear map $d : A \to A$ such that for all $x, y, z \in A$,

$$d(xy) = d(x)y + xd(y).$$

A triple (D, E, F), where D, E, F are linear maps, is called a *ternary derivation* if

$$D(xy) = E(x)y + xF(y)$$

for any $x, y \in A$. In this case the map D is called the *principal component* of a ternary derivation. The map D is called a *generalized derivation* if it is the principal component of some ternary derivation. A *Brešar generalized derivation* is the principal component of a ternary derivation (D, D, δ), where δ is a derivation. Also we will use the notation (D, δ) for Brešar generalized derivations.

Let A be a unital associative commutative algebra and (D, E, F) a ternary derivation. Then the map δ given by the rule $\delta = D - R_{D(1)}$, where R_a is the operator of multiplication by an element a, is a derivation. Moreover, (D, δ) is a Brešar generalized derivation.

LEMMA 1. Let A be an associative commutative algebra with a ternary derivation (D, E, F). The following identities are equivalent:

$$x(D - E)(y) = y(D - E)(x), \tag{1}$$

$$x(D-F)(y) = y(D-F)(x). \tag{2}$$

Proof. By commutativity,

$$E(x)y + xF(y) = D(xy) = D(yx) = E(y)x + yF(x),$$

whence

$$E(x)y - E(y)x = F(x)y - F(y)x. \ \square$$

COROLLARY 1. Let (D, δ) be a Brešar generalized derivation on a commutative algebra. Then identities (1) and (2) hold for the corresponding ternary derivation (D, D, δ).

From this moment on, we will assume that A is a nonunital associative commutative algebra with a ternary derivation (D, E, F) and that identities (1) and (2) hold. Then for any $x, y \in A$ we have

$$\begin{aligned} D(xy) &= E(x)y + xF(y) \\ &= D(x)y + (E-D)(x)y + xF(y) \\ &= D(x)y + x(E+F-D)(y) \\ &= D(x)y + x\Delta(y), \end{aligned}$$

where $\Delta = E+F-D$. In particular, (D, D, Δ) will be a ternary derivation. Note that if the algebra A is unital then the map Δ is a derivation, $\Delta = D - R_{D(1)}$, and (D, Δ) is a Brešar generalized derivation.

Let S be a multiplicatively closed set in A and $S^{-1}A$ the localization of the algebra A relative to the set S, i.e. a set of fractions of the form $\frac{a}{s}$, where $a \in A$ and $s \in S$. The fractions $\frac{a}{s}$ and $\frac{b}{t}$ are equal if

$$p(at - bs) = 0$$

for some $p \in S$. The operations are given as follows:

$$\alpha\frac{a}{s} + \beta\frac{b}{t} = \frac{\alpha at + \beta bs}{st}, \qquad \frac{a}{s}\cdot\frac{b}{t} = \frac{ab}{st},$$

where $\alpha, \beta \in \mathbb{F}$ and $\frac{a}{s}, \frac{b}{t} \in S^{-1}A$. The map $\varphi : a \mapsto \frac{as}{s}$ is a homomorphism from A to $S^{-1}A$. If S does not contain zero divisors, then φ is an embedding. For brevity, we denote elements of the form $\frac{as}{s}$ by a. This expression does not depend on the choice of s.

Let $f : A \to A$ be a function. We say that a function $\widehat{f}$ is an extension of f to $S^{-1}A$ if for all $a \in A$ we have

$$(\varphi \circ \widehat{f})(a) = (f \circ \varphi)(a),$$

where $\circ$ is a composition of functions, i.e. $(g \circ h)(x) = h(g(x))$. In particular, if φ is an embedding, then $\widehat{f}$ is the usual extension of f.

A map $\widehat{D}$ on $S^{-1}A$ is defined by setting

$$\widehat{D}\left(\frac{a}{s}\right) = \frac{qsE(a) + asD(q) - aE(qs)}{qs^2} \tag{3}$$

for all $a \in A$ and all $s \in S$.

LEMMA 2. The map $\widehat{D}$ defined by identity (3) does not depend on the choice of q, is well defined, and $\widehat{D}$ is an extension of the map D to $S^{-1}A$.

Proof. Consider $q, p \in S$. First we show that

$$\frac{qsE(a) + asD(q) - aE(qs)}{qs^2} = \frac{psE(a) + asD(p) - aE(ps)}{ps^2}.$$

This equality follows from

$$\begin{aligned}
(qsE(a) &+ asD(q) - aE(qs))p - (psE(a) + asD(p) - aE(ps))q \\
&= aspD(q) - apE(qs) - asqD(p) + aqE(ps) \\
&= a(spD(q) - D(pqs) + qsF(p) - sqD(p) + D(pqs) - psF(q)) \\
&= a(spD(q) + qsF(p) - sqD(p) - psF(q)) \\
&= as(p(D - F)(q) - q(D - F)(p)) = 0.
\end{aligned}$$

We verify the soundness of our definition. Let $\frac{a}{s} = \frac{b}{t}$. Then $q(at - bs) = 0$ for some $q \in S$. Consider the following expression:

$$\begin{aligned}
&\widehat{D}\left(\frac{a}{s}\right) - \widehat{D}\left(\frac{b}{t}\right) \\
&\quad = \frac{qsE(a) + asD(q) - aE(qs)}{qs^2} - \frac{qtE(b) + btD(q) - bE(qt)}{qt^2} \\
&\quad = \frac{qst^2E(a) + ast^2D(q) - at^2E(qs) - qts^2E(b) - bts^2D(q) + bs^2E(qt)}{qt^2s^2} \\
&\quad = \frac{qst^2E(a) - at^2E(qs) - qts^2E(b) + bs^2E(qt)}{qt^2s^2} + \left(\frac{a}{s} - \frac{b}{t}\right)\frac{D(q)}{q} \\
&\quad = \frac{q^2st^2E(a) - aqt^2E(qs) - q^2ts^2E(b) + bqs^2E(qt)}{q^2t^2s^2}.
\end{aligned}$$

Consider the numerator

$$\begin{aligned}
q^2st^2E(a) &- aqt^2E(qs) - q^2ts^2E(b) + bqs^2E(qt) \\
&= q^2st^2E(a) - atqD(tqs) + atqsF(tq) - q^2ts^2E(b) \\
&\quad + qbsD(sqt) + bsqtF(sq) \\
y &= qstD(atq) - qtsD(bsq) - (at + bs)qD(tqs) \\
&= qstD((at - bs)q) - (at + bs)qD(tqs) = 0.
\end{aligned}$$

Thus

$$\widehat{D}\Big(\frac{a}{s}\Big) = \widehat{D}\Big(\frac{b}{t}\Big),$$

and the map is well defined.

It remains to verify the last property. Let $a \in A$. Then for some $s \in S$ we obtain

$$\begin{aligned}(\varphi \circ \widehat{D})(a) - (D \circ \varphi)(a) &= \widehat{D}\left(\frac{as}{s}\right) - \varphi(D(a)) \\ &= \frac{s^2E(as) + as^2D(s) - asE(s^2)}{s^3} - \frac{sD(a)}{s} \\ &= \frac{sE(as) + asD(s) - aE(s^2) - s^2D(a)}{s^2}.\end{aligned}$$

Consider the numerator

$$\begin{aligned}&sE(as) + asD(s) - aE(s^2) - s^2D(a) \\ &\quad = D(as^2) - asF(s) + asD(s) - D(as^2) + s^2F(a) - s^2D(a) \\ &\quad = s(a(D - F)(s) - s(D - F)(a)) = 0. \ \square\end{aligned}$$

Define also extensions of the functions E and F as follows:

$$\widehat{E}\left(\frac{a}{s}\right) = \frac{sD(a) - aF(s)}{s^2}, \qquad \widehat{F}\left(\frac{a}{s}\right) = \frac{sD(a) - aE(s)}{s^2}. \tag{4}$$

LEMMA 3. The maps $\widehat{E}$ and $\widehat{F}$ are well defined and are extensions of E and F to $S^{-1}A$.

Proof. Let $\frac{a}{s} = \frac{b}{t}$ for $a, b \in A$ and $s, t \in S$. Then $q(at - bs) = 0$ for some $q \in S$. Consider the following expression:

$$\begin{aligned}\widehat{E}\left(\frac{a}{s}\right) - \widehat{E}\left(\frac{b}{t}\right) &= \frac{sD(a) - aF(s)}{s^2} - \frac{tD(b) - bF(t)}{t^2} \\ &= \frac{q^2(st^2D(a) - at^2F(s) - ts^2D(b) + bs^2F(t))}{q^2s^2t^2}.\end{aligned}$$

Consider the numerator

$$\begin{aligned}&q^2(st^2D(a) - at^2F(s) - ts^2D(b) + bs^2F(t)) \\ &\quad = q(qst^2D(a) - tD(qat^2s) + stE(aqt) - qts^2D(b) \\ &\qquad + sD(bqs^2t) - tsE(bqs)) \\ &\quad = q(qst^2D(a) - tD(qat^2s) - qts^2D(b) + sD(bqs^2t) \\ &\qquad - (stE((at - bs)))) \\ &\quad = q(qst^2D(a) - qt^2sE(a) + taF(qt^2s) - qts^2D(b) \\ &\qquad + qts^2E(b)) - sbF(qs^2t)) \\ &\quad = q(qst^2(D - E)(a) - qts^2(D - E)(b))) + q(ta - sb)F(qt^2s) \\ &\quad = q(at(D - E)(qst) - bs(D - E)(qst))) = 0.\end{aligned}$$

Thus $\widehat{E}$ is well defined. Now we look at the expression

$$
\begin{aligned}
(E\circ\varphi-\varphi\circ\widehat{E})(a) &= \frac{sE(a)}{s}-\widehat{E}\left(\frac{as}{s}\right)\\
&= \frac{sE(a)}{s}-\frac{sD(as)-asF(s)}{s^2}\\
&= \frac{s(sE(a)+aF(s)-D(as))}{s^2}=0.
\end{aligned}
$$

Hence $\widehat{E}$ is an extension of E to $S^{-1}A$. The proof for F is analogous. □

LEMMA 4. The triple $(\widehat{D},\widehat{E},\widehat{F})$ is a ternary derivation of the algebra S^1A.

Proof. We write the left-and right-hand sides separately and compare them.

$$
\begin{aligned}
\widehat{D}\left(\frac{a}{s}\cdot\frac{b}{t}\right) &= \frac{qstE(ab)+abstD(q)-abE(qst)}{qs^2t^2};\\
\widehat{E}\left(\frac{a}{s}\right)\frac{b}{t}+\frac{a}{s}\widehat{F}\left(\frac{b}{t}\right) &= \frac{(sD(a)-aF(s))b}{s^2t}+\frac{a(tD(b)-bE(t))}{st^2}\\
&= \frac{(sD(a)-aF(s))bt+as(tD(b)-bE(t))}{s^2t^2}\\
&= \frac{q(bstD(a)+astD(b)-abD(st))}{qs^2t^2}.
\end{aligned}
$$

Then the difference of the numerators in the first and second expressions takes the form

$$
\begin{aligned}
qstE(ab)&+abstD(q)-abE(qst)-qbstD(a)-qastD(b)+qabD(st)\\
&= qD(abst)-qabF(st)+abstD(q)-aD(qbst)+qastF(b)\\
&\quad -qbstD(a)-qastD(b)+qabD(st)\\
&= \Big(qD(abst)+abstD(q)-aD(qbst)-qbstD(a)\Big)\\
&\quad -\Big(qast(D-F)(b)-qab(D-F)(st)\Big).
\end{aligned}
$$

By virtue of (1), we have

$$
\begin{aligned}
qD(abst)&+abstD(q)-aD(qbst)-qbstD(a)\\
&= qE(a)bst+qaF(bst)+abstD(q)-aE(q)bst\\
&\quad -aqF(bst)-qbstD(a)\\
&= bst(a(D-E)(q)-q(D-E)(a))=0.
\end{aligned}
$$

In view of (2), we obtain

$$
qast(D-F)(b)-qab(D-F)(st)=abst(D-F)(q)+abst(D-F)(q)=0.
$$

Hence $(\widehat{D},\widehat{E},\widehat{F})$ is a ternary derivation of the algebra $S^{-1}A$. □

THEOREM 1. Let A be an associative commutative algebra with a ternary derivation (D, E, F), $s \in A$ not be a zero divisor, and $S = \{s^n \mid n \in \mathbb{N} \setminus \{0\}\}$, i.e. A embeds in $S^{-1}A$. An extension of (D, E, F) to $S^{-1}A$ exists if and only if identities (1) and (2) hold.

Proof. As shown in Lemma 4, if (1) and (2) are satisfied, then there exists an extension of (D, E, F) to $S^{-1}A$.

Conversely, it there exists an extension $\widehat{D}$ of a generalized derivation D to $S^{-1}A$, then for $q, s \in S$ we have

$$\frac{sD(q)}{s} = \varphi(D(q)) = \widehat{D}\left(\frac{qs}{s}\right) = \frac{E(qs)}{s} + qsF\left(\frac{1}{s}\right).$$

In other words,

$$F\left(\frac{1}{s}\right) = \frac{D(q)}{qs} - \frac{E(qs)}{qs^2} = \frac{sD(q) - E(qs)}{qs^2}.$$

$\widehat{D}$ is given by formula (3), and

$$\widehat{D}\left(\frac{a}{s}\right) = \frac{qsE(a) + asD(q) - aE(qs)}{qs^2}$$

for some $q \in S$.

Consider $a \in A$. The function $\widehat{D}$ is an extension of D, i.e.

$$\begin{aligned} 0 &= \frac{sD(a)}{s} - \widehat{D}\left(\frac{as}{s}\right) \\ &= \frac{sD(a)}{s} - \frac{s^2E(a) + asD(s) - aE(s^2)}{s^3} \\ &= \frac{s^2D(a) - sE(as) - asD(s) + aE(s^2)}{s^2} \\ &= \frac{s^2D(a) + asF(s) - asD(s) - s^2F(a)}{s^2}. \end{aligned}$$

From this, we deduce that $s(D-F)(a) = a(D-F)(s)$ for any $a \in A$. It follows that for all $a, b \in A$,

$$as(D - F)(b) = ab(D - F)(s) = bs(D - F)(a);$$

i.e., $a(D - F)(b) = b(D - F)(a)$ and identity (2) is satisfied. □

Note that Theorem 1 holds also for a Brešar generalized derivation because identities (1) and (2) are satisfied. Localization is of interest to us for it gives the possibility of adjoining unity. Below is an example showing that this is not always possible.

Example 1. Consider a two-dimensional algebra B with basis e_1, e_2 and multiplication given as follows:

$$e_1^2 = e_1e_2 = 0, \quad e_2^2 = e_1.$$

Obviously, it is an associative commutative algebra. Define the operations D and Δ by setting

$$D(e_1) = D(e_2) = \Delta(e_1) = e_1, \quad \Delta(e_2) = e_2.$$

Then

$$\begin{aligned}
D(e_1^2) &= 0 = D(e_1)e_1 + e_1\Delta(e_1),\\
D(e_1e_2) &= 0 = D(e_1)e_2 + e_1\Delta(e_2),\\
D(e_2e_1) &= 0 = D(e_2)e_1 + e_2\Delta(e_1),\\
D(e_2e_2) &= D(e_1) = e_1 = e_1e_2 + e_2(e_1+e_2)\\
&= D(e_2)e_2 + e_2\Delta(e_2);
\end{aligned}$$

i.e., (D,Δ) is a ternary derivation. Note that

$$\Delta(e_2^2) = \Delta(e_1) = e_1 \neq 2e_1 = 2e_2(e_1+e_2) = 2e_2\Delta(e_2);$$

i.e., Δ is not a derivation, while (D,Δ) is not a Brešar generalized derivation.

Suppose that there exists an embedding of the algebra B into the unital algebra $\widehat{B}$ with an extension of the ternary derivation $(\widehat{D},\widehat{\Delta})$. Then

$$\begin{aligned}
\widehat{D}(1\cdot e_2) &= \widehat{D}(e_2) = e_1,\\
\widehat{D}(1)e_2 + 1\Delta(e_2) &= \widehat{D}(1)e_2 + e_1 + e_2,\\
\widehat{D}(1\cdot e_1) &= \widehat{D}(e_1) = e_1,\\
\widehat{D}(1)e_1 + 1\cdot\Delta(e_1) &= \widehat{D}(1)e_1 + e_1.
\end{aligned}$$

In other words, $\widehat{D}(1)e_2 = -e_2$ and $\widehat{D}(1)e_1 = 0$. By multiplying the left- and right-hand sides of the first equality by e_2, we obtain

$$\begin{aligned}
\widehat{D}(1)e_2^2 &= \widehat{D}(1)e_1 = 0,\\
e_2e_2 &= e_1 \neq 0.
\end{aligned}$$

An algebra A with a set of linear maps φ_i, $i \in I$, is simple if $A\cdot A = A$ and A does not contain ideals invariant under φ_i. This means that D-, E-, and F-invariant ideals are missing for algebras with ternary and generalized derivations, and that D- and δ-invariant ideals are missing for algebras with Brešar generalized derivations.

It is not hard to verify that for unital algebras, the simplicity conditions for an algebra with ternary derivation (D,E,F) or Brešar generalized derivation (D,δ) are equivalent to there being no ideals invariant with respect to just one of the components. Indeed, in this case they differ from each other only in the right multiplication operator.

For any algebra A, its annihilator $Ann_A = \{x \in A \mid xA = 0\}$ is an ideal. Moreover, the following holds:

LEMMA 5. Let A be a commutative algebra with a ternary derivation (D,E,F) and let identities (1) and (2) hold. Then Ann_A is invariant with respect to each of the components D, E, and F. In particular, the annihilator of a (D,E,F)-simple algebra is equal to zero.

Proof. Let $x \in Ann_A$ and $a \in A$. Then

$$0 = D(xa) = E(x)a + xF(a) = E(x)a,$$
$$0 = D(ax) = E(a)x + aF(x) = aF(x).$$

And finally, in view of identity (1) we have

$$D(x)a = (D - E)(a)x + aE(x) = 0. \square$$

2. NOVIKOV–POISSON ALGEBRAS

Below we assume that the characteristic of the base field is not two. Let $\langle A, \cdot, \circ\rangle$ be an algebra with a couple of multiplications $\cdot$ and $\circ$ such that $\langle A, \cdot\rangle$ is an associative commutative algebra and the following identities hold:

$$xy \circ z = x(y \circ z), \tag{5}$$
$$xz \circ y - x \circ yz = yz \circ x - y \circ xz, \tag{6}$$
$$(x \circ y) \circ z = (x \circ z) \circ y, \tag{7}$$
$$(x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z). \tag{8}$$

Then $\langle A, \cdot, \circ\rangle$ is a Novikov–Poisson algebra.

Example 2. Consider an associative commutative algebra A with derivation d and define the operation $\circ$ as follows:

$$a \circ b = ad(b) + \gamma ab, \tag{9}$$

where $\gamma \in A$. Then $\langle A, \cdot, \circ\rangle$ is a Novikov–Poisson algebra, which we will call a Novikov–Poisson algebra of vector type.

PROPOSITION 1. Let A be an associative commutative algebra with ternary derivation (D, E, F). Define multiplication $\circ$ as follows:

$$a \circ b = aD(b). \tag{10}$$

If identities (1) and (2) hold, then $\langle A, \cdot, \circ\rangle$ is a Novikov–Poisson algebra.

Let A contain at least one element that is not a zero divisor. Then the converse is also true: if $\langle A, \cdot, \circ\rangle$ is a Novikov–Poisson algebra, then identities (1) and (2) hold.

Proof. We verify satisfaction of (5)-(8). Identities (5) and (7) are satisfied in view of $\langle A, \cdot\rangle$ being commutative. Consider identity (6):

$$xz \circ y - x \circ yz - yz \circ x + y \circ xz$$
$$= xzD(y) - xD(yz) - yzD(x) + yD(xz)$$

$$= xzD(y) - xE(y)z - xyF(z) - yzD(x) - yE(x)z + yxF(z)$$
$$= z(x(D - E)(y) - y(D - E)(x)) = 0.$$

Consider identity (8):

$$\begin{aligned}
&(x \circ y) \circ z - x \circ (y \circ z) - (y \circ x) \circ z + y \circ (x \circ z) \\
&\quad = xD(y)D(z) - xD(yD(z)) - yD(x)D(z) + yD(xD(z)) \\
&\quad = xD(y)D(z) - xE(y)D(z) - xyF(D(z)) \\
&\qquad - yD(x)D(z) + yE(x)D(z) + yxF(D(z)) \\
&\quad = D(z)(x(D - E)(y) - y(D - E)(x)) = 0.
\end{aligned}$$

If A contains at least one element that is not a zero divisor, then, as noted, identity (6) reduces to

$$z(x(D - E)(y) - y(D - E)(x)) = 0$$

for all $x, y, z \in A$. Choosing z to be an element which is not a zero divisor, we obtain the result required. □

Note that not all Novikov–Poisson algebras are given by formula (9). A series of examples of such algebras is presented in [21]. Consider a two-dimensional Novikov–Poisson algebra B from that paper, defined as follows:

$$e_2 \cdot e_2 = e_1, \quad e_2 \circ e_1 = -e_1,$$

and the other products are zero. This algebra is not given by (9).

PROPOSITION 2. An algebra B does not contain a ternary derivation (D, E, F) such that Novikov multiplication is given by formula (10).

Proof. Let multiplication be given by formula (10) of some ternary derivation (D, E, F). Then

$$e_2 \circ e_1 = e_2 D(e_1) = -e_1.$$

Consequently, $D(e_1) = \alpha e_1 - e_2$. Therefore,

$$E(e_2)e_2 + e_2 F(e_2) = D(e_2 e_2) = D(e_1) = \alpha e_1 - e_2.$$

On the other hand,

$$E(e_2)e_2 + e_2 F(e_2) \subseteq B \cdot B \subseteq \mathbb{F} e_1,$$

a contradiction. Hence the algebra B is not given by (10). □

PROPOSITION 3. An algebra B does not embed in a Novikov–Poisson algebra with unital commutative part. In particular, it does not embed in a Novikov–Poisson algebra of vector type.

Proof. Suppose that the given Novikov–Poisson algebra embeds in a Novikov–Poisson algebra with unital commutative part. By identity (6),

$$e_2 e_2 \circ 1 - e_2 \circ e_2 = e_2 \cdot 1 \circ e_2 - 1 \circ e_2 e_2.$$

In other words,

$$e_1 \circ 1 = -1 \circ e_1.$$

By multiplying via commutative multiplication the left and right parts by e_2, we obtain

$$0 = e_2 e_1 \circ 1 = -e_2 \circ e_1 = e_1,$$

a contradiction. Hence B does not embed in a Novikov–Poisson algebra with unital commutative part. □

COROLLARY 2. Novikov–Poisson algebras do not embed in Novikov–Poisson algebra of vector type.

Let $\langle A, \cdot, \circ\rangle$ be a Novikov–Poisson algebra of vector type with multiplication defined by identity (9). If $\langle A, \cdot\rangle$ is differentially simple with respect to a derivation d, then the algebra $\langle A, \cdot\rangle$ is unital [22] and the algebra $\langle A, \circ\rangle$ is simple [18]. The converse is also true: if $\langle A, \circ\rangle$ is simple and $A \cdot A = A$, then $\langle A, \cdot\rangle$ is unital and differentially simple [19].

LEMMA 6. Let $\langle A, \cdot\rangle$ be an associative commutative algebra with ternary derivation (D, E, F) such that identities (1) and (2) hold, $A \cdot A = A$, and with multiplication $\circ$ given by rule (10). If $\langle A, \cdot\rangle$ has no nontrivial D-invariant ideals, then the proper ideals of $\langle A, \circ\rangle$ lie in the annihilator of $\langle A, \cdot\rangle$.

Proof. (1) Let $\langle A, \cdot\rangle$ be (D, E, F)-simple. Consider a proper ideal I of the algebra $\langle A, \circ\rangle$. Let $K = AI$. Then $AK \subseteq K$. Take $k \in K$. Then $k = \sum a_i p_i$, where $a_i \in A$ and $p_i \in I$. Hence

$$\begin{aligned} D(k) &= D\left(\sum a_i p_i\right) \\ &= \sum \left(E(a_i)p_i + a_i F(p_i)\right) \\ &= \sum (E(a_i)p_i + a_i D(p_i) - p_i(D - F)(a_i)) \\ &= \sum (E(a_i)p_i + a_i \circ p_i - p_i(D - F)(a_i)). \end{aligned}$$

Since $A \cdot A = A$, we have $a_i = \sum b_j c_j$ and

$$a_i \circ p_i = \left(\sum b_j c_j\right) \circ p_i = \sum b_j (c_j \circ p_i) \in K.$$

Hence $D(K) \subseteq K$. Then $K = A$ or $K = 0$. Let $K = A$, i.e. $IA = A$, but $I \lhd A$, i.e. $A \subseteq I$, whence $I = A$. Thus $K = 0$, i.e. $AI = 0$ and I lies in the annihilator of $\langle A, \cdot\rangle$. □

THEOREM 2. Let $\langle A, \cdot\rangle$ be an associative commutative algebra with Brešar generalized derivation (D, δ), $A \cdot A = A$, and with multiplication $\circ$ given by rule (10). Then the following conditions are equivalent:

(1) $\langle A, \cdot\rangle$ is a (D, δ)-simple algebra;

(2) $\langle A, \cdot\rangle$ is a δ-simple algebra;

(3) $\langle A, \circ\rangle$ is simple.

Proof. (1)→(2) Let $\langle A, \cdot\rangle$ be (D,δ)-simple. We show that $\langle A, \cdot\rangle$ is δ-simple. Take a δ-invariant ideal I of the algebra $\langle A, \cdot\rangle$. Put $K = IA$. Then $KA \subseteq K$ and

$$\delta(IA) \subseteq \delta(I)A + I\delta(A) \subseteq IA.$$

We have

$$D(K) = D(IA) = D(AI) \subseteq D(A)I + A\delta(I) \subseteq AI = K.$$

Thus K is a D- and δ-invariant ideal. It follows that $K = A$ or $K = 0$. Let $K = A$, i.e. $A = IA \subseteq I$ since $I \lhd A$. Then $A \subseteq I$, whence $I = A$. Hence $K = 0$, i.e. $AI = 0$ and I lies in the annihilator of $\langle A, \cdot\rangle$. By Lemma 5, the annihilator is a D- and δ-invariant ideal, i.e. $I = 0$.

(2)→(1) Is obvious.

(2)→(3) The algebra A is δ-simple and hence unital by virtue of [22], and the algebra $\langle A, \circ\rangle$ is simple in view of [18].

(3)→(2) If $\langle A, \circ\rangle$ is simple and $A\cdot A = A$, then $\langle A, \cdot\rangle$ is unital. The algebra $\langle A, \cdot\rangle$ has no proper ideals invariant under the map ∂, where

$$\partial(a) = 1\circ a - a\circ 1 = D(a) - a(1\circ 1).$$

Hence any D-invariant ideal is ∂-invariant; i.e., the algebra does not contain proper ideals invariant with respect to D. We conclude that $\langle A, \cdot\rangle$ is (D,δ)-simple. $\square$

We have already mentioned that differential algebras are connected with Jordan superalgebras. Let A ba an associative commutative algebra with a skew-symmetric bilinear operation $\{\,,\}$, which we call a bracket. Multiplication on $J = A + A\xi$ is defined as follows:

$$x\cdot y = xy, \quad x\xi\cdot y = x\cdot y\xi = xy\xi, \quad x\xi\cdot y\xi = \{x,y\},$$

where ab is multiplication in A. The algebra obtained is called a *Kantor double* and is denoted by $J(A,\{\,,\})$.

A bracket $\{\,,\}$ given on an associative commutative algebra A is called *Jordan* if the Kantor double $J(A,\{\,,\})$ is a Jordan superalgebra. A criterion for a bracket defined on a unital algebra A to be Jordan was found in [23, 24]. For a nonunital algebra, relations assume the following form (see [25]):

$$\begin{aligned}\{x,y\}\{z,t\} - \{x,t\}\{z,y\} + \{y,t\}\{z,x\}& \\ = \{\{x,y\}z,t\} - \{\{x,t\}z,y\} + \{\{y,t\}z,x\}&,\end{aligned} \tag{11}$$

$$\{xy,z\}t - \{xt,z\}y = xy\{z,t\} - xt\{z,y\}, \tag{12}$$

$$\{xy,z\}t + \{xt,z\}y + \{x,yzt\} = \{xy,zt\} + \{xdt,yz\} + yt\{x,z\} \tag{13}$$

for all $x,y,z,t\in A$.

Example 3. Let A be an associative commutative algebra with derivation d. Define the bracket as follows:

$$\{x,y\} = xd(y) - d(x)y.$$

This bracket is Jordan and is called a *bracket of vector type*. The obtained Jordan superalgebra $J(A, \{\,,\})$ is called a Jordan *superalgebra of vector type*.

For an algebra A with a ternary derivation (D, E, F), we define the bracket

$$\{x, y\} = xD(y) - D(x)y. \tag{14}$$

Example 4. Consider a Novikov–Poisson algebra $\langle A, \cdot, \circ \rangle$ and define the bracket

$$\{x, y\} = x \circ y - y \circ x.$$

In [19] it was shown that this bracket is Jordan. If $\langle A, \cdot \rangle$ is unital, then the map ∂ given by the rule

$$\partial(x) = 1 \circ x - x \circ 1$$

is a derivation and the following equality holds:

$$\{x, y\} = x\partial(y) - \partial(x)y;$$

i.e., $\{\,,\}$ is a bracket of vector type.

PROPOSITION 3. Let A be an associative commutative algebra with a ternary derivation (D, E, F). If identities (1) and (2) are satisfied, then the bracket defined via rule (14) is Jordan. If A contains at least one element which is not a zero divisor, then the converse is also true: if the bracket defined via rule (14) is Jordan then identities (1) and (2) are satisfied.

Proof. By Proposition 1, the algebra $\langle A, \cdot, \circ \rangle$ will be a Novikov–Poisson algebra, and hence $\{\,,\}$ will be a Jordan bracket.

Conversely, let $\{\,,\}$ be Jordan. Then identity (12) holds, i.e.

$$\begin{aligned}
\{xy, z\}t - \{xt, z\}y - xy\{z, t\} + xt\{z, y\} \\
&= (D(xy)z - xyD(z))t - (D(xt)z - xtD(z))y \\
&\quad - xy(D(z)t - zD(t)) + xt(D(z)y - zD(y)) \\
&= E(x)yzt + F(y)xzt - E(x)yzt - F(t)xyz + D(t)xyz - D(y)xzt \\
&= xz(y(D - F)(t) - t(D - F)(y)).
\end{aligned}$$

Choosing a and c to be elements that are not zero divisors, we obtain identity (2) and hence identity (1). □

Note that if A is an associative commutative algebra with a Brešar generalized derivation (D, δ), then the brackets relative to D and δ coincide; i.e., $J(A, D)$ will be a Jordan superalgebra of vector type.

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