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PERIODIC GROUPS SATURATED WITH FINITE SIMPLE GROUPS $L_4(q)$

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If M is a set of finite groups, then a group G is said to be saturated with the set M (saturated with groups in M) if every finite subgroup of G is contained in a subgroup isomorphic to some element of M. It is proved that a periodic group with locally finite centralizers of involutions, which is saturated with a set consisting of groups $L_4(q)$, where q is odd, is isomorphic to $L_4(F)$ for a suitable field F of odd characteristic.

Let M be a nonempty set of finite groups. By definition, a group G is *saturated with groups in M*, or *saturated with the set M*, if every finite subgroup of G lies in a subgroup isomorphic to some element of the set M.

Our goal is to prove the following:

THEOREM. Let G be a periodic group saturated with groups in the set

$$M = \{L_4(q) \mid q \text{ is odd}\}.$$

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If all centralizers of involutions in G are locally finite, then G is isomorphic to the group $L_4(F)$ for some locally finite field F of odd characteristic.

1. NOTATION AND PRELIMINARY RESULTS

We will use the notation from [1]. For a number $q = p^s$, where p is an odd prime, by P we denote a field $GF(q)$ of order q. Let $\tilde{L} = SL_4(q)$ be the special linear group of dimension 4 over P, i.e., a group of (4×4)-matrices over P with determinants equal to 1, and

$$\tilde{C} = \left\langle \begin{bmatrix} \alpha & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \alpha^{-1} & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \middle|\, \alpha \in P,\, \alpha \neq 0 \right\rangle,$$

$$\tilde{i}_1 = \begin{bmatrix} -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} = \begin{bmatrix} -E & 0 \\ 0 & E \end{bmatrix},\ \tilde{i}_2 = -\tilde{i}_1,\ \tilde{t} = \begin{bmatrix} 0 & E \\ E & 0 \end{bmatrix},$$

$$\tilde{S}_1 = \left\langle \begin{bmatrix} X & 0 \\ 0 & E \end{bmatrix} \middle|\, X \in SL_2(P) \right\rangle,\ \tilde{S}_2 = \left\langle \begin{bmatrix} E & 0 \\ 0 & X \end{bmatrix} \middle|\, X \in SL_2(P) \right\rangle,$$

$$\tilde{H} = \tilde{H}(q) = \langle \tilde{S}_1, \tilde{S}_2, \tilde{C}, \tilde{t} \rangle.$$

An easy check shows that $\tilde{C}$ is a cyclic group of order $q-1$, $\tilde{i}_1$ and $\tilde{i}_2$ are involutions, $\langle \tilde{C}, \tilde{t} \rangle = (q-1) : 2$ is a dihedral group, and

$$\tilde{H} = \langle \tilde{S}_1, \tilde{S}_2, \tilde{C}, \tilde{t} \rangle = \langle \tilde{S}_1, \tilde{S}_2 \rangle \cdot \langle \tilde{C}, \tilde{t} \rangle = (SL_2(q) \times SL_2(q)) : (q-1) : 2.$$

Denote by H, i_1, i_2, C, S_1, S_2, and t the corresponding images of $\tilde{H}$, $\tilde{i}_1$, $\tilde{i}_2$, $\tilde{C}$, $\tilde{S}_1$, $\tilde{S}_2$, and $\tilde{t}$ under the canonical homomorphism of $SL_4(q)$ onto $L_4(q) = PSL_4(q)$.

LEMMA 1. (1) $S_1 \simeq \tilde{S}_1 \simeq \tilde{S}_2 \simeq S_2 \simeq SL_2(q)$, $\langle S_1, S_2 \rangle = S_1 \circ S_2$ is a central product of the groups S_1 and S_2, $i = i_1 = i_2$ is the unique involution in the center of $S_1 S_2$, $S_2 = S_1^t$, $S_2^t = S_1$, and $C_H(S_1) = S_2 Z$, where Z is the image of a cyclic subgroup

$$\left\langle \begin{bmatrix} \alpha & \cdot & \cdot & \cdot \\ \cdot & \alpha & \cdot & \cdot \\ \cdot & \cdot & \alpha^{-1} & \cdot \\ \cdot & \cdot & \cdot & \alpha^{-1} \end{bmatrix} \middle|\, \alpha \in P,\, \alpha \neq 0 \right\rangle$$

under the canonical homomorphism of $SL_4(q)$ onto $L_4(q)$, and $C_H(S_2) = S_1 Z$.

(2a) If $r \in S_1 S_2$ and $r^2 = i$, then $r \in S_1 \cup S_2$.

(2b) H contains a subgroup isomorphic to $SL_2(3)$. If D is one of these subgroups, then the subgroup $O_2(D)$ isomorphic to the quaternion group of order 8, is contained in S_j for some $j \in \{1, 2\}$, and $C_H(O_2(D)) = C_H(S_j)$. If $D_1 \simeq D$ and $[O_2(D), O_2(D_1)] = 1$, then $C_H(O_2(DD_1)) = Z$.

(3) H contains an elementary Abelian subgroup A of order 16 satisfying the following conditions:

(3a) $N_L(A)$ contains a subgroup $K \geq A$ of index at most 2, where K/A is isomorphic to the alternating group Alt (6) for $q \equiv 1 (\mathrm{mod}\, 4)$, and to the alternating group Alt (5) for $q \equiv -1 (\mathrm{mod} 4)$;

(3b) $\langle H, K \rangle = L$ for $q > 3$.

(4a) An involution t is conjugate in L to i. For $q > 3$, H is a maximal subgroup of L which coincides with $C_L(i)$.

(4b) If $q \equiv 1 (\bmod 8)$, then every involution of L is conjugate to i; otherwise, the centralizer of every involution that is not conjugate to i in L contains a section isomorphic to $L_2(q^2)$. If U is a subgroup of L isomorphic to $SL_2(3) \circ SL_2(3)$, then an involution lying in the center of U is conjugate to i.

Proof. (1) Is straightforward.

(2a) Let $r = s_1 s_2$, where $s_1 \in S_1$, $s_2 \in S_2$, and $r^2 = i$. Then $r^2 = s_1^2 s_2^2 = i$, whence $S_1 \ni s_1^2 = s_2^{-2} i \in S_2$ and $s_1^2 \in S_1 \cap S_2 = \langle i \rangle \ni s_2^{-2} i$. Suppose that $s_1^2 = 1$. Since i is the only involution in S_1, we have $s_1 \in \langle i \rangle \leq S_2$ and $r \in S_2$. Suppose now that $s_1^2 \neq 1$. Then $s_1^2 = i = s_2^{-2} i$, $s_2^2 = 1$, and $r \in S_1$.

(2b) Let D be a subgroup of H isomorphic to $SL_2(3)$. Since H/S_1S_2 is a dihedral group, $O_2(D) \leq S_1 S_2$.

If D does not contain i, then $D \cap S_1 = 1$, which is impossible because in that case $O_2(D)$ would be isomorphically embeddable in a dihedral group, a Sylow 2-subgroup of the group $L_2(q) \simeq S_1S_2/S_1$. Let $\langle r_1 \rangle$, $\langle r_2 \rangle$, and $\langle r_3 \rangle$ be different subgroups of order 4 in $O_2(D)$. Then $r_1^2 = r_2^2 = r_3^2 = i$. By (2a), two of the three subgroups lie in S_j for some $j \in \{1, 2\}$ and generate $O_2(D)$, i.e., $O_2(D) \leq S_j$. We have $C_{S_j}(O_2(D)) = \langle i \rangle$, therefore $C_H(O_2(D)) = C_H(S_j)$. Besides, $O_2(D_1) \leq S_{3-j}$, which implies that $C_H(O_2(DD_1)) = C_H(S_1S_2) = Z$.

(3) The group $\tilde{H} \leq SL_4(q)$ contains a subgroup $\tilde{A}$ of symplectic type, which is isomorphic to the extraspecial group 2^{1+4} for $q \equiv -1 (\bmod 4)$, and to the group $4 \circ 2^{1+4}$ for $q \equiv 1 (\bmod 4)$. The image N of the subgroup $N_{SL_4(q)}(\tilde{A})$ in $L_4(q)$ under the natural homomorphism of $SL_4(q)$ onto $L_4(q)$ is equal to $A : A_5$ or $A : S_5$ for $q \equiv -1 (\bmod 4)$, and for $q \equiv 1 (\bmod 4)$, it is equal to $A : A_6$ or $A : S_6$, where A is the image of $\tilde{A}$ (see [2, Sec. 4.6; 3, Sec. 2.2.6, Table 8.8]).

For $q > 3$, the subgroup H contains a Sylow 2-subgroup of the group $L_4(q)$, in which case we may assume that $A = O_2(N) \leq H$. It is clear that $K = [N, N] \not\leq H$. Since H for $q > 3$ is a maximal subgroup in $L_4(q)$, we have $\langle H, K \rangle = L_4(q)$.

(4a) Follows from [3, Table 8.8].

(4b) See the proof in [4, pp. 424-427]. The lemma is completed.

2. MAIN RESULT

Proof of the theorem. If an involution in L is such that its centralizer contains a subgroup isomorphic to $SL_2(3) \circ SL_2(3)$, then we will say that it is an involution of principal type in L.

Let G be a periodic group saturated with the set $M = \{L_4(q) \mid q \text{ is odd}\}$, in which the centralizer of every involution is locally finite. Denote by $M(G)$ the set of subgroups of G isomorphic to elements in M.

If $M(G) = \{L_4(3)\}$, then the theorem is true [5]. Below, therefore, we will assume that $M(G) \neq \{L_4(3)\}$.

We take in $M(G)$ a subgroup $L \simeq L_4(q)$ with $q \equiv 1 (\mathrm{mod}\, 4)$. If, however, $M(G)$ does not have such subgroups, then as $L \simeq L_4(q)$ we take a subgroup in $M(G)$ for which $q > 3$.

We fix L till the end of the proof of the theorem and keep the notation from Section 1 for its subgroups and elements. Fix a subgroup $D_1 \simeq SL_2(3)$ from S_1. Let $D_2 = D_1^t$. Note that $D_1D_2 = D_1 \circ D_2$ and, in particular, $[D_1, D_2] = 1$.

LEMMA 2. $C = C_G(O_2(D_1D_2))$ is a locally cyclic group. If D is a subgroup of $C_G(i)$ isomorphic to $SL_2(3) \circ SL_2(3)$, then $C_G(O_2(D)) = C$ and, in particular, $C \trianglelefteq C_G(i)$.

Proof. Let $x, y \in C$. Then $X = \langle D_1D_2, x, y\rangle \leq C_G(i)$, and therefore X is finite. Let $L^* \simeq L_4(q^*)$ be a subgroup in $M(G)$ that contains X, and $H^* = C_{L^*}(i)$. Since H^* contains $D_1D_2 \simeq$ $\simeq SL_2(3) \circ SL_2(3)$, it follows by Lemma 1(4b) that H^* has a normal subgroup $S_1^* \circ S_2^*$, where $S_1^* \simeq S_2^* \simeq SL_2(q^*)$. In view of Lemma 1(2b), the elements x and y belong to a cyclic group $C_{H^*}(O_2(D_1D_2))$, and C is a locally cyclic group.

Suppose that $x \in C_G(O_2(D))$ and $y \in C_G(O_2(D_1D_2))$. Then $\langle x, y, D, D_1D_2\rangle$ is a finite subgroup of $C_G(i)$ lying in some element L^* of the set $M(G)$. As above, i is an involution of principal type in L^*, and if $H^* = C_{L^*}(t)$, then $C_{H^*}(O_2(D)) = C_{H^*}(O_2(D_1D_2))$ and $x, y \in C_G(O_2(D_1D_2))$. Hence $C = C_G(O_2(D)) = C_G(O_2(D_1D_2))$.

If now $x \in C_G(i)$, then $C^x = C_G(O_2(D_1^xD_2^x)) = C$. The lemma is proved.

LEMMA 3. $C_G(O_2(D_1))/C \simeq L_2(F^*)$ for a locally finite field F^* of odd characteristic.

Proof. By Lemma 2, $C \trianglelefteq C_G(i)$. In virtue of the fact that $C \leq C_G(O_2(D_1)) \leq C_G(i)$, we have $C \trianglelefteq C_G(O_2(D_1))$. Let X be a finite subgroup in $C_G(O_2(D_1))/C$ containing $O_2(D_1)C/C$ and let U be its finite preimage in $C_G(O_2(D_1))$ containing $O_2(D_1)$. By hypothesis, $U \leq L^* \simeq L_4(q^*)$ for some q^*, and hence $U \leq L^* \cap C_G(D_1) \simeq SL_2(q^*)$. In view of $(L^* \cap C_G(D_1))C/C \simeq L_2(q^*)$ and [6], we conclude that $C_G(O_2(D_1))/C \simeq L_2(F^*)$ for some locally finite field F^* of odd characteristic. The lemma is proved.

LEMMA 4. The subgroup $C_G(i)$ is countable.

Proof. Since $O_2(D_2) = O_2(D_1)^t$ and C is invariant with respect to t, Lemma 3 implies that $C_G(O_2(D_2))/C \simeq L_2(F^*)$.

We want to show that $[C_G(O_2(D_1)), C_G(O_2(D_2))] = 1$. Let $x \in C_G(O_2(D_1))$, $y \in C_G(O_2(D_2))$, and $U = \langle x, y, D_1, D_2\rangle$. Because $U \leq C_G(i)$, U is a finite group by assumption. It lies in a subgroup $L^* \cap C_G(i)$, where $L^* \simeq L_4(q^*)$ for some q^*, and contains D_1D_2.

In view of Lemma 2, $[x, y] = 1$.

We show that $N = C_G(O_2(D_1))C_G(O_2(D_2))$ is a subgroup of finite index in $C_G(i)$.

First, $N \trianglelefteq C_G(i)$. Indeed, suppose that $g \in C_G(i)$. Then $O_2(D_1)^g \leq C_G(O_2(D_1))$ or $O_2(D_1)^g \leq C_G(O_2(D_2))$, and $O_2(D_1)^g \leq N$. Similarly, $O_2(D_2)^g \leq N$. This implies $\langle O_2(D_1)^{C_G(i)}, O_2(D_2)^{C_G(i)}, C\rangle = N$.

If $g \in C_G(i)$, then $C_G(O_2(D_1))^g = C_G(O_2(D_j))$, where $j \in \{1,2\}$, and

$$|C_G(i) : N_{C_G(i)}(C_G(O_2(D_1))| = 2.$$

In $L_2(F^*)$, there are at most two classes of nonconjugate noncyclic subgroups of order 4, hence N is a subgroup of finite index in $C_G(i)$. Since C is a countable group and N/C too is countable, $C_G(i)$ is a countable group. The lemma is proved.

LEMMA 5. The subgroup $C_G(i)$ lies in a subgroup R of the group G isomorphic to $L_4(F)$ for a locally finite field F of odd characteristic.

Proof. By Lemma 4, $C_G(i)$ is countable, and $C_G(i) = \{x_j \mid x_j \in C_G(i),\ j = 1, 2, \dots\}$. Let $H_1 = C_L(i)$, where L is a subgroup which we fixed at the beginning of the section. Suppose that for $k \in \mathbb{N}$, the subgroup H_k is already defined, and take the first of the elements x_j not in H_k to be y_k. The subgroup $\langle H_k, y_k\rangle$ is finite and is therefore contained in some subgroup $L_{k+1} \simeq L_4(q_{k+1})$ of G. Set $H_{k+1} = C_{L_{k+1}}(i)$. In view of Lemma 1(3), $L_{k+1} = \langle H_{k+1}, K\rangle$, where K is as in Lemma 1(3). Then $L_k \le L_{k+1}$. The union $R = \bigcup_k L_k$ is a locally finite group and it contains $C_G(i)$. By [6], $R \simeq L_4(F)$ for some locally finite field F of odd characteristic. The lemma is proved.

LEMMA 6. R contains every involution of G that is conjugate to i, and such an involution is conjugate to i in R.

Proof. Let $g \in G$ and $X = \langle i, i^g\rangle$. The subgroup X is finite, and by hypothesis, lies in a subgroup $L_* \simeq L_4(q_*)$ for some q_*. We define a graph Γ whose vertices are the involutions of L_* conjugate in G to i, and we connect two involutions in Γ by an edge if they commute.

We claim that Γ is connected. First, we check that L_* acts transitively on the set of vertices of Γ, i.e., two involutions of L_* that are conjugate to i in G will be conjugate to i in L_*. Suppose that this is not true. Let u be a vertex of Γ not conjugate to i in L_*. By [4, 6(5.2)], $C_{L_*}(u)$ contains a subgroup $E\langle r\rangle$, where $E/Z(E) \simeq L_2(q_*^2)$, r is an involution normalizing E, and r induces under conjugation in $E/Z(E)$ an automorphism mapping every element of $E/Z(E)$ into its image under the action of a nontrivial element of the Galois group of the extension of $GF(q^2)/GF(q)$.

On the other hand, u is conjugate in G to i, therefore $C_{L_*}(u)$ lies in some finite subgroup H^* of $C_G(u)$ isomorphic to $SL_2(q^*) \circ SL_2(q^*) : (q^* - 1) : 2$ (cf. Lemma 1), which does not have subgroups isomorphic to $E\langle r\rangle$. The contradiction obtained shows that every vertex of Γ is conjugate in L_* to i and L_* acts under conjugation transitively on the vertices of Γ. If $q_* > 3$, then $C_{L_*}(i)$ is a maximal subgroup in L_* (cf. Lemma 1) and it contains at least one involution t^* which is a conjugate of i and commutes with it. Then a connected component of Γ that contains i coincides with the whole set of vertices in Γ. And if $q_* = 3$, then straightforward calculations in GAP [7] show that Γ is connected.

Let $i = i_0, i_1, \dots, i_s = i^g$, where (i_k, i_{k+1}) is an edge in Γ for all $k = 0, 1, \dots, s-1$. We use induction on s to show that the involution i_s lies in R and is conjugate in R to i. If $s = 1$, then $i_1 \in R$ by Lemma 5, and in this case L_* can be chosen in R. In view of the above, $i = i_1^r$ for some

$r \in R$ and $\{i = i_1^r, i_2^r, \ldots, i_s^r\}$ is a chain of length $s - 1$ in Γ. By induction, i^{gr} lies in R and is conjugate in R to i. The lemma is proved.

LEMMA 7. We have $R = G$.

Proof. By virtue of Lemma 6, $\langle i^G \rangle \leq R$, and since R is simple, $R = \langle i^G \rangle \trianglelefteq G$. If $g \in G$, then $i^g \in R$ and there is $x \in R$ such that $i^{gx} = i$. Since $C_G(i) \leq R$, we have $gx \in R$ and $g \in R$. The lemma, together with the theorem, is proved.

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