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ABSTRACT RELATIONS BETWEEN FUNCTIONAL CLONES

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Functional clones on a set A are investigated at an abstract level, i.e., up to isomorphism of universal algebras $\langle A; F\rangle$, with their signature treated as an unindexed set. Abstract relations are introduced on a collection F_A of all functional clones on A, and the question of their coincidence is discussed.

Recall that a *functional clone on a set A* is an arbitrary collection of functions on A which is closed under superposition and includes all selector functions $e_n^i(x_1, \ldots, x_n) = x_i$ $(1 \leq i \leq n)$ on the set A. As a rule, a collection F_A of all functional clones on A is treated as an ordered set (a lattice) under set-theoretic inclusion $\subseteq$ (see, e.g., [1]). In [2], we proposed a natural topology τ on F_A, which transforms a lattice $\langle F_A; \wedge, \vee\rangle$ into a continuous lattice $\langle F_A; \wedge, \vee, \tau\rangle$. For any collection S of functions on a set A, by $\langle S\rangle$ we denote the least functional clone on A including the collection S.

A classical and, in essence, exhaustive example of functional clones on a set A is given by collections $Tr(\mathfrak{A})$ of termal functions of universal algebras $\mathfrak{A} = \langle A; \sigma\rangle$ with universe A (for any $F \in F_A$, we have $F = Tr(\langle A; F\rangle)$; here, the collection of signature functions of an algebra $\langle A; F\rangle$ consists of all functions occurring in the clone F). Therefore, it seems natural to examine functional clones on a set A at an abstract level—up to isomorphism of universal algebras $\langle A; F\rangle$, with their signature treated as an unindexed set. In fact, this corresponds to a rational equivalence relation for universal algebras, which was introduced by Mal'tsev in [3]. On a collection F_A of functional clones on a set A, we introduce an equivalence relation $\sim$ corresponding to such an approach. Namely, for $F_1, F_2 \in F_A$, we set

$F_1 \sim F_2$ if there exists a permutation π on A conjugating the collections F_1 and F_2, i.e., $F_1 = \pi F_2 \pi^{-1} = \{\pi f(\pi^{-1}x_1, \ldots, \pi^{-1}x_n) \mid f \in F_2\}$.

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Such an approach suggests introduction of another relation $\ll$ on F_A, i.e., for $F_1, F_2 \in F_A$, we set $F_1 \ll F_2$ (F_1 is poorer than F_2; in other words, F_1 is not richer than F_2) if there exists a permutation π on A such that $\pi F_1 \pi^{-1} \subseteq F_2$.

The relation $\ll$ is a quasiorder relation, inducing an equivalence relation $\approx$ on F_A corresponding to a given quasiorder, i.e., for $F_1, F_2 \in F_A$, we set $F_1 \approx F_2$ if $F_1 \ll F_2$ and $F_2 \ll F_1$. The relation $\sim$ implies the relation $\approx$, and so it is natural to suppose that $\sim$ and $\approx$ coincide, as conjectured in [4]. However, this is not so. The following holds:

THEOREM 1. The relations $\sim$ and $\approx$ coincide on a collection F_A of all functional clones on a set A if and only if A is finite.

Proof. That $\sim$ and $\approx$ on F_A coincide for finite sets A was proved in [4]. We cite the proof here to make our discussion self-contained. Recall that an *n-fragment* $F_{(n)}$ of a clone F ($n \in \omega$) is a collection of all not more than n-ary functions from F.

Let A be a finite set, $F', F'' \in F_A$, and $F' \approx F$"; i.e., there exist permutations π_1 and π_2 on A such that $\pi_1 F' \pi_1^{-1} \subseteq F$" and $\pi_2 F\text{"} \pi_2^{-1} \subseteq F'$. Then $\pi_1 F'_{(n)} \pi_1^{-1} \subseteq F\text{"}_{(n)}$ and $\pi_2 F\text{"}_{(n)} \pi_2^{-1} \subseteq F'_{(n)}$ for any $n \in \omega$; i.e., $\pi_2 \pi_1 F'_{(n)} \pi_1^{-1} \pi_2^{-1} \subseteq \pi_2 F''_{(n)} \pi_2^{-1} \subseteq F'_{(n)}$ for any n. For finite sets A, n-fragments of clones on A are finite, and conjugation by permutations on A acts on the fragments of clones injectively. Therefore, the following equalities hold:

$$\pi_2 \pi_1 F'_{(n)} \pi_1^{-1} \pi_2^{-1} = \pi_2 F''_{(n)} \pi_2^{-1} = F'_{(n)}.$$

Since $F' = \bigcup\limits_{n \in \omega} F'_{(n)}$ and $F'' = \bigcup\limits_{n \in \omega} F''_{(n)}$, it is also true that $\pi_2 F''_{(n)} \pi_2^{-1} = F'$; i.e., $F' \sim F''$ in this case.

Now we show that for infinite sets A, there exist clones F' and F'' on A such that $F' \approx F''$ and $F' \nsim F''$. To do this, we make use of a sequence of functions f_i ($i = 2, 3, \ldots$) on a set $3 = \{0, 1, 2\}$, which were derived in [5] (see also [6]), in constructing on $3 = \{0, 1, 2\}$ a clone with countable basis.

For any $i = 2, 3, \ldots$, we put

$$f_i(x_1, \ldots, x_i) = \begin{cases} 1 & \text{if } x_1 = \ldots = x_{j-1} = x_{j+1} = \ldots = x_i = 2, \\ & x_j = 1 \text{ for some } 1 \leq j \leq i; \\ 0 & \text{otherwise.} \end{cases}$$

Let $G_n = \langle \{f_2, f_3, \ldots, f_{n+1}\} \rangle$. We show that a chain of clones $G_1 \subseteq G_2 \subseteq \ldots \subseteq G_n \subseteq G_{n+1} \subseteq \ldots$ is strictly increasing, and that no permutation π on $3 = \{0, 1, 2\}$ conjugates the clones G_n and G_m for $n \neq m$. Assume to the contrary that there exists a permutation π on $\{0, 1, 2\}$ such that $\pi G_m \pi^{-1} = \{\pi g(\pi^{-1} x_1, \ldots, \pi^{-1} x_n) \mid g(x_1, \ldots, x_n) \in G_m\} = G_n$, where $n < m$. The ranges of functions f_i generating clones G_j coincide with a set $\{0, 1\}$, and so the permutation π conjugating the clones G_m and G_n either is identical or is the transpose $(0, 1)$. In any case, for $i = 2, 3, \ldots$, the function $\pi f_i(\pi^{-1} x_1, \ldots, \pi^{-1} x_i)$ will not be identically equal to zero on $\{0, 1, 2\}$. The equality

$\pi G_m \pi^{-1} = G_n$, as well as the inclusion $\pi\{f_m\}\pi^{-1} \in G_n$, ensures that

$$\pi f_m(\pi^{-1}x_1, \ldots, \pi^{-1}x_m) = t(x_1, \ldots, x_m), \qquad (*)$$

where $t(x_1, \ldots, x_m)$ is a term of a signature $\langle f_2, \ldots, f_{n+1}\rangle$, i.e., $t(x_1, \ldots, x_m) = f_r(t_1, \ldots, t_r)$, where $r < m$ and t_i are terms of the same signature. The function f_m essentially depends on its arguments $x_1, \ldots, x_m$, and so the situation where all terms $t_1, \ldots, t_r$ are variables is excluded. Also impossible is the situation in which some pair of the terms $t_1, \ldots, t_r$ is distinct from variables; otherwise (values for that pair will be just 0 or 1) the right-hand side of equality $(*)$ is identically equal to zero, whereas the function $\pi f_m(\pi^{-1}x_1, \ldots, \pi^{-1}x_m)$ is not. Thus, exactly one of the terms $t_1, \ldots, t_r$ is distinct from variables. Since $r \geq 2$, a term t has the form $f_r(t_1, x_j, \ldots)$, where $1 \leq j \leq m$. If $x_j = 1$ or $x_j = 0$, then the right-hand side of $(*)$ assumes value 0. On the other hand, if π is an identical permutation on $\{0, 1, 2\}$, then the left-hand side assumes value 1, provided that $x_j = 1$ and $x_i = 2$ for $i \neq j$. For the case where $\pi = (0, 1)$, the same fact holds with $x_j = 0$ and $x_i = 2$ for $i \neq j$.

Therefore, the chain of clones $G_1 \subset G_2 \subset \ldots \subset G_n \subset G_{n+1} \subset \ldots$ on the set $\{0, 1, 2\}$ is strictly increasing, and its constituent clones are pairwise not conjugate via permutations on $\{0, 1, 2\}$.

Now we consider a partition of an infinite set A into subsets A_i $(i \in \omega + 1)$ such that $A_0 = \{a, b, c, d\}$, $A_n = \{0^n, 1^n, 2^n\}$ for $n \in \omega \backslash \{0\}$, and $|A_\omega| = |A|$. Here the elements $a, b, c, d, 0^n, 1^n, 2^n$ are pairwise disjoint.

For any function $g(x_1, \ldots, x_n)$ on $\{0, 1, 2\}$ and any $m \in \omega \backslash \{0\}$, on the set A we define a function $g^m(x_1, \ldots, x_n)$ such that for $b_1, \ldots, b_n \in A$,

$$g^m(b_1, \ldots, b_n) = \begin{cases} k^m & \text{if } \{b_1, \ldots, b_n\} \subseteq A_m, \langle b_1, \ldots, b_n\rangle = \langle e_1^m, \ldots, e_n^m\rangle, \\ & \text{and } g(e_1, \ldots, e_n) = k; \\ a & \text{otherwise, if in addition } \{b_1, \ldots, b_n\} \neq \{a\}, \{b\}, \{c\}, \{d\}, \\ b & \text{if } \{b_1, \ldots, b_n\} = \{a\}, \\ c & \text{if } \{b_1, \ldots, b_n\} = \{b\}, \\ d & \text{if } \{b_1, \ldots, b_n\} = \{c\}, \\ a & \text{if } \{b_1, \ldots, b_n\} = \{d\}. \end{cases}$$

Let I' and I'' be infinite subsets of the set $\omega \backslash \{0\}$ such that $I' = \{i_1 < i_2 < \ldots < i_n < i_{n+1} < \ldots\}$, $I'' = \{j_1 < j_2 < \ldots < j_n < j_{n+1} < \ldots\}$, and $I' \nsubseteq I''$.

We denote by F' a functional clone on a set A generated by the following collection of functions:

$$\bigcup_{m=1}^{\infty} \{g^m(x_1, \ldots, x_n) \mid g(x_1, \ldots, x_n) \in G_{i_m}, m \in \omega \backslash \{0\}\}$$
$$\cup \{e_n^i(x_1, \ldots, x_n) \mid 1 \leq i \leq n, n \in \omega\}.$$

In a similar way, we define a clone F'' using the set I'' instead of I'.

For any sets $B \subseteq C$ and any functional clone F on C, if the set B is closed with respect to functions in F, then we write $F \upharpoonright B$ to denote a functional clone on B which consists of restrictions of the functions in F to the set B. Thus, a bijection $\varphi_m : \{0,1,2\} \to \{0^m, 1^m, 2^m\}$ such that $\varphi_m(i) = i^m$ will conjugate G_{i_m} with $F' \upharpoonright A_m$ and G_{j_m} with $F'' \upharpoonright A_m$.

By the definition of clones F' and F'', the remark on restrictions of F' and F'' to sets A_n $(n \in \omega\backslash\{0\})$, and the condition that for any m there are $m' \in I'$ and $m'' \in I''$ for which $G_{i_m} \subseteq G_{j_{m'}}$ and $G_{j_m} \subseteq G_{i_{m''}}$, the relations $F' \ll F''$ and $F'' \ll F'$ hold.

It remains to observe that $F' \not\approx F''$. Assume the contrary, letting π be some permutation on A such that $\pi F' \pi^{-1} = F''$. The only three-element subsets of A closed with respect to functions in F' (in F'') are the sets A_n $(n \in \omega\backslash\{0\})$. Thus, for any $k \in \omega\backslash\{0\}$ there exists $s \in \omega\backslash\{0\}$ such that $\pi(\{0^k, 1^k, 2^k\}) = \{0^s, 1^s, 2^s\}$. By the choice of sets I' and I'', there is i_n for which $i_n \in I'$ and $i_n \notin I''$. In view of our remark on three-element subsets of A, there is s such that $\pi F' \upharpoonright A_{i_n} \pi^{-1} = F'' \upharpoonright A_{j_s}$, and $\pi \upharpoonright A_{i_n}$ will be a bijection of the set $\{0^{i_n}, 1^{i_n}, 2^{i_n}\}$ onto the set $\{0^{j_s}, 1^{j_s}, 2^{j_s}\}$. In this case $F' \upharpoonright A_{i_n}$ is conjugate to a clone G_{i_n} via a map $\varphi_{i_n}^{-1}$. The same holds for $F'' \upharpoonright A_{j_s}$ and G_{j_s}. As a result, we conclude that the clones G_{i_n} and G_{j_s} are conjugate for $i_n \neq j_s$, which is a contradiction with the initial choice of clones G_j $(j \in \omega)$. Hence $F' \not\approx F''$. The theorem is proved.

The proof of Theorem 1 (due to the proper choice of sets I' and I'') implies the following:

COROLLARY. For infinite sets A, there exist clones F on A such that the class $F/\approx$ contains at least continuum many pairwise non $\sim$-equivalent clones on A.

Let us dwell on one more question concerning the relation $\sim$ on the collection F_A, the answer to which will separate situations with finite and infinite A.

THEOREM 2. The relations $F' \subseteq F''$ and $F' \sim F''$ for functional clones on a set A imply that the clones F' and F'' will be equal if and only if A is finite.

Proof. For the case of a finite A, the relations $F' \subseteq F''$ and $F' \sim F''$ for $F', F'' \in F_A$ imply the same relations for n-fragments of clones F' and F'' with any $n \in \omega$. However, the relations $F'_{(n)} \subseteq F''_{(n)}$ and $F'_{(n)} \sim F''_{(n)}$, in view of the fact that the collections $F'_{(n)}$ and $F''_{(n)}$ are finite, imply the equality $F'_{(n)} = F''_{(n)}$ and hence also $F' = F''$.

Now we give an example of clones F' and F'' on an infinite set A such that $F' \subseteq F''$, $F' \sim F''$, and $F' \neq F''$. Consider a partition of A into subsets A_n $(n \in \omega + 1)$ such that $A_0 = \{a\}$, $A_n = \{0^n, 1^n, 2^n\}$ $(n \in \omega\backslash\{0\})$, $|A_n| = 3$ $(n \in \omega\backslash\{0\})$, and $|A_\omega| = |A|$. Let φ_n be a bijection of the set $\{0,1,2\}$ onto the set A_n as defined in the proof of Theorem 1.

Let $G_1 \subset G_2 \subset \ldots \subset G_n \subset G_{n+1} \subset \ldots$ be a sequence of functional clones on A as specified in the proof of Theorem 1.

For any function $g(x_1, \ldots, x_n)$ on a set $\{0,1,2\}$ and any $m \in \omega\backslash\{0\}$, on A we define a function

$g^m(x_1,\ldots,x_n)$ such that for $b_1,\ldots,b_n \in A$,

$$g^m(b_1,\ldots,b_n) = \begin{cases} k^m & \text{if } \{b_1,\ldots,b_n\} \subseteq A_m, \langle b_1,\ldots,b_n\rangle = \langle e_1^m,\ldots,e_n^m\rangle, \\ & \text{and } g(e_1,\ldots,e_n) = k; \\ a & \text{otherwise.} \end{cases}$$

Let F' be a clone on A generated by a collection of functions $\bigcup\limits_{m=2}^{\infty} \{g^m(x_1,\ldots,x_n) \mid g(x_1,\ldots,x_n) \in G_{i_{m-1}},\ m \in \omega\backslash\{0\}\} \cup \{e_n^i(x_1,\ldots,x_n) \mid 1 \le i \le n,\ n \in \omega\}$, and let F'' be a clone generated by the functions in $\bigcup\limits_{m=1}^{\infty} \{g^m(x_1,\ldots,x_n) \mid g(x_1,\ldots,x_n) \in G_{i_m},\ m \in \omega\backslash\{0\}\} \cup \{e_n^i(x_1,\ldots,x_n) \mid 1 \le i \le n,\ n \in \omega\}$.

The inclusion $F' \subseteq F''$ is obvious, as is the inequality $F' \ne F''$. Now we let π be a permutation on a set A, which is a bijection of A_ω onto $A_\omega \cup A_1$, and let $\pi(0^n) = 0^{n+1}$, $\pi(1^n) = 1^{n+1}$, and $\pi(2^n) = 2^{n+1}$ for $n \in \omega\backslash\{0\}$. Clearly, π conjugates the clones F' and F''. The theorem is proved.

To sum up, we will collect together the known properties of collections F_A of functional clones on sets A, which distinguish between the cases of finite A and infinite A.

Claim. (1) The relations $\sim$ and $\approx$ coincide on F_A if and only if A is finite.

(2) The conjunction of $\sim$ and $\subseteq$ on F_A coincides with the equality relation $=$ if and only if A is finite.

(3) The topological space $\langle F_A; \tau\rangle$ is compact if and only if A is finite.

Recall that a topology τ on F_A, which was introduced in [2], is induced by a metric on F_A based on the minimal arity of functions distinguishing one clone from another.

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