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$(2,3)$-GENERATED GROUPS WITH SMALL ELEMENT ORDERS

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A periodic group is called an OC_n-group if the set of its element orders consists of all natural numbers from 1 to some natural n. W. Shi posed the question whether every OC_n-group is locally finite. Until now, the case $n = 8$ remains open. Here we prove that if a group is generated by an involution and an element of order 3, and its element orders do not exceed 8, then it is finite. Thereby we obtain an affirmative answer to Shi's question for $n = 8$ for $(2,3)$-generated groups.

A periodic group is called an *OC_n-group* if the set of its element orders consists of all natural numbers from 1 to some natural n. All *finite* OC_n-groups were classified by R. Brandl and W. Shi in [1]. In particular, it was proved that finite OC_n-groups do not exist for $n > 8$. In 1995, Shi posed the following question:

Is every OC_n-group locally finite? (See 2, Quest. 13.64.)

In [3-8], this question was answered in the affirmative for $n \leq 7$. The case $n = 8$ remains open.

In the present paper, we prove that if a group is generated by an involution and an element of order 3, and its element orders do not exceed 8, then it is finite. Thereby we obtain an affirmative answer to Shi's question for $n = 8$ for $(2,3)$-generated groups. This result can be used for studying subgroups of OC_8-groups generated by involutions (see [5-8]). New obstacles pose the following questions:

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Let i be an involution of an OC_8-group G, which does not invert elements of odd order. Will then $H = \langle i^G \rangle$ be a 2-group? Is H locally finite?

THEOREM. Let K be a group generated by an involution t and an element x of order 3, and let K not contain elements of order more than 8. Then K is finite and one of the following statements holds:

(a) $(xt)^3 = 1$, and K is isomorphic to A_4;

(b) $(xt)^4 = 1$, and K is isomorphic to S_3 or S_4;

(c) $(xt)^5 = 1$, and K is isomorphic to A_5;

(d) $(xt)^6 = 1$, and K is a homomorphic image of an extension of a direct product of two cyclic groups of order k by a cyclic group of order 6, where k is the order of $[x,t]$;

(e) $(xt)^7 = [x,t]^8 = 1$, and K is a homomorphic image of $2^6.PSL_2(7)$, and if in addition $[x,t]^4 = 1$, then $K \simeq PSL_2(7)$;

(f) $(xt)^8 = 1$, and K either is isomorphic to $PGL_2(7)$ or is a solvable $\{2,3\}$-group.

In proving, we apply computer calculations in GAP [9] using the enumeration coset algorithm. The first several cases of the theorem are well known. Their proof is given in full, does not rely on computer calculations, and demonstrates the idea of selecting suitable relations.

By Γ_n we denote the set of elements of order n in the group K. By writing $a \sim b$ we mean that the orders of elements a and b in K are equal. The proof of the theorem reduces to examining the respective cases.

Proof. (a) Let $(xt)^3 = 1$. Then

$$tt^x = tx \cdot xtx \sim xtx \cdot tx = (xt)^3 t = t.$$

Thus, the order of the product of two involutions tt^x equals 2, and so $[t, t^x] = 1$. Consequently, K is an extension of a nontrivial normal elementary Abelian 2-subgroup $\langle t^K \rangle$ by a cyclic subgroup $\langle x \rangle$. Note that $tt^x t^{x^2} = (tx^{-1})^3 = 1$, and so the group $\langle t^K \rangle$ is generated by two elements and has order 4, while the element x acts fixed-point-freely on $\langle t^K \rangle$. Hence K is isomorphic to A_4, and we can construct an isomorphism as follows: $x \mapsto (1,2,3)$ and $t \mapsto (1,2)(3,4)$.

(b) Let $(xt)^4 = 1$. Put $a = (tx)^2$. If $a = 1$, then $xx^t = (xt)^2 = 1$. In other words, an involution t inverts an element x, and K is isomorphic to S_3.

Now let a be an involution. Note that

$$at \sim ta = t \cdot (tx)^2 = xtx \sim tx^{-1} \sim xt \in \Gamma_4.$$

Therefore, $aa^t = (at)^2$ has order 2 and $[a, a^t] = 1$. Similarly, $aa^x = (tx^{-1})^2$ has order 2 and $[a, a^x] = 1$. Thus, K is an extension of a nontrivial normal elementary Abelian 2-group $\langle a^K \rangle$ by a group isomorphic to S_3. Finally,

$$aa^x a^{x^2} = (ax^{-1})^3 = (x^t)^3 = 1,$$

and so the group $\langle a^K \rangle$ is generated by two elements and has order 4. Hence K is isomorphic to S_4, and we can construct an isomorphism as follows: $x \mapsto (2,3,4)$ and $t \mapsto (1,2)$.

(c) Let $(xt)^5 = 1$. Put $a = tx^{-1}$; then $a^5 = 1$. Note that

$$t \cdot x^a = t \cdot xtxtx \cdot x = x^{-1}tx^{-1}t \cdot x = (x^{-1})^{tx} \in \Gamma_3.$$

By item (a), $H = \langle t, x^a \rangle \simeq A_4$. Moreover, $atH = aH$ and $axH = H$. Consequently, $|K : H| = 5$ and $|K| = |A_5|$. Thus, a permutation representation of K with respect to H defines an isomorphism between K and A_5, in which case $x \mapsto (1,2,3)$ and $t \mapsto (2,4)(3,5)$.

(d) Let $(xt)^6 = 1$. Denote by a the commutator $[x,t] = x^{-1}txt$, and by k the order of the element a. Note that

$$a^t = t \cdot x^{-1}txt \cdot t = tx^{-1}tx = a^{-1}.$$

Moreover,

$$aa^x a^{x^2} = (ax^{-1})^3 = x^{-1}(tx)^6 x = 1,$$
$$a^{xt} = a^{tx[x,t]} = (a^{-1})^{xa}.$$

Thus, the subgroup $H = \langle a, a^x \rangle$ is normal in $K = \langle x, t \rangle$. Direct computations show that $[a, a^x] = (tx)^6 = 1$ and H is Abelian. The images of the elements t and x in the factor group G/H commute, so G/H is a cyclic group of order 6.

(e) Let $(xt)^7 = 1$.

First suppose that $tt^x \in \Gamma_5$ and define $a = t$ and $b = t^x$. Then $ab \in \Gamma_5$. We point out the following chain of equalities:

$$a^x = b = a^{(ba)^2}.$$

Consequently, $c = (ba)^2 x^{-1}$ centralizes the involution a. We prove that the element c has order 7, thus obtaining a contradiction. In fact,

$$c = (x^{-1}txt)^2 x^{-1} \sim (xt)^3 x^{-1} t = (tx^{-1})^4 x^{-1} t \sim tx^{-1}t \cdot x^{-1}tx^{-1} \cdot txt \sim xt.$$

Therefore, $|c| = |xt| = 7$.

Next,

$$x \cdot t^{xt} \sim (tx)^3 \cdot x = (x^{-1}t)^4 \cdot x \sim (x^{-1}t)^2 x^{-1} \sim tx^{-1}tx = tt^x.$$

If the order of tt^x equals 6, then $H = \langle x, t^{xt} \rangle$ is a finite group such as in item (d). Enumerating the cosets of G with respect to H, we arrive at a permutation representation of the group $PSL_2(13)$ on 14 elements:

$$t \mapsto (1,2)(3,8)(4,5)(9,11)(10,14)(12,13);$$
$$x \mapsto (2,3,6)(4,9,8)(5,11,13)(7,12,10).$$

The group $PSL_2(13)$ has elements of order 13, so it does not satisfy the hypotheses of the theorem.

The case $tt^x \in \Gamma_7$ is also impossible since the group

$$\langle x, t \mid x^3, t^2, (xt)^7, [x,t]^7 \rangle$$

is isomorphic to $PSL_2(13)$ [10. p. 96].

Consider the case where the order of tt^x is equal to 8. Let $a = t$, $b = t^x$, $c = t^{x^2}$, and $i = (ab)^4$ be a central involution in the dihedral group $\langle a, b \rangle$. Since x permutes a, b, and c, we conclude that i^x is a central involution in $\langle b, c \rangle$. In particular, the subgroup $\langle i, i^x \rangle$ centralizes b, and hence the order of ii^x is not equal to 5 and 7. Suppose first that $(x^{-1}t)^2(xt)^2 \in \Gamma_6$. Notice that

$$\begin{aligned} ii^x = (x^{-1}txt)^4 x^{-1}(tx^{-1}tx)^4 x &\sim (txtx^{-1})^4(tx^{-1}tx)^4 = (txtx^{-1})^4(x^{-1}txt)^4 \\ &\sim (x^{-1}t)^3(xtx^{-1}t)^2(xt)^3(x^{-1}txt)^2 \\ &\sim [(x^{-1}t)^2(xt)^2]^3. \end{aligned}$$

The order of ii^x is equal to 2, and $[i, i^x] = 1$. Consequently, $\langle i, x \rangle$ is a homomorphic image of $2 \times A_4$. In particular, $ii^x = i^t i^x \sim ii^{xt}$ and $[i, i^{xt}] = 1$. Since i^{x^2} is a central involution in $\langle a, c \rangle$, it follows that $ii^{(xt)^2} = i(tx^{-1})^2 i(xt)^2 \sim (xt)^2 i(tx^{-1})^2 i = xti^{x^{-1}}tx^{-1}i = i^x i$ is an involution. Thus, the group $H = \langle i, xt \rangle$ is an extension of an elementary Abelian 2-group by a cyclic group of order 7.

Calculations in GAP show that H has index 24, and the elements x and t act on the cosets as the following permutations:

$$x_1 = (1,3)(2,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24);$$

$$t_1 = (1,3,2)(4,6,10)(5,8,14)(7,12,18)(9,16,11)(13,15,20)(17,22,21)(19,24,23).$$

In this case x_1 and t_1 generate a subgroup of $PSL_2(7)$ of order $168 = 24 \cdot 7$; so elements i^K generate the kernel of a permutation representation of order 2^6 and the conclusion of the theorem holds.

Finally, let $(x^{-1}t)^2(xt)^2 \in \Gamma_8$. Then $\langle x, t \rangle$ is a homomorphic image of

$$G = \langle x, t \mid x^3, t^2, (xt)^7, (t^x t)^8, ((x^{-1}t)^2(xt)^2)^8 \rangle.$$

Calculations in GAP show that the group G is trivial.

(f) Below we assume that $(xt)^8 = 1$.

First let the order of $t^x t$ divide 6. Then $i = (t^x t)^3$ is an element of order dividing 2. Moreover, $xi = txt(x^{-1}txt)^2 \sim (x^{-1}t)^3(xt) = (tx)^5(xt) \sim x^{-1}txt$ is of order dividing 6. According to item (d), $H = \langle x, i \rangle$ is finite and is a homomorphic image of an extension of an elementary Abelian group of order k by a cyclic group of order 6, where k is the order of an element $i^x i$. Next,

$$i^x i \sim (xt)^3 x^{-1}txt(x^{-1}t)^3 xtx^{-1}t \sim (xt)^3 x^{-1}txt(tx)^5 xtx^{-1}t = [(xt)^3(x^{-1}t)^2]^2,$$

and hence ii^x cannot be of order 8 or 6. Therefore, it seems convenient to assume that the group K is a homomorphic image of the group

$$G(k) = \langle x, t \mid x^3, t^2, (xt)^8, [x,t]^6, ((xt)^3(x^{-1}t)^2)^k \rangle,$$

where $k \leq 8$. Our computations show that the index $|G(k) : \langle x, i\rangle|$ is always a finite $\{2,3\}$-number and that it is greater than 1 if k is a $\{2,3\}$-number. Thus, in this case K is a finite solvable $\{2,3\}$-group.

If $tt^x \in \Gamma_5$, then, reasoning as in the respective case in item (e), we see that the element c centralizes the involution t, and

$$c = (x^{-1}txt)^2x^{-1} \sim (xt)^3x^{-1}t \sim (tx^{-1})^5x^{-1}t \sim tt^x$$

has order 5, a contradiction.

Let the order of $[x,t]$ be equal to 7. Then $\langle x, t\rangle$ is a homomorphic image of

$$K(i_1, i_2, i_3) = \langle x, t \mid x^3, t^2, (xt)^8, [x,t]^7, ((xt)^2x^2t)^{i_1}, ((xt)^2(x^2t)^2)^{i_2}, (xtx^{-1}t(xt)^2(x^{-1}t)^3)^{i_3}\rangle,$$

for some $i_1, i_2, i_3 \in \{5,6,7,8\}$. Computations in GAP show that $K(8,6,8) \simeq PGL(2,7)$ and the order of $K(i_1,i_2,i_3)$ divides 2 for other values of the parameters.

Now let $[x,t]^8 = 1$. Put $z = x^t$. Note that

$$1 = (tx)^8 = (x^tx)^4 = (zx)^4 \text{ and } (xz^{-1})^8 = 1.$$

Therefore, $\langle x, z\rangle$ is a homomorphic image of

$$K(k,l,m,n) = \langle a, b \mid a^3, b^3, (ab)^4, (ab^{-1})^8, (abab^{-1})^k, (baba^{-1})^l, (bab^{-1}a^{-1})^m, (aba^{-1}b^{-1}ab^{-1})^n\rangle,$$

where $k,l,m,n \in \{5,6,7,8\}$. Computations in GAP show that nontrivial groups are obtained only for the following two sets of parameters: $K(8,8,8,6) \simeq PSL_3(3)$, which is impossible since K does not contain elements of order 13, and $K(8,8,7,7) \simeq PSL_2(7)$. Hence xz^{-1} has order 4. In other words, $[x,t]^4 = 1$, and $\langle x, t\rangle$ is a homomorphic image of the group

$$\langle x, t \mid x^3, t^2, (xt)^8, [x,t]^4\rangle \simeq PGL_2(7). \ \square$$

The theorem implies the following:

COROLLARY. Let $a, b, c \in \Gamma_2$, $ab \in \Gamma_3$, $[a,c] = 1$, and $H = \langle a, b, c\rangle$ have no elements of order greater than 8. Then H is finite, and if H does not contain elements of order 8, then H has no elements of order 7.

Proof. Let $x = ab$ and $t = c$. Then $t^b = t^x$. Thus, $K = \langle x, t\rangle$ satisfies the hypotheses of the theorem and is a normal subgroup of H of index dividing 2. Hence H is finite. Moreover,

$$[x,t] = [ab, c] = bacabc = (bc)^2,$$

and so item (e) of the theorem is possible for K only if bc has order 8 and $H = PGL_2(7)$. $\square$

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