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COORDINATE GROUPS OF IRREDUCIBLE ALGEBRAIC SETS OVER DIVISIBLE METABELIAN r-GROUPS

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We describe coordinate groups of generalized rigid metabelian groups in which, whenever a group is noncommutative, the second factor of a rigid series is a divisible R-module over an appropriate integral domain R.

INTRODUCTION

A definition of an r-group, which generalizes the concept of a rigid solvable group, was given in [1]. We recall the definition.

Let a group G have a normal series of the form

$$G = G_1 > G_2 > \ldots > G_m > G_{m+1} = 1, \tag{1}$$

whose quotients G_i/G_{i+1} are Abelian. The action of G on G_i by conjugations $x \to x^g = g^{-1}xg$ defines on G_i/G_{i+1} the structure of a (right) module over the group ring $\mathbb{Z}[G/G_i]$. Denote by R_i the quotient ring of $\mathbb{Z}[G/G_i]$ with respect to the annihilator G_i/G_{i+1}. Then G_i/G_{i+1} can be treated as a right R_i-module. It is required that the module G_i/G_{i+1} be R_i-torsion free, and that a canonical mapping $\mathbb{Z}[G/G_i] \to R_i$ be injective on the group G/G_i. It was stated that the ring R_i associated with the quotient G_i/G_{i+1} is a two-sided Ore domain, the derived length of G is exactly m, and series (1), if it exists, is uniquely defined by G. A series such as in (1) is said to be *rigid* and its members are denoted $G_i = \rho_i(G)$. We note from the outset that for the ring R_1 associated

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with the first quotient, there are only two possibilities: $R_1 = \mathbb{Z}$ or $R_1 = F_p$ is a field consisting of p elements. In particular, Abelian r-groups are exhausted by torsion-free groups and groups of prime period p. An r-group G is *divisible* if every quotient $\rho_i(G)/\rho_{i+1}(G)$ is a divisible R_i-module on which, in this case, the structure of a vector space is defined over the division ring of quotients of R_i. Again, in the Abelian case, the property of being divisible is shared by any Abelian groups of period p and torsion-free divisible Abelian groups, which, as is known, are isomorphic to direct sums of copies of the additive group of the field of rational numbers, $\mathbb{Q}$.

The rigid groups mentioned are obtained if rings R_i coincide with group rings $\mathbb{Z}[G/\rho_i(G)]$. Algebraic geometry and model theory for rigid groups were thoroughly studied in a series of papers due to the author and A. G. Myasnikov. In particular, [2] contains a description of coordinate groups of irreducible algebraic sets over divisible rigid groups. In the present paper, we describe coordinate groups of irreducible algebraic sets over metabelian r-groups, in which the subgroup $\rho_2(G)$ is a divisible R_2-module whenever G is a noncommutative group. The class of such groups contains divisible metabelian r-groups, for which it is also required that the module $\rho_1(G)/\rho_2(G)$ be divisible.

1. ALGEBRAIC GEOMETRY OVER GROUPS

Algebraic geometry over groups was developed in [3, 4]. We recall some definitions and facts.

For a given group G, a *G-group* is any group containing G as a fixed subgroup. A *category of G-groups* arises, in which morphisms are homomorphisms that act identically on G.

Denote by $F = G * \langle x_1, \ldots, x_n \rangle$ the free product of a group G and a free group with basis $\{x_1, \ldots, x_n\}$. An expression of the form $v(x) = 1$, where $x = (x_1, \ldots, x_n)$ and $v(x) \in F$, is called an *equation over G*. Sometimes, by an equation we mean the element $v(x)$ itself. It is supposed that the variables $x_1, \ldots, x_n$ assume values in G. The set $S \subseteq G^n$ of all solutions for some system $\{v_i(x) = 1 \mid i \in I\}$ of equations is called an *algebraic subset* in an affine space G^n. Let $\Theta(S) = \{v(x) \in F \mid v(s) = 1, \ s \in S\}$ be the *annihilator* of a nonempty algebraic set S. The *coordinate group* of this set is the quotient group $F/\Theta(S) = \Gamma(S)$. A group G is embedded in that quotient group and $\Gamma(S)$, being a G-group, is generated by the images of the elements $x_1, \ldots, x_n$.

A group F can be treated as a group of equations in x with coefficients from G. More generally, a *group of equations over G* is any group D which is generated by its subgroup G and by a set of elements $\{x_1, \ldots, x_n\}$ if it satisfies the condition that every mapping $x \to (a_1, \ldots, a_n) \in G^n$ defines a G-epimorphism $D \to G$. Figuratively speaking, the variables $x_1, \ldots, x_n$ in this group can be assigned any values from G. A group D can be represented as the quotient group F/H. Among such D is a group with maximal H equal to $\Theta(G^n)$. This is $\Gamma(G^n)$, and so D covers $\Gamma(G^n)$. The meaning of this notion is as follows. It is often convenient to consider as equations not elements of the group F and not elements of the group $\Gamma(G^n)$, which it is sometimes difficult to represent, but elements of some intermediate group D.

Let S be a nonempty algebraic subset of G^n. If, in the above definition of a group D of equations, we confine ourselves to mappings $x \to (a_1, \dots, a_n) \in S$, then we arrive at a definition of a *group of equations on* S, or, in other words, a *group of equations over* G *given that* $x \in S$. Such a group covers $\Gamma(S)$. Let D be a group of equations over G provided that $x \in S$. Every G-epimorphism $D \to G$ defined by a mapping $x \to (a_1, \dots, a_n) \in S$ is called a *specialization*, and the image of an element $v(x_1, \dots, x_n)$ in D is denoted by $v(a_1, \dots, a_n)$.

On a set G^n, the Zariski topology can be defined by taking all algebraic sets to be a subbasis for a family of closed sets. If this topology is Noetherian, then every nonempty closed set is uniquely representable as a noncancellable union of finitely many irreducible components. A group G is said to be *equationally Noetherian* if, for every n, any system of equations in n variables over G is equivalent to some of its finite subsystems. Being equationally Noetherian for a group is equivalent to being Noetherian for the Zariski topology on G^n with all n [3].

We say that a G-group H_1 is *separated by a* G*-group* H_2 if for any nontrivial element $h \in H_1$ there exists a G-homomorphism $\varphi : H_1 \to H_2$ such that $h\varphi \neq 1$.

We say that a G-group H_1 is *discriminated by a* G*-group* H_2 if for any finite set of distinct elements in H_1 there exists a G-homomorphism $\varphi : H_1 \to H_2$ which is injective on the set under consideration. This is equivalent to the condition that for any finite set of nontrivial elements in H_1 there exists a G-homomorphism $\varphi : H_1 \to H_2$ under which the images of the elements of a given set remain to be nontrivial.

PROPOSITION 1 [3]. Suppose that H is generated as a G-group by a finite set of elements $x = (x_1, \dots, x_n)$. H is separated by a group G if and only if it is the coordinate group of some nonempty algebraic set of G^n in the variables x.

Remark. If we add the premise that H is G-separated by G to the conditions of Proposition 1, then the appropriate algebraic set will consist of exactly those points $(g_1, \dots, g_n)$ of an affine space G^n for which a mapping $x \to (g_1, \dots, g_n)$ defines a G-epimorphism $H \to G$.

PROPOSITION 2 [3]. Suppose that a group G is equationally Noetherian and H, being a G-group, is generated by a finite set of elements $x = (x_1, \dots, x_n)$. Then the following conditions are equivalent:

(1) H is G-discriminated by G;

(2) H is the coordinate group of some irreducible algebraic set of G^n in the variables x;

(3) the universal theories of G and H with constants in G coincide.

From [5, 6], we derive the following:

PROPOSITION 3. (1) Every metabelian r-group is equationally Noetherian.

(2) Every group whose universal theory coincides with the universal theory of an m-step solvable r-group is itself an m-step solvable r-group.

(3) Every Abelian group whose universal theory coincides with the universal theory of some group in the class of 2-step solvable r-groups the first factor of a rigid series of which is torsion-free belongs to the same class.

2. ABELIAN GROUPS

Consider the case of Abelian r-groups. Recall that every Abelian group is equationally Noetherian [3].

Let A be an Abelian group, S be a nonempty algebraic set over A in an affine space A^n, and $H = \Gamma(S)$ be its coordinate group (which is of course Abelian). Since there exist A-epimorphisms $H \to A$ (specializations), the subgroup A is distinguished in H by a direct factor, i.e., $H = A \times B$. Obviously, B is a finitely generated subgroup.

Note that if A is a finite group then the Zariski topology on A^n is discrete, irreducible sets are points, and $\Gamma(S) = A$ for these.

PROPOSITION 4. Suppose that A is an infinite Abelian group, and that either (1) A is torsion-free or (2) A is a group of prime period p. Then all nonempty algebraic sets over A are irreducible, and their coordinated groups are exhausted by groups of the form $A \times B$, where in case (1) B is a free Abelian group of finite rank, and in case (2) B is a finite Abelian group of period p.

Proof. As mentioned, the coordinate group $H = \Gamma(S)$ of a nonempty algebraic set $S \subseteq A^n$ has the form $A \times B$, where B is a finitely generated Abelian group. The group H is A-separated by A, and so in case (1) B is torsion-free and is a free Abelian group of finite rank, and in case (2) B should be a finite Abelian group of period p. It is easy to verify that in both of these cases the group H of the form $A \times B$ is not only A-separated by A but it is also A-discriminated by A. This fact, combined with the equational Noetherianess of A and Proposition 2, implies that H is the coordinate group of an irreducible algebraic set over A. The proposition is proved.

Remark. For an infinite Abelian group A of period p^2, it is no longer necessary that algebraic sets over A are irreducible. As an example we take a group which in the variety of Abelian groups is generated by elements $a, a_1, \ldots, a_n, \ldots$ and is defined by relations $a^p = 1,\ a_1^p = a, \ldots, a_n^p = a, \ldots$. Then the group is itself representable as the union of p algebraic sets which are defined by the respective equations $x^p = 1,\ x^p = a, \ldots, x^p = a^{p-1}$.

3. 2-STEP SOLVABLE GROUPS, PROBLEM SETTING

Below we consider an infinite 2-step solvable r-group G in which $\rho_2(G)$ is a divisible R_2-module. We fix the notation $A = \rho_1(G)/\rho_2(G)$, and use R in place of R_2. Thus, with G we uniquely associate a pair (A, R), where R is a commutative integral domain and A is a subgroup of the group R^* of invertible elements which generates R as a ring. Let K be the ring of quotients of R. By [5, Thm. 2], the subgroup $\rho_2(G)$ splits off in G, and then the group G has the following matrix representation:

$$\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix},$$

where T is a vector space over the field K. Let $\{t_i \mid i \in I\}$ be its basis. In the above representation, $\rho_2(G) = \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$, while the matrix group $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ can be identified with A.

By analogy with rigid groups, we will describe coordinate groups of irreducible algebraic sets over G in special variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. This implies that x_i assume values in A, and y_i assume values in $\rho_2(G)$, or, in essence, in T. Ordinary variables can be represented as products of the special, i.e., $x_i y_i$. Conversely, special variables can be conceived of as the ordinary, satisfying extra equations $[x_i, a] = 1$ and $[y_i, g] = 1$ for two fixed elements $1 \neq a \in A$ and $1 \neq g \in \rho_2(G)$. Of course, this definition of special variables is associated with a fixed decomposition of a group G.

For brevity, let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. We construct a group of equations over G in the above special variables, i.e., a group that contains G as a subgroup, has tuples x and y which generate the group over G, and is such that every mapping $x \to A^n$, $y \to (\rho_2(G))^n$ extends to a G-epimorphism of this group on G. First, we assume that a group $A[x]$ is equal to $A \times \langle x_1, \ldots, x_n \rangle$, the direct product of a group A and a free Abelian group with basis $\{x_1, \ldots, x_n\}$. Denote by $R(x)$ the ring of Laurent polynomials in x with coefficients in the ring R, and by $K(x)$ the ring of Laurent polynomials in x with coefficients in the field K. Consider an $R(x)$-module

$$\sum t_i K(x) + y_1 R(x) + \ldots + y_n R(x),$$

which is the direct sum of a free $K(x)$-module with basis $\{t_i \mid i \in I\}$ and a free $R(x)$-module with basis $\{y_1, \ldots, y_n\}$. The desired group of equations is

$$G[x, y] = \begin{pmatrix} A[x] & 0 \\ \sum t_i K(x) + y_1 R(x) + \ldots + y_n R(x) & 1 \end{pmatrix}.$$

It contains G, and its variables x_j are identified with matrices $\begin{pmatrix} x_j & 0 \\ 0 & 1 \end{pmatrix}$, while y_j are identified with $\begin{pmatrix} 1 & 0 \\ y_j & 1 \end{pmatrix}$. Clearly, $G[x, y]$ is generated as a G-group by the tuples x and y, and every mapping $x \to A^n$, $y \to T^n$ extends to a G-epimorphism $G[x, y] \to G$, as desired.

If we take an arbitrary equation $\begin{pmatrix} f(x) & 0 \\ w(x, y) & 1 \end{pmatrix} = 1$, where the left part is in $G[x, y]$, then it splits, i.e., is equivalent to a system of two equations $f(x) = 1$ and $w(x, y) = 0$. The coordinate groups of irreducible algebraic sets $S \subseteq A^n \times T^n$ will be some quotient groups $G[x, y]$, which we will describe below.

Recall that for A, there are two possibilities: either $|A| = p$ or A is torsion-free. We consider them separately.

4. 2-STEP SOLVABLE GROUPS, THE CASE WHERE $|A| = p$

In this section, $A = \langle a \rangle$ is a cyclic group of prime order p. Then R will be the quotient ring of a group ring $\mathbb{Z}A$. Since $(a - 1)(1 + a + \ldots + a^{p-1}) = 0$ and $a \neq 1$ in R, it follows that R is in fact the quotient ring of a ring $\mathbb{Z}[\xi]$, where ξ is a primitive pth root of unity in the field of complex numbers.

LEMMA 1. Every nonzero prime ideal J of the ring $\mathbb{Z}[\xi]$ has a nonzero intersection with $\mathbb{Z}$, and so the quotient ring $\mathbb{Z}[\xi]/J$ is a finite field.

Proof. Let $0 \neq \alpha \in J$. We will consider α as an element of the cyclotomic extension $\mathbb{Q}(\xi)/\mathbb{Q}$ and take its norm $N(\alpha) = \alpha\beta \in \mathbb{Q}$. There is a natural number m such that $mN(\alpha) \in \mathbb{Z}$ and $m\beta \in \mathbb{Z}[\xi]$. Then $0 \neq m\alpha\beta \in J \cap \mathbb{Z}$. The lemma is proved.

With due regard for the lemma, we have two versions. In the first version $R = \mathbb{Z}[\xi]$ and $K = \mathbb{Q}(\xi)$. In the second version $R = K = F_q(a) = F_{q^l}$ is a finite field, where q is a prime and p divides $q^l - 1$ with l minimal with this condition, and since G is an infinite group, the space T should have an infinite basis $\{t_i \mid i \in I\}$.

Let $\Gamma(S)$ be the coordinate group of an irreducible algebraic set S in an affine space $A^n \times T^n$. The space itself is representable as the union of finitely many algebraic sets $\{(a_1, \ldots, a_n)\} \times T^n$ over all tuples $(a_1, \ldots, a_n) \in A^n$, and so S is in one of these sets, i.e., all elements of S satisfy the equations $x_1 = a_1, \ldots, x_n = a_n$ corresponding to some tuple $(a_1, \ldots, a_n)$. Therefore, $\Gamma(S)$ is the quotient group of a group

$$G[y] = \begin{pmatrix} A & 0 \\ T + y_1R + \ldots + y_nR & 1 \end{pmatrix},$$

which will the image of $G[x, y]$ after we substitute $x_1 = a_1, \ldots, x_n = a_n$.

The theorem below describes coordinate groups of irreducible algebraic sets in special variables for the case of an infinite 2-step solvable r-group G treated in this section, with $\rho_2(G)$ a divisible R-module and $|A| = p$.

THEOREM 1. (1) Let p and q be primes, K a finite field of characteristic q, $A = \langle a \rangle$ a cyclic subgroup of K^* of order p, $K = F_q(a)$, T a vector space over K with infinite basis $\{t_i \mid i \in I\}$, $T_y = T + y_1K + \ldots + y_nK$ a vector space over K with basis $\{t_i \mid i \in I\} \cup \{y_1, \ldots, y_n\}$, and

$$G = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}.$$

Then the coordinate groups of irreducible algebraic sets in the special variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ corresponding to the above decomposition of G are exhausted by groups of the form $\begin{pmatrix} A & 0 \\ T_y/L & 1 \end{pmatrix}$, where $x = (x_1, \ldots, x_n)$ is set equal to some fixed tuple $(a_1, \ldots, a_n) \in A^n$ and L is an arbitrary subspace of T_y provided that $T \cap L = 0$.

(2) Let a be a primitive root of unity of prime power p in the field of complex numbers, $R = \mathbb{Z}[a]$, $K = \mathbb{Q}(a)$, $A = \langle a \rangle$ be a cyclic subgroup in R^* generated by an element a, T be a vector space over K with basis $\{t_i \mid i \in I\}$, $T_y = T + y_1R + \ldots + y_nR$ be an R-module equal to the direct sum of T and a free R-module with basis $\{y_1, \ldots, y_n\}$, and

$$G = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}.$$

Then the coordinate groups of irreducible algebraic sets in the special variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ corresponding to the above decomposition of G are exhausted by groups of the form $\begin{pmatrix} A & 0 \\ T_y/L & 1 \end{pmatrix}$, where $x = (x_1, \dots, x_n)$ is set equal to some fixed tuple $(a_1, \dots, a_n) \in A^n$ and L is an arbitrary isolated R-submodule in T_y provided that $T \cap L = 0$.

Proof. As noted, the coordinate group $\Gamma(S)$ of an irreducible algebraic set S in an affine space $A^n \times T^n$ is representable as the quotient group of a group

$$G[y] = \begin{pmatrix} A & 0 \\ T + y_1R + \dots + y_nR & 1 \end{pmatrix}$$

with respect to a normal subgroup, which obviously has the from

$$\begin{pmatrix} 1 & 0 \\ L & 1 \end{pmatrix},$$

where L is an R-submodule of $T + y_1R + \dots + y_nR$. Intersection of the annihilator of S with G is equal to the trivial subgroup, and so $T \cap L = 0$. We show that L is an isolated R-submodule. Let $0 \neq u \in T + y_1R + \dots + y_nR$, $0 \neq \alpha \in R$, and $u\alpha \in L$. Consider an arbitrary specialization $(x, y) \to S$, which induces an R-module homomorphism $\varphi : T + y_1R + \dots + y_nR \to T$. We have $(u\alpha)\varphi = 0$, with $(u\alpha)\varphi = (u\varphi)\alpha$. Then $u\varphi = 0$ for all specializations, and $u \in L$. In both of the cases treated in the theorem, therefore, the coordinate group of an irreducible algebraic set has the desired form.

To prove the converse statement, we need

LEMMA 2. Let K be a field, A a subgroup in K^*, T an infinite vector space over K with basis $\{t_i \mid i \in I\}$, $T + z_1K + \dots + z_mK$ a vector space over K with basis $\{t_i \mid i \in I\} \cup \{z_1, \dots, z_m\}$, and $G = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$. Then

$$H = \begin{pmatrix} A & 0 \\ T + z_1K + \dots + z_mK & 1 \end{pmatrix}$$

is G-discriminated by G.

Proof. The identity mapping $A \to A$ and every T-epimorphism of vector spaces $T + z_1K + \dots + z_mK \to T$ define a group G-epimorphism $H \to G$. Therefore, it suffices to show that the space $T + z_1K + \dots + z_mK$ is T-discriminated by the space T. By induction on m, the problem reduces to the case $m = 1$. In this case we assume that there exists a finite set $\{u_1 + z_1\alpha_1, \dots, u_r + z_1\alpha_r\}$, where $\alpha_j \in K$ and $u_j \in T$, of nonzero elements of the space $T + z_1K$. Since T is infinite, there is an element $t \in T$ for which all $u_1 + t\alpha_1, \dots, u_r + t\alpha_r$ are other than zero. Hence a specialization $z_1 \to t$ discriminates the set mentioned. The lemma is proved.

We come back to the proof of Theorem 1. First we show that $H = \begin{pmatrix} A & 0 \\ T_y/L & 1 \end{pmatrix}$, where T_y and L satisfy the conditions formulated in items (1) or (2) of the theorem, is G-discriminated by G. In the conditions of item (1), the claim follows from Lemma 2 since the space T_y/L is representable as $T + z_1K + \ldots + z_mK$, where $m \leq n$ and elements $z_1, \ldots, z_m$ complement the basis of T to a basis for T_y/L; these, for instance, can be chosen among $y_1, \ldots, y_n$. Now let L satisfy the conditions formulated in item (2). We embed an R-module T_y in a vector space $V = T + y_1K + \ldots + y_nK$ over a field K, and denote by $\overline{L}$ the subspace generated by L. Since the submodule L is isolated in T_y, it follows that $T_y \cap \overline{L} = L$ and $T \cap \overline{L} = 0$. Hence the group G embeds in a matrix group $\begin{pmatrix} A & 0 \\ V/\overline{L} & 1 \end{pmatrix}$, which, by Lemma 2, is G-discriminated by the group G. Then its subgroup $H = \begin{pmatrix} A & 0 \\ T_y/L & 1 \end{pmatrix}$ is also G-discriminated by G.

Now, relying on Proposition 2, we assert that H is the coordinate group of an irreducible algebraic set S in variables x and y in an affine space $G^n \times G^n$. By the remark to Proposition 1, the set S consists of tuples $(g_1, \ldots, g_n, g_{n+1}, \ldots, g_{2n})$, for which the mapping $x \to (g_1, \ldots, g_n)$, $y \to (g_{n+1}, \ldots, g_{2n})$ defines a G-epimorphism $H \to G$. Now, for x, the specific value $(a_1, \ldots, a_n) \in A^n$ is fixed, and so it is always true that $(g_1, \ldots, g_n) = (a_1, \ldots, a_n)$. Note also that H is itself a 2-step solvable r-group, in which $\rho_2(H) = \begin{pmatrix} 1 & 0 \\ T_y/L & 1 \end{pmatrix}$ coincides with the centralizer of any nontrivial element in $\rho_2(G) = \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$. Therefore, for any G-epimorphism $H \to G$, the image of $\rho_2(H)$ is $\rho_2(G)$, and we identify the latter with T. Thus $y_1, \ldots, y_n$ must assume values in T, and $S \subseteq \{(a_1, \ldots, a_n)\} \times T^n$ is an irreducible algebraic set in the special variables x and y. The theorem is proved.

5. UNIVERSAL THEORIES OF r-PAIRS

Universal theories of r-groups were studied in [7], from which we borrow some definitions and facts. More precisely, we update them to the case of theories with constants.

An *r-pair* is (A, R), where R is a commutative integral domain and A is a torsion-free nontrivial multiplicative subgroup in R^* generating a ring R. Such a pair is associated with every 2-step solvable r-group the first factor of a rigid series in which is torsion-free.

An r-pair (A, R) is *finitely generated* if A is a finitely generated group.

An r-pair (A, R) is a *subpair of an r-pair* (A', R') if R is a subring in R' and A is a subgroup in A'. If $G \leqslant G'$ are 2-step solvable r-groups then, for appropriate pairs, (A, R) will be a subpair in (A', R').

A *morphism* $(A', R') \to (A, R)$ between two r-pairs is a ring homomorphism $R' \to R$ mapping A' to A. A morphism of pairs is *essential* if the image of A' is nontrivial. A homomorphism $G' \to G$ of 2-step solvable r-groups is *essential* if the image of G' is non-Abelian.

PROPOSITION 5 [6]. Let $G' \to G$ be an essential homomorphism (epimorphism) between 2-step solvable r-groups. Then it induces an essential morphism (epimorphism) between appropriate pairs.

For a given r-pair (A, R), we take a tuple of variables $(x_1, \ldots, x_m, y_1, \ldots, y_n)$, assuming that the variables x_i take values in A, and y_j take values in R. As terms we consider elements of the free right module

$$y_1 \cdot R(x_1, \ldots, x_m) + \ldots + y_n \cdot R(x_1, \ldots, x_m)$$

with basis $\{y_1, \ldots, y_n\}$ over the ring $R(x_1, \ldots, x_m)$ of Laurent polynomials with coefficients from R in the variables $x_1, \ldots, x_m$. In a natural manner, we define first-order formulas in the variables $(x_1, \ldots, x_m, y_1, \ldots, y_n)$, in which the quantifier-free part is a Boolean combination of equalities of the form $w(x_1, \ldots, x_m, y_1, \ldots, y_n) = 0$, where w is a term. As usual, by the universal theory of a pair (A, R) with constants in A (equivalently, in R) we mean a collection of $\forall$-sentences valid on that pair.

PROPOSITION 6. Let (A, R) be a subpair of an r-pair (A', R'). The universal theories of these two pairs with constants in A coincide if and only if (A', R') is locally (A, R)-discriminated by (A, R).

Proof. Let (A', R') be locally (A, R)-discriminated by (A, R). Coincidence of $\forall$-theories is equivalent to coincidence of $\exists$-theories. If some $\exists$-sentence (with constants in A) is satisfied on a pair (A, R), then it is obviously satisfied on the pair (A', R') containing (A, R). Now let an $\exists$-sentence Φ be satisfied on (A', R'). Take all terms $w_1(x_1, \ldots, x_m, y_1, \ldots, y_n), \ldots, w_s(x_1, \ldots, x_m, y_1, \ldots, y_n)$ involved in the formula Φ. There are values $x_1 = a'_1, \ldots, x_m = a'_m$ and $y_1 = u_1, \ldots, y_n = u_n$, where $a'_1, \ldots, a'_m \in A'$ and $u_1, \ldots, u_n \in R'$, realizing Φ. Let $\overline{w}_1, \ldots, \overline{w}_s$ be the corresponding values of the terms. In the group A', there is a finite set of elements such that an A-subgroup generated by this set contains $a'_1, \ldots, a'_m$, and an R-subring generated by it contains $u_1, \ldots, u_n$. Let (A_1, R_1) be a subpair generated over (A, R) by the set mentioned. Clearly, the sentence Φ is satisfied on (A_1, R_1). We choose an (A, R)-epimorphism $\varphi : (A_1, R_1) \to (A, R)$ for which $\overline{w}_i\varphi \neq 0$ if $\overline{w}_i \neq 0$. The values

$$x_1 = a'_1\varphi, \ \ldots, \ x_m = a'_m\varphi, \ y_1 = u_1\varphi, \ \ldots, \ y_n = u_n\varphi$$

realize the formula Φ.

Now let the $\exists$-theories of pairs (A, R) and (A', R') with constants in A be equal. Then so will be the $\exists$-theory of any intermediate pair. Hence it suffices to show that if the pair (A', R') is itself generated over (A, R) by a finite set of elements $\{a'_1, \ldots, a'_m\} \subseteq A'$, then it is (A, R)-discriminated by (A, R). For convenience, we will suppose that the set $\{a'_1, \ldots, a'_m\}$ is closed under taking inverse elements, and then $R' = R[a'_1, \ldots, a'_m]$. Denote by K the ring of quotients of R. A ring $K[a'_1, \ldots, a'_m]$ is represented as the quotient ring of a polynomial ring $K[x_1, \ldots, x_m]$ with respect to an ideal generated by some finite set of elements $\{f_1(x_1, \ldots, x_m), \ldots, f_s(x_1, \ldots, x_m)\}$. There is no loss of generality in assuming that the polynomials f_i have coefficients in R. We may assert that if $(a_1, \ldots, a_m)$ is a tuple of elements in A, and $f_i(a_1, \ldots, a_m) = 0$ for all $i = 1, \ldots, s$, then the mapping $(a'_1, \ldots, a'_m) \to (a_1, \ldots, a_m)$ defines an R-epimorphism of the ring R' onto R, which induces an A-epimorphism of the group A' onto A. That is, we obtain an (A, R)-epimorphism of the pair (A', R') onto (A, R). In the ring R', we take a finite set $\{v_1(a'_1, \ldots, a'_m), \ldots, v_n(a'_1, \ldots, a'_m)\}$ of nonzero elements, which we want to discriminate, and associate with it an $\exists$-sentence

$$\exists x_1 \ldots \exists x_m \exists y((y \neq 0) \wedge (y \cdot f_1(x_1, \ldots, x_m) = 0) \wedge \ldots \wedge (y \cdot f_s(x_1, \ldots, x_m) = 0)$$
$$\wedge (y \cdot v_1(x_1, \ldots, x_m) \neq 0) \wedge \ldots \wedge (y \cdot v_n(x_1, \ldots, x_m) \neq 0)).$$

The sentence is satisfied on (A', R'), and hence on (A, R). Let $x_1 = a_1, \ldots, x_m = a_m$, and let $y = w$ be the corresponding realization. Then the mapping $(a'_1, \ldots, a'_m) \to (a_1, \ldots, a_m)$ defines an (A, R)-epimorphism of (A', R') onto (A, R), which discriminates the set $\{v_1(a'_1, \ldots, a'_m), \ldots, v_n(a'_1, \ldots, a'_m)\}$. The proposition is proved.

6. 2-STEP SOLVABLE GROUPS, THE CASE WHERE A IS A TORSION-FREE GROUP

In this section, we consider the main case where A is torsion free.

THEOREM 2. Let R be a commutative integral domain, A be a torsion-free nontrivial multiplicative subgroup in R^* which generates R as a ring, K be the field of quotients of R, T be a vector space over K with basis $\{t_i \mid i \in I\}$, and G be a group equal to $\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$. Then the coordinate group H of an irreducible algebraic set over G in the special variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ corresponding to the decomposition

$$G = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$$

will have the following form.

There exists an r-pair (A', R') which contains (A, R) as a subpair and is such that A' is generated over A by the elements $x_1, \ldots, x_n \in A'$, and the universal theories (with constants in A) of (A, R) and (A', R') coincide. Let K' be the field of quotients of R', KR' be a subring in K' generated by K and R', and $T' = \sum t_i \cdot KR' + y_1 R' + \ldots + y_n R'$ be an R'-module equal to the direct sum of a free KR'-module with basis $\{t_i \mid i \in I\}$ and a free R'-module with basis $\{y_1, \ldots, y_n\}$. The group H is representable as $\begin{pmatrix} A' & 0 \\ T'/L & 1 \end{pmatrix}$, where L is an isolated R'-submodule in T' with the condition $L \cap \sum t_i \cdot KR' = 0$.

Conversely, any group H represented as shown above is the coordinate group of an irreducible algebraic set over G in the special variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Proof. Let H be the coordinate group of an irreducible algebraic set $S \subseteq A^n \times T^n$ in the special variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. By Proposition 2 and 3, its universal theory with constants in G coincides with the universal theory of G, and H is itself a 2-step solvable r-group the first factor of a rigid series in which is torsion-free.

Let H comply with (A', R'). Since G is contained in H, (A, R) will be a subpair of (A', R'). In Sec. 3, we noted that H is representable as the quotient group of the constructed group $G[x, y]$ of equations with respect to the annihilator $\Theta(S)$ of a set S, which splits itself; i.e., $\Theta(S) = \begin{pmatrix} \Theta_1 & 0 \\ \Theta_2 & 1 \end{pmatrix}$, where $\Theta_1 \leqslant A \times \langle x_1, \ldots, x_n \rangle$. Therefore, $H = \begin{pmatrix} A' & 0 \\ E & 1 \end{pmatrix}$. The group $A' = (A \times \langle x_1, \ldots, x_n \rangle)/\Theta_1$ is torsion-free, contains A as a subgroup, and is generated over A by the images of $x_1, \ldots, x_n$, for which the notation is left unchanged.

Every specialization $(x, y) \to S$ defines a G-epimorphism $H \to G$, and by Proposition 5, an (A, R)-epimorphism $(A', R') \to (A, R)$. Then the pair (A', R') is (A, R)-discriminated by such epimorphisms, and by virtue of Proposition 6, the universal theories of (A', R') and (A, R) with constants in A will coincide. We represent R' as the quotient ring $R(x)/J$, and then factor $G[x, y]$ with respect to the normal subgroup

$$\begin{pmatrix} \Theta_1 & 0 \\ \sum t_i K(x)J + y_1 R(x)J + \ldots + y_n R(x)J, & 1 \end{pmatrix},$$

which is contained in $\Theta(S)$. As a result, we obtain exactly the group $\begin{pmatrix} A' & 0 \\ T' & 1 \end{pmatrix}$. The group H derives from it as $\begin{pmatrix} A' & 0 \\ T'/L & 1 \end{pmatrix}$, where L is an R'-submodule of T'. The module T'/L is R'-torsion free, and so L is an isolated submodule.

We show that $L \cap \sum t_i \cdot KR' = 0$. Assume the contrary, i.e., $0 \neq \sum t_i u_i \in L$, where $u_i \in KR'$. There exists $0 \neq u \in R$ for which all $u_i u$ are in R'. The ring R' is R-discriminated by R via specializations $(x, y) \to S$, and so there exists a specialization for which the images of all nonzero $u_i u$ are distinct from zero. By assumption, the image of any element in L should be equal to zero, a contradiction. Thus the coordinate group of an irreducible algebraic set $S \subseteq A^n \times T^n$ in the special variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ has the desired form.

In order to prove the converse, in view of Propositions 2 and 3, it suffices to state that H, whose construction is presented in the formulation of the theorem, is G-discriminated by G.

By assumption, the R'-modules $T' = \sum t_i \cdot KR' + y_1 R' + \ldots + y_n R'$ and T'/L are torsion-free, and so they are complemented to vector spaces over the field K'. Since $L \cap \sum t_i \cdot KR' = 0$, the projection of L onto $y_1 R' + \ldots + y_n R'$ is an embedding, and the rank m of the module L does not exceed n. We can assert (using only elementary transformations over R') that L contains a maximal system of elements $h_1, \ldots, h_m$ linearly independent over R', which, up to renumbering $y_1, \ldots, y_n$, has the following form: $h_1 = y_1 u + f_1, \ \ldots, \ h_m = y_m u + f_m$, where

$$0 \neq u \in R', \ f_1, \ldots, f_m \in y_{m+1} R' + \ldots + y_n R' + \sum t_i \cdot KR'.$$

LEMMA 3. Suppose that there is an R-epimorphism $\varphi : R' \to R$, with $u\varphi \neq 0$. This induces a module epimorphism $T' \to T + y_1R + \ldots + y_nR$, which we denote by the same symbol φ. Then $L\varphi$ is contained in the R-isolator of a submodule $h_1\varphi \cdot R + \ldots + h_m\varphi \cdot R$ of $T + y_1R + \ldots + y_nR$.

Proof. Indeed, let

$$v = \sum t_i\alpha_i + y_1\beta_1 + \ldots + y_n\beta_n \in L.$$

Then

$$vu = h_1\beta_1 + \ldots + h_m\beta_m, \; v\varphi \cdot u\varphi = h_1\varphi \cdot \beta_1\varphi + \ldots + h_m\varphi \cdot \beta_m\varphi.$$

Since $u\varphi \neq 0$, it follows that $v\varphi$ is in the isolator of the submodule $h_1\varphi \cdot R + \ldots + h_m\varphi \cdot R$. The lemma is proved.

Suppose that we want to discriminate some finite set of nontrivial elements of H. Each of the elements under consideration will be decomposed into the product of an element in $\begin{pmatrix} A' & 0 \\ 0 & 1 \end{pmatrix}$ and an element in $\begin{pmatrix} 1 & 0 \\ T'/L & 1 \end{pmatrix}$. Since T embeds in T'/L, it follows that $a' \neq 1 \Leftrightarrow t_i(a'-1) \neq 0$ for $a' \in A'$. This allows us to reduce the problem to the case where a discriminated set is contained in $\begin{pmatrix} 1 & 0 \\ T'/L & 1 \end{pmatrix}$. In this case the result follows from the following:

LEMMA 4. Let W be a finite set of elements in $T' \setminus L$. Then there exist a pair (A, R)-epimorphism $(A', R') \to (A, R)$ and a mapping $y \to T^n$, which, together with a mapping $t_i \to t_i$ ($i \in I$), define a module T-epimorphism $T' \to T$ with kernel containing L, and moreover, the images of all elements in W are distinct from zero.

Proof. Every element $w \in W$ can be represented as $w = y_1\alpha_1 + \ldots + y_n\alpha_n + w'$, where $\alpha_j \in R'$ and $w' \in \sum t_i \cdot KR'$. Then

$$wu - h_1\alpha_1 - \ldots - h_m\alpha_m \in y_{m+1}R' + \ldots + y_nR' + \sum t_i \cdot KR'.$$

If we replace w by $wu - h_1\alpha_1 - \ldots - h_m\alpha_m$ we reduce the problem to the case where

$$W \subseteq y_{m+1}R' + \ldots + y_nR' + \sum t_i \cdot KR'.$$

Suppose that $w \in W$ and $w = y_{m+1}\alpha_{m+1} + \ldots + y_n\alpha_n + \sum t_i\beta_i$ in this case. Choose a pair (A, R)-epimorphism $\varphi : (A', R') \to (A, R)$ such that $u\varphi \neq 0$ and $\alpha_j\varphi \neq 0$, if $\alpha_j \neq 0$, and $\beta_i\varphi \neq 0$ if $\beta_i \neq 0$. The pair isomorphism induces a module epimorphism

$$y_{m+1}R' + \ldots + y_nR' + \sum t_i \cdot KR' \to y_{m+1}R + \ldots + y_nR + T,$$

which discriminates the set W. We can then select a tuple of values $v_{m+1}, \ldots, v_n$ for $y_{m+1}, \ldots, y_n$, so that for a composition epimorphism

$$y_{m+1}R' + \ldots + y_nR' + \sum t_i \cdot KR' \to T,$$

which we denote by ψ, the images of all elements of W would also be other than zero. Recall that L is the isolator in T' of an R'-submodule $h_1R' + \ldots + h_mR'$. We write

$$y_1 \cdot u\varphi + f_1\psi = 0, \ldots, y_m \cdot u\varphi + f_m\psi = 0$$

and find in T values $y_1 = v_1, \ldots, y_m = v_m$ such that the above equalities are satisfied. As a result, ψ is lifted to a module epimorphism

$$\psi : T' = y_1R' + \ldots + y_mR' + y_{m+1}R' + \ldots + y_nR' + \sum t_i \cdot KR' \to T,$$

complying with a ring epimorphism $\varphi : R' \to R$ which discriminates W and is such that $h_1\psi = 0, \ldots, h_m\psi = 0$. Relying on Lemma 3, we can assert that $L\psi = 0$. Lemma 4, together with Theorem 2, is proved.

REFERENCES

1. N. S. Romanovskii, "Generalized rigid groups: definitions, basic properties, and problems," *Sib. Math. J.*, **59**, No. 4, 705-709 (2018).
2. N. S. Romanovskii, "Irreducible algebraic sets over divisible decomposed rigid groups," *Algebra and Logic*, **48**, No. 6, 449-464 (2009).
3. G. Baumslag, A. Myasnikov, and V. Remeslennikov, "Algebraic geometry over groups. I: Algebraic sets and ideal theory," *J. Alg.*, **219**, No. 1, 16-79 (1999).
4. A. Myasnikov and V. N. Remeslennikov, "Algebraic geometry over groups. II. Logical foundations," *J. Alg.*, **234**, No. 1, 225-276 (2000).
5. N. S. Romanovskii, "Generalized rigid metabelian groups," *Sib. Math. J.*, **60**, No. 1, 148-152 (2019).
6. N. S. Romanovskii, "Equational Noethericity of metabelian r-groups," *Sib. Math. J.*, **61**, No. 1, 154-158 (2020).
7. N. S. Romanovskii, "On the universal theories of generalized rigid metabelian groups," *Sib. Math. J.*, **61**, No. 5, 878-883 (2020).