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A BASIS FOR THE COMMUTATOR SUBGROUP OF A PARTIALLY COMMUTATIVE METABELIAN PRO-p-GROUP

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A basis for the commutator subgroup of a partially commutative metabelian pro-p-group is described.

INTRODUCTION

Combinatorial group theory deals with groups defined by generators and relations. Often defining relations are associated with a certain graph. A typical example is a *partially commutative group* F_Γ defined by a finite undirected graph $\Gamma = (V; E)$ without loops, where $V = \{v_1, \ldots, v_n\}$ is the vertex set of the graph and E is its edge set. The group F_Γ has the following representation:

$$F_\Gamma = \langle v_1, \ldots, v_n \mid v_i v_j = v_j v_i \text{ if } (v_i, v_j) \text{ is an edge} \rangle.$$

Thus F_Γ is the quotient group of a free group F generated by a set of elements $x_1, \ldots, x_n$ with respect to a normal subgroup N_Γ generated by those commutators $[x_i, x_j] = x_i^{-1} x_j^{-1} x_i x_j$ for which v_i and v_j are adjacent vertices of the graph, i.e., $(v_i, v_j) \in E$. We call Γ a *defining graph* for F_Γ. The group F_Γ is also referred to as a *free partially commutative group*. It has numerous applications and has been well studied (see [1]).

Along with F_Γ, which can be called a *partially commutative group of the variety $\mathfrak{O}$ of all groups*, partially commutative groups of varieties $\mathfrak{M}$ distinct from $\mathfrak{O}$ are examined. A partially commutative group of a variety $\mathfrak{M}$ is obtained in the same way as F_Γ, but based not on F but on a free

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group $F(\mathfrak{M})$ of the variety. Most thoroughly studied among such groups are partially commutative metabelian groups, and there are some results concerning partially commutative nilpotent groups. In dealing with partially commutative groups other than F_Γ, methods of combinatorial group theory are inefficient. Approaches developed for solvable and nilpotent groups turn out useful in examining partially commutative groups in solvable and nilpotent varieties. For a review of results on partially commutative solvable and nilpotent groups, we ask the reader to consult [2].

In [3], a basis for the commutator subgroup of a partially commutative metabelian group was defined, and, as a consequence, the canonical representation of elements of the group was obtained. In [4], a Mal'tsev basis for a partially commutative metabelian nilpotent group was described.

Research on partially commutative topological groups—more exactly, partially commutative metabelian pro-p-groups—was initiated in [5]. In this paper, we continue to study partially commutative metabelian pro-p-groups and describe a basis for the commutator subgroup of a partially commutative metabelian pro-p-group, as was done in [3] for a partially commutative group. As a corollary, we obtain a description of a basis for the commutator subgroup of a partially commutative group in the variety of nilpotent metabelian pro-p-groups.

Here pro-p-groups are examined. By a homomorphism, a subgroup, a generating set, etc., we mean a continuous homomorphism, a closed subgroup, a generating set in the topological sense, etc. Relevant information on pro-p-groups is contained in [6, 7].

We define a partially commutative group of the variety $\mathfrak{A}^2$ of metabelian pro-p-groups, also referred to as a partially commutative metabelian pro-p-group M_Γ with defining graph Γ. Let M be a free metabelian pro-p-group with basis $X = \{x_1, \dots, x_n\}$. The group M_Γ is obtained from M by imposing extra relations $[x_i, x_j] = 1$ if the vertices v_i and v_j are adjacent in Γ, i.e., $M_\Gamma = M/R_\Gamma$, where R_Γ is a normal subgroup of M generated by commutators $[x_i, x_j]$ for which $(v_i, v_j) \in E$. We have the following short exact sequence:

$$1 \longrightarrow R_\Gamma \longrightarrow M \longrightarrow M_\Gamma \longrightarrow 1.$$

The images of elements x_i in M_Γ are denoted $y_i = x_i R_\Gamma$.

The quotient group M_Γ with respect to the commutator subgroup $M'_\Gamma = [M_\Gamma, M_\Gamma]$ will be a free Abelian pro-p-group A with basis $\{a_1, \dots, a_n\}$, which is the image of X under the natural homomorphism $M \to M_\Gamma \to M_\Gamma/M'_\Gamma$. The group A is isomorphic to a direct sum of n copies of the additive group of the ring of p-adic integers $\mathbb{Z}_p$.

The action of M_Γ on M'_Γ by conjugations

$$x \to x^g = g^{-1}xg,\ x \in M'_\Gamma,\ g \in M_\Gamma,$$

endows M'_Γ with the structure of a right module over the completed group algebra $\mathbb{Z}_p[[A]]$, which is identified with the algebra of power series $\mathbb{Z}_p[[b_1, \dots, b_n]]$ provided that $b_i = a_i - 1$. Therefore, every element $g \in M_\Gamma$ can be represented as

$$g = y_1^{l_1} \dots y_n^{l_n} \prod_{1 \le i < j \le n, (v_i, v_j) \notin E} [y_i, y_j]^{\alpha_{ij}}, \qquad (1)$$

where $l_i \in \mathbb{Z}_p$ and $\alpha_{ij} \in \mathbb{Z}_p[[b_1, \ldots, b_n]]$.

The main result of the paper is the following:

THEOREM. Let M be a free metabelian pro-p-group with basis $X = \{x_1, \ldots, x_n\}$, and let $M_\Gamma = M/R_\Gamma$, $y_i = x_i R_\Gamma$, $A = M_\Gamma/M'_\Gamma$, $a_i = x_i M'_\Gamma$, and $b_i = a_i - 1$. On the vertex set $V = \{v_1, \ldots, v_n\}$ of the graph Γ, an order $<$ is defined. Then the set $\mathcal{B}(M_\Gamma)$ of elements w of the form

$$w = [y_i, y_j]^{b_{j_1}^{s_1} \ldots b_{j_m}^{s_m}}, \ \{s_1, \ldots, s_m\} \subset \mathbb{N}, \tag{2}$$

satisfying the conditions

(1) $v_j \leq v_{j_1} < \ldots < v_{j_m}$, $v_j < v_i$,

(2) the vertices v_i and v_j are in distinct connected components of the graph Γ_w generated by all distinct vertices in the set $\{v_i, v_j, v_{j_1}, \ldots, v_{j_m}\}$, and

(3) v_i is the greatest vertex in its connected component in Γ_w

will constitute a basis for the commutator subgroup M'_Γ over the ring $\mathbb{Z}_p$.

Let $\mathfrak{N}_c$ be a variety of nilpotent pro-p-groups of nilpotency class $\leq c$. A partially commutative group in the variety $\mathfrak{A}^2 \cap \mathfrak{N}_c$ is denoted by $M_{c,\Gamma}$. It is isomorphic to the quotient group $M_\Gamma/\gamma_{c+1}(M_\Gamma)$, where $\gamma_2(M_\Gamma) = M'_\Gamma$ and $\gamma_{c+1}(M_\Gamma) = [\gamma_c(M_\Gamma), M_\Gamma]$.

The images of elements y_i in $M_{c,\Gamma}$ are denoted $u_i^{(c)}$. Let

$$U^{(c)} = \left\{u_1^{(c)}, \ldots, u_n^{(c)}\right\}.$$

On the vertex set V of Γ, some order $<$ is defined. For any $d \in \{2, \ldots, c\}$, we specify in $M_{c,\Gamma}$ a set $\mathcal{B}_d(M_{c,\Gamma})$ of left-normed commutators

$$u = \left[u_{j_1}^{(c)}, u_{j_2}^{(c)}, \ldots, u_{j_d}^{(c)}\right],$$

for which the following conditions hold:

(1) $v_{j_2} \leq \ldots \leq v_{j_d}, v_{j_2} < v_{j_1}$;

(2) the vertices v_{j_1} and v_{j_2} are in distinct connected components of the graph Γ_u generated by all distinct vertices in the set $\{v_{j_1}, \ldots, v_{j_d}\}$;

(3) v_{j_1} is greatest in its connected component in Γ_u.

COROLLARY 1. A set of commutators $\bigcup_{d=2}^{c} \mathcal{B}_d(M_{c,\Gamma})$ form a basis for the commutator subgroup of a pro-p-group $M_{c,\Gamma}$.

1. PRELIMINARY INFORMATION AND RESULTS

The group $M_c = M/\gamma_{c+1}(M)$ is a free group in the variety $\mathfrak{A}^2 \cap \mathfrak{N}_c$. It is generated (freely in this variety) by the images $z_i^{(c)}$ of elements $x_i \in M$ under the natural homomorphism $M \to M_c$. Denote by

$$Z^{(c)} = \left\{z_1^{(c)}, \ldots, z_n^{(c)}\right\}$$

a basis for the group M_c. The set $Z^{(c)}$ will be ordered in some way.

Let $\mathcal{B}_d(M_c)$, $2 \le d \le c$, be a set of left-normed commutators of the form

$$\left[z_{j_1}^{(c)}, z_{j_2}^{(c)}, \ldots, z_{j_d}^{(c)}\right], \tag{3}$$

where $z_{j_2}^{(c)} < z_{j_1}^{(c)}$ and $z_{j_2}^{(c)} \le z_{j_3}^{(c)} \le \ldots \le z_{j_d}^{(c)}$.

PROPOSITION 1. For $c \ge 2$, a set of commutators $\mathcal{B}_c(M_c)$ is independent over $\mathbb{Z}_p$.

Proof. For the commutator

$$z = \left[z_{j_1}^{(c)}, z_{j_2}^{(c)}, \ldots, z_{j_c}^{(c)}\right] \tag{4}$$

and for any $\lambda \in \{1, \ldots, n\}$, by $\lambda(z)$ we denote the number of elements $z_\lambda^{(c)}$ in the sequence

$$z_{j_1}^{(c)}, z_{j_2}^{(c)}, \ldots, z_{j_c}^{(c)}. \tag{5}$$

Let

$$\pi(z) = (1(z), 2(z), \ldots, n(z))$$

and let $\sigma(z)$ denote the set of different elements in (5).

Suppose that among the elements $z_1, \ldots, z_m \in \mathcal{B}_c(M_c)$ there is a nontrivial dependency

$$z_1^{\gamma_1} \ldots z_m^{\gamma_m} = 1, \tag{6}$$

where all $\gamma_i \in \mathbb{Z}_p$ are not equal to zero.

For any integers $l_1, \ldots, l_n$,

$$\left[z_{j_1}^{(c)l_{j_1}}, z_{j_2}^{(c)l_{j_2}}, \ldots, z_{j_c}^{(c)l_{j_c}}\right] = \left[z_{j_1}^{(c)}, z_{j_2}^{(c)}, \ldots, z_{j_c}^{(c)}\right]^{l_{j_1} \ldots l_{j_c}}.$$

The mapping

$$\varphi = \left\{z_i^{(c)} \mapsto z_i^{(c)l_i},\ i = 1, \ldots, n\right\} \tag{7}$$

extends to an endomorphism φ of the group M_c. Apply φ to (6). Instead of (6), generally speaking, we obtain a new dependency such that exponents γ_i and γ_j at z_i and z_j having equal collections $\pi(z_i)$ and $\pi(z_j)$ will be multiplied by equal numbers. Choosing suitable integers l_i, we may obtain a nontrivial dependency among those elements in $\{z_1, \ldots, z_m\}$ that have equal collections $\pi(z_i)$. Suppose that the dependency in (6) has the property

$$\pi(z_1) = \ldots = \pi(z_m).$$

Hence $\sigma(z_1) = \ldots = \sigma(z_m)$. We may assume that $\sigma(z_i) = Z^{(c)}$ for $i = 1, \ldots, m$.

The definition of a set $\mathcal{B}_c(M_c)$ (see the first condition) implies that for any commutator of form (4) in $\mathcal{B}_c(M_c)$, the element $z_{j_2}^{(c)}$ is least among $Z^{(c)}$. We can suppose that $j_2 = 1$ for all commutators $z_1, \ldots, z_m$.

There is no loss of generality in assuming that

$$z_1 = \left[z_2^{(c)}, z_1^{(c)}, z_{j_3}^{(c)}, \ldots, z_{j_c}^{(c)}\right].$$

For all elements $z_1, \ldots, z_m$, the values $\pi(z_i)$ are equal, and so among the commutators $z_1, \ldots, z_m$, only z_1 starts with the letter $z_2^{(c)}$. Hence, under the endomorphism

$$\psi = \left\{z_1^{(c)} \mapsto z_1^{(c)},\ z_2^{(c)} \mapsto z_2^{(c)},\ z_i^{(c)} \to z_1^{(c)},\ i = 3, \ldots, n\right\},$$

all commutators $z_2, \ldots, z_m$ will be mapped to 1. In a group M_c of rank two with basis $\left\{z_1^{(c)}, z_2^{(c)}\right\}$, therefore, we obtain a dependency of the form

$$\left[z_2^{(c)}, z_1^{(c)}, z_1^{(c)}, \ldots, z_1^{(c)}, z_2^{(c)}, \ldots, z_2^{(c)}\right]^\gamma = 1, \tag{8}$$

where $0 \neq \gamma \in \mathbb{Z}_p$.

We show that this is impossible. To do so, we use an embedding of the group M_c in a matrix pro-p-group. First we recollect a Magnus embedding for a free metabelian pro-p-group M.

Let T be a right topological free $\mathbb{Z}_p[[b_1, \ldots, b_n]]$-module with basis $\{t_1, \ldots, t_n\}$. Consider a matrix pro-p-group

$$\mathcal{M} = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}.$$

Then an extension of the mapping

$$\mu : x_i \longmapsto \begin{pmatrix} a_i & 0 \\ t_i & 1 \end{pmatrix},\ i = 1, \ldots, n,$$

is called a Magnus embedding of M in $\mathcal{M}$. (For more details about the Magnus embedding for pro-p-groups, see [7-9].)

Denote by B an ideal generated by elements $b_1, \ldots, b_n$ in the ring $\mathbb{Z}_p[[b_1, \ldots, b_n]]$. Consider a matrix pro-$p$-group

$$\mathcal{N} = \begin{pmatrix} A & 0 \\ T/TB^c & 1 \end{pmatrix}$$

and a natural homomorphism ν from $\mathcal{M}$ to $\mathcal{N}$. Let $\lambda = \nu\mu$ be a mapping from M to $\mathcal{N}$. It is easy to calculate that $\gamma_{c+1}(M)$ lies in $\ker\lambda$. For a group M of rank 2, we can readily verify the reverse inclusion $\ker\lambda \subseteq \gamma_{c+1}(M)$.

Indeed, suppose that an element of the form

$$g = \begin{pmatrix} a & 0 \\ t_1\alpha_1 + t_2\alpha_2 & 1 \end{pmatrix} \in M$$

is in $\ker\lambda$. Then $a = 1$ and $g \in M'$. Hence $g = [x_1, x_2]^\delta$ and $\delta \in \mathbb{Z}_p[[b_1, b_2]]$. It is not hard to calculate that $\alpha_1 = \delta b_2$. By virtue of $g \in \ker\lambda$, it is true that $\delta b_2 \in B^c$. Therefore, $\delta \in B^{c-1}$. Hence $g = [x_1, x_2]^\delta \in \gamma_{c+1}(M)$.

Suppose that in the commutator $\left[z_2^{(c)}, z_1^{(c)}, z_1^{(c)}, \dots, z_1^{(c)}, z_2^{(c)}, \dots, z_2^{(c)}\right]$, the element $z_1^{(c)}$ occurs r_1 times, while $z_2^{(c)}$ occurs r_2 times. Consider the image of the element IN (8) under the homomorphism λ. It is easy to calculate that $\alpha_1 = \gamma b_2 b_1^{r_1-1} b_2^{r_2-1}$. Since $r_1 + r_2 = c - 1$, it follows that $\alpha_1 \notin B^c$. The proposition is proved.

COROLLARY 2. A set of commutators $\bigcup\limits_{d=2}^{c} \mathcal{B}_d(M_c)$ is independent over $\mathbb{Z}_p$.

Recall that by $u_i^{(c)}$, $1 \le i \le n$, we denote the image of an element $z_i^{(c)}$ under the natural homomorphism $M_\Gamma \to M_{c,\Gamma}$, and by Γ_u, where $u = \left[u_{j_1}^{(c)}, u_{j_2}^{(c)}, \dots, u_{j_c}^{(c)}\right]$, the subgraph of Γ generated by different vertices in the set $v_{j_1}, \dots, v_{j_c}$.

Necessary and sufficient conditions for a commutator of weight c in $M_{c,\Gamma}$ to be equal to 1 are given by the following:

LEMMA 1. A commutator of the form

$$u = [u_{j_1}^{(c)}, u_{j_2}^{(c)}, \dots, u_{j_c}^{(c)}] \in M_{c,\Gamma} \tag{9}$$

equals 1 if and only if the vertices v_{j_1} and v_{j_2} lie in the same connected component of the graph Γ_u.

Proof. Let v_{j_1} and v_{j_2} be in one connected component of Γ_u. Suppose that $\{v_{j_1}, v_{i_1}, \dots, v_{i_q}, v_{j_2}\}$ is a path (without repetitions) between v_{j_1} and v_{j_2} in Γ_u. The natural homomorphism $M_\Gamma \to M_{c,\Gamma}$ maps an element $[y_{j_1}, y_{j_2}]^{b_{j_3} \dots b_{j_c}} \in M_\Gamma$ to an element $u \in M_{c,\Gamma}$. From [5, Prop. 3], it follows that $[y_{j_1}, y_{j_2}]^{b_{i_1} \dots b_{i_q}} = 1$. If we interchange the elements in u occupying places starting from the third, the commutator does not change, and so $u = 1$.

Assume that the vertices v_{j_1} and v_{j_2} lie in distinct connected components of Γ_u. Suppose that $\Gamma_1, \dots, \Gamma_m$ are all the connected components of Γ_u, with $v_{j_1} \in \Gamma_1$ and $v_{j_2} \in \Gamma_2$. Fix a vertex v_{j_1} in Γ_1 and a vertex v_{j_2} in Γ_2. In each of the remaining connected components, we fix one (arbitrary) vertex. A vertex $v_i \in V$ will be assigned an element $u_i^{(c)} \in M_{c,\Gamma}$.

The set of fixed vertices is associated with a subset of elements in $W^{(c)}$. We name elements that correspond to fixed vertices by the fixed elements.

On a set of fixed elements, we specify an order under which $u_{j_2}^{(c)}$ is the least element. Obviously, the fixed elements generate a free group G in the variety of metabelian nilpotent pro-p-groups of class at most c. Consider a homomorphism φ of M_{c,Γ_u} onto G under which elements that correspond to vertices in any connected component of Γ_u are mapped to a fixed element in the same connected component, while unfixed elements are mapped to 1. If we interchange the letters occupying places starting from the third in commutator (9) we obtain an equal commutator. Under the homomorphism φ, therefore, the commutator u is mapped to v, which is equal to some commutator in $\mathcal{B}_c(G)$. As follows from Proposition 1, $v \ne 1$ in G. Hence $u \ne 1$ in M_{c,Γ_u}. On the other hand, M_{c,Γ_u} is a subgroup of $M_{c,\Gamma}$. The lemma is proved.

For $l = 1, \dots, n$, we denote by $l(u)$ the number of elements $u_l^{(c)}$ occurring in the representation of u in (9). Put

$$p(w) = (1(u), \dots, n(u)).$$

LEMMA 2. Let $\{g_i \mid i \in I\}$ be a set of commutators of the form

$$\left[u_{j_1}^{(c)}, u_{j_2}^{(c)}, \ldots, u_{j_c}^{(c)}\right]$$

in the group $M_{c,\Gamma}$. If among the elements g_i there is a nontrivial dependency over $\mathbb{Z}_p$, then there exists a nontrivial dependency among those elements g_j, $j \in J \subseteq I$, for which the collections $p(g_j)$ coincide.

Proof. For any $h_s \in M_{c,\Gamma}$ and any integers l_s, $s = 1, \ldots, c$,

$$\left[h_1^{l_1}, \ldots, h_c^{l_c}\right] = [h_1, \ldots, h_c]^{l_1 \ldots l_c}. \tag{10}$$

Suppose that

$$\prod g_i^{r_i} = 1 \tag{11}$$

is a nontrivial dependency over $\mathbb{Z}_p$. We extend the mapping

$$\left\{u_q^{(c)} \to u_q^{(c)\,l_i},\ q = 1, \ldots, n\right\}$$

to an endomorphism φ of $M_{c,\Gamma}$. Apply then φ to the dependency in (11). From (10) with $l_i \neq 0$, we derive a new nontrivial relation among the elements g_i. In this case the exponents r_i at g_i are multiplied by the same number if the collections $p(g_i)$ for these elements are equal. If we choose different values for l_i we obtain a nontrivial dependency among the elements g_j having equal collections $p(g_j)$. The lemma is proved.

PROPOSITION 2. A set of commutators $\mathcal{B}_c(M_{c,\Gamma})$ is independent over $\mathbb{Z}_p$.

Proof. By Lemma 2, it suffices to show that those elements u_q in $\mathcal{B}_c(M_{c,\Gamma})$ are independent for which $p(u_q)$ coincide. Denote this set of elements as $W = \{u_q, q \in Q\}$. Let

$$\prod u_q^{\alpha_q} = 1,\ \alpha_q \in \mathbb{Z}_p,$$

be a nontrivial dependency. Suppose that $u_{i_1}^{(c)}, \ldots, u_{i_s}^{(c)}$ are all distinct elements in $U^{(c)}$ that occur in the representation of a commutator $u = \left[u_{j_1}^{(c)}, u_{j_2}^{(c)}, \ldots, u_{j_c}^{(c)}\right]$.

Define

$$s(u) = \{v_{i_1}, \ldots, v_{i_s}\} \subseteq V.$$

Note that the equality of all $p(u_q)$, $q \in Q$, implies that all $s(u_q)$ are equal. For any two elements

$$w = \left[u_{j_1}^{(c)}, u_{j_2}^{(c)}, \ldots, u_{j_c}^{(c)}\right] \quad \text{and} \quad v = \left[u_{i_1}^{(c)}, u_{i_2}^{(c)}, \ldots, u_{i_c}^{(c)}\right]$$

in W, we have $i_2 = j_2$ since the vertices v_{i_2} and v_{j_2} are equal (they are least in the set $s(v)$ which coincides with $s(w)$ in view of condition (1) in the definition of $\mathcal{B}_c(M_{c,\Gamma})$).

Suppose that for elements v and w, the vertices v_{i_1} and v_{j_1} are in one connected component of a graph Γ_v equal to Γ_u. The first vertices will also coincide, since condition (3) in the definition of $\mathcal{B}_c(M_{c,\Gamma})$ implies that each of these vertices is greatest in the same connected component of Γ_v.

Then v and w are equal commutators (as words in letters of $U^{(c)}$), which clashes with the definition of $\mathcal{B}_c(M_{c,\Gamma})$.

All graphs Γ_{u_q}, $q \in Q$, coincide. Denote this graph by Δ.

In each connected component of Δ we fix one vertex. As noted, vertices that correspond to first letters of commutators in W lie in distinct connected components of Δ. In the connected components containing the vertices corresponding to the first letters of the commutators in W, we fix just these vertices. In each of the remaining connected components, we fix an arbitrary vertex. Consider a subgraph Δ_1 of Δ generated by the fixed vertices. A subgroup generated by elements of $U^{(c)}$ that correspond to vertices in Δ_1 is free in the variety $\mathfrak{A}^2 \cap \mathfrak{N}_c$. Denote this group by G.

Consider a retraction of $M_{c,\Gamma}$ onto $M_{c,\Delta}$ under which all elements of $W^{(c)}$ corresponding to vertices in Δ are mapped onto themselves, while the rest are mapped into identity elements. We extend this homomorphism to a homomorphism ϕ of $M_{c,\Delta}$ onto G under which all elements of $W^{(c)}$ that correspond to vertices in one connected component of Δ are mapped onto an element corresponding to the fixed vertex in the same connected component. The images of different commutators $u \in W$ under ψ are different commutators, since the first and second letters of the commutators are preserved, and the second letter in each is least. Interchanging in our commutators letters starting from the third if necessary, we obtain different commutators in $\mathcal{B}_c(M_c)$. By Proposition 1, they are independent in the pro-p-group G. The proposition is proved.

Under the homomorphism $M_{c,\Gamma} \to M_{c-1,\Gamma} \cong M_{c,\Gamma}/\gamma_c(M_{c,\Gamma})$, the set $\mathcal{B}_d(M_{c,\Gamma})$ is mapped onto a set $\mathcal{B}_d(M_{c-1,\Gamma})$, $d = 2, \ldots, c-1$. Therefore, Proposition 2 implies

COROLLARY 3. A set of elements $\bigcup_{d=2}^{c} \mathcal{B}_d(M_{c,\Gamma})$ is independent over $\mathbb{Z}_p$.

Given a graph Γ and any of its vertices v_i and v_j, we define an ideal $\mathcal{A}^{\Gamma}_{i,j}$ of the algebra $\mathbb{Z}_p[[y_1, \ldots, y_n]]$. If v_i and v_j are in distinct connected components of Γ, then we put $\mathcal{A}^{\Gamma}_{i,j} = 0$. If v_i and v_j are in one connected component, then we consider each path $\{v_i, v_{i_1}, \ldots, v_{i_r}, v_j\}$ (without recurrences) between these vertices. With such a path we associate the product $b_{i_1} \ldots b_{i_r}$ whenever the length of the path is greater than the identity element, or 1, whenever it is equal to the identity element. The ideal $\mathcal{A}^{\Gamma}_{i,j}$ is generated by all such elements. In particular, if v_i and v_j are joined by an edge, then $\mathcal{A}^{\Gamma}_{i,j}$ contains 1 and, therefore, coincides with the whole of $\mathbb{Z}_p[[y_1, \ldots, y_n]]$.

PROPOSITION 3 [7]. Let $v_1, \ldots, v_n$, $n \geq 2$, be vertices of a defining graph Γ for a partially commutative metabelian pro-p-group M_Γ. Then, for $i \neq j$, the annihilator of a commutator $[x_i, x_j] \in M_\Gamma$ in $\mathbb{Z}_p[[b_1, \ldots, b_n]]$ coincides with the ideal $\mathcal{A}^{\Gamma}_{i,j}$.

2. PROVING THE THEOREM AND COROLLARY 1

Proof of the theorem. A set $\mathcal{C}(G) = \{c_i, i \in I\}$ of elements of a metabelian pro-p-group G constitute a basis of its commutator subgroup G' over $\mathbb{Z}_p$ if the set $\mathcal{C}(G)$ converges to zero, $\mathcal{C}(G)$ generates (topologically) G', and the elements of $\mathcal{C}(G)$ are independent over $\mathbb{Z}_p$. We verify that a set $\mathcal{B}(M_\Gamma)$ possesses these properties.

The set $\mathcal{B}(M_\Gamma)$ converges to 1 since the set

$$\left\{ b_{j_1}^{s_1} \dots b_{j_m}^{s_m} \mid v_{j_1} < v_{j_2} < \dots < v_{j_m} \right\}$$

converges to zero in the ring $\mathbb{Z}[[b_1, \dots, b_n]]$.

We prove that $\mathcal{B}(M_\Gamma)$ generates the commutator subgroup M'_Γ over $\mathbb{Z}_p$. Any element $w \in M'_\Gamma$ is written in the form

$$w = \prod_{1 \le i < j \le n} [y_i, y_j]^{\beta_{i,j}},$$

where $\beta_{i,j} \in \mathbb{Z}_p[[b_1, \dots, b_n]]$. We represent the series $\beta_{i,j}$ as an infinite sum $\beta_{i,j} = \sum_{m=1}^{\infty} \beta_{i,j}^{(m)}$, where $\beta_{i,j}^{(m)}$ is a homogeneous component of degree m in $\beta_{i,j}$. Therefore, every element $[y_i, y_j]^{\beta_{i,j}}$ is the limit of elements $[y_i, y_j]^{\beta_{i,j}^{(m)}}$. Note that any permutation of $b_{j_1}^{l_1}, \dots, b_{j_m}^{l_m}$ does not alter the element $[y_i, y_j]^{b_{j_1}^{l_1} \dots b_{j_m}^{l_m}}$. Consequently, it suffices to prove that every element of the form

$$u = [y_i, y_j]^{b_{j_1}^{l_1} \dots b_{j_m}^{l_m}}, \tag{12}$$

where $v_{j_1} < \dots < v_{j_m}$, $v_j < v_i$, l_i are positive integers, belongs to a subgroup generated (algebraically) by the set $\mathcal{B}(M_\Gamma)$.

Suppose that u does not satisfy the first condition of the theorem, i.e., $v_{j_1} < v_j$. Every metabelian group satisfies the Jacobi identity

$$[x, y, z][y, z, x][z, x, y] = 1.$$

From

$$[y_i, y_j]^{b_{j_1}} = [y_i, y_j, y_{j_1}]$$

and the Jacobi identity, it follows that

$$\begin{aligned} [y_i, y_j]^{b_{j_1}} &= [y_j, y_{j_1}, y_i]^{-1} [y_{j_1}, y_i, y_j]^{-1} = [y_j, y_{j_1}]^{-b_i} [[y_{j_1}, y_i]^{-1}, y_j] \\ &= [y_j, y_{j_1}]^{-b_i} [y_i, y_{j_1}, y_j] = [y_j, y_{j_1}]^{-b_i} [y_i, y_{j_1}]^{b_j}. \end{aligned}$$

In view of $v_{j_1} < v_j < v_i$, the elements $[y_j, y_{j_1}]^{b_i}$ and $[y_i, y_{j_1}]^{b_j}$ satisfy the first condition. Therefore, we can express the element u via elements of form (12), for which $v_j \le v_{j_1} < v_{j_2} \dots < v_{j_m}$, $v_j < v_i$.

Assume now that $u = [y_i, y_j]^{b_{j_1}^{l_1} \dots b_{j_m}^{l_m}}$ satisfies the first condition of the theorem, but v_i and v_j are in one connected component of the graph Γ_u generated by the vertices $v_i, v_j, v_{j_1}, \dots, v_{j_m}$. Suppose that L is a path between v_i and v_j in Γ_u and that $v_{q_1}, \dots, v_{q_r}$ are all different intermediate vertices occurring in the path L. It follows from [5, Prop. 3] that

$$[y_i, y_j]^{b_{q_1} \dots b_{q_r}} = 1.$$

Hence $u = 1$.

It remains to consider the case where the element $u = [x_i, x_j]^{b_{j_1}^{l_1} \dots b_{j_m}^{l_m}}$ satisfies the first two conditions of the theorem, but the vertex v_i is not maximal in its connected component in Γ_u.

Assume that $v_i \neq v_q$ and v_q is the greatest vertex in the connected component of Γ_u containing v_i. Consider a path $\{v_i, v_{p_1}, \dots, v_{p_s} = v_q\}$ between v_i and v_q in Γ_u. Let $b_{j_1}^{l_1} \dots b_{j_m}^{l_m} = b_{p_1} \dots b_{p_s} b'$. We have

$$[y_i, y_j]^{b_{p_1} \dots b_{p_s} b'} = [y_i, y_j, y_{p_1}]^{b_{p_2} \dots b_{p_s} b'} = ([y_j, y_{p_1}, y_i]^{-1} [y_{p_1}, y_i, y_j]^{-1})^{b_{p_2} \dots b_{p_s} b'}.$$

With $[y_i, y_{p_1}] = 1$ in mind, we obtain

$$u = [y_i, y_j]^{b_{p_1} \dots b_{p_s} b'} = [y_{p_1}, y_j]^{b_i b_{p_2} \dots b_{p_s} b'} = \dots = [y_q, y_j]^{b_i b_{p_1} \dots b_{p_{s-1}} b'} \in \mathcal{B}(M_\Gamma).$$

Thus the set $\mathcal{B}(M_\Gamma)$ generates the commutator subgroup M'_Γ.

We show that the elements of $\mathcal{B}(M_\Gamma)$ are independent over $\mathbb{Z}_p$. Suppose that there exists a nontrivial dependency for a finite subset of elements $w_1, \dots, w_q$ in $\mathcal{B}(M_\Gamma)$. For every element w_j, $j = 1, \dots, q$, of form (2), we calculate $c_j = s_1 + \dots + s_m$. From the numbers c_j, we choose a greatest one and denote it by c. Let $d = c + 2$. Consider a natural homomorphism of M_Γ onto $M_{d,\Gamma}$. That the elements of $\mathcal{B}(M_\Gamma)$ are independent over $\mathbb{Z}_p$ follows from Corollary 3. The theorem is proved.

Proof of Corollary 1. Consider a homomorphism of M_Γ onto $M_{c,\Gamma}$ under which x_i is mapped onto $u_i^{(c)}$, $i = 1, \dots, n$. The image of $\mathcal{B}(M_\Gamma)$ under this homomorphism is the set $\bigcup_{d=2}^{c} \mathcal{B}_d(M_{c,\Gamma})$. Hence $\bigcup_{d=2}^{c} \mathcal{B}_d(M_{c,\Gamma})$ generates $M_{c,\Gamma}$ and converges to 1. By Corollary 2, the elements of this set are independent over $\mathbb{Z}_p$. The corollary is proved.

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