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ALGEBRAS OF BINARY FORMULAS FOR COMPOSITIONS OF THEORIES

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We consider algebras of binary formulas for compositions of theories both in the general case and as applied to $\aleph_0$-categorical, strongly minimal, and stable theories, linear preorders, cyclic preorders, and series of finite structures. It is shown that e-definable compositions preserve isomorphisms and elementary equivalence and have basicity formed by basic formulas of the initial theories. We find criteria for e-definable compositions to preserve $\aleph_0$-categoricity, strong minimality, and stability. It is stated that e-definable compositions of theories specify compositions of algebras of binary formulas. A description of forms of these algebras is given relative to compositions with linear orders, cyclic orders, and series of finite structures.

Algebras of binary formulas are a tool for describing connections between realizations of types at a binary level relative to a superposition of binary definable sets. These algebras are characterized for the general case in [1-3] and for natural classes of theories in [4-11].

In the present paper, we consider specific features and describe properties and algebras for compositions of theories both in the general case and as applied to $\aleph_0$-categorical, strongly minimal, and stable theories, linear orders, cyclic orders, and series of finite structures.

Our plan is as follows. In Sec. 1, we set out the notation and preliminary concepts. Namely, we define the following: an I-groupoid for an axiomatization of algebras of binary isolating formulas, a composition for graphs and monoids, and a composition of some natural monoids with groups

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generating algebras of binary isolating formulas for suitable theories (Thms. 1.4, 1.5). In Sec. 2, we introduce natural concepts of compositions for structures and theories that generalize the corresponding notions for graphs, as well as e-definable compositions $\mathcal{M}[\mathcal{N}]$ which allow us to construct the desired structures by substituting definable equivalence classes with copies of a structure $\mathcal{N}$ for elements of a structure $\mathcal{M}$. We show that compositions preserve uniqueness of 1-types (Prop. 2.2), and that e-definable compositions preserve isomorphisms and elementary equivalence (Thm. 2.4), allow reconstructing the given theories from their e-definable compositions (Cor. 2.5), and have basicity formed by basic formulas of the initial theories (Prop. 2.8). We find criteria for e-definable compositions to preserve $\aleph_0$-categoricity, strong minimality, and stability (Thms. 2.9-2.11). It is stated that e-definable compositions specify compositions of algebras of binary formulas for families of 1-types (Thm. 2.12) and for fixed 1-types (Cor. 2.13). In Secs. 3-5, we describe forms of these algebras relative to compositions with linear orders, cyclic orders, and for series of finite structures. It is proved that every composition of an algebra of binary isolating formulas for dense linear orders with a given algebra having nonnegative labels is realized by some theory (Thm. 3.1). Similarly, there exist realizations in which linear orders are replaced with an Ehrenfeucht example (Thm. 3.2), with a discrete infinite linear order (Thm. 3.4), and with dense and discrete cyclic orders (Thms. 4.5, 4.6), including values of the number of labels and rules for the labels (Thms. 4.3, 4.4). In Sec. 5, we describe rules for algebras of compositions of finite complete graphs and of compositions of cycles (Examples 5.1-5.3).

1. PRELIMINARY NOTIONS, THE NOTATION, AND STATEMENTS

Without further comment, we use the terminology adopted in [1-3]. In treating algebras of binary formulas, we describe, as usual, their restrictions to corresponding algebras of binary isolating formulas.

Recall basic properties of groupoids $\mathfrak{P}_{\nu(p)}$ for 1-types p.

Definition 1.1 [1, 2]. Let $U = U^- \dot\cup \{0\} \dot\cup U^+$ be some alphabet consisting of a set U^- of *negative elements*, a set U^+ of *positive elements*, and zero (0). We will write $u < 0$ for any element $u \in U^-$, $u > 0$ for any element $u \in U^+$, $u \leq 0$ for elements $u \in U^- \cup \{0\}$, and $u \geq 0$ for elements $u \in U^+ \cup \{0\}$. Thus the symbols $<$, $>$, $\leq$, and $\geq$ are used to compare labels in U with zero. In treating the operation $\cdot$ on the set $\mathcal{P}(U) \setminus \{\varnothing\}$, we will use $u \cdot v$ instead of $\{u\} \cdot \{v\}$.

A groupoid $\mathfrak{P} = \langle \mathcal{P}(U) \setminus \{\varnothing\}; \cdot \rangle$ is called an *I-groupoid* if the following conditions hold:

the set $\{0\}$ is the identity element of $\mathfrak{P}$;

the operation $\cdot$ on $\mathfrak{P}$ is generated by the function $\cdot$ on elements in U, which assigns a nonempty set $(u \cdot v) \subseteq U$ to any elements $u, v \in U$; for any sets $X, Y \in \mathcal{P}(U) \setminus \{\varnothing\}$, the following relation holds:

$$X \cdot Y = \bigcup \{x \cdot y \mid x \in X, y \in Y\};$$

if $u < 0$, then the sets $u \cdot v$ and $v \cdot u$ consist of negative elements for any $v \in U$;

if $u > 0$ and $v > 0$, then the set $u \cdot v$ consists of nonnegative elements;

for any element $u > 0$, there exists a unique *inverse* element $u^{-1} > 0$ for which $0 \in (u \cdot u^{-1}) \cap (u^{-1} \cdot u)$;

if a positive element u belongs to the set $v_1 \cdot v_2$, then the element u^{-1} belongs to a set $v_2^{-1} \cdot v_1^{-1}$;

for any elements $u_1, u_2, u_3 \in U$, the following inclusion holds:

$$(u_1 \cdot u_2) \cdot u_3 \supseteq u_1 \cdot (u_2 \cdot u_3)$$

in which case the strict inclusion

$$(u_1 \cdot u_2) \cdot u_3 \supset u_1 \cdot (u_2 \cdot u_3)$$

is possible only if $u_1 < 0$ and the set $u_2 \cdot u_3$ has infinite cardinality;

$\mathfrak{P}$ contains the *deterministic* subgroupoid $\mathfrak{P}_d^{\geq 0}$ (which is a monoid) with universe $\mathcal{P}(U_d^{\geq 0}) \setminus \{\varnothing\}$, where

$$U_d^{\geq 0} = \{u \in U^{\geq 0} \mid u^{-1} \cdot u = \{0\}\};$$

moreover, $|u \cdot v| = 1$ for any $u, v \in U_d^{\geq 0}$.

By definition, every I-groupoid $\mathfrak{P}$ contains I-subgroupoids $\mathfrak{P}^{\leq 0}$ and $\mathfrak{P}^{\geq 0}$ with universes $\mathcal{P}(U^- \cup \{0\}) \setminus \{\varnothing\}$ and $\mathcal{P}(U^+ \cup \{0\}) \setminus \{\varnothing\}$, respectively. The structure $\mathfrak{P}^{\geq 0}$ is a monoid.

Using syntactic generic constructions, we can establish the following representation theorems.

THEOREM 1.2 [1, 2]. For any I-groupoid $\mathfrak{P}$, there is a theory T with a type $p(x) \in S^1(T)$ and a regular labeling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)} = \mathfrak{P}$. If the alphabet is at most countable and the operation in $\mathfrak{P}$ does not induce continuum many types, then T is a small theory.

Recall [12, 13] that a *composition* $\Gamma_1[\Gamma_2]$ of graphs $\Gamma_1 = \langle X_1; R_1 \rangle$ and $\Gamma_2 = \langle X_2; R_2 \rangle$ is a graph $\langle X_1 \times X_2; R \rangle$ in which $((a_1, b_1), (a_2, b_2)) \in R$ iff one of the following conditions is satisfied:

(1) $(a_1, a_2) \in R_1$;

(2) $a_1 = a_2$ and $(b_1, b_2) \in R_2$.

A composition of monoids is defined in a similar way. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be monoids for which 0 is an identity element, and let $S_1 \subseteq U^{\leq 0}$ and $S_2 \subseteq U^{\geq 0}$. A *composition*, or *sequentially annihilating band* $\mathcal{S}_1[\mathcal{S}_2]$ [14, 15], of monoids $\mathcal{S}_1$ and $\mathcal{S}_2$ is an algebra $\langle S_1 \cup S_2; \odot \rangle$ where $\langle S_1 \cup S_2; \odot \rangle \upharpoonright S_i = \mathcal{S}_i$, for $i = 1, 2$, and $u \odot v = v \odot u = u$ for $u < 0$ and $v > 0$.

PROPOSITION 1.3 [15]. Every sequentially annihilating band $\mathcal{S}_1[\mathcal{S}_2]$ is a monoid.

Below are two theorems which show that compositions of graphs, as well as compositions of monoids and their associated compositions of algebras of binary formulas, can be realized both for dense linear orders and for a monoid on natural numbers.

THEOREM 1.4 [1, 2]. For any group $\langle G; * \rangle$ the universe of which consists of nonnegative elements and whose identity element is 0 and for a monoid $\langle \{-1, 0\}; + \rangle$ with a zero element 0 and an idempotent element -1, there is a theory T with a type $p \in S(T)$ and a regular labeling function $\nu(p)$ such that the monoid $\mathfrak{P}'_{\nu(p)}$ coincides with $\langle \{-1, 0\}; + \rangle[\langle G; * \rangle]$.

THEOREM 1.5 [1, 2]. For any group $\langle G; *\rangle$ the universe of which consists of nonnegative elements and whose identity element is 0 and for a monoid $\langle\omega^*; +\rangle$ of nonpositive integers, there is a theory T with a complete type $p \in S(T)$ and a regular labeling function $\nu(p)$ such that the monoid $\mathfrak{P}'_{\nu(p)}$ coincides with $\langle\omega^*; +\rangle[\langle G; *\rangle]$.

2. COMPOSITIONS OF STRUCTURES AND COMPOSITIONS OF THEORIES

We generalize the concept of a composition $\Gamma_1[\Gamma_2]$ of graphs as follows. Let $\mathcal{M}$ and $\mathcal{N}$ be structures in relational signatures $\Sigma_{\mathcal{M}}$ and $\Sigma_{\mathcal{N}}$, respectively. The *composition* $\mathcal{M}[\mathcal{N}]$ of structures $\mathcal{M}$ and $\mathcal{N}$ is defined via the following rules:

(1) $\Sigma_{\mathcal{M}[\mathcal{N}]} = \Sigma_{\mathcal{M}} \cup \Sigma_{\mathcal{N}}$;

(2) $M[N] = M \times N$, where $M[N]$, M, and N are universes of the structures $\mathcal{M}[\mathcal{N}]$, $\mathcal{M}$, and $\mathcal{N}$, respectively;

(3) for $R \in \Sigma_{\mathcal{M}} \setminus \Sigma_{\mathcal{N}}$ and $\mu(R) = n$, put $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if $(a_1, \ldots, a_n) \in R_{\mathcal{M}}$;

(4) for $R \in \Sigma_{\mathcal{N}} \setminus \Sigma_{\mathcal{M}}$ and $\mu(R) = n$, put $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if $a_1 = \ldots = a_n$ and $(b_1, \ldots, b_n) \in R_{\mathcal{N}}$;

(5) for $R \in \Sigma_{\mathcal{M}} \cap \Sigma_{\mathcal{N}}$ and $\mu(R) = n$, put $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if $(a_1, \ldots, a_n) \in R_{\mathcal{M}}$ or $a_1 = \ldots = a_n$ and $(b_1, \ldots, b_n) \in R_{\mathcal{N}}$.

Note that the condition $a_1 = \ldots = a_n$ in the definition somewhat modifies the predicates R by adding sets of elements to diagonals. Also there are other approaches to such a definition of a composition. With several possibilities for defining a composition at hand, we consider a composition with diagonals for the sake of convenience and based on the initial definition for graphs.

By definition, the composition $\mathcal{M}[\mathcal{N}]$ is obtained by replacing each element in $\mathcal{M}$ with a copy of the structure $\mathcal{N}$.

The notion of a composition of structures is naturally carried over to a composition $\mathfrak{P}_1[\mathfrak{P}_2]$ of algebras $\mathfrak{P}_1$ and $\mathfrak{P}_2$ of binary isolating formulas.

The theory $T = \mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is the *composition* $T_1[T_2]$ of theories $T_1 = \mathrm{Th}(\mathcal{M})$ and $T_2 = \mathrm{Th}(\mathcal{N})$.

In what follows, we consider a special form of the composition of theories T_1 and T_2 for which $T_1[T_2]$ does not depend on the choice of structures $\mathcal{M} \models T_1$ and $\mathcal{N} \models T_2$. At the same time, the remark below shows that such independence may not be the case in general.

Remark 2.1. If $\mathcal{M} \simeq \mathcal{M}'$ and $\mathcal{N} \simeq \mathcal{N}'$, then $\mathcal{M}[\mathcal{N}] \simeq \mathcal{M}'[\mathcal{N}']$, but the reverse implication is untrue. Indeed, if we consider the structures $\mathcal{M}$ and $\mathcal{N}$ in the empty signature we obtain the composition $\mathcal{M}[\mathcal{N}]$ of the empty signature with $|M| \times |N|$ elements. However, the cardinality $|M| \times |N|$ does not recover cardinalities $|M|$ and $|N|$. In particular, the infinite cardinality $|M \times N|$ may be obtained with some finite set M and infinite N, as well as with infinite M and N. Therefore, there are $\mathcal{M} \not\equiv \mathcal{M}'$ and $\mathcal{N} \not\equiv \mathcal{N}'$ such that $\mathcal{M}[\mathcal{N}] \simeq \mathcal{M}[\mathcal{N}']$ and $\mathcal{M}[\mathcal{N}] \simeq \mathcal{M}'[\mathcal{N}']$.

PROPOSITION 2.2. If structures $\mathcal{M}$ and $\mathcal{N}$ have transitive automorphism groups, then $\mathcal{M}[\mathcal{N}]$ also has a transitive automorphism group.

Proof. By hypothesis, any two elements of $\mathcal{N}$ are connected by an automorphism, and there are automorphisms connecting copies of the structure $\mathcal{N}$ in $\mathcal{M}[\mathcal{N}]$. Therefore, choosing elements a and b in the copies $\mathcal{N}'$ and $\mathcal{N}''$ of $\mathcal{N}$, respectively, we can map $\mathcal{N}'$ to $\mathcal{N}''$ by some automorphism f and then map $f(a)$ to b by some automorphism g extending an automorpism of the structure $\mathcal{N}''$ and fixing all elements of $M[N] \setminus N''$. Thus $f \circ g$ is an automorphism of the structure $\mathcal{M}[\mathcal{N}]$ that connects a with b. $\square$

By Proposition 2.2, $T = \mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is a *transitive* theory, i.e., T has a unique complete 1-type $p_0(x)$, and therefore the operation $\mathcal{M}[\mathcal{N}]$ can be treated as a version of the transitive arrangement of structures [16].

Definition 2.3. A composition $\mathcal{M}[\mathcal{N}]$ is said to be *e-definable* or equ-*definable* if $\mathcal{M}[\mathcal{N}]$ has an $\varnothing$-definable equivalence relation E the E-classes of which are universes of copies of the structure $\mathcal{N}$ forming $\mathcal{M}[\mathcal{N}]$. If an equivalence relation E is fixed, then an e-definable composition is said to be *E-definable*.

By definition, every E-definable composition $\mathcal{M}[\mathcal{N}]$ can be represented as an E-combination [17] of copies of $\mathcal{N}$ together with an additional structure which is generated by predicates on $\mathcal{M}$ and connects elements of the copies of $\mathcal{N}$.

Clearly, there are compositions which are not e-definable. For example, compositions of structures of the empty signature which are not e-definable are presented in Remark 2.1.

The construction for proving Theorem 1 specifies an E-definable composition $\mathcal{M}[\mathcal{N}]$, where $\mathcal{M}$ forms a dense linear order, $\mathcal{N}$ is a binary structure representing a group, and E is a definable relation of belonging to a common antichain:

$$xEy \Leftrightarrow x = y, \text{ or } x \not< y \text{ and } y \not< x.$$

At the same time, the construction for proving Theorem 1.5 specifies a composition which is not e-definable, producing a transitive arrangement of structures [16] on an exact pseudoplane $\mathcal{M}$ with a contourless directed graph $\langle M; Q\rangle$. In these structures, copies of the structure $\mathcal{N}$ are recovered in the form of structures on lines of the structure $\mathcal{M}$ where the lines are defined as sets of solutions for formulas $Q(a, y)$, $a \in M$.

THEOREM 2.4. If $\mathcal{M}[\mathcal{N}]$ is not an e-definable composition, then the following relations hold:

(1) $\mathcal{M}[\mathcal{N}] \simeq \mathcal{M}'[\mathcal{N}']$ if and only if $\mathcal{M} \simeq \mathcal{M}'$ and $\mathcal{N} \simeq \mathcal{N}'$;

(2) $\mathcal{M}[\mathcal{N}] \equiv \mathcal{M}'[\mathcal{N}']$ if and only if $\mathcal{M} \equiv \mathcal{M}'$ and $\mathcal{N} \equiv \mathcal{N}'$.

Proof. Let $\mathcal{M}[\mathcal{N}]$ be an E-definable composition.

If $\mathcal{M}[\mathcal{N}] \simeq \mathcal{M}'[\mathcal{N}']$, then $\mathcal{M}[\mathcal{N}]$ consists of E-classes which are copies of $\mathcal{N}$, and $\mathcal{M}'[\mathcal{N}']$ consists of E'-classes of copies of $\mathcal{N}'$ for which the relations E and E' are defined by the same formula, and $\mathcal{N} \simeq \mathcal{N}'$ holds. The structure $\mathcal{M}$ is isomorphic to $\mathcal{M}_0$, which is obtained from $\mathcal{M}[\mathcal{N}]$ by replacing

E-classes with elements, and similarly the factor structure $\mathcal{M}'[\mathcal{N}']$ with respect to E'-classes defines a structure $\mathcal{M}'_0$ isomorphic to $\mathcal{M}'$. Since $\mathcal{M} \simeq \mathcal{M}_0 \simeq \mathcal{M}'_0 \simeq \mathcal{M}'$, we obtain $\mathcal{M} \simeq \mathcal{M}'$.

Conversely, if $\mathcal{M} \simeq \mathcal{M}'$ and $\mathcal{N} \simeq \mathcal{N}'$, then $\mathcal{M}[\mathcal{N}] \simeq \mathcal{M}'[\mathcal{N}']$ by the definition of a composition. Moreover, this implication holds in the general case, without the presupposition of e-definability.

Based on the same argument using e-definability, we state that $\mathcal{M}[\mathcal{N}] \equiv \mathcal{M}[\mathcal{N}]$ iff $\mathcal{M} \equiv \mathcal{M}'$ and $\mathcal{N} \equiv \mathcal{N}'$. □

Theorem 2.4 immediately implies

COROLLARY 2.5. If $\mathcal{M}[\mathcal{N}]$ is an e-definable composition, then the theory $\mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is uniquely determined by $\mathrm{Th}(\mathcal{M})$ and $\mathrm{Th}(\mathcal{N})$, while the theories $\mathrm{Th}(\mathcal{M})$ and $\mathrm{Th}(\mathcal{N})$ are uniquely determined by $\mathrm{Th}(\mathcal{M}[\mathcal{N}])$.

Corollary 2.5 allows us to uniquely determine the composition $T_1[T_2] = \mathrm{Th}(\mathcal{M}[\mathcal{N}])$ given theories $T_1 = \mathrm{Th}(\mathcal{M})$ and $T_2 = \mathrm{Th}(\mathcal{N})$, and to determine T_1 and T_2 given $T_1[T_2]$.

Note that in view of Remark 2.1, Theorem 2.4(1), (2) and Corollary 2.5 may fail to hold if $\mathcal{M}[\mathcal{N}]$ is not an e-definable composition.

Definition 2.6 [18]. A theory T is said to be Δ-*based*, where Δ is a set of formulas without parameters if every formula of T is equivalent in T to a Boolean combination of formulas in Δ.

For Δ-based theories T, we also say that T admits *elimination of quantifiers* or *reduction of quantifiers* relative to Δ.

Let Δ be a set of formulas of a theory T and let $p(\bar{x})$ be a type of the theory T belonging to $S(T)$. We say that $p(\bar{x})$ is Δ-*based* if $p(\bar{x})$ is isolated by a set of formulas $\varphi^{\delta} \in p$, where $\varphi \in \Delta$ and $\delta \in \{0, 1\}$.

LEMMA 2.7 [18]. A theory T is Δ-based if and only if $\mathrm{tp}(\bar{a})$ is a Δ-based type for any tuple $\bar{a}$ of any (some) weakly saturated model of the theory T.

Suppose now that $\mathcal{M}[\mathcal{N}]$ is E-definable, $T_1 = \mathrm{Th}(\mathcal{M})$ is Δ_1-based, and $T_2 = \mathrm{Th}(\mathcal{N})$ is Δ_2-based.

Using Lemma 2.7, we describe a set Δ for which $T = T_1[T_2]$ is a Δ-based. For this goal to be met, we consider a tuple $\bar{a}$ of a weakly saturated model of a theory T. By Theorem 2.4, we may assume that this model is again a composition $\mathcal{M}[\mathcal{N}]$. Since the composition is E-definable, the tuple $\bar{a}$ is partitioned into nonintersecting parts $\bar{a}_i$ so that every part belongs to some E-class and different parts belong to different E-classes. Include in Δ the formulas $E'(x, y)$ witnessing the partition mentioned. Now, by the definition of an E-definable composition, every type $\mathrm{tp}(\bar{a}_i)$ is $(\Delta_1 \cup \Delta_2 \cup \widetilde{E})$-based, where $\widetilde{E}$ consists of formulas $E'(x, y)$ since this type is defined by the description of 'belonging to a common E-class,' by a Boolean combination of formulas in Δ_2 describing $\mathrm{tp}_{\mathcal{N}_i}(\bar{a}_i)$ for a copy $\mathcal{N}_i$ of a structure $\mathcal{N}$ containing $\bar{a}_i$, and by a Boolean combination of 1-formulas in Δ_1 describing $\mathrm{tp}_{\mathcal{M}}(b_i)$ for an element $b_i \in M$, which is replaced by a structure $\mathcal{N}'$ in constructing $\mathcal{M}[\mathcal{N}]$.

Moreover, the type $\mathrm{tp}(\bar{a})$ is again $(\Delta_1 \cup \Delta_2 \cup \widetilde{E})$-based since it is derived in T from the description of 'belonging to a common E-class' for every $\bar{a}_i$, "belonging to different E-classes' for different $\bar{a}_i$, and from Boolean combinations of formulas in Δ_2 describing types $\mathrm{tp}_{\mathcal{N}_i}(\bar{a}_i)$ together with Boolean combinations of formulas in Δ_1 describing a type $\mathrm{tp}_{\mathcal{M}}(\bar{b})$ for a tuple $\bar{b} \in M$, whose coordinates are replaced by the structures $\mathcal{N}_i$ considered in constructing $\mathcal{M}[\mathcal{N}]$.

Applying Lemma 2.7, we obtain the following:

PROPOSITION 2.8. If $\mathcal{M}[\mathcal{N}]$ is an E-definable composition, then $T = \mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is a $(\Delta_1 \cup \Delta_2 \cup \widetilde{E})$-based theory, where $\mathrm{Th}(\mathcal{M})$ is Δ_1-based, $\mathrm{Th}(\mathcal{N})$ is Δ_2-based, and $\widetilde{E}$ is a set of formulas $E'(x, y)$ witnessing that E is a definable equivalence relation.

By virtue of Theorem 2.4 and Proposition 2.8, a number of model-theoretic properties are preserved for e-definable compositions $\mathcal{M}[\mathcal{N}]$. Below we deal with characterizations for $\aleph_0$-categorical, (strongly) minimal, and stable compositions $\mathcal{M}[\mathcal{N}]$.

THEOREM 2.9. The theory $T_1[T_2]$ of an infinite E-definable composition $\mathcal{M}[\mathcal{N}]$, where $T_1 = \mathrm{Th}(\mathcal{M})$ and $T_2 = \mathrm{Th}(\mathcal{N})$, is $\aleph_0$-categorical if and only if one of the structures $\mathcal{M}$ and $\mathcal{N}$ is finite and the other is $\aleph_0$-categorical, or these structures are both $\aleph_0$-categorical.

Proof. If $T_1[T_2]$ is $\aleph_0$-categorical, then it follows by the Ryll-Nardzewski theorem that $T_1[T_2]$ has finitely many n-types for any $n \in \omega$. In view of the fact that $\mathcal{M}[\mathcal{N}]$ is E-definable and Theorem 2.4, n-types for $T_1[T_2]$ specify both n-types for T_2, where all coordinates of realizing tuples are related via E, and n-types for T_1, where all coordinates of realizing tuples are not related via E. Consequently, the number of n-types is finite both for T_1 and T_2. Applying the Ryll-Nardzewski theorem again, we conclude that $\mathcal{M}$ and $\mathcal{N}$ are either finite or $\aleph_0$-categorical. Since $\mathcal{M}[\mathcal{N}]$ is infinite, we see that either one of the structures $\mathcal{M}$ and $\mathcal{N}$ is finite and the other is $\aleph_0$-categorical, or these structures are both $\aleph_0$-categorical.

Now we suppose that each of the structures $\mathcal{M}$ and $\mathcal{N}$ either is finite or $\aleph_0$-categorical. By the Ryll-Nardzewski theorem, T_1 and T_2 have finitely many n-types for any $n \in \omega$. In view of Proposition 2.8, there are finitely many options for n-types of the theory $T_1[T_2]$. If again we apply the Ryll-Nardzewski theorem we conclude that $T_1[T_2]$ is $\aleph_0$-categorical. □

Recall [19, 20] that an infinite structure $\mathcal{M}$ is (*definably*) *minimal* if any definable subset in M is either finite or cofinite.

Following [21], we call $\mathcal{M}$ a *strongly minimal* structure if any definable subset of the universe of a structure $\mathcal{N} \equiv \mathcal{M}$ is either finite or cofinite.

THEOREM 2.10. An infinite E-definable composition $\mathcal{M}[\mathcal{N}]$ is (strongly) minimal if and only if one of the following conditions is satisfied:

(1) $\mathcal{M}$ is a singleton and $\mathcal{N}$ is (strongly) minimal;

(2) $\mathcal{M}$ is (strongly) minimal and $\mathcal{N}$ is finite and has no proper $\varnothing$-definable subsets.

Proof. By Theorem 2.4, it suffices to show that $\mathcal{M}[\mathcal{N}]$ is a minimal composition iff $\mathcal{M}$ is a singleton and $\mathcal{N}$ is minimal, or $\mathcal{M}$ is minimal and $\mathcal{N}$ is finite, and the latter has no proper $\varnothing$-definable subsets.

Let $\mathcal{M}[\mathcal{N}]$ be minimal. By virtue of Proposition 2.8, every E-class cannot be divided by definable sets into two infinite parts, and definable sets of E-classes cannot be divided into two infinite parts. Consequently, each of the structures $\mathcal{M}$ and $\mathcal{N}$ is either finite or minimal. In this case the finite structure $\mathcal{N}$ should not have proper $\varnothing$-definable subsets, since otherwise an infinite number of copies of $\mathcal{N}$ in $\mathcal{M}[\mathcal{N}]$ would allow us to partition the universe into two infinite definable parts. It remains to observe that if the structure $\mathcal{M}$ is not a singleton and $\mathcal{N}$ is minimal, then $\mathcal{M}[\mathcal{N}]$ has at least two infinite E-classes, which contradicts the minimality.

Conversely, having considered a definable subset X in $M[N]$, we note that X is representable as the union of definable subsets in E-classes. If the structure $\mathcal{M}$ is a singleton and $\mathcal{N}$ is minimal, then there exists a unique E-class, and the set X is finite or cofinite since $\mathcal{N}$ is minimal. If $\mathcal{M}$ is minimal and $\mathcal{N}$ is finite, then it follows by Proposition 2.8 that the set X contains elements of a finite or cofinite family of E-classes. The E-classes are finite and $\mathcal{N}$ has no proper $\varnothing$-definable subsets, so X is either finite or cofinite. Since the set X has been chosen arbitrarily, the structure $\mathcal{M}[\mathcal{N}]$ is minimal. □

Recall [22] that a formula $\varphi(\bar{x}, \bar{y})$ of a theory T is *stable* if there exist no tuples $\bar{a}_n$ and $\bar{b}_n$, $n \in \omega$, such that $\models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j$. A theory T is *stable* if every formula of T is stable.

In [23], it was proved that every Boolean combination of stable formulas is again stable. Based on Proposition 2.8, we can therefore state the following:

THEOREM 2.11. For every E-definable composition $\mathcal{M}[\mathcal{N}]$, the theory $\mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is stable if and only if $\mathrm{Th}(\mathcal{M})$ and $\mathrm{Th}(\mathcal{N})$ are stable.

Proof. Let $\mathrm{Th}(\mathcal{M}[\mathcal{N}])$ be a stable theory. Then every formula $\varphi(\bar{x})$ of the theory $\mathrm{Th}(\mathcal{M})$ or $\mathrm{Th}(\mathcal{N})$ is representable as the restriction of some formula $\psi(\bar{x})$ of the theory $\mathrm{Th}(\mathcal{M}[\mathcal{N}])$ to some E-class or to definable relations between E-classes, i.e., as the conjunction $\psi(\bar{x}) \wedge \bigwedge\limits_{x_i, x_j \in \bar{x}} E^{\delta_{ij}}(x_i, x_j)$, $\delta_{ij} \in \{0, 1\}$. Formulas $\psi(\bar{x})$ and $E(x_i, x_j)$ are stable, so $\varphi(\bar{x})$ is also stable.

If the theories $\mathrm{Th}(\mathcal{M})$ and $\mathrm{Th}(\mathcal{N})$ are stable, then it follows by Proposition 2.8 that every formula of $\mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is equivalent to a Boolean combination of stable formulas of $\mathrm{Th}(\mathcal{M})$ and $\mathrm{Th}(\mathcal{N})$, and also of stable formulas $E'(x, y)$ for an equivalence relation E. Thus $\mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is a stable theory. □

THEOREM 2.12. If $\mathcal{M}[\mathcal{N}]$ is an E-definable composition, then an algebra $\mathfrak{P}_T$ of binary isolating formulas for a theory $T = \mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is isomorphic to a composition $\mathfrak{P}_{T_1}[\mathfrak{P}_{T_2}]$ of algebras $\mathfrak{P}_{T_1}$ and $\mathfrak{P}_{T_2}$ of binary isolating formulas for theories $T_1 = \mathrm{Th}(\mathcal{M})$ and $T_2 = \mathrm{Th}(\mathcal{N})$.

The **proof** follows from Corollary 2.5 and Proposition 2.8, since relations via isolating formulas are representable as compositions of relations on E-classes forming $\mathfrak{P}_{T_1}$, and of relations inside E-classes forming $\mathfrak{P}_{T_2}$. □

Proposition 2.2 and Theorem 2.12 can be combined to yield

COROLLARY 2.13. If $\mathcal{M}[\mathcal{N}]$ is an E-definable composition, and $T_1 = \mathrm{Th}(\mathcal{M})$ and $T_2 = \mathrm{Th}(\mathcal{N})$ are transitive theories with algebras $\mathfrak{P}_{\nu(p)}$ and $\mathfrak{P}_{\nu'(p')}$, respectively, then the theory $T_1[T_2]$ has an algebra $\mathfrak{P}_{\nu''(p'')}$ with a unique 1-type p'' which is isomorphic to a composition $\mathfrak{P}_{\nu(p)}[\mathfrak{P}_{\nu'(p')}]$.

Remark 2.14. Theorems 2.9, 2.10, and 2.12 allow us to apply the descriptions [4, 5] of algebras of binary formulas for E-definable compositions of $\aleph_0$-categorical structures and for E-definable compositions of strongly minimal structures.

Remark 2.15. In view of Remark 2.1, Theorem 2.12 and Corollary 2.13 may fail to hold, if $\mathcal{M}[\mathcal{N}]$ is not an E-definable composition. In particular, if $\mathcal{M}$ and $\mathcal{N}$ are infinite structures in the empty signature, then, for the theories $T_1 = \mathrm{Th}(\mathcal{M})$, $T_2 = \mathrm{Th}(\mathcal{N})$, and $T = \mathrm{Th}(\mathcal{M}[\mathcal{N}])$, the algebras $\mathfrak{P}_{T_1}$, $\mathfrak{P}_{T_2}$, and $\mathfrak{P}_T$ are pairwise isomorphic, with two labels 0 and 1 satisfying the rules $0 \cdot u = u \cdot 0 = \{u\}$, for $u \in \{0, 1\}$, and $1 \cdot 1 = \{0, 1\}$.

3. I-GROUPOIDS FOR LINEAR PREORDERS

In this section, we generalize Theorem 1.4 for I-groupoids, which are treated instead of groups $\langle G; *\rangle$.

Let λ be a positive cardinality, suppose that $\mathcal{M}_\lambda = \langle M_\lambda, <\rangle$ is a dense linearly preordered set such that all maximal antichains A have the same cardinality λ, and assume that $\mathcal{M}_\lambda/{\sim}$ has no terminal elements, where $x \sim y \Leftrightarrow x = y$ or $x \not< y$ and $y \not< x$. This means that $\mathcal{M}_\lambda$ is obtained from $\mathcal{M}_1$ by replacing every element with a copy of an antichain A.

Obviously, $T_\lambda = \mathrm{Th}(\mathcal{M}_\lambda)$ is a transitive theory.

The structure $\mathcal{M}_\lambda$ is linearly ordered iff $\lambda = 1$. In this case the algebra $\mathfrak{P}_0 = \mathfrak{P}_{\nu(p_0)}$ consists of three labels 0, 1, and 2 corresponding to formulas $a \approx y$, $a < y$, and $y < a$, respectively. An operation $\cdot$ on the Boolean of a set of labels is defined via the following rules: $u \cdot 0 = 0 \cdot u = \{u\}$ for $u \in \{0, 1, 2\}$, $1 \cdot 1 = \{1\}$, $2 \cdot 2 = \{2\}$, and $1 \cdot 2 = 2 \cdot 1 = \{0, 1, 2\}$.

For $\lambda = 2$, the algebra $\mathfrak{P}_\lambda$ is placed on every antichain A and has two labels, say, 0 and 3, corresponding to formulas $a \approx y$ and $\neg(a \approx y)$, with the following rules: $u \cdot 0 = 0 \cdot u = \{u\}$ for $u \in \{0, 3\}$, and $3 \cdot 3 = \{0\}$. For $\lambda > 2$, $\mathfrak{P}_\lambda$ has the same labels with the following rules: $u \cdot 0 = 0 \cdot u = \{u\}$ for $u \in \{0, 3\}$, and $3 \cdot 3 = \{0, 3\}$.

Thus the algebra $\mathfrak{P}_{\nu(p_0)}$ (of cardinality λ in the general case) is representable as a composition $\mathfrak{P}_0[\mathfrak{P}_\lambda]$, where 0 is the unique common label for $\mathfrak{P}_0$ and $\mathfrak{P}_\lambda$ with condition $0 \cdot u = u \cdot 0 = \{u\}$ for any label u. If $u \neq 0$ is a label in $\mathfrak{P}_0$ and v is one in $\mathfrak{P}_\lambda$, then $u \cdot v = v \cdot u = \{u\}$.

We will place isomorphic structures $\mathcal{N}$ having transitive theories on every antichain A of the structure $\mathcal{M}_\lambda$, so as to satisfy Theorem 1.2 with condition $\mathfrak{P}_{\nu(p)} = \mathfrak{P}$ for a given I-groupoid $\mathfrak{P}$ with nonnegative labels. The resulting structure is the composition $\mathcal{M}_1[\mathcal{N}]$. By Proposition 2.2, $\mathrm{Th}(\mathcal{M}_1[\mathcal{N}])$ is a transitive theory.

Consider an algebra $\mathfrak{P}_0[\mathfrak{P}_\lambda]$. If we replace the algebra $\mathfrak{P}_\lambda$ by $\mathfrak{P}$ we obtain a composition $\mathfrak{P}_0[\mathfrak{P}]$. This algebra coincides with an algebra $\mathfrak{P}_{\nu(q)}$ of binary isolating formulas for a transitive theory $T = \mathrm{Th}(\mathcal{M}_1[\mathcal{N}])$ with a unique 1-type q.

Thus the following holds:

THEOREM 3.1. For any I-groupoid $\mathfrak{P}$ consisting of nonnegative labels, there exists a theory T with a type $p \in S(T)$ and a regular labeling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)} = \mathfrak{P}_0[\mathfrak{P}]$.

Now we consider a modification of Theorems 1.4 and 3.1 by introducing an algebra $\widehat{\mathfrak{P}}_0 = \mathfrak{P}_{\nu(q)}$ for a unique nonisolated 1-type q of the Ehrenfeucht example $\langle \mathbb{Q}; <, c_k\rangle_{k\in\omega}$, $c_k < c_{k+1}$, $k \in \omega$. The algebra $\widehat{\mathfrak{P}}_0$ has two labels, say, 0 and -1, with the following rules: $(-1)\cdot 0 = 0\cdot(-1) = (-1)\cdot(-1) = \{-1\}$.

THEOREM 3.2. For every I-groupoid $\mathfrak{P}$ consisting of nonnegative labels, there exists a theory T with a type $p \in S(T)$ and a regular labeling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)} = \widehat{\mathfrak{P}}_0[\mathfrak{P}]$.

Proof. We construct a structure $\mathcal{M}_1[\mathcal{N}]$ for which $T = \mathrm{Th}(\mathcal{M}_1[\mathcal{N}])$ has a type $p(x) \in S(T)$ and a regular labeling function $\nu(p)$ with $\mathfrak{P}_{\nu(p)} = \widehat{\mathfrak{P}}_0[\mathfrak{P}]$. For this goal to be met, we consider the Ehrenfeucht example $\langle \mathbb{Q}; <, c_k\rangle_{k\in\omega}$, $c_k < c_{k+1}$, $k \in \omega$, and replace every element a by a $<$-antichain consisting of λ elements, where λ is the cardinality of the universe of a structure $\mathcal{N}$ that realizes an I-groupoid $\mathfrak{P}$. Moreover, we replace every constant c_k by a unary predicate R_k consisting of λ elements. As a result, we obtain a composition $\langle \mathbb{Q}; <\rangle[\mathcal{N}]$ enriched with relations R_k, $k \in \omega$, $(x < y)$, $\neg(x < y) \wedge \neg(y < x)$. A unique nonprincipal 1-type $p(x)$ is isolated by a set of formulas $\exists y(R_k(y) \wedge (y < x))$, $k \in \omega$. For any realization a of the type p, a list of pairwise nonequivalent isolating formulas $\varphi(a, y)$ with $\varphi(a, y) \vdash p(y)$ is exhausted by a formula $(a < y)$ and isolating relations inside $\mathcal{N}$. We will define a regular labeling function $\nu(p)$ so that the formula $(a < y)$ would have label -1 and formulas $\varphi(a, y)$ for isolating relations inside $\mathcal{N}$ would have nonnegative labels u: $\varphi(a, y) = \theta_u(a, y)$. Since $< \circ < = <$ and $< \circ\, \theta_u = \theta_u \circ < = <$ for any label u, and relations between elements of $\rho^{\geq 0}_{\nu(p)}$ are defined by $\mathfrak{P}$, the algebra $\mathfrak{P}_{\nu(p)}$ coincides with $\widehat{\mathfrak{P}}_0[\mathfrak{P}]$. $\square$

Example 3.3. Let $\mathcal{N}$ be a model of the theory of pure equality with $\lambda \geq 2$ elements, and let $\mathcal{M} = \langle \mathbb{Q}, ; <, c_k\rangle_{k\in\omega}$ be an Ehrenfeucht example, where $c_k < c_{k+1}$ and $k \in \omega$. Then the algebra $\mathfrak{P}_{\nu(p)}$ for a unique 1-type $p \in S_1(\varnothing)$ in $\mathrm{Th}(\mathcal{N})$ has two labels: 0 $(x \approx y)$ and 1 $(\neg(x \approx y))$, and the algebra $\mathfrak{P}_{\nu(q)}$ for a unique nonisolated 1-type $q \in S_1(\varnothing)$ in $\mathrm{Th}(\mathcal{M})$ has the following labels: 0 $(x \approx y)$ and -1 $(x < y)$.

If we replace every element in $\mathcal{M}$ by a copy of the structure $\mathcal{N}$, then the algebra $\mathfrak{P}_{\nu(q')}$ for a unique nonisolated 1-type $q' \in S_1(\varnothing)$ in their composition $\mathcal{M}[\mathcal{N}]$ has three labels: 0 $(x \approx y)$, -1 $(x < y)$, and 1 $(x \diamond y)$, where

$$x \diamond y := \neg(x \approx y) \wedge \neg(x < y) \wedge \neg(y < x).$$

The following rules are valid: $0 \cdot 1 = 1 \cdot 0 = \{1\}$, $1 \cdot 1 = \{0\}$ for $\lambda = 2$, $1 \cdot 1 = \{0, 1\}$ for $\lambda > 2$, and $1 \cdot (-1) = (-1) \cdot 1 = (-1) \cdot (-1) = \{-1\}$.

Now we consider compositions of a linearly ordered set $\mathcal{M} = \langle \mathbb{Z}; \leq\rangle$ with structures $\mathcal{N}$ having transitive theories, and given I-groupoids $\mathfrak{P}$. Obviously, every such composition $\mathcal{M}[\mathcal{N}]$ is E-definable, where an equivalence relation E connotes the 'belonging to a common antichain.' By Proposition 2.2, $\mathrm{Th}(\mathcal{M}_1[\mathcal{N}])$ is a transitive theory. Note that the algebra $\mathfrak{P}_{\mathbb{Z}}$ of binary isolating formulas of $\mathrm{Th}(\mathcal{M})$ is generated by a monoid $\mathfrak{P}'_{\mathbb{Z}} = \langle \mathbb{Z}; +\rangle$ under the assumption that all labels for $\mathbb{Z}$ are nonnegative. Thus $\mathfrak{P}'_{\mathbb{Z}}$ is used in constructing the algebra $\mathfrak{P}_{\mathbb{Z}}[\mathfrak{P}]$, which is an algebra of binary isolating formulas for $\mathrm{Th}(\mathcal{M}[\mathcal{N}])$. Hence the following holds:

THEOREM 3.4. For any I-groupoid $\mathfrak{P}$ consisting of nonnegative labels, there exists a theory T with a type $p \in S(T)$ and a regular labeling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)} = \mathfrak{P}_{\mathbb{Z}}[\mathfrak{P}]$.

Example 3.5. Let $\mathcal{N}$ be an infinite model of the theory of pure equality, and let $\mathcal{M} = \langle \mathbb{Z}, ; \leq \rangle$. Then the algebra $\mathfrak{P}_{\mathbb{Z}}$ for a unique 1-type $q \in S_1(\varnothing)$ in $\mathrm{Th}(\mathcal{M})$ has a countable number of labels: 0 ($x \approx y$), 1 ($S_1(x, y)$), 2 ($P_1(x, y)$), ..., $2n - 1$ ($S_n(x, y)$), $2n$ ($P_n(x, y)$), ..., $n \in \omega$, $n \geq 2$, where

$$S_1(x, y) := \text{“}y \text{ is an immediate successor of } x\text{,”}$$

$$P_1(x, y) := \text{“}y \text{ is an immediate predecessor of } x\text{,”}$$

$$S_n(x, y) := x < y \wedge \exists t_1, t_2, \ldots, t_{n-1} \Big[x < t_1 < t_2 < \ldots < t_{n-1} < y \\ \wedge S_1(x, t_1) \wedge \bigwedge_{i=1}^{n-2} S_1(t_i, t_{i+1}) \wedge S_1(t_{n-1}, y) \Big], \; n \geq 2,$$

$$P_n(x, y) := y < x \wedge \exists t_1, t_2, \ldots, t_{n-1} \Big[y < t_1 < t_2 < \ldots < t_{n-1} < x \\ \wedge P_1(y, t_1) \wedge \bigwedge_{i=1}^{n-2} P_1(t_i, t_{i+1}) \wedge P_1(t_{n-1}, x) \Big], \; n \geq 2.$$

If every element of $\mathcal{M}$ is replaced by a copy of the structure $\mathcal{N}$, then the algebra $\mathfrak{P}_{\nu(q')}$ for a unique 1-type $q' \in S_1(\varnothing)$ in their composition $\mathcal{M}[\mathcal{N}]$ will also have countably many labels: all labels in $\mathfrak{P}_{\mathbb{Z}}$, as well as one extra label for a formula $x \diamond y$.

4. I-GROUPOIDS FOR CYCLIC PREORDERS

We will deal with some types of cyclic preorders $\mathcal{C}_\lambda$, which are obtained from cyclic orders $\mathcal{C}$ [24-28] by replacing elements on an antichain of fixed cardinality λ.

Definition 4.1. A *cyclic order* is a ternary relation K satisfying the following conditions:
(co1) $\forall x \forall y \forall z (K(x, y, z) \to K(y, z, x))$;
(co2) $\forall x \forall y \forall z (K(x, y, z) \wedge K(y, x, z) \Leftrightarrow x = y \vee y = z \vee z = x)$;
(co3) $\forall x \forall y \forall z (K(x, y, z) \to \forall t [K(x, y, t) \vee K(t, y, z)])$;
(co4) $\forall x \forall y \forall z (K(x, y, z) \vee K(y, x, z))$.

If $\mathcal{M} := \langle M, <, \ldots \rangle$ is a linearly ordered structure, and K is a ternary relation defined by a relation $\leq$ via the rule

$$K(x, y, z) :\Leftrightarrow (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x), \tag{1}$$

then K is cyclic order relation on M, i.e., $\langle M; K \rangle$ is a cyclically ordered structure. Note that there exist cyclically ordered structures which are not linearly ordered.

Let $A \subseteq M$, where $\mathcal{M}$ is a cyclically ordered structure. A set A is said to be *convex* if, for any $a, b \in A$, the following property holds: for every $c \in M$ with condition $K(a, c, b)$, it is true that $c \in A$, or for every $c \in M$ with $K(b, c, a)$, we have $c \in A$. A structure $\mathcal{M}$ is said to be *weakly cyclically minimal* if every definable (with parameters) subset M is a finite union of convex sets.

Let $\mathcal{M}$ be a countably categorical weakly cyclically minimal structure, and let $G := \mathrm{Aut}(\mathcal{M})$. Following standard group-theoretic terminology, we say that a group G is *k-homogeneous*, where $k \in \omega$, if for every two k-element sets $A, B \subseteq M$ there exists $g \in G$ for which $g(A) = B$. A group G is *strongly homogeneous* if G is k-homogeneous for all $k \in \omega$. A group G is *k-transitive* if for any pairwise distinct $a_1, a_2, \ldots, a_k \in M$ and any pairwise distinct $b_1, b_2, \ldots, b_k \in M$ there is $g \in G$ for which $g(a_1) = b_1, g(a_2) = b_2, \ldots, g(a_k) = b_k$. A *congruence* on $\mathcal{M}$ is any G-invariant equivalence relation on $\mathcal{M}$. A group G is said to be *primitive* if G is 1-transitive and there do not exist nontrivial proper congruences on $\mathcal{M}$.

If $\mathcal{M}$ is a countably categorical weakly cyclically minimal structure with a primitive automorphism group, then only the following options are available [24]:

(1) $\mathrm{Aut}(\mathcal{M})$ is 2-transitive;

(2) $\mathrm{Aut}(\mathcal{M})$ is 2-homogeneous but not 2-transitive;

(3) $\mathrm{Aut}(\mathcal{M})$ is primitive but not 2-homogeneous.

Consider a structure $\mathcal{M} := \langle \mathbb{Q}, < \rangle$. Obviously, $\mathcal{M}$ is a countably categorical o-minimal structure. If we replace the linear order relation $<$ with the ternary relation K defined by the relation $\leq$ via rule (1) we obtain a structure $\mathbb{Q}_1^* := \langle \mathbb{Q}, K \rangle$ which is countably categorical weakly cyclically minimal and has a 2-transitive group of automorphisms. The corresponding algebra $\mathfrak{P}_{\mathbb{Q}_1}$ of binary isolating formulas has two labels: 0 for a formula $x \approx y$, and 1 for a formula $\neg(x \approx y)$, in which case $0 \cdot 0 = \{0\}$, $1 \cdot 0 = 0 \cdot 1 = \{1\}$, and $1 \cdot 1 = \{0, 1\}$.

Example 4.2 [29, 30]. Let n be a positive integer with $n \geq 2$, let $L = \{\sigma_0, \ldots, \sigma_{n-1}\}$, where $\sigma_0, \ldots, \sigma_{n-1}$ are binary predicate symbols, and let $\mathbb{Q}_n^*$ be the structure $\langle Q_n, K, L \rangle$ for which the following conditions are satisfied:

(i) the universe Q_n is a countable densely ordered subset of the unit circle, and no two points form a central angle $2\pi k/n$, where k runs over integers;

(ii) for different $x, y \in Q_n$, $(x, y) \in \sigma_i \Leftrightarrow 2\pi i/n < \arg(x/y) < 2\pi(i+1)/n$, where $\arg(x/y)$ stands for the magnitude of a central angle between x and y in the clockwise direction.

In [24], it was sown that $\mathbb{Q}_n^*$ is a countably categorical weakly cyclically minimal structure with a primitive automorphism group $\mathrm{Aut}(\mathbb{Q}_n^*)$.

The structure $\mathbb{Q}_2^*$ is properly a *countable homogeneous local order*, or a *cyclic tournament* (see [30, 31]).

Let $\mathcal{M}$ and $\mathcal{N}$ be cyclically ordered structures. A 2-*reduct* of the structure $\mathcal{M}$ is a cyclically ordered structure having the universe of $\mathcal{M}$ and containing a predicate symbol for every $\varnothing$-definable relation on $\mathcal{M}$ of arity at most 2 as well as a ternary predicate symbol K for a cyclic order, but not containing other predicate symbols of higher arity. We say that $\mathcal{M}$ *isomorphic to* $\mathcal{N}$ *up to binarity* if a 2-reduct of the structure $\mathcal{M}$ is isomorphic to the structure $\mathcal{N}$.

By virtue of [24, Thm. 7.14], the structure $\mathbb{Q}_2^*$ describes (up to binarity) countably categorical weakly cyclically minimal structures with a 2-homogeneous but not 2-transitive group of automorphisms, provided that these structures are not linearly ordered. Notice that the corresponding algebra $\mathfrak{P}_{\mathbb{Q}_2}$ of binary isolating formulas has three labels: 0 for $x \approx y$, 1 for $\sigma_0(x, y)$, and 2 for $\sigma_1(x, y)$, with the following rules: $0 \cdot 0 = \{0\}$, $1 \cdot 0 = 0 \cdot 1 = \{1\}$, $2 \cdot 0 = 0 \cdot 2 = \{2\}$, $1 \cdot 1 = \{1, 2\}$, $2 \cdot 2 = \{1, 2\}$, and $1 \cdot 2 = 2 \cdot 1 = \{0, 1, 2\}$.

In view of [24, Thm. 7.15], the structures $\mathbb{Q}_n^*$, $n \geq 3$, describe (up to binarity) $\aleph_0$-categorical weakly cyclically minimal structures with a primitive but not 2-homogeneous automorphism group. Notice that the corresponding algebra $\mathfrak{P}_{\mathbb{Q}_n}$ of binary isolating formulas has $n+1$ labels: 0 for $x \approx y$, 1 for $\sigma_0(x, y)$, 2 for $\sigma_1(x, y)$, $\ldots$, and n for $\sigma_{n-1}(x, y)$.

Thus the following holds:

THEOREM 4.3. For any natural $n \geq 1$, there is a countably categorical weakly cyclically minimal structure $\mathbb{Q}_n$ with a primitive automorphism group, and the corresponding algebra $\mathfrak{P}_{\mathbb{Q}_n}$ of binary isolating formulas has exactly $n + 1$ labels.

THEOREM 4.4. The algebra $\mathfrak{P}_{\mathbb{Q}_n}$ of binary isolating formulas possesses the following multiplication rules:

(1) for any label k with $0 \leq k \leq n$, we have $0 \cdot k = k \cdot 0 = \{k\}$;

(2) for any labels k_1 and k_2 with $1 \leq k_1, k_2 \leq n$, the following conditions hold:

(2a) if $k_1 + k_2 \leq n$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{k_1 + k_2 - 1, k_1 + k_2\}$;

(2b) if $k_1 + k_2 - n = 1$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{0, 1, n\}$;

(2c) if $k_1 + k_2 - n = m$ for some $m \geq 2$, then $k_1 \cdot k_2 = k_2 \cdot k_1 = \{m - 1, m\}$.

Proof. (1) Since $\exists t[x = t \wedge t = y] \equiv x = y$ and $\exists t[x = t \wedge \sigma_i(t, y)] \equiv \sigma_i(x, y)$ for $0 \leq i \leq n-1$, it follows that $0 \cdot k = k \cdot 0 = \{k\}$ for any k with $0 \leq k \leq n$.

(2) Take arbitrary labels k_1 and k_2 with $1 \leq k_1, k_2 \leq n$. Then k_1 is a label for $\sigma_{k_1-1}(x, y)$ and k_2 is one for $\sigma_{k_2-1}(x, y)$. We have

$$\sigma_{k_1-1}(x, t) \Leftrightarrow \frac{2\pi(k_1 - 1)}{n} < \arg(x/t) < \frac{2\pi k_1}{n},$$
$$\sigma_{k_2-1}(t, y) \Leftrightarrow \frac{2\pi(k_2 - 1)}{n} < \arg(t/y) < \frac{2\pi k_2}{n}.$$

Consequently,

$$\frac{2\pi(k_1 + k_2 - 2)}{n} < \arg(x/y) < \frac{2\pi(k_1 + k_2)}{n}.$$

Case (2a). Let $k_1 + k_2 \leq n$. Then one of the following relations holds:

$$\frac{2\pi(k_1 + k_2 - 2)}{n} < \arg(x/y) < \frac{2\pi(k_1 + k_2 - 1)}{n},$$
$$\frac{2\pi(k_1 + k_2 - 1)}{n} < \arg(x/y) < \frac{2\pi(k_1 + k_2)}{n},$$

whence $k_1 \cdot k_2 = k_2 \cdot k_1 = \{k_1 + k_2 - 1, k_1 + k_2\}$.

Case (2b) Let $k_1 + k_2 - n = 1$. Then

$$\frac{2\pi(k_1 + k_2 - 2)}{n} = \frac{2\pi(k_1 + k_2 - n - 2) + 2\pi n}{n} = 2\pi - \frac{2\pi}{n},$$

$$\frac{2\pi(k_1+k_2)}{n}=\frac{2\pi(k_1+k_2-n)+2\pi n}{n}=2\pi+\frac{2\pi}{n}.$$

If $\arg(x/t)=2\pi-\frac{2\pi}{n}$ and $\arg(t/y)=2\pi+\frac{2\pi}{n}$, then $\arg(x/y)=4\pi$, i.e., x may be equal to y. Hence $k_1\cdot k_2=k_2\cdot k_1=\{0,1,n\}$.

Case (2c) Let $k_1+k_2-n=m$ for some $m\geq 2$. In virtue of $k_1+k_2\leq 2n$, it is true that $m\leq n$. Then

$$\frac{2\pi(k_1+k_2-2)}{n}=\frac{2\pi(k_1+k_2-n-2)+2\pi n}{n}=2\pi+\frac{2\pi(m-2)}{n},$$
$$\frac{2\pi(k_1+k_2)}{n}=\frac{2\pi(k_1+k_2-n)+2\pi n}{n}=2\pi+\frac{2\pi m}{n}.$$

Thus $k_1\cdot k_2=k_2\cdot k_1=\{m-1,m\}$. $\Box$

Repeating the argument used for dense linear orders and applying it to an algebra $\mathfrak{P}_{\mathrm{dco}}$ of binary isolating formulas on dense cyclic preorders, we validate the following:

THEOREM 4.5. For any I-groupoid $\mathfrak{P}$ consisting of nonnegative labels, there exists a theory T with a type $p\in S(T)$ and a regular labeling function $\nu(p)$ for which $\mathfrak{P}_{\nu(p)}=\mathfrak{P}_{\mathrm{dco}}[\mathfrak{P}]$.

Now we consider an infinite discrete cyclic order $\mathcal{C}$ which is obtained from a discrete order on $\mathbb{Z}$. If we replace elements of $\mathcal{C}$ by copies of a structure $\mathcal{N}$ with a transitive automorphism group we obtain an E-definable composition $\mathcal{C}[\mathcal{N}]$ with a transitive theory $T=\mathrm{Th}(\mathcal{C}[\mathcal{N}])$. If $\mathrm{Th}(\mathcal{N})$ has an algebra $\mathfrak{P}$ of binary isolating formulas, then T has a unique type $p\in S(T)$ and a regular labeling function $\nu(p)$ for which $\mathfrak{P}_{\nu(p)}=\mathfrak{P}_{\mathbb{Z}}[\mathfrak{P}]$. Thus compositions of discrete cyclic orders $\mathcal{C}$ with structures $\mathcal{N}$ also confirm Theorem 3.4.

At the moment, we consider a natural finite cyclic order $\mathcal{C}$ on $\mathbb{Z}_n$, $n\geq 2$, and replace elements of $\mathcal{C}$ by copies of a structure $\mathcal{N}$ with a transitive automorpism group. As a result, we obtain an E-definable composition $\mathcal{C}[\mathcal{N}]$ with a transitive theory $T=\mathrm{Th}(\mathcal{C}[\mathcal{N}])$. An algebra $\mathfrak{P}_{\mathbb{Z}_n}$ of binary isolating formulas for $\mathrm{Th}(\mathcal{C})$ is generated by a group $\mathbb{Z}_n$. The group $\mathbb{Z}_n$ and the algebra $\mathfrak{P}$ of binary isolating formulas for $\mathrm{Th}(\mathcal{N})$ generate the composition $\mathfrak{P}_{\mathbb{Z}_n}[\mathfrak{P}]$, which is an algebra of binary isolating formulas for $\mathrm{Th}(\mathcal{C}[\mathcal{N}])$. Thus the following holds:

THEOREM 4.6. For any I-groupoid $\mathfrak{P}$ consisting of nonnegative labels and for an arbitrary natural $n\geq 2$, there exists a theory T with a type $p\in S(T)$ and a regular labeling function $\nu(p)$ for which $\mathfrak{P}_{\nu(p)}=\mathfrak{P}_{\mathbb{Z}_n}[\mathfrak{P}]$.

5. EXEMPLIFYING COMPOSITIONS OF FINITE STRUCTURES AND COMPOSITIONS OF FINITE ALGEBRAS OF BINARY FORMULAS

In this section, we illustrate with examples compositions of finite structures and compositions of finite algebras of binary formulas. The compositions of finite structures are finite, so their algebras of binary formulas are also finite.

Example 5.1. Let $\mathcal{M}_0$ be a two-element graph consisting of a single edge. The theory $T_0 = \mathrm{Th}(\mathcal{M}_0)$ is transitive, and for a unique 1-type $p_0 \in S(T)$, as well as for its algebra $\mathfrak{L} = \mathfrak{P}_{\nu(p_0)}$, we have a unique nonzero label, say, 1, satisfying the following table:

$\cdot$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0\}$

A composition $\mathcal{M}_0[\mathcal{M}_0]$ represents a 4-element complete graph K_4, is not E-definable, but, nevertheless, defines a transitive theory T_1 with a unique 1-type $p_1 \in S(T)$ and an algebra $\mathfrak{LL} = \mathfrak{P}_{\nu(p_1)}$ having the following table:

$\cdot$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0,1\}$

More generally, for any m-complete graph K_m and for an n-element complete graph K_n of the same signature, where $m, n \geq 2$, $K_m[K_n] \simeq K_{mn}$ and the theory $\mathrm{Th}(K_m[K_n])$ has an algebra $\mathfrak{LL}$ of binary isolating formulas.

Example 5.2. We modify Example 5.1, by treating the graphs K_m and K_n so that edges $e_1 \in K_m$ and $e_2 \in K_n$ have different colors. For an algebra $\mathfrak{P}$ of binary isolating formulas for $\mathrm{Th}(K_m[K_n])$, we have several options.

(1) If $m = n = 2$, then the algebra $\mathfrak{P} = \mathfrak{L}[\mathfrak{L}']$, where $\mathfrak{L}' \simeq \mathfrak{L}$, has the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

(2) If $m = 2$ and $n > 2$, then $\mathfrak{P} = \mathfrak{L}[\mathfrak{LL}]$ has the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

(3) If $m > 2$ and $n = 2$, then $\mathfrak{P} = \mathfrak{LL}[\mathfrak{L}]$ has the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$

(4) If $m > 2$ and $n > 2$, then $\mathfrak{P} = \mathfrak{LL}[\mathfrak{LL}']$, where $\mathfrak{LL}' \simeq \mathfrak{LL}$, has the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$

Example 5.3. Let C_m and C_n be undirected graphs forming cycles of length $m \geq 2$ and $n \geq 2$, respectively. Every theory $\mathrm{Th}(C_m)$ has diameter $d_m = \left[\frac{m}{2}\right]$ and contains an algebra $\mathfrak{P}_{d_m}$ of binary isolating formulas with labels $0, 1, \ldots, d_m$, and also with the following rules for labels u and v:

$u \cdot v = |u \pm v| (\mathrm{mod}\, d_m)$ if either m is even, or m is odd and $u + v \leq m$,

$u \cdot v = \{(u + v - 1)(\mathrm{mod}\, d_m), |u - v|(\mathrm{mod}\, d_m)\}$ if m is odd and $u + v > m$.

Having considered the graph $C_m[C_n]$, we obtain a non-E-definable combination of diameter d_m, with every copy C_n being of diameter $\min\{d_n, 2\}$. For the theory $\mathrm{Th}(C_m[C_n])$, its algebra $\mathfrak{P}_{d_m,d_n}$ of binary isolating formulas has labels $0, 1, \ldots, d_m$ and the following rule: for any labels u and v, $u \cdot v$ consists of respective values for $\mathrm{Th}(C_m)$, and includes a label u, if $v = 1$ or $v = 2$, and a label v if $u = 1$ or $u = 2$.

If C_m and C_n consist of edges of different colors, then $C_m[C_n]$ is an E-definable combination, while algebras $\mathfrak{P}_1$ and $\mathfrak{P}_2$ for theories $T_1 = \mathrm{Th}(C_m)$ and $T_2 = \mathrm{Th}(C_n)$, respectively, form an algebra $\mathfrak{P}_1[\mathfrak{P}_2]$ for $T_1[T_2]$, in accordance with Theorem 2.12.

REFERENCES

1. S. V. Sudoplatov, *Classification of Countable Models of Complete Theories*, 2nd ed., Novosibirsk State Tech. Univ., Novosibirsk (2018).

2. I. V. Shulepov and S. V. Sudoplatov, "Algebras of distributions for isolating formulas of a complete theory," *Sib. El. Mat. Izv.*, **11**, 380-407 (2014); http://semr.math.nsc.ru/v11/p380-407.pdf.

3. S. V. Sudoplatov, "Algebras of distributions for semi-isolating formulas of a complete theory," *Sib. El. Mat. Izv.*, **11**, 408-433 (2014); `http://semr.math.nsc.ru/v11/p408-433.pdf`.

4. S. V. Sudoplatov, "Algebras of distributions for binary semi-isolating formulas for families of isolated types and for countably categorical theories," *Int. Math. Forum*, **9**, No. 21, 1029-1033 (2014).

5. S. V. Sudoplatov, "Forcing of infinity and algebras of distributions of binary semi-isolating formulas for strongly minimal theories," *Math. Stat.*, **2**, No. 5, 183-187 (2014).

6. B. Sh. Kulpeshov and S. V. Sudoplatov, "Algebras of distributions of binary formulas for quite *o*-minimal theories," *News Nat. Acad. Sci. Rep. Kazakhstan, Phys.-Math. Ser.*, **2**, No. 300, 5-13 (2015).

7. D. Yu. Emel'yanov, "Algebras of distributions for binary formulas of theories of unars," *Izv. Irkutsk Gos. Univ., Mat.*, **17**, 23-36 (2016).

8. D. Yu. Emel'yanov, B. Sh. Kulpeshov, and S. V. Sudoplatov, "Algebras of distributions for binary formulas in countably categorical weakly o-minimal structures," *Algebra and Logic*, **56**, No. 1, 13-36 (2017).
9. D. Yu. Emel'yanov and S. V. Sudoplatov, "On deterministic and absorbing algebras of binary formulas of polygonometrical theories," *Izv. Irkutsk Gos. Univ., Mat.*, **20**, 32-44 (2017).
10. K. A. Baikalova, D. Yu. Emel'yanov, B. Sh. Kulpeshov, E. A. Palyutin, and S. V. Sudoplatov, "On algebras of distributions of binary isolating formulas for theories of Abelian groups and their ordered enrichments," *Izv. Vyssh. Uch. Zav., Mat.*, No. 4, 3-15 (2018).
11. D. Yu. Emel'yanov, B. Sh. Kulpeshov, and S. V. Sudoplatov, "Algebras of distributions of binary isolating formulas for quite o-minimal theories," *Algebra and Logic*, **57**, No. 6, 429-444 (2018).
12. F. Harary, *Graph Theory, Addison-Wesley Ser. Math.*, Reading, Addison-Wesley, Mass. (1969).
13. S. V. Sudoplatov and E. V. Ovchinnikova, *Discrete Mathematics* [in Russian], 5th ed., Yurait, Moscow (2020).
14. V. A. Artamonov, V. N. Salii, L. A. Skornyakov, et al., *General Algebra* [in Russian], Vol. 2, Nauka, Moscow (1991).
15. E. S. Lyapin, *Semigroups*, Gos. Izd. Fiz.-Mat. Lit., Moscow (1960).
16. S. V. Sudoplatov, "Transitive arrangements of algebraic systems," *Sib. Math. J.*, **40**, No. 6, 1142-1145 (1999).
17. S. V. Sudoplatov, "Combinations of structures," *Izv. Irkutsk Gos. Univ., Mat.*, **24**, 82-101 (2018).
18. Yu. Saffe, E. A. Palyutin, and S. S. Starchenko, "Models of superstable Horn theories," *Algebra and Logic*, **24**, No. 3, 171-210 (1985).
19. J. Reineke, "Minimale Gruppen," *Z. Math. Log. Grund. Math.*, **21**, 357-359 (1975).
20. Y. R. Baisalov, K. A. Meirembekov, and A. T. Nurtazin, "Definably minimal models," in *Model Theory and Algebra, France-Kazakhstan Conf. on Model Theory and Algebra*, Astana (2005), pp. 8-11.
21. J. T. Baldwin and A. T. Lachlan, "On strongly minimal sets," *J. Symb. Log.*, **36**, No. 1, 79-96 (1971).
22. S. Shelah, *Classification Theory and the Number of Non-Isomorphic Models*, 2nd edn., *Stud. Log. Found. Math.*, **92**, North-Holland, Amsterdam (1990).
23. V. Harnik and L. Harrington, "Fundamentals of forking," *Ann. Pure Appl. Log.*, **26**, No. 3, 245-286 (1984).

24. B. Sh. Kulpeshov and H. D. Macpherson, "Minimality conditions on circularly ordered structures," *Math. Log. Q.*, **51**, No. 4, 377-399 (2005).

25. B. Sh. Kulpeshov, "On $\aleph_0$-categorical weakly circularly minimal structures," *Math. Log. Q.*, **52**, No. 6, 555-574 (2006).

26. B. Sh. Kulpeshov, "Definable functions in the $\aleph_0$-categorical weakly circularly minimal structures," *Sib. Math. J.*, **50**, No. 2, 282-301 (2009).

27. B. Sh. Kulpeshov, "On indiscernibility of a set in circularly ordered structures," *Sib. El. Mat. Izv.*, **12**, 255-266 (2015); `http://semr.math.nsc.ru/v12/p255-266.pdf`.

28. B. Sh. Kulpeshov, "On almost binarity in weakly circularly minimal structures," *Euras. Math. J.*, **7**, No. 2, 38-49 (2016).

29. M. Droste, M. Giraudet, H. D. Macpherson, and N. Sauer, "Set-homogeneous graphs," *J. Comb. Theory, Ser. B*, **62**, No. 1, 63-95 (1994).

30. P. J. Cameron, "Orbits of permutation groups on unordered sets. II," *J. London Math. Soc., II. Ser.*, **23**, No. 2, 249-264 (1981).

31. A. H. Lachlan, "Countable homogeneous tournaments," *Trans. Am. Math. Soc.*, **284**, 431-461 (1984).